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Constraints on profit income distribution and production efficiency in private ownership economies with Ramsey taxation[*]

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Abstract

In economies with Ramsey taxation, decreasing returns to scale, and private ownership, we show that second-best production efficiency is desirable when profit tax rates vary across groups of firms provided that the institutional rules which define profit incomes of consumers depend on the distribution of profits across these groups of firms. The classic results of Dasgupta and Stiglitz [1972] (of firm-specific profit taxation) and Diamond and Mirrlees [1971] and Guesnerie [1995] (of uniform one-hundred percent profit taxation) follow as special cases of our model. Moreover, second-best analysis suggests the desirability of proportionate taxation of inter-firm transactions in the absence of profit taxes. Alternatively, it recommends profit taxation as a perfect substitute for intermediate-input taxation. The analysis also suggests that, combined with the knowledge of the distribution of profit incomes in the economy, profit taxation can promote both efficiency and redistributive objectives of the government.

Journal of economic literature classification number: H21

Keywords: Ramsey taxation, private ownership, profit taxation, production inefficiency, general equilibrium.

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1. Introduction.

 Diamond and Mirrlees (DM) [1971] revisited the problem first posed by Ramsey [1927] about alternative policy instruments that can be employed when there are informational constraints on the implementation of the second-welfare theorem. Using a general equilibrium model they showed that, when personalized lump-sum transfers are not available to the government as redistributive devices, commodity (Ramsey) taxation can be employed as an alternative, albeit second-best, means of redistribution. They showed that production efficiency was desired by the second-best optimal commodity tax system and that taxation of inter-firm transactions was not required.

The general equilibrium model they employed was one where technologies of firms exhibited constant returns to scale. Thus, consumers received no profit incomes. An extension of this model to decreasing returns to scale technologies (see *e.g.,* Guesnerie [1995] and Weymark [1979]) led to similar results regarding production efficiency when private firms were subject to one-hundred percent profit taxation with the proceeds going back to consumers as a uniform lump-sum transfer (also called a demogrant).[1]

To check the robustness of the result on second-best production efficiency under more general settings, the DM model was extended to allow consumers to receive profit incomes in proportion to the shares that they owned in the private firms.[2] A series of papers pioneered by Dasgupta and Stiglitz (DS) [1972] and followed up by Mirrlees [1972], Hahn [1973], Sadka [1977], and Munk [1980] concluded that, in such models of privately owned firms, second-best production efficiency continues to be desirable if the government can implement firm-specific profit taxes. Challenges in the general proof of this result were brought to light by these papers and, more recently, by Reinhorn [2010].

In general, the ability of the government to implement a firm-specific system of profit taxation is questionable. More realistic scenarios, one could argue, may be those where profit tax rates vary only across groups of private firms, *i.e.,* the number of profit tax rates the government can peg may be smaller than the number of private firms. Further, the rigid institutional rules by which profit incomes are distributed among consumers may be more general than the ones considered in the literature cited above. This paper extends the above results in these directions.

In particular, we show that second-best production efficiency remains desirable when profit tax rates can vary across groups of firms provided that the institutional rules which

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[1] For an excellent exposition of these results see Myles [1995].
[2] As in a Arrow-Debreu private-ownership economy.

define profit incomes of consumers depend on the distribution of profits across these groups of firms. We show that all the previous results on second-best production efficiency follow as special cases of our result. On one extreme is the DS case, which involves the finest partition of the set of private firms–each firm-group in this partition comprises of a single firm and hence the case of firm-specific profit taxation. On the other extreme is the DM and Guesnerie [1995] case, which involves the coarsest partition of the set of private firms–only one firm-group that contains all the firms and government implementing a single (uniform) rate of profit taxation. Then there are cases that lie in between these two extremes.

Our strategy of proof is different from the earlier papers. Production inefficiency at a tax equilibrium implies that the aggregate supply of the firms is not on the frontier of the aggregate technology of the economy. With profit maximizing private and public-sector firms, this implies that the producer price vectors faced by private and public firms in such a tax equilibrium are not proportional, implying that there are differences in the marginal rates of technical substitution across these two types of firms. Hence, an increase in aggregate supply is technically feasible by reallocating production across these firms. In fact, we show that, at such a tax equilibrium, there are changes in the price vectors faced by private and public firms which will ensure that an increase in aggregate supply is consistent with profit maximizing choices of firms at these new prices. Ceteris paribus, such an increase in aggregate supply implies an increase in the aggregate income in the economy. The question is can this increased aggregate income be distributed to consumers in a manner that respects the existing institutional rules of income distribution in the economy and improves the welfare of all consumers? In general, the institutional rules of income distribution and the restrictions on the ability of the government to implement profit taxation can constrain severely such welfare improvements. However, we show that our institutional set-up that we described above permits the distribution of the increased aggregate income across consumers in a welfare improving manner.

In our proof, we take recourse to an intermediate construct of an economy with producer price vectors varying across (groups of) private-sector firms.[3] Hence, implicitly, this implies a wedge between price vectors faced by firms and, hence, the taxation of transactions between firms are allowed. We show that this economy exhibits second-best production efficiency, albeit, unlike in the DM model, this means that firms are subject to proportionate rates of intermediate-input taxation. It is shown that the second-best allocations of this economy are the same as those of an economy where all private firms face the same price vector but are subject to profit taxes. This implies that second-best equilibrium allocations with intermediate-input taxation can also be decentralized as

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[3] This is as in the earlier literature.

second-best tax equilibrium allocations with no intermediate-input taxation but with firms being subject to profit taxation.

The results on second-best production efficiency and intermediate-input taxation are important for three reasons: Second-best production efficiency supports the use of producer prices as the right proxies for the generally unobservable shadow prices for the cost-benefit evaluation of public sector projects.[4] The recommendation of either proportionate intermediate-goods taxation or profit taxation with no intermediate-goods taxation minimizes also the practical administrative costs of implementing a system of Ramsey taxes, *e.g.,* it supports tax structures like VAT. Our analysis also suggests how, by understanding the rules of profit income distribution in the economy, the government can potentially design a system of profit taxation that can further both its redistributive and efficiency objectives.

In Section 2, we lay out our general equilibrium model. We define two types of profit-making economies–those with firm-group specific profit taxes and those with firm group specific prices. In Section 3 we prove a preliminary lemma. In Section 4, we employ this lemma to prove Theorem 1 that states that second-best production efficiency is desirable in profit-making economies with firm-group specific prices. In Section 5, we obtain as corollaries of Theorem 1 two results: one, the desirability of second-best production efficiency in profit-making economies with firm-group specific profit taxes and two, the case for either proportionate intermediate-input taxation in profit-making economies or profit taxation with no intermediate goods taxation. In Section 6, we conclude.

2. The model.

There are N commodities indexed by k, H consumers indexed by h, and $I+1$ firms indexed by i. We denote the index set of consumers as $\mathcal{H} = \{1, \ldots, H\}$ and the index set of firms as $\mathcal{I} = \{0, \ldots, I\}$. We assume that firm 0 is a public sector firm, while all others are private firms.

For every firm $i \in \mathcal{I}$, the technology is denoted by $Y^i \subset \mathbf{R}^N$. The aggregate technology is $Y = \sum_i Y^i$.

For all $h \in \mathcal{H}$, the *gross* consumption set of consumer h is $X^h \subseteq \mathbf{R}^N_+$ and the preferences over X^h are represented by continuous, quasi-concave, and locally nonsatiated utility functions $u^h : X^h \to \mathbf{R}$ with images $u^h(x^h)$. The endowment vector of consumer h is denoted by $e^h \in \mathbf{R}^N_+$.

Let $E = ((X^h, u^h, e^h)_{h \in \mathcal{H}}, (Y^i)_{i \in \mathcal{I}})$ denote the vector of economic fundamentals specified above.

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[4] See Little and Mirrlees [1974], Boadway [1975], and Dréze and Stern [1987].

We assume that all firms are price-takers. For all $i \in \mathcal{I}$ define the set of prices for which profit maximization is well defined as

$$B^i := \{p \in \mathbf{R}_+^N \setminus \{0^N\} |\ p \cdot y \text{ is bounded from above for all } y \in Y^i\} \qquad (2.1)$$

and define the profit function $\pi^i : B^i \to \mathbf{R}$ with image

$$\pi^i(p) = \sup_{y^i} \{p \cdot y^i |\ y^i \in Y^i\}. \qquad (2.2)$$

The supply vectors of firm $i \in \mathcal{I}$ are obtained from the solution mapping of (2.2) as $y^i : B^i \to \mathbf{R}^N$ with image $y^i(p)$.[5] For all $i \in \mathcal{I}$ define the frontier of Y^i as

$$\hat{Y}^i = \{y^i \in Y^i |\ \bar{y}^i \geq y^i \text{ implies that } \bar{y}^i \notin Y^i\}. \qquad (2.3)$$

Similarly, we can define $\hat{Y}$ as the frontier of the aggregate technology Y.

The vector of consumer prices is denoted by $q \in \mathbf{R}_+^N$. Income of every consumer $h \in \mathcal{H}$ is denoted by $m^h \in \mathbf{R}_+$. Let the mapping $x^h : \mathbf{R}_+^N \times \mathbf{R}_+ \to \mathbf{R}^N$ with image $x^h(q, m^h)$ denote the Marshallian demand vector of consumer h and let the mapping $V^h : \mathbf{R}_+^N \times \mathbf{R}_+ \to \mathbf{R}$ with image $V^h(q, m^h)$ be the corresponding indirect utility function. Every consumer receives a demogrant, which is denoted by $m \in \mathbf{R}$. The income of consumer $h \in \mathcal{H}$ in economy E comprises of his endowment income, profit income that he receives from the private firms, and the demogrant that he receives from the government. A partition of $\mathcal{I} \setminus \{0\}$ (the index set of all private sector firms) is denoted by $\mathcal{P} = \{P_1, \ldots, P_T\}$. Members of a partition will be indexed by t or l.

2.1. A profit-making economy with firm-group specific profit taxes.

A profit-making economy with firm-group specific profit taxes is an economy derived from E where the private firms are partitioned into various groups, the government can implement profit taxation with the profit tax rates varying across firms depending on the groups to which they belong, and the consumers receive profit incomes which depend on the distribution of firm-group profits. If $\mathcal{P} = \{P_1, \ldots, P_T\}$ is such a partition, then we denote the vector of firm-group specific profit tax rates by $\tau = \langle \tau^1, \ldots, \tau^T \rangle \in \mathbf{R}^T$, *i.e.*, all firms in each firm-group $t = 1, \ldots, T$ are subject to the same profit tax rate τ^t so that the net of tax profit of firm-group t is $(1 - \tau^t) \sum_{i \in P_t} \pi^i(p)$, which we denote in the definition below by the function a^t. The profit incomes that consumers receive in such economies assume the form below. The form is very general requiring only that profit incomes of all consumers should add up to the total net of tax profits in the economy (condition (i) in the definition below) and that the profit income received by each consumer should be

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[5] In general, $y^i(p)$ need not be a singleton set, *i.e.*, the solution mapping $y^i(p)$ need not be a function.

non-negative (condition (ii) in the definition below). Examples 1 to 3 below consider a well-known functional form and some of its special cases that satisfy these conditions.

Definition. A *map of profit incomes with firm-group specific profit taxes* that is associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ is a continuous vector-valued map

$$r_{\mathcal{P},\tau} : \mathbf{R}^T \longrightarrow \mathbf{R}^H \tag{2.4}$$

with image

$$\begin{bmatrix} r^1 = r^1_{\mathcal{P},\tau}(a^1, \ldots, a^T) \\ \vdots \\ r^H = r^H_{\mathcal{P},\tau}(a^1, \ldots, a^T) \end{bmatrix}, \tag{2.5}$$

where $a = \langle a^1, \ldots, a^T \rangle$ is obtained as the vector-valued mapping $a : \mathbf{R}^T \times \cap_{i \in \mathcal{I} \setminus \{0\}} B^i \longrightarrow \mathbf{R}^T$ with image

$$\begin{bmatrix} a^1(\tau^1, \ldots, \tau^T, p) = (1 - \tau^1) \sum_{i \in P_1} \pi^i(p) \\ \vdots \\ a^T(\tau^1, \ldots, \tau^T, p) = (1 - \tau^T) \sum_{i \in P_T} \pi^i(p) \end{bmatrix} \tag{2.6}$$

such that

(i) $\sum_{h \in \mathcal{H}} r^h_{\mathcal{P},\tau}\big((1-\tau^1) \sum_{i \in P_1} \pi^i(p), \ldots, (1-\tau^T) \sum_{i \in P_T} \pi^i(p) \big) = \sum_{P_t \in \mathcal{P}} (1-\tau^t) \sum_{i \in P_t} \pi^i(p)$ and

(ii) for all $h \in \mathcal{H}$, $r^h_{\mathcal{P},\tau}\big((1 - \tau^1) \sum_{i \in P_1} \pi^i(p), \ldots, (1 - \tau^T) \sum_{i \in P_T} \pi^i(p) \big) \geq 0$.

Definition. Let $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ denote a profit-making economy with firm-group specific profit taxes associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ and a map of profit incomes $r_{\mathcal{P},\tau}$. A *tax equilibrium* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ is a configuration $\langle q, p, p^0, \tau^1, \ldots, \tau^T, m \rangle \in \mathbf{R}^{3N}_+ \times \mathbf{R}^{T+1}$ such that[6]

$$\sum_{h \in \mathcal{H}} x^h(q, m^h) \leq \sum_{t=1}^{T} \sum_{i \in P_t} y^i((1 - \tau^t)p) + y^0(p^0) + \sum_{h \in \mathcal{H}} e^h \text{ and}$$

$$m^h = r^h_{\mathcal{P},\tau}((1 - \tau^1) \sum_{i \in P_1} \pi^i(p), \ldots, (1 - \tau^T) \sum_{i \in P_T} \pi^i(p)) + m + qe^h, \ \forall \ h \in \mathcal{H}. \tag{2.8}$$

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[6] Vector notation: for x and y in $\mathbf{R}^n$,

$$x \geq y \Leftrightarrow x_i \geq y_i, \ \forall i = 1, \ldots, n,$$
$$x > y \Leftrightarrow x \neq y \text{ and } x_i \geq y_i, \ \forall i = 1, \ldots, n, \tag{2.7}$$
$$x \gg y \Leftrightarrow x_i > y_i, \ \forall i = 1, \ldots, n.$$

Also, recall that the income m^h of each consumer h comprises of his profit income, endowment income, and the demogrant received from the government.

Some examples.

Consider the standard case where the profits are distributed to consumers according to an exogeneous $H \times I$-dimensional matrix of shares Θ with typical element $\theta_i^h \geq 0$ which denotes the share of consumer h in the profit of the private firm i. Thus, we require $\sum_h \theta_i^h = 1$ for all $i \in \mathcal{I}$ and $h \in \mathcal{H}$. Let $\mathcal{O}$ denote the set of matrices Θ with these properties.

Example 1. Let $\Theta \in \mathcal{O}$. If $\mathcal{P} = \big\{\{1\}, \ldots, \{I\}\big\}$ (*i.e.*, $\mathcal{P}$ is the finest partition of the set of all private firms), then the map of profit incomes with firm-group specific profit taxes is given by

$$r^h = r^h_{\mathcal{P},p}((1-\tau^1)\pi^1(p), \ \ldots \ , (1-\tau^I)\pi^I(p)) = \theta_1^h(1-\tau^1)\pi^1(p)+\ldots+\theta_I^h(1-\tau^I)\pi^I(p), \ \forall \, h \in \mathcal{H}.$$
(2.9)

This is the DS case of firm-specific profit taxation.

Example 2. If $\Theta \in \mathcal{O}$ is such that $\theta_i^h = \theta^h$ for all $h \in \mathcal{H}$ and $i \in \mathcal{I} \setminus \{0\}$, then the coarsest partition $\mathcal{P}$ of $\mathcal{I} \setminus \{0\}$ that can be used to define a map of profit incomes with firm-group specific profit taxes is $\mathcal{P} = \{\mathcal{I} \setminus \{0\}\}$ and the map of profit incomes is given by

$$r^h = r^h_{\mathcal{P},p}((1 - \tau) \sum_{i \in \mathcal{I} \setminus \{0\}} \pi^i(p)) = \theta^h(1 - \tau) \sum_{i \in \mathcal{I}} \pi^i(p), \ \forall \, h \in \mathcal{H}. \tag{2.10}$$

A special case of this example is $\theta^h = \frac{1}{H}$ and $\tau = 0$. This is equivalent to the Guesnerie [1995] and Weymark [1979] case, where the profits of all firms are subject to a uniform (one-hundred percent) profit tax rate and the government returns its profit tax revenue as a uniform lump-sum transfer to all consumers.[7]

Example 3. $\mathcal{P} = \{P_1, \ldots, P_T\}$. Let $\Theta \in \mathcal{O}$ be such that $\theta_i^h = \theta^{ht}$ for all $h \in \mathcal{H}, t = 1, \ldots, T$, and $i \in P_t$. Then, the map of profit incomes with firm-group specific profit taxes is given by

$$r^h = r^h_{\mathcal{P},p}(\sum_{i \in P_1}(1 - \tau^1)\pi^i(p), \ \ldots \ , \sum_{i \in P_T}(1 - \tau^T)\pi^i(p))$$
$$= \theta^{h1}(1 - \tau^1) \sum_{i \in P_1} \pi^i(p) + \ldots + \theta^{hT}(1 - \tau^T) \sum_{i \in P_T} \pi^i(p), \ \forall \, h \in \mathcal{H}. \tag{2.11}$$

This covers the case where government can implement more than a single but not quite firm-specific profit tax rates, *i.e.*, $0 < T < I$, and where profit incomes of consumers depend on distribution of profits in the T groups of firms.

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[7] This is also trivially the DM case where constant returns to scale is assumed, so that profits of firms are zero.

Definition. Let $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ be a profit-making economy with firm-group specific profit taxes associated with a partition $\mathcal{P} = \langle P_1, \ldots, P_T \rangle$ of $\mathcal{I} \setminus \{0\}$ and a map of profit incomes $r_{\mathcal{P},\tau}$. A tax equilibrium $\langle q, p, p^0, \tau^1, \ldots, \tau^T, m \rangle$ of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ is a *production efficient tax equilibrium* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ if $\sum_{i=1}^I y^i(p) + y^0(p^0) \in \hat{Y}$.

The second-best problem for $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ is to find the mapping $\mathcal{V}_{\mathcal{P},r_{\mathcal{P},\tau}} : \Delta_{H-1} \to \mathbf{R}$ with image[8]

$$\mathcal{V}_{\mathcal{P},r_{\mathcal{P},\tau}}(s^1, \ldots, s^H) := \max_{q, \ p, \ p^0, \ \langle \tau^1, \ldots, \tau^T \rangle, \ m} \sum_h s_h V^h(q, m^h)$$

$$\text{subject to} \tag{2.12}$$

$$\langle q, p, p^0, \tau^1, \ldots, \tau^T, m \rangle \text{ satisfying } (2.8).$$

A solution to (2.12) is called a *second-best tax equilibrium* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.

Definition. A second-best tax equilibrium $\langle \overset{*}{q}, \overset{*}{p}, \overset{*}{p}^0, \overset{*}{\tau}^1, \ldots, \overset{*}{\tau}^T, \overset{*}{m} \rangle$ of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ is a *production efficient second-best* if it is production efficient. If all second-best tax equilibria of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ are production efficient, then $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ *exhibits second-best production efficiency* or *production efficiency is desirable at the second-best* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.

2.2. A profit-making economy with firm-group specific prices.

In this paper, we show that production efficiency is desirable at the second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$. To do so, we consider a more general institutional structure than a profit-making economy with firm-group specific profit taxes. This is an economy with profit incomes where the government can implement firm-group specific prices. We denote such an economy by $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$. The set of tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ turns out to be a subset of the set of tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$. Moreover, we show in Section 4 that $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ exhibits second-best production efficiency. In Section 5 we show that all the second-best tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ can be decentralized as tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$. The desirability of production efficiency at the second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ will hence follow.

Let $\mathcal{P} = \{P_1, \ldots, P_T\}$ be a partition of $\mathcal{I} \setminus \{0\}$. For all $t = 1, \ldots, T$, we denote $\cap_{i \in P_t} B^i$ by B_t and the price vector faced by firm-group P_t is denoted by $p^t \in \mathbf{R}_+^N$.

Definition. A *map of profit incomes with firm-group specific prices* that is associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ is a continuous vector-valued map

$$r_{\mathcal{P},p} : \mathbf{R}^T \longrightarrow \mathbf{R}^H \tag{2.13}$$

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[8] Δ_{H-1} is the $H-1$-dimensional unit simplex in $\mathbf{R}^H$. $\langle s^1, \ldots, s^H \rangle \in \Delta_{H-1}$ can be interpreted as a vector of welfare weights attached to consumer utilities. The second-best utility possibility frontier is obtained by solving the second-best optimization for all possible vectors of welfare weights.

with image

$$
\begin{bmatrix}
r^1 = r^1_{\mathcal{P},p}(a^1, \ldots, a^T) \\
\vdots \\
r^H = r^H_{\mathcal{P},p}(a^1, \ldots, a^T)
\end{bmatrix},
\tag{2.14}
$$

where $a = \langle a^1, \ldots, a^T \rangle$ is obtained as the vector-valued mapping $a : \prod_{t=1}^{T} B_t \longrightarrow \mathbf{R}^T$ with image

$$
\begin{bmatrix}
a^1(p^1, \ldots, p^T) = \sum_{i \in P_1} \pi^i(p^1) \\
\vdots \\
a^T(p^1, \ldots, p^T) = \sum_{i \in P_T} \pi^i(p^T)
\end{bmatrix}
\tag{2.15}
$$

such that

(i) $\sum_{h \in \mathcal{H}} r^h_{\mathcal{P},p}(\sum_{i \in P_1} \pi^i(p^1) , \ldots , \sum_{i \in P_T} \pi^i(p^T)) = \sum_{P_t \in \mathcal{P}} \sum_{i \in P_t} \pi^i(p^t)$ and

(ii) for all $h \in \mathcal{H}$, $r^h_{\mathcal{P},p}(\sum_{i \in P_1} \pi^i(p^1) , \ldots, \sum_{i \in P_T} \pi^i(p^T)) \geq 0$.

Definition. Let $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ denote a profit-making economy with firm-group specific prices associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ and a map of profit incomes $r_{\mathcal{P},p}$. A *tax equilibrium* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ is a configuration $\langle q, p^1, \ldots, p^T, m \rangle \in \mathbf{R}_+^N \times \prod_{t=1}^{T} B_t \times \mathbf{R}$ such that

$$
\sum_{h \in \mathcal{H}} x^h(q, m^h) \leq \sum_{t=1}^{T} \sum_{i \in P_t} y^i(p^t) + y^0(p^0) + \sum_{h \in \mathcal{H}} e^h \text{ and}
$$
$$
m^h = r^h_{\mathcal{P},p}(\sum_{i \in P_1} \pi^i(p^1), \ldots, \sum_{i \in P_T} \pi^i(p^T)) + m + q e^h, \ \forall \ h \in \mathcal{H}.
\tag{2.16}
$$

As in the previous section we can define an efficient tax equilibrium of economy $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$. Note that the system of equations (2.16) is homogeneous of degree zero in in $p^1, \ldots, p^T$, q, and m and is homogeneous of degree zero in p^0. Hence, it admits two normalization rules.[9] Under the maintained assumptions on consumers' preferences, the budget constraints hold as equalities under utility maximization, that is, for all h, we have

$$
q \cdot x^h(q, m^h) = m^h.
\tag{2.17}
$$

To show that at a tax equilibrium of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$, the government budget is balanced, we multiply both sides of the first part (a vector of inequalities) of (2.16) by q and employ

---

[9] For example, we can adopt the normalization rules $p_1^1 = 1$ and $p_1^0 = 1$.

(2.17) to obtain

$$q \cdot \sum_h x^h(q, m^h) \leq q \cdot \sum_{t=1}^{T} \sum_{i \in P_t} y^i(p^t) + q \cdot y^0(p^0) + q \cdot \sum_{h \in \mathcal{H}} e^h$$

$$\Rightarrow \sum_{t=1}^{T} \sum_{i \in P_t} \pi^i(p^t) + Hm + q \cdot \sum_{h \in \mathcal{H}} e^h \leq$$

$$\sum_{t=1}^{T} \sum_{i \in P_t} p^t \cdot y^i(p^t) + q \cdot y^0(p^0) + \sum_{t=1}^{T} \sum_{i \in P_t} [q - p^t] \cdot y^i(p^t) + q \cdot \sum_{h \in \mathcal{H}} e^h$$

$$\Rightarrow Hm \leq \sum_{t=1}^{T} \sum_{i \in P_t} [q - p^t] \cdot y^i(p^t) + q \cdot y^0(p^0)$$

$$\Rightarrow m \leq \frac{\sum_{t=1}^{T} \sum_{i \in P_t} [q - p^t] \cdot y^i(p^t) + q \cdot y^0(p^0)}{H}$$

$$\Rightarrow m \leq \frac{\sum_{t=1}^{T} \sum_{i \in P_t} [q - p^t] \cdot y^i(p^t) + [q - p^0] \cdot y^0(p^0) + p^0 \cdot y^0(p^0)}{H}.$$

(2.18)

Condition (2.18), which is an implication of Walras law, says that the demogrant is financed from the government's revenue from indirect taxation and the sale of publicly produced goods to the consumers.

The second-best problem for $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ is to find the mapping $\mathcal{V}_{\mathcal{P},r_{\mathcal{P},p}} : \Delta_{H-1} \to \mathbf{R}$ with image

$$\mathcal{V}_{\mathcal{P},r_{\mathcal{P},p}}(s^1, \ldots, s^H) := \max_{q, \ p^0, \ \langle p^1, \ldots, p^T \rangle, \ m} \sum_h s_h V^h(q, m^h)$$

subject to

$$\langle q, p^0, p^1, \ldots, p^T, m \rangle \text{ satisfying (2.16).}$$

(2.19)

A solution to (2.19) is called a *second-best tax equilibrium* of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$. As in the previous section we can define a production efficient second-best tax equilibrium of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ and the desirability of production efficiency at the second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$.

3. A preliminary lemma.

Assumptions 1 and 2 stated below are regularity assumptions on the technologies of firms. They are similar to the ones employed in the previous literature on this topic.

Assumption 1. For all $i \in \mathcal{I}$, Y^i is closed, convex, contains 0, is not a cone, and satisfies $Y^i + \mathbf{R}_-^N \subset Y^i$.

Assumption 2. For all $i \in \mathcal{I}$, the set B^i is non-empty and there exists $\rho \in \mathbf{R}_{++}^N \cap B^i$.

Assumption 1 excludes firms that exhibit constant returns-to-scale. This exclusion seems without loss of generality as these firms are associated with zero profits and hence the presence of such firms offers no constraints on the distribution of profits in the economy (the issue of focus in this paper). Note that under Assumptions 1 and 2, for all i, π^i is continuous, non-negative valued, linearly homogeneous, and convex on the set B^i.

Let $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ denote a profit-income economy with firm-group specific prices associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$. For all $t = 1, \ldots, T$ and $p^t \in B_t$, we denote the supply of firms in $P_t \in \mathcal{P}$ as $y^t(p^t)$, *i.e.*, $y^t(p^t) = \sum_{i \in P_t} y^i(p^t)$. For all $t = 1, \ldots, T$, the frontier of $Y^t = \sum_{i \in P_t} Y^i$ is denoted by $\hat{Y}^t$.[10]

Let $\langle p^1, \ldots, p^T \rangle \in \prod_{t=1}^T B_t$ and $p^0 \in B^0$. Assumption 3 (below) restricts our analysis (which we claim is without loss of generality[11]) to the case of technologies with smooth frontiers.

Remark 1 (below) presents the well-known fact that, under our assumptions, the vector of aggregate supply $\sum_{t=1}^T y^t(p^t) + y^0(p^0)$ lies in $\hat{Y}$ if the price vectors $p^0, \ldots, p^T$ are proportional.

Lemma 1 shows that if there exist $t, t' \in \{0, 1, \ldots, T\}$ such that p^t and $p^{t'}$ are not proportional,[12] then there exist changes in price vectors faced by firms in groups t and t' that can strictly increase the aggregate supply (*i.e.*, lead to a higher aggregate supply than $\sum_{l=1}^T y^l(p^l) + y^0(p^0)$) given the profit maximizing behavior of our price-taking firms and, hence, $\sum_{l=1}^T y^l(p^l) + y^0(p^0)$ is not in $\hat{Y}$. Before formally stating Lemma 1, we first present two examples to illustrate the point made in this lemma. Example 4 (below) considers the case of smooth production frontiers, while Example 5 shows that the argument extends to the non-smooth case. In both the examples, it is assumed that $N = 2$, there is no public production, $I = 2$, and $\mathcal{P} = \{\{1\}, \{2\}\}$. Good two is the output and good one is the input of these firms, so that if $y = \langle y_1, y_2 \rangle \in \mathbf{R}^2$ is a production vector, then $y_2 \geq 0$ and $y_1 \leq 0$. Let us also denote a hyperplane with normal p and constant a by $H(p, a)$ and its lower and strictly lower half-spaces by $H_{\leq}(p, a)$ and $H_{<}(p, a)$, respectively. Similarly we can define the upper and strictly upper half-spaces of $H(p, a)$.

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[10] Note the slight abuse of notation: the technology, its frontier, and a production vector corresponding to any firm $i \in \mathcal{I}$ are denoted by Y^i, $\hat{Y}^i$, and y^i, respectively, while the aggregate technology, its frontier, and a production vector obtained by summing over all firms in P_t for $t = 1, \ldots, T$ are denoted by Y^t, $\hat{Y}^t$, and y^t, respectively. However, in what follows, it will be clear always whether we are referring to a firm in $\mathcal{I}$ or to a group of firms P_t.

[11] We defend this claim with an example below.

[12] Or, in the non-smooth case, if the sets of support prices of $y^t(p^t)$ and $y^{t'}(p^{t'})$ do not intersect.

Example 4. Suppose technology of Firm 1 is $Y^1 = \{y^1 \in \mathbf{R}^2 | \ y_2^1 \leq (-y_1^1)^{\frac{1}{2}}\}$ and $Y^2 = \{y^2 \in \mathbf{R}^2 | \ y_2^2 \leq (-y_1^2)^{\frac{1}{3}}\}$. Suppose Firms 1 and 2 face price vectors $p^1 = \langle 1, \frac{1}{2} \rangle$ and $p_2 = \langle 1, \frac{1}{12} \rangle$, respectively. It can be verified that the profit maximizing production vector of Firm 1 will be $y^1(p^1) = \langle -1, 1 \rangle$, while that of Firm 2 will be $y^2(p^2) = \langle -8, 2 \rangle$. Since p^1 and p^2 are not proportional, it is well-known (and can be easily verified) that the aggregate production vector $y := y^1(p^1) + y^2(p^2) = \langle -9, 3 \rangle$ must lie in the interior of $Y := Y^1 + Y^2$. In fact, at these production vectors, the marginal productivity of input in Firm 1 is 2, which is less than the marginal productivity of input equal to 12 in Firm 2. This suggests that reallocating some of the input from Firm 1 to Firm 2 will result in an increase in the aggregate output. The question is whether there exist such reallocations which can also be supported as profit maximization choices of firms. We show below that this is true.

Since p^1 and p^2 are not collinear, Figure 1 shows that there exist changes $\Delta y^1 = \langle \frac{1}{2}, -2 \rangle$ and $\Delta y^2 = \langle -\frac{1}{4}, \frac{5}{2} \rangle$ in production vectors of Firms 1 and 2, respectively, such that $p^1 \cdot \Delta y^1$ and $p^2 \cdot \Delta y^2$ are non-positive and the aggregate change in production is positive, *i.e.*, $\Delta y = \Delta y^1 + \Delta y^2 = \langle \frac{1}{4}, \frac{1}{2} \rangle \gg 0$. In particular, as seen in the figure, $\Delta y^1 \in H_{\leq}(p^1, 1) \cap H_{>}(p^2, 0)$. Similarly $\Delta y^2 \in H_{\leq}(p^2, 0) \cap H_{>}(p^1, 0)$.[13]

Note however, that such changes may not be technologically feasible, *e.g.*, for Firm 2, $y^2(p^2) + \Delta y^2 = \langle -8.25, 4.5 \rangle$ and $4.5 > (8.25)^{\frac{1}{3}}$. Nevertheless, Figure 2 shows that Δy^2 can be suitably scaled so that it becomes technologically feasible with respect to Y^2, *e.g.*, $\bar{y}^2 := y^2(p^2) + \frac{1}{3.32347}\Delta y^2 = \langle -2.482, 1.354 \rangle$ lies in $\hat{Y}^2$. By scaling down Δy^2 further, we obtain points that lie on the line segment joining $\bar{y}^2$ and $y^2(p^2)$. Each such point, when added to $y^1(p^1) + \Delta y^1$, results in higher aggregate output that y. For any such point that lies in the interior of of Y^2, Figure 2 shows that there exist production points in the frontier $\hat{Y}^2$ that are bigger. Since these points are on the frontier, there will exist producer prices that support them as profit maximizing choices for the private-sector. $p^{2^{\nu_1}}$ and $p^{2^{\nu_2}}$ are two such price vectors. In fact, a sequence of such price vectors $\{p^{2^{\nu}}\}$ converging to p^2 can be constructed.[14] A similar sequence of price vectors for firm 1 $\{p^{1^{\nu}}\}$ converging to p^1 can also be constructed using Δy^1. It is clear from such a construction that, for all ν, we will have $y^1(p^{1^{\nu}}) + y^2(p^{2^{\nu}}) \gg y$.

Example 5. Suppose technology of Firms 1 and 2 are (see also Figure 3)

$$Y^1 = \{y^1 \in \mathbf{R}^2 | \ y_2^1 \leq 5(-y_1^1), \qquad 0 \leq (-y_1^1) \leq 1$$
$$\leq 3 + 2(-y_1^1), \qquad (-y_1^1) \geq 1\}, \text{ and} \tag{3.1}$$

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[13] Note that these changes imply reducing the input usage of Firm 1 and increasing the input usage of Firm 2.

[14] For example, these can be support vectors for a sequence of production vectors converging to $y^2(p^2)$ and lying on the part of the frontier $\hat{Y}^2$ that is on the left of $y^2(p^2)$.
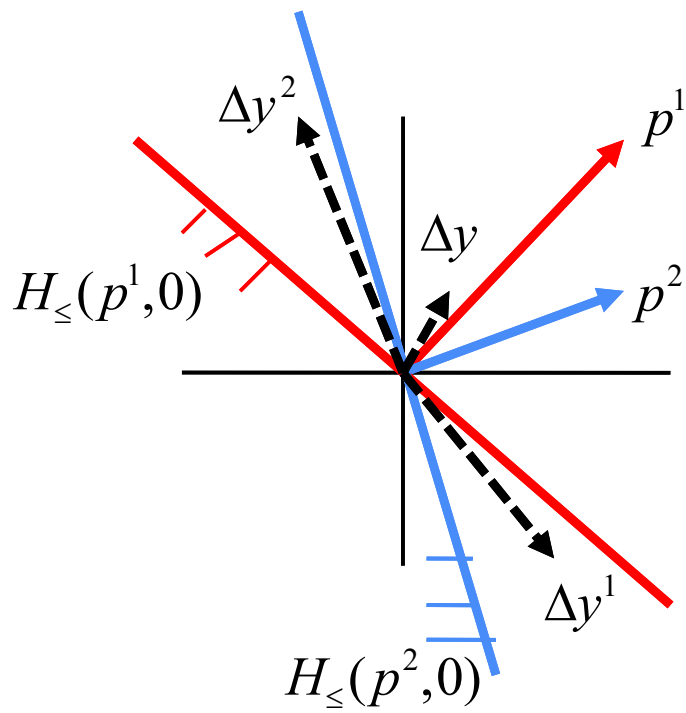
$$\Delta y = \Delta y^1 + \Delta y^2 >> 0$$
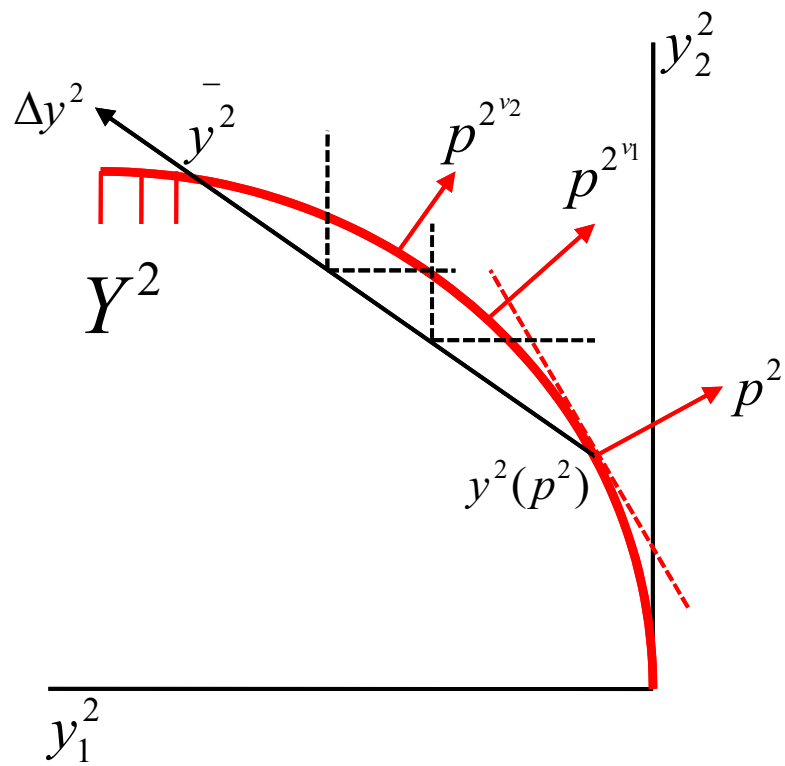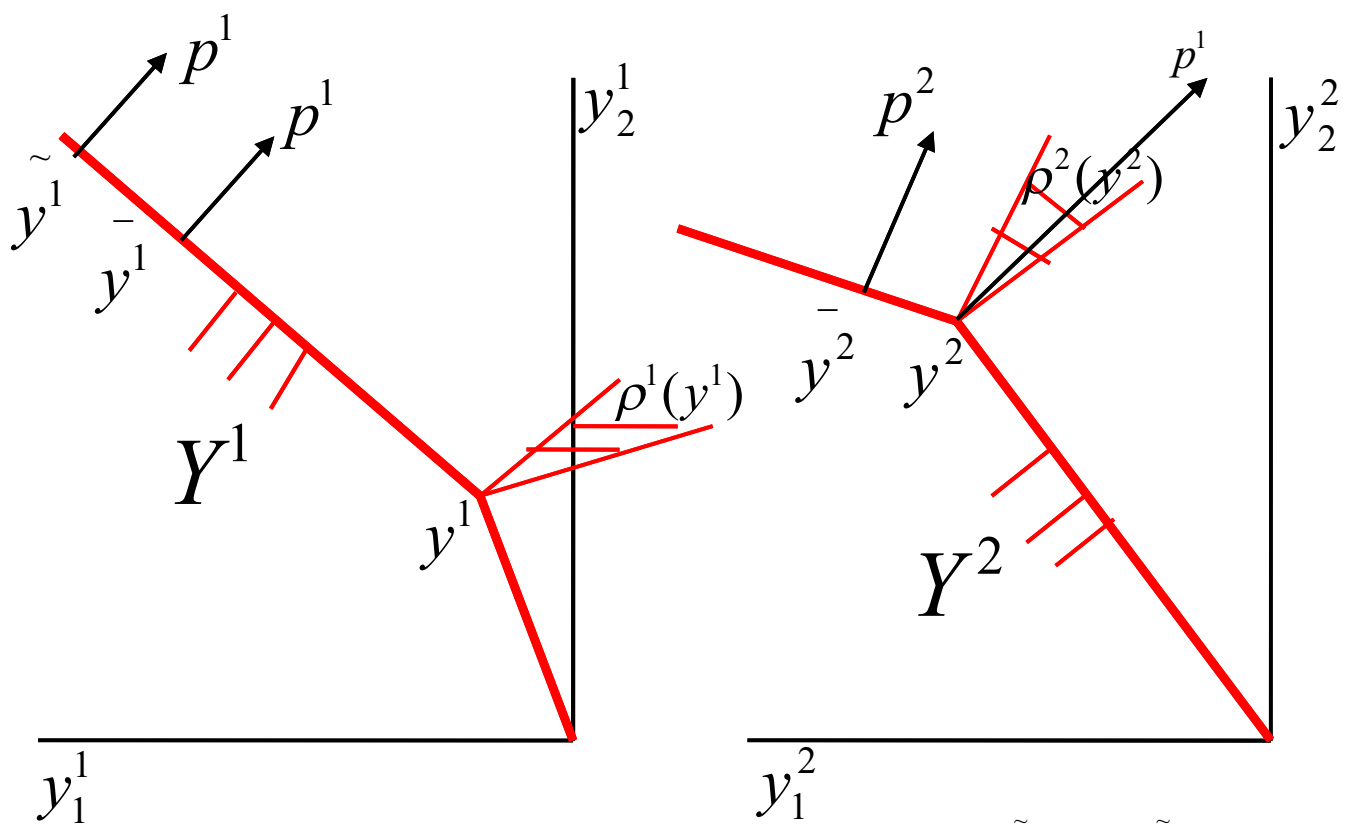
Figure 1.

Figure 2.

Figure 3.

$\tilde{y}^2 = y^2, \tilde{p}^2 = p^1$

$\rho^1(y^1) \bigcap \rho^2(y^2) \neq \varnothing$

$$Y^2 = \{y^2 \in \mathbf{R}^2|\ y_2^2 \leq 3(-y_1^2), \qquad 0 \leq (-y_1^2) \leq 4$$
$$\leq 8 + (-y_1^2), \qquad (-y_1^2) \geq 4\}, \tag{3.2}$$

so that the aggregate technology is

$$Y = \{y \in \mathbf{R}^2|\ y_2 \leq 5(-y_1), \qquad 0 \leq (-y_1) \leq 1,$$
$$\leq 2 + 3(-y_1), \qquad 1 \leq (-y_1) \leq 5 \tag{3.3}$$
$$\leq 7 + 2(-y_1), \qquad (-y_1) \geq 5\}.$$

Any production vector y such that $y = y^1 + y^2$ where $y^1 \in \hat{Y}^1$ and $y^2 \in \hat{Y}^2$ lies in $\hat{Y}$ if and only if the sets of support prices of y^1 and y^2 have a non-empty intersection, *e.g.*, consider $y = \langle -5, 17 \rangle \in \hat{Y}$, $y^1 = \langle -1, 5 \rangle \in \hat{Y}^1$, and $y^2 = \langle -4, 12 \rangle \in \hat{Y}^2$. The set of support prices of y^1 is (see Figure 3)

$$\rho^1(y^1) = \{p \in \mathbf{R}_+^2|\ p = \lambda_1\langle 5, 1 \rangle + \lambda_2\langle 2, 1 \rangle,\ \forall \lambda_1 \geq 0,\ \lambda_2 \geq 0\} \tag{3.4}$$

and the set of support prices of y^2 is

$$\rho^2(y^2) = \{p \in \mathbf{R}_+^2|\ p = \lambda_1\langle 3, 1 \rangle + \lambda_2\langle 1, 1 \rangle,\ \forall \lambda_1 > 0,\ \lambda_2 > 0\}. \tag{3.5}$$

It is clear that $\rho^1(y^1) \cap \rho^2(y^2) \neq \emptyset$.

Suppose Firms 1 and 2 face price vectors $p^1 = \langle 2, 1 \rangle$ and $p^2 = \langle 1, 1 \rangle$ and suppose Firms 1 and 2 choose bundles $\bar{y}^1 = \langle -6, 15 \rangle \in y^1(p^1)$ and $\bar{y}^2 = \langle -5, 13 \rangle \in y^2(p^2)$ as their profit maximizing choices. (See Figure 3.) The set of support prices $\rho^1(\bar{y}^1)$ is the set of all (non-zero) prices proportional to the vector $p^1 = \langle 2, 1 \rangle$, while the set of support prices $\rho^2(\bar{y}^2)$ is the set of all (non-zero) prices proportional to the vector $p^2 = \langle 1, 1 \rangle$. It is clear that $\rho^1(\bar{y}^1) \cap \rho^2(\bar{y}^2) = \emptyset$ and, hence, $\bar{y} = \bar{y}^1 + \bar{y}^2 = \langle -11, 28 \rangle$ does not lie in $\hat{Y}$. If the price vector faced by Firm 2 is changed to $\tilde{p}^2 = p^1$, while the price vector faced by Firm 1 is unchanged, then the profit maximizing choice of Firm 2 changes to $y^2(\tilde{p}^2) = \tilde{y}^2 = y^2 = \langle -4, 12 \rangle \in \hat{Y}^2$. For Firm 1, both the bundles $\tilde{y}^1 = \langle -6.8, 16.6 \rangle$ and $\bar{y}^1$ lie in $y^1(p_1)$ (see Figure 3). If Firm 1 can be made to change its profit maximizing choice to $\tilde{y}^1$ from $\bar{y}^1$, then the new aggregate supply vector is $\tilde{y} = \langle -10.8, 28.6 \rangle$, which is strictly bigger than $\bar{y}$.

Assumption 3 below will be made to restrict our analysis to the case of smooth and strictly convex technologies.[15] Though the examples above illustrate the generality of the conclusion of Lemma 1 for convex, smooth, and non-smooth production technologies, the apparatus required to prove the non-smooth case is more elaborate, while it will add nothing new to the general intuition.[16]

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[15] The restriction to strictly convex technologies implies that the supply mappings of firms are functions.

[16] The general non-smooth and convex case will require working, for example, with normal and tangent cones (see, *e.g.,* Rockafellar [1978] and Cornet [1989]) to define the cones of support prices for points on the frontiers of the technology sets.

Assumption 3. For all $i \in \mathcal{I}$, there exist smooth and strictly quasi-convex functions $f^i : \mathbf{R}^N \to \mathbf{R}$ with images $f^i(y)$ such that $Y^i = \{y \in \mathbf{R}^N | \ f^i(y) \leq 0\}$.[17]

Remark 1. Let Assumptions 1, 2, and 3 hold and let $\mathcal{P} = \{P_1, \dots, P_T\}$ be a partition of $\mathcal{I} \setminus \{0\}$. Suppose $p^t \in B_t$ for all $t = 1, \dots, T$ and $p^0 \in B^0$. If $p^0, p^1, \dots, p^T$ are proportional to each other then $\sum_{t=1}^T y^t(p^t) + y^0(p^0) \in \hat{Y}$.[18]

Lemma 1: *Suppose Assumptions 1, 2, and 3 hold and let $\mathcal{P} = \{P_1, \dots, P_T\}$ be a partition of $\mathcal{I} \setminus \{0\}$. Suppose $\bar{p}^t \in B_t$ for all $t = 1, \dots, T$ and $\bar{p}^0 \in B^0$. Suppose there exist $t, t' \in \{0, 1, \dots, T\}$ such that $\bar{p}^t$ is not proportional to $\bar{p}^{t'}$. Let $\bar{y} := \sum_{l=0}^T y^l(\bar{p}^l)$. Then there exist sequences $\{p^{tv}\} \to \bar{p}^t$ and $\{p^{t'v}\} \to \bar{p}^{t'}$ and an integer $\hat{v}$ such that $\sum_{l \neq t, t'} y^l(\bar{p}^l) + y^t(p^{tv}) + y^{t'}(p^{t'v}) \gg \bar{y}$ for all $v > \hat{v}$.*

4. Desirability of production efficiency at the second-best of a profit-making economy with firm-group specific prices.

The following theorem proves that production efficiency is desirable at any second-best of a profit-making economy with firm-group specific prices. Conditions (a) and (b) of the theorem ensure that local Pareto nonsatiation, as defined in Hahn [1973], always holds. In particular, condition (b) is the DM version of local Pareto nonsatiation, *i.e.,* there exists a good that is in positive (or negative) *net* demand by all consumers.[19]

Theorem 1: *Suppose $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ is a profit-making economy with firm-group specific prices associated with a partition $\mathcal{P} = \{P_1, \dots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ and a map of profit incomes $r_{\mathcal{P},p}$. Let Assumptions 1, 2, and 3 hold. Suppose either*

(a) there exists h such that u^h is strictly monotonic or

(b) there exists a commodity k and a consumer $h' \in \mathcal{H}$ such that

    (i) $x_k^{h'}(q, m^{h'}) - e_k^{h'} > 0$ and $x_k^h(q, m^h) - e_k^h \geq 0$, $\forall \ h \in \mathcal{H}$, $q \in \mathbf{R}_+^N$, and $m^{h'}, m^h \in \mathbf{R}$

    or

    (ii) $x_k^{h'}(q, m^{h'}) - e_k^{h'} < 0$ and $x_k^h(q, m^h) - e_k^h \leq 0$, $\forall \ h \in \mathcal{H}$, $q \in \mathbf{R}_+^N$, and $m^{h'}, m^h \in \mathbf{R}$

holds. Then production efficiency is desirable at the second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$.

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[17] A function $f : \mathbf{R}^N \to \mathbf{R}$ is smooth if it is C^∞, *i.e.,* its partial derivatives of all orders exist. Note that under Assumptions 1 and 2, the set Y^i has a functional representation for all $i \in \mathcal{I}$. Assumption 3 only ensures that this functional representation is smooth and strictly quasi-convex.

[18] This follows from Koopmans' well-known result on interchangeability of set summation and optimization. In the non-smooth case, this generalizes to the sets of support prices of $y^0(p^0), y^1(p^1), \dots, y^T(p^T)$ having a non-empty intersection.

[19] See Weymark [1978] for a generalization of condition (b).

Proof: Define $P_0 = \{0\}$ so that $\{P_0, P_1, \ldots, P_T\}$ is a partition of $\mathcal{I}$. Suppose $\bar{s} := \langle \bar{q}, \bar{p}^1, \ldots, \bar{p}^T, \bar{m} \rangle \in \mathbf{R}^{N+T+1}$ is a solution to (2.12) but $\bar{y} := \sum_{t=0}^{T} y^t(\bar{p}^t) \notin \hat{Y}$. Remark 1 implies that there exist $t, t' \in \{0, \ldots, T\}$ such that $\bar{p}^{t'} \neq \kappa \bar{p}^t$ for any $\kappa \geq 0$.

Step 1: *We show that, starting from $\bar{p}^t$ and $\bar{p}^{t'}$, there exist changes in producer prices of firm-groups t and t' which lead to an aggregate supply greater than $\bar{y}$.*

This is true because Lemma 1 demonstrates that there exist sequences $\{p^{tv}\} \to \bar{p}^t$ and $\{p^{t'v}\} \to \bar{p}^{t'}$ such that $y^t(p^{tv}) + y^{t'}(p^{t'v}) \gg \hat{y}$ for all $v > \hat{v}$, where $\hat{v}$ is defined as in Lemma 1.

This implies that the aggregate income or the value of aggregate output measured using consumer prices $\bar{q}$ increases when producer prices $\bar{p}^t$ and $\bar{p}^{t'}$ change to p^{tv} and $p^{t'v}$ for all $v > \hat{v}$, that is,

$$M^v = \bar{q} \cdot \left[\sum_{l \neq t, t'} y^l(\bar{p}^l) + y^t(p^{tv}) + y^{t'}(p^{t'v}) \right] > \bar{q} \cdot \bar{y} =: \bar{M}, \ \forall v > \hat{v}. \tag{4.1}$$

Step 2: *We show that there exists a $\overset{*}{v} > \hat{v}$, two scalars $\lambda^t \geq 0$ and $\lambda^{t'} \geq 0$ and two new price vectors for firm-groups t and t' defined by $\overset{*}{p}^t := \lambda^t p^{t\overset{*}{v}}$ and $\overset{*}{p}^{t'} := \lambda^{t'} p^{t'\overset{*}{v}}$ such that, for all $h \in \mathcal{H}$, the profit incomes defined at the old and new price vectors do not change, i.e.,* [20]

$$\overset{*}{r}^h := r^h_{\mathcal{P},p}\left(\left(\sum_{i \in P_l} \pi^i(\bar{p}^l) \right)_{l \neq t, t'}, \sum_{i \in P_t} \pi^i(\overset{*}{p}^t), \sum_{i \in P_t} \pi^i(\overset{*}{p}^{t'}) \right)$$
$$= r^h_{\mathcal{P},p}\left(\sum_{i \in P_1} \pi^i(\bar{p}^1), \ldots, \sum_{i \in P_T} \pi^i(\bar{p}^T) \right) =: \bar{r}^h. \tag{4.2}$$

The continuity of the profit functions π^i implies that $\sum_{i \in P_t} \pi^i(p^{tv}) \to \sum_{i \in P_t} \pi^i(\bar{p}^t)$. If $\sum_{i \in P_t} \pi^i(\bar{p}^t) = 0$ then choose *any* $\epsilon > 0$. If $\sum_{i \in P_t} \pi^i(\bar{p}^t) > 0$ then choose ϵ such that $0 < \epsilon < \sum_{i \in P_t} \pi^i(\bar{p}^t)$. Then, there exists v^t such that for all $v > v^t$ we have $|\sum_{i \in P_t} \pi^i(p^{tv}) - \sum_{i \in P_t} \pi^i(\bar{p}^t)| < \epsilon$. Our choice of ϵ implies that, for every $v > v^t$, the sign of $\sum_{i \in P_t} \pi^i(p^{tv})$ is the same as the sign of $\sum_{i \in P_t} \pi^i(\bar{p}^t)$: if $\sum_{i \in P_t} \pi^i(\bar{p}^t) = 0$ then $\sum_{i \in P_t} \pi^i(p^{tv}) \geq 0$ and if $\sum_{i \in P_t} \pi^i(\bar{p}^t) > 0$ then $\sum_{i \in P_t} \pi^i(p^{tv}) > 0$. Similarly, we can define $v^{t'}$.

---

[20] If any one of t or t' is 0, say t', then $\pi^{t'}$ will not be an argument of the income map $r_{\mathcal{P},p}$.

Pick $\overset{*}{v}$ to be any $v > \max\{v^t, v^{t'}, \hat{v}\}$. Choose scaling factors λ^t and $\lambda^{t'}$ such that

$$\lambda^t \sum_{i \in P_t} \pi^i(p^{t\overset{*}{v}}) = \sum_{i \in P_t} \pi^i(\bar{p}^t) \text{ and}$$

$$\lambda^{t'} \sum_{i \in P_{t'}} \pi^i(p^{t'\overset{*}{v}}) = \sum_{i \in P_{t'}} \pi^i(\bar{p}^{t'}). \tag{4.3}$$

This is possible, *e.g.*, if $\sum_{i \in P_t} \pi^i(\bar{p}^t) = 0$, then $\lambda^t = 0$ needs to be chosen. If $\sum_{i \in P_t} \pi^i(\bar{p}^t) > 0$, then λ^t is given by

$$\lambda^t = \frac{\sum_{i \in P_t} \pi^i(\bar{p}^t)}{\sum_{i \in P_t} \pi^i(p^{t\overset{*}{v}})}, \tag{4.4}$$

which is well defined as $\sum_{i \in P_t} \pi^i(p^{t\overset{*}{v}}) \neq 0$. (Note, $\lambda^t \geq 0$ and $\lambda^{t'} \geq 0$.) Then (4.2) follows from (4.3) and the linear homogeneity of the profit functions. Thus, the profit incomes of individual consumers do not change when $\bar{p}^t$ and $\bar{p}^{t'}$ change to $\overset{*}{p}^t$ and and $\overset{*}{p}^{t'}$.

Step 3: *We show that Step 2 implies that the move to $\overset{*}{p}^t$, $\overset{*}{p}^{t'}$ results in no change in the aggregate profit income generated in the economy and that this implies that the increase in aggregate income from $\bar{M}$ to $M^{\overset{*}{v}}$ must show up as an increase in the government's tax revenue.*

Summing (4.2) over all h and including the profits of the public sector, we obtain[21]

$$\sum_{l \neq t, t'} \sum_{i \in P_l} \pi^i(\bar{p}^l) + \sum_{i \in P_t} \pi^i(\overset{*}{p}^t) + \sum_{i \in P_{t'}} \pi^i(\overset{*}{p}^{t'}) = \sum_{l=0}^{T} \sum_{i \in P_l} \pi^i(\bar{p}^l). \tag{4.5}$$

Thus, the aggregate profits of firms do not change when $\bar{p}^t$ and $\bar{p}^{t'}$ change to $\overset{*}{p}^t$ and and $\overset{*}{p}^{t'}$. Define $\overset{*}{y}^t := y^t(\overset{*}{p}^t)$, and $\overset{*}{y}^{t'} := y^{t'}(\overset{*}{p}^{t'})$.

$$M^{\overset{*}{v}} =: \overset{*}{M} = \bar{q} \cdot \Big[\sum_{l \neq t, t'} y^l(\bar{p}^l) + \overset{*}{y}^t + \overset{*}{y}^{t'} \Big]$$

$$= \sum_{l \neq t, t'} [(\bar{q} - \bar{p}^l) \cdot y^l(\bar{p}^l)] + [\bar{q} - \overset{*}{p}^t] \cdot \overset{*}{y}^t + [\bar{q} - \overset{*}{p}^{t'}] \cdot \overset{*}{y}^{t'} + \sum_{l \neq t, t'} \bar{p}^l \cdot y^l(\bar{p}^l) + \overset{*}{p}^t \cdot \overset{*}{y}^t + \overset{*}{p}^{t'} \cdot \overset{*}{y}^{t'}$$

$$= \sum_{l \neq t, t'} [(\bar{q} - \bar{p}^l) \cdot y^l(\bar{p}^l)] + [\bar{q} - \overset{*}{p}^t] \cdot \overset{*}{y}^t + [\bar{q} - \overset{*}{p}^{t'}] \cdot \overset{*}{y}^{t'}$$

$$+ \sum_{l \neq t, t'} \sum_{i \in P_l} \pi^i(\bar{p}^l) + \sum_{i \in P_t} \pi^i(\overset{*}{p}^t) + \sum_{i \in P_{t'}} \pi^i(\overset{*}{p}^{t'})$$

$$\tag{4.6}$$

---

[21] Note that this is true for both cases: (i) both t and t' are not equal to 0 and (ii) one of t or t' is 0.

Similarly,

$$\bar{M} = \sum_{l=0}^{T} [(\bar{q} - \bar{p}^l) \cdot y^l(\bar{p}^l)] + \sum_{l=0}^{T} \sum_{i \in P_l} \pi^i(\bar{p}^l). \tag{4.7}$$

Since $\overset{*}{M} - \bar{M} > 0$, it follows from (4.5) that the government's revenue from commodity taxes is higher when we move to $\overset{*}{p}^t, \overset{*}{p}^{t'}$ keeping consumer prices and producer prices for the firm groups other than t and t' unchanged, that is,

$$\overset{*}{G} := \sum_{l \neq t,t'} [(\bar{q} - \bar{p}^l) \cdot y^l(\bar{p}^l)] + [\bar{q} - \overset{*}{p}^t] \cdot \overset{*}{y}^t + [\bar{q} - \overset{*}{p}^{t'}] \cdot \overset{*}{y}^{t'} > \sum_{l=1}^{T} [(\bar{q} - \bar{p}^l) \cdot y^l(\bar{p}^l)] =: \bar{G}.$$
$$\tag{4.8}$$

Step 4: *We show that the increase in the government's revenue can be used to construct another tax equilibrium where utility of at least one consumer is higher, with no loss in utility for the others: this is obtained by either increasing the demogrant by an appropriate amount (this is possible if (a) holds) or by decreasing (increasing) the consumer price of, and hence the tax on, the kth commodity by an appropriate amount (this is possible if b(i) (b(ii)) holds).*

For all h define $x^h(\bar{q}, \bar{r}^h + \bar{m} + \bar{q} \cdot e^h) =: \bar{x}^h$, $\sum_h \bar{x}^h =: \bar{x}$, and $e := \sum^h e^h$. (4.2) implies that for all h, $\bar{x}^h = x^h(\bar{q}, \overset{*}{r}^h + \bar{m} + \bar{q} \cdot e^h)$. Since $\bar{x} \leq \bar{y} + e \ll \sum_{l \neq t,t'} y^l(\bar{p}^l) + \overset{*}{y}^t + \overset{*}{y}^{t'} + e =: \overset{*}{y} + e$, we have $\bar{x} \in \{\overset{*}{y} + e\} + \mathbf{R}^N_{--}$.

Since x^h is a continuous function of q_k for all h, clearly, if condition b(i) or b(ii) hold, we can apply the DM argument to find $\overset{*}{q} := \langle \bar{q}_{-k}, \overset{*}{q}_k \rangle$ and $\overset{*}{\epsilon}$ such that (1) $\sum_h x^h(\overset{*}{q}, \overset{*}{r}^h + \bar{m} + \overset{*}{q} \cdot e^h)) =: \sum_h \overset{*}{x}^h \in N_{\overset{*}{\epsilon}}(\bar{x}) \subset \{\overset{*}{y} + e\} + \mathbf{R}^N_{--}$ and (2) $u^h(\overset{*}{x}^h) \geq u^h(\bar{x}^h)$ for all h and $u^h(\overset{*}{x}^h) > u^h(\bar{x}^h)$ for some h.[22] This implies that $\overset{*}{s} := \langle \overset{*}{q}, (\bar{p}^l)_{l \neq t,t'}, \overset{*}{p}^t, \overset{*}{p}^{t'}, \bar{m} \rangle$ is another tax equilibrium configuration of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ that Pareto dominates $\bar{s}$.[23] This contradicts the fact that $\bar{s}$ is a solution to (2.12).

If condition (a) holds, then we can exploit the continuity of x^h in m for all h to find $\hat{m} > \bar{m}$ and $\hat{\epsilon}$ such that (1) $\sum_h x^h(\bar{q}, \overset{*}{r}^h + \hat{m} + \bar{q} \cdot e^h)) =: \sum_h \hat{x}^h \in N_{\hat{\epsilon}}(\bar{x}) \subset \{\overset{*}{y} - e\} + \mathbf{R}^N_{--}$ and (2) $u^h(\hat{x}^h) \geq u^h(\bar{x}^h)$ for all h and $u^h(\overset{*}{x}^h) > u^h(\bar{x}^h)$ for some h.[24] This again leads to a tax equilibrium that Pareto dominates $\bar{s}$, which once again contradicts the hypothesis of the theorem.[25] ∎

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[22] Note, this implies reducing (increasing) the consumer price on commodity k that every one likes and has a non-negative net demand (dislikes and has a non-positive net demand).

[23] Note that it is always possible to make the new tax equilibrium configuration conform to the normalization rules adopted, *e.g.*, if the normalization rules are $p_1^1 = 1$ and $p_1^0 = 1$ and if $t \neq 1, 0$ and $t' \neq 1, 0$, then divide $\langle \overset{*}{q}, (\bar{p}^l)_{l \neq t,t',0}, \overset{*}{p}^t, \overset{*}{p}^{t'}, \bar{m} \rangle$ by $\bar{p}_1^1$ and $\bar{p}^0$ by $\bar{p}_1^0$.

[24] Note, this is made possible by the fact that $\overset{*}{G} > \bar{G}$ in (4.8), so that it is possible to distribute all or a part of this increased government budget-surplus as a higher demogrant.

[25] Once again, the new tax equilibrium can be reconfigured to conform to our normalization rules.

5. Corollaries of Theorem 1:

Two results follow as corollaries of Theorem 1.

Corollary 1 of Theorem 1: *Production efficiency is desirable at the second-best of a profit-making economy with firm-group specific profit taxes.*

Proof:
Step 1: *We show that the set of tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ is a subset of the set of tax equilibrium allocations of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$.*

Let $\bar{\alpha}_\tau := \langle \bar{q}, \bar{p}, \bar{p}^0, \bar{\tau}^1, \ldots, \bar{\tau}^T, \bar{m} \rangle$ be a tax equilibrium of a profit-making economy with firm-group specific profit taxes $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$ associated with a partition $\mathcal{P} = \{P_1, \ldots, P_T\}$ of $\mathcal{I} \setminus \{0\}$ and a map of profit incomes $r_{\mathcal{P},\tau}$. Define $\bar{\alpha}_p := \langle \bar{q}, \bar{p}^0, \bar{p}^1, \ldots, \bar{p}^T, \bar{m} \rangle$ with $\bar{p}^t := (1 - \bar{\tau}^t)\bar{p}$ for all $t = 1, \ldots, T$. Let $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ be a profit-making economy with firm-group specific prices, where[26]

$$r_{\mathcal{P},p}(a) = r_{\mathcal{P},\tau}(a). \tag{5.1}$$

For all $t = 1, \ldots, T$, the linear homogeneity of the profit function implies $\sum_{i \in P_t} \pi^i(\bar{p}^t) = (1 - \bar{\tau}^t) \sum_{i \in P_t} \pi^i(\bar{p})$, and for all $h \in \mathcal{H}$, we have

$$r^h_{\mathcal{P},p}\Big(\sum_{i \in P_1} \pi^i(\bar{p}^1), \ldots, \sum_{i \in P_T} \pi^i(\bar{p}^T) \Big) = r^h_{\mathcal{P},\tau}\Big((1 - \bar{\tau}^1) \sum_{i \in P_1} \pi^i(\bar{p}), \ldots, (1 - \bar{\tau}^T) \sum_{i \in P_T} \pi^i(\bar{p}) \Big). \tag{5.2}$$

This implies that $\bar{\alpha}_p$ is a tax equilibrium of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ that results in the same allocation as $\bar{\alpha}_\tau$ in economy $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.

Step 2: *We show that every second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$ can be decentralized as a tax equilibrium of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.*

Theorem 1 implies that if $\langle q, p^0, p^1, \ldots, p^T, m \rangle$ is a second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$, then there exist positive scalars $\lambda^2, \ldots, \lambda^T$ such that $p^t = \lambda^t p^1$ for all $t = 2, \ldots, T$. Choose $p = p^1$, $\tau^1 = 0$, and $\tau^t = 1 - \lambda^t$ for all $t = 2, \ldots, T$. Then $\langle q, p, p^0, \tau^1, \ldots, \tau^T, m \rangle$ is a tax equilibrium of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.

Steps 1 and 2 imply that production efficiency is desirable at the second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.[27] ∎

Corollary 2 of Theorem 1: *If profit taxes cannot be implemented then intermediate-inputs should be taxed proportionately. Alternatively, firm-specific profit taxation justifies not taxing inter-firm transactions in profit making economies.*

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[26] Recall the definitions of maps of profit incomes for firm-group specific profit taxes and firm-group specific prices in Sections 2.

[27] Of course, this follows under the conditions laid down in Theorem 1 as translated to economy $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},\tau})$.

Proof: Consider a profit-making economy with firm-group specific prices $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$. Let $\mathcal{P}$ be a partition of $\mathcal{I} \setminus \{0\}$. Thus, the government can implement firm-group specific prices in such an economy. This means that non-zero wedges are allowed between price vectors faced by any two groups of private firms in this economy, *e.g.*, $p^t - p^{t'}$ is the wedge between the price vectors $p^t \in \mathbf{R}_+^N$ and $p^{t'} \in \mathbf{R}_+^N$ faced by private firms in firm-groups t and t', respectively. Thus, transactions that happen between these two groups of private firms are effectively taxed. Thus, this economy permits a very general system of intermediate input taxation. Theorem 1 demonstrates, however, that at the second-best of such an economy producer prices are proportional to each other, *e.g.*, if $\langle q, p^0, p^1, \ldots, p^T, m \rangle$ is a second-best of $\mathcal{E}(E, \mathcal{P}, r_{\mathcal{P},p})$, then there exist positive scalars $\lambda^2, \ldots, \lambda^T$ such that $p^t = \lambda^t p^1$ for all $t = 2, \ldots, T$, where p^1 is the price vector for firms in group 1. Thus, the wedge between price vectors of firm-groups t and t' at this second-best is $p^1(\lambda^t - \lambda^{t'})$. Transactions in commodities between these two firm-groups are, hence, taxed proportionately at a rate $\lambda^t - \lambda^{t'}$. Alternatively, from Corollary 1 of Theorem 1 (above) it follows that this second-best can be decentralized as a second-best tax equilibrium of a profit-making economy with firm-group specific profit taxes. But this precludes intermediate goods taxation as all firm-groups face the same producer price p^1 in this economy. Rather, firm-groups are subject to firm-group specific profit tax rates $\tau^t = 1 - \lambda^t$ for all $t = 1, \ldots, T$. It is in this sense that Corollary 2 of Theorem 1 follows. ■

As a special case, the famous implication with regards to intermediate input taxation for DM and Guesnerie [1995] models follows:[28] If government can implement one-hundred percent profit taxation and redistribute proceeds to consumers as a demogrant, then intermediate input taxation is not desirable.

6. Conclusions.

There is a classic literature that studies the desirability of production efficiency in economies with Ramsey taxation where firms make positive profits which can potentially be partly taxed away and partly distributed back to consumers. The results of this classic literature are often invoked to justify the use of producer prices as proxies for shadow prices in cost-benefit tests of marginal public sector projects. We show that the desirability of second-best production efficiency depends on the link between the constraints on government's profit taxation power and the institutional rules by which (profit) incomes are distributed in the economy. We generalize results in the literature by showing that second-best production efficiency is desirable whenever firms can be organized into groups such that (i) profit tax rates vary across these groups and (ii) consumer incomes depend on

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[28] Recall, this case is equivalent to the case where $\mathcal{P}$ is the coarsest partition of private firms, *i.e.*, $\mathcal{P} = \{\mathcal{I} \setminus \{0\}\}$ and $\theta_i^h = \frac{1}{H}$ for all h and $i \in \mathcal{I} \setminus \{0\}$.

the distribution of profits across these groups. Thus, the fewer (larger) the number of firm-groups, the lesser (more) are the profit tax rates that are required to ensure second-best production efficiency. The two cases studied in the literature of firm-specific and uniform (*e.g.,* one-hundred percent) profit taxation follow as two special cases of our general result.

The result follows because, at any production inefficient status-quo of such economies, the private sector producer price vector and the prices in the public sector (the latter reflect the true shadow prices in the economy) are not proportional. The differences in the marginal rates of substitution in the private and public sectors imply that production can be reallocated between these sectors to increase aggregate output and income in the economy. Lemma 1 proves that, in an institutional structure where producers are price takers and maximize profits, small changes in the two price vectors can be constructed that ensure that this potential increase in the aggregate output can be supported as profit maximizing choices of firms. The continuity and linear homogeneity of the profit function imply that by implementing firm-group specific profit taxation, the government can effectively implement further (but proportionate) changes in producer prices that ensure that the net of tax firm-group profits, and hence the profit incomes of the consumers that depend on the distribution of the firm-group profits, remain at the status-quo levels, while at the same time there are no further changes in the (increased) supply by firms. This must mean that the increase in aggregate income shows up as an increase in the tax and public sector incomes of the government, which can be used to change commodity taxes or to increase demogrant incomes of people in a Pareto improving way. Thus, there are always Pareto improvements at any production inefficient status-quo of such economies.

The mechanism suggests why this strategy does not work, generally, in most private ownership economies when restrictions on profit taxation are not consistent with the rules of income distribution as outlined above in (i) and (ii). This is because, while a production inefficient status-quo suggests that there are changes in producer prices that can increase the aggregate output in the economy, any attempt by the government to change profit tax rates to maintain net of tax profits at the status-quo levels, may not translate into maintaining profit incomes of the consumers at the status-quo levels. Thus, all of the increased output may not, in general, become available to the government for designing Pareto improving changes in taxes and demogrant. Private ownership diverts some of the increased resources from the government coffers and puts it into the hands of consumers as profit incomes. But the private ownership structure could be such that it may lead to an inequitable distribution of profit incomes and a decrease in welfare of some consumers, which no government policy may be able to correct with the remaining resources, that is, there may exist no directions of change in the government policy instruments that are Pareto-improving, equilibrium preserving, and compatible with the existing private ownership structure. Our analysis hence suggests how, by understanding the rules of

profit income distribution in the economy, the government can potentially design profit taxation that can promote both its redistributive and efficiency objectives.

We also show that in economies with Ramsey taxation where consumers also receive profit incomes, proportionate intermediate input taxation is recommended in the absence of profit taxation. Alternatively, profit taxation is a perfect substitute for intermediate input taxation. The classic result of DM (extended as in Guesnerie [1995] to take account of profit making economies) on no intermediate input taxation follows as a special case of our model with profit taxation where all firms are subject to one-hundred percent profit taxation with the tax proceeds being redistributed back to consumers as a demogrant. In this sense, the recommended structure of intermediate input taxation also serves both efficiency and redistributive objectives of the government and supports tax systems such as VAT.

APPENDIX

Proof of Lemma 1. Smoothness of $\hat{Y}^t$ and $\hat{Y}^{t'}$ implies that $H(\bar{p}^t, \bar{p}^t \cdot y^t(\bar{p}^t))$ and $H(\bar{p}^{t'}, \bar{p}^{t'} \cdot y^{t'}(\bar{p}^{t'}))$ are unique supporting hyperplanes for Y^t and $Y^{t'}$ at $y^t(\bar{p}^t)$ and $y^{t'}(\bar{p}^{t'})$, respectively.

Step 1. Since $\bar{p}^t$ and $\bar{p}^{t'}$ are not collinear, $H(\bar{p}^t, 0)$ is not a supporting hyperplane for $H_{\geq}(\bar{p}^{t'}, 0)$ and $H(\bar{p}^{t'}, 0)$ is not a supporting hyperplane for $H_{\geq}(\bar{p}^t, 0)$ at 0^N. This implies that there exist $\Delta y^t \in \mathbf{R}^N$ and $\Delta y^{t'} \in \mathbf{R}^N$ such that the following is true:[29]

$$\Delta y^t \in H_{<}(\bar{p}^t, 0) \cap H_{\geq}(\bar{p}^{t'}, 0),$$

$$\Delta y^{t'} \in H_{<}(\bar{p}^{t'}, 0) \cap H_{\geq}(\bar{p}^t, 0), \text{ and} \qquad (A.1)$$

$$\Delta y^t + \Delta y^{t'} \gg 0^N.$$

This implies that $y^t(\bar{p}^t) + y^{t'}(\bar{p}^{t'}) + \Delta y^t + \Delta y^{t'} \gg \bar{y}$. Denote $y^t(\bar{p})$ by $\bar{y}^t$ and $y^{t'}(\bar{p}^{t'})$ by $\bar{y}^{t'}$. Since $\bar{y}^t$ and $\bar{y}^{t'}$ belong to $\hat{Y}^t$ and $\hat{Y}^{t'}$, Assumption 3 implies that $f^t(\bar{y}^t) = 0$ and $f^{t'}(\bar{y}^{t'}) = 0$.

Step 2. Recall that $\nabla f^t(\bar{y}^t)$ is defined as the linear mapping such that for all $\{h^v\} \to 0^N$, we have

$$\lim_{h^v \to 0^N} \frac{f^t(\bar{y}^t + h^v) - [f^t(\bar{y}^t) + \nabla f^t(\bar{y}^t)h^v]}{|h^v|} \equiv \lim_{h^v \to 0^N} \frac{e(h^v, \bar{y}^t)}{|h^v|} = 0, \qquad (A.2)$$

where $e(h^v, \bar{y}^t) = f^t(\bar{y}^t + h^v) - [f^t(\bar{y}^t) + \nabla f^t(\bar{y}^t)h^v]$. We show that there exists $\gamma^t > 0$ such that $\nabla f^t(\bar{y}^t) = \gamma^t \bar{p}^t$. Take any point $y \in Y^t$ such that $y \neq \bar{y}^t$. Then, from the convexity

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[29] The intuition becomes clear when one sees Figure 1.

of Y^t, $f^t(\bar{y}^t + \lambda(y - \bar{y}^t)) \leq 0$ for all $\lambda \in [0,1]$. Using (A.2) and the fact that $f^t(\bar{y}^t) = 0$, we have

$$\frac{\nabla f^t(\bar{y}^t)(y - \bar{y}^t)}{|y - \bar{y}^t|} = \lim_{\lambda \to 0} \frac{f^t(\bar{y}^t + \lambda(y - \bar{y}^t)) - f^t(\bar{y}^t)}{|y - \bar{y}^t|\lambda} = \lim_{\lambda \to 0} \frac{f^t(\bar{y}^t + \lambda(y - \bar{y}^t))}{|y - \bar{y}^t|\lambda} \leq 0. \quad \text{(A.3)}$$

Since this is true for all $y \in Y^t$, $\nabla f^t(\bar{y}^t)$ is a normal to a supporting hyperplane of Y^t at $\bar{y}^t$. Since, $\hat{Y}^t$ is smooth and $H(\bar{p}^t, \bar{p}^t \cdot \bar{y}^t)$ is also a supporting hyperplane of Y^t at $\bar{y}^t$, there must exist $\gamma^t > 0$ such that $\nabla f^t(\bar{y}^t) = \gamma^t \bar{p}^t$. Similarly, we can prove that there exists $\gamma^{t'} > 0$ such that $\nabla f^{t'}(\bar{y}^{t'}) = \gamma^{t'} \bar{p}^{t'}$.

Step 3. (A.3) implies that $\Delta y^t \cdot \nabla f^t(\bar{p}^t) < 0$ and $\Delta y^{t'} \cdot \nabla f^{t'}(\bar{p}^{t'}) < 0$. Choose a sequence $\{\lambda^v\}$ such that $\lambda^v \Delta y^t \to 0$ and $\lambda^v > 0$ for all v. We now show that there exists v' such that for all $v > v'$, we have $y^{tv} := \bar{y}^t + \lambda^v \Delta y^t \in Y^t$. From (A.2) we have

$$\lim_{\lambda^v \to 0} \frac{f^t(\bar{y}^t + \lambda^v \Delta y^t) - f^t(\bar{y}^t) - \lambda^v \nabla f^t(\bar{y}^t)\Delta y^t}{|\Delta y^t|\lambda^v} = 0. \quad \text{(A.4)}$$

Since $f^t(\bar{y}^t) = 0$ and $\nabla f^t(\bar{y}^t)\Delta y^t < 0$ (from Step 2), we have

$$\lim_{\lambda^v \to 0} \frac{f^t(\bar{y}^t + \lambda^v \Delta y^t)}{|\Delta y^t|\lambda^v} = \frac{\nabla f^t(\bar{y}^t)\Delta y^t}{|\Delta y^t|} < 0. \quad \text{(A.5)}$$

Hence, there exists a large enough v' such that for all $v > v'$, we have $f^t(\bar{y}^t + \lambda^v \Delta y^t) < 0$, and hence $y^{tv} := \bar{y}^t + \lambda^v \Delta y^t \in int Y^t$ for all $v > v'$.[30] Similarly, we can prove that there exists v'' such that for all $v > v''$, we have $y^{t'v} := \bar{y}^{t'} + \lambda^v \Delta y^{t'} \in int Y^{t'}$.

Step 4. We now show that there exist sequences $\{p^{tv}\}$ and $\{p^{t'v}\}$, and a positive integer $\hat{v}$ such that for all $v > \hat{v}$, we have $y^t(p^{tv}) + y^{t'}(p^{t'v}) \gg \bar{y}^t + \bar{y}^{t'}$. Define $\hat{v} = \max\{v', v''\}$. For every $v > \hat{v}$, $y^{tv} \in Y^t$. It can therefore be shown that there are continuous maps $\kappa^t(y^{tv}) := \max_\kappa \{\kappa \geq 0 | [y^{tv} + \kappa \mathbf{1}^N] \in Y^t\}$ and $\kappa^{t'}(y^{t'v}) := \max_\kappa \{\kappa \geq 0 | [y^{t'v} + \kappa \mathbf{1}^N] \in Y^{t'}\}$.[31] For all $v > \hat{v}$, it is clear that (i) $y^{tv} + y^{t'v} \gg \bar{y}^t + \bar{y}^{t'}$ and so $(y^{tv} + \kappa^t(y^{tv})\mathbf{1}^N) + (y^{t'v} + \kappa^{t'}(y^{t'v})\mathbf{1}^N) \gg \bar{y}^t + \bar{y}^{t'}$, (ii) $(y^{tv} + \kappa^t(y^{tv})\mathbf{1}^N)$ and $(y^{t'v} + \kappa^{t'}(y^{t'v})\mathbf{1}^N)$ belong to $\hat{Y}^t$ and $\hat{Y}^{t'}$, respectively, and (iii) $\{y^{tv} + \kappa^t(y^{tv})\mathbf{1}^N\} \to \bar{y}^t$ and $\{y^{t'v} + \kappa^{t'}(y^{t'v})\mathbf{1}^N\} \to \bar{y}^{t'}$. Define $p^{tv} = \frac{1}{\gamma^t}\nabla f^t(y^{tv} + \kappa^t(y^{tv})\mathbf{1}^N)$ and $p^{t'v} = \frac{1}{\gamma^{t'}}\nabla f^{t'}(y^{t'v} + \kappa^{t'}(y^{t'v})\mathbf{1}^N)$. The smoothness of functions f^t and $f^{t'}$ imply that $\{p^{tv}\} \to \bar{p}^t$ and $\{p^{t'v}\} \to \bar{p}^{t'}$. Clearly, $y^t(p^{tv}) = y^{tv} + \kappa^t(y^{tv})\mathbf{1}^N$ and $y^{t'}(p^{t'v}) = y^{t'v} + \kappa^{t'}(y^{t'v})\mathbf{1}^N$, so that for all $v > \hat{v}$, we have $y^t(p^{tv}) + y^{t'}(p^{t'v}) \gg \bar{y}^t + \bar{y}^{t'}$.

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[30] For any set $A \subset \mathbf{R}^n$, $int A$ is the interior of A relative to $\mathbf{R}^n$.

[31] Assumptions 1 and 2 guarantee the existence of such maps.

Hence, for all $v > \hat{v}$, the conclusions of the lemma follow for sequences $\{p^{t^v}\}$ and $\{p^{t'^v}\}$. ∎

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