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Unit Versus Ad Valorem Taxes: The Private Ownership of Monopoly In General Equilibrium*

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Abstract

Employing a general equilibrium framework, Blackorby and Murty [2007] prove that, with a monopoly and under one hundred percent profit taxation and uniform lump-sum transfers, the utility possibility sets of economies with unit and ad valorem taxes are identical. This welfare-equivalence is in contrast to most previous studies, which demonstrate the superiority of the ad valorem tax in a partial equilibrium framework. In this paper we relax the assumption of one hundred percent profit taxation and allow the consumers to receive profit incomes from ownership of shares in the monopoly firm. We find that, under certain regularity conditions, for any fixed vector of profit shares, the utility possibility sets of economies with unit and ad valorem taxes are not generally identical. But it does not imply that one completely dominates the other. Rather, the two utility possibility frontiers cross each other. Additionally, employing a standard partial equilibrium welfare analysis, we show that the Marshallian social surpluses resulting from the two tax structures are identical when the government can implement unrestricted transfers.

JEL classification: H21

Keywords: Ad Valorem taxes, unit taxes, monopoly, private ownership economy, general equilibrium, second-best Pareto optimality.

Unit Versus Ad Valorem Taxes: The Private Ownership of Monopoly In General Equilibrium

by

Charles Blackorby and Sushama Murty

1. Introduction

In a recent paper[1] we showed, in the context of a general equilibrium model with a monopoly sector, that the utility possibility frontier in the face of ad valorem taxes is identical to the utility possibility frontier with unit taxes. This result is contrary to almost all of the previous literature, which demonstrates the welfare superiority of the ad valorem tax.[2] The characteristic of Blackorby and Murty [2007] model that generates the contradictory result is the assumption that the government levied profit taxes of one hundred percent rebating any resulting surplus as a *uniform* lump-sum transfer (also called a demogrant), which is a standard assumption in the general equilibrium literature on indirect taxes.[3]

It could be argued that the differences in the earlier results and Blackorby and Murty [2007] are because, in contrast to the general equilibrium approach of the latter paper, much of the earlier literature employs a partial-equilibrium framework. The models in this literature are usually silent about the end use of the monopolist's profit and the government's revenue. For this reason, perhaps, these analyses could be interpreted as being performed in an institutional structure where the government has control of and can do unrestricted redistribution of the available economic resources among consumers.[4]

Suits and Musgrave [1955] showed that for every ad valorem tax, there exists an *equivalent* unit tax that can support the profit maximizing output of the monopolist under the ad valorem tax, and vice-versa. The asymmetry between the unit and the ad valorem taxes arises because the monopolist's profits and the government's indirect tax revenues under an ad valorem tax and the equivalent unit tax are not equal.[5] Blackorby and Murty [2007] showed, however, that the sum of the government's revenue and monopoly profit does not change in the move from the ad valorem tax to the equivalent unit-tax.

This must imply that, even in a standard partial equilibrium welfare analysis conducted in an institutional setting where the government can implement personalized lump-sum transfers, the Marshallian social surpluses should be the same across both the tax systems. This result seems not to have been demonstrated by the earlier literature.

[1] Blackorby and Murty [2007].

[2] See Cournot [1838, 1960], Wicksell [1896, 1959], Suits and Musgrave [1955], Skeath and Trandel [1994], Keen [1998], and Delipalla and Keen [1992]. Lockwood [2004] is an exception.

[3] See, for example, Guesnerie [1995] and Guesnerie and Laffont [1978].

[4] *I.e.,* it can implement personalized lump-sum transfers.

[5] If the ad valorem tax is positive, the government revenue (the monopoly profit) is higher (lower) under the ad valorem tax as compared to the equivalent unit tax.

This must also imply that, in an institutional setting where the government's redistributive ability is constrained and the total governmental revenue from profit and indirect taxation is rebated to consumers as uniform lump-sum transfers, the ad valorem tax and the equivalent unit tax result in, not only identical monopoly output and consumer prices, but also identical consumer incomes and demands. Thus, every ad valorem-tax equilibrium also has a unit-tax equilibrium representation. The converse is also true. This is the welfare equivalence of ad valorem and unit taxes demonstrated in the Blackorby and Murty model.

In this paper we relax the assumption that government can tax the profits of firms, and allow consumers to benefit directly from profit incomes from their ownership of shares in firms. The problem raised by private ownership is that, given that the monopolist's profits and the government's indirect tax revenues under an ad valorem tax and the *equivalent* unit-tax are different, for a fixed vector of profit shares, the profit incomes and the demogrant incomes of the consumers change when moving from a system of ad valorem taxes to an *equivalent* system of unit taxes; hence, in general, a given ad valorem-tax equilibrium is not a unit-tax equilibrium of the same private ownership economy. Thus, there is no direct way to compare the set of unit-tax equilibria with the set of ad valorem-tax equilibria for a given private ownership economy. To circumvent this problems, in this paper, we have resorted to an indirect and somewhat novel procedure which draws heavily on earlier work in second-best economies by Guesnerie [1980] and Quinzi [1992] in a somewhat different context. Our method exploits the Suits and Musgrave result and the continuity of the difference in the consumer incomes under unit (respectively, ad valorem) taxes and equivalent ad valorem (respectively, unit) taxes.

In this paper, we first conduct a standard partial equilibrium welfare analysis employing an example with quasi-linear preference to demonstrate the equivalence of the Marshallian social surpluses that result from unit and ad valorem taxation of a monopoly when the government can do unrestricted personalized transfers. We also use this simple example to demonstrate the central problem created by private-ownership in the welfare comparison of unit and ad valorem taxes when there are restrictions of government's redistributive ability, namely, it can only implement a demogrant. Further, we use this example to outline the strategy that we adopt in this paper to facilitate such a comparison and show that it works for our example. The rest of the paper provides the proof of the above claim in a rather general model.

Our main result is that, under certain regularity assumptions, private ownership of the monopoly firm implies that the unit-tax utility possibility frontier and the ad valorem utility possibility frontier must cross each other. That is, there is a region where unit taxes Pareto dominate ad valorem taxes and another where ad valorem taxes dominate unit taxes.

2. A Two-Good Example

Consider a two-good economy, where the good indexed by zero is supplied by a monopoly, while the unindexed good is a non-produced good that is consumed by the consumers and used as an input by the monopoly.

2.1. Preferences and technology.

Suppose there are H consumers in the economy. Each consumer h has quasi-linear preferences that are linear in the non-produced good, which are represented by utility function $u^h = x^h + \phi^h\left(x_0^h\right) := x^h + b^h x_0^h - \left(x_0^h\right)^2,\ 0 \leq x_0^h \leq \frac{b^h}{2},\ x^h \geq 0$. Then the individual and aggregate Marshallian demands for the monopoly good are independent of consumer incomes and the aggregate demand for the non-produced good depends only upon the aggregate income and not upon its distribution. Thus, the individual and aggregate demand functions, as a function of consumer prices (q_0, q) and consumer incomes, $\langle w_h \rangle$, are[6]

$$x_0^h\left(q_0, q\right) = \frac{b^h - q_0}{2}, \quad x^h(q_0, q, w_h) = \frac{1}{q}[w_h - q_0\frac{b^h - q_0}{2}], \tag{2.1}$$

$$x_0\left(q_0, q\right) = \frac{b - Hq_0}{2}, \text{ and } \sum_h x^h\left(q_0, q, w_h\right) = \frac{1}{q}[\sum_h w_h - q_0\frac{b - Hq_0}{2}], \tag{2.2}$$

where $b := \sum_h b^h$. Assume also that the monopolist faces constant marginal costs, so that his cost function is $C(y_0^u, q) = cqy_0^u$. His demand for input is thus cy_0^u. This framework lends itself to a usual partial equilibrium welfare analysis.[7]

2.2. The case of unit taxation and personalized lump-sum transfers.

Consider the case of an economy where (i) the government can implement a unit tax on the monopolist, (ii) the monopolist takes the price of the input as given, (iii) the initial endowment of the input is ω, (iv) there is no initial endowment of the monopoly good, (v) there is no tax on the competitive commodity, and (vi) government has full discretion to distribute, as personalized lump-sum transfers (w_h), its revenue from taxation of the monopoly good, the monopolist's profits[8], and the endowment of the non-produced good that is unused by the monopolist . If the monopolist is subject to a unit tax t_0 then the net-of-tax (producer) price that he receives is $p_0^u = q_0 - t_0$. The monopolist solves the problem

$$\Pi^{mu}\left(t_0, q\right) := \max_{p_0^u} p_0^u x_0\left(p_0^u + t_0, q\right) - cx_0\left(p_0^u + t_0, q\right) q, \tag{2.3}$$

which yields the profit maximizing price, quantity, and profits of the monopolist and the consumer price as the functions

$$\begin{aligned} p_0^u = P_0^u(t_0, q) = \frac{b - Ht_0 + Hcq}{2H}, \quad y_0^u = x_0\left(q_0, q\right) = \frac{b - Hcq - Ht_0}{4}, \\ \Pi^{mu}(t_0, q) = \frac{\left(b - Hcq - Ht_0\right)^2}{8H}, \text{ and } q_0 = P_0^u(t_0, q) + t_0 = \frac{b + Ht_0 + Hcq}{2H}. \end{aligned} \tag{2.4}$$

6 We use the notation $\langle w_h \rangle$ is used to denote a vector $(w_1, \ldots, w_H)$.
7 See, for example, Chapter 10 in Mas-Colell et al [1995].
8 *I.e.*, it can implement one-hundred percent taxation of profits

Note that for the profit maximizing price and quantity to be non-negative, we require that

$$t_0 \leq \frac{b - Hcq}{H} \leq \frac{b + Hcq}{H}. \tag{2.5}$$

An equilibrium in this economy with taxation of monopoly profits, unit taxation of the monopoly good, and personalized lump-sum transfer is described below:

$$\begin{aligned} x_0(p_0^u + t_0, q) &= y_0^u \\ p_0^u &= P_0^u(t_0, q) \\ \sum_h x^h(p_0^u + t_0, q, w_h) &= \omega - cy_0^u. \end{aligned} \tag{2.6}$$

It follows from Walras law that at any such equilibrium, the government's budget is balanced:

$$\begin{aligned} \sum_h w_h &= \Pi^{mu}(t_0, q) + t_0 y_0^u + q\omega \\ &= \frac{(b - Hcq - Ht_0)(b - Hcq + Ht_0)}{8H} + q\omega. \end{aligned} \tag{2.7}$$

(2.6) is a system of 3 equations in $4 + H$ unknowns, namely, $p_0^u, y_0^u, q, t_0, w_1, \ldots, w_H$. Also it is homogeneous of degree zero in $p_0^u, q, t_0, w_1, \ldots, w_H$. Hence it admits a harmless normalization, say $q = 1$. The non-produced good then can be interpreted as a numeraire commodity. With this normalization, there are H degrees of freedom in choosing tax equilibria in this economy. For example, an equilibrium specified by (2.6) is uniquely determined once we fix t_0 and $w_h \geq 0$ for $h = 1, \ldots, H-1$ such that (2.5) holds and $\sum_{h=1}^{H-1} w_h \leq \frac{(b-Hc-Ht_0)(b-Hc+Ht_0)}{8H} + \omega$, *i.e.,* fixing these variables fixes the equilibrium levels of all the remaining variables, *i.e.,* under the normalization adopted, (2.6) is characterized by

$$\begin{aligned} y_0^u = \frac{b - Hc - Ht_0}{4}, \quad p_0^u = \frac{b - Ht_0 + Hc}{2H} \\ w_H = \frac{(b - Hc - Ht_0)(b - Hc + Ht_0)}{8H} + \omega - \sum_{h=1}^{H-1} w^h. \end{aligned} \tag{2.8}$$

Thus, equilibria described by (2.6) are fully parametrized by the H variables $t_0, w_1, \ldots, w_{H-1}$. In particular, note that if we fix t_0, we get a whole set of equilibria corresponding to different distributions of the total income. In each of these equilibria, since t_0 is fixed, (2.8) implies that p_0^u, the individual and aggregate demands for the monopoly good (and hence the individual and aggregate expenditure on the monopoly good), and the demands for input by the monopolist are the same as these variables are independent of consumer incomes. The total amount of the numeraire good that is available for distribution to consumers is the initial endowment of this good minus the amount used as input by the monopolist: $\sum_h \omega^h - cy_0^u = \omega - \frac{c[b-Hc-Ht_0]}{4}$. Hence, these equilibria will differ only with respect to consumptions of (and expenditures on) the numeraire good by different consumers. The quasi-linear structure of the preferences implies that equilibrium allocation corresponding to a given t_0 are obtained by transferring the available amount of the numeraire good unit for unit between consumers. Depending on the amount of the numeraire good, x^h, that he receives, the utility of consumer h for a fixed t_0 is given by $u_h = \frac{(2b^h H - b - Ht_0 - Hc)(2b^h H + b + Ht_0 + Hc)}{16H^2} + x^h$. We define the frontier of equilibrium utility profiles of this economy that are made possible when t_0 is fixed as

$$\mathcal{U}^u(t_0) := \{\langle u_1, \ldots, u_H \rangle | \; \forall\; h = 1, \ldots, H, \exists\; x^h \geq 0 \text{ such that } \sum_h x^h = \omega - c[\frac{b - Hc - Ht_0}{4}]$$
$$\text{and } u_h = \frac{(2b^h H - b - Ht_0 - Hc)(2b^h H + b + Ht_0 + Hc)}{16H^2} + x^h\}. \tag{2.9}$$

For all $\langle u_1, \ldots, u_H \rangle \in \mathcal{U}^u(t_0)$, we have

$$\begin{aligned}
\sum_h u_h &= \sum_h \phi^h \left(x_0^h (p_0^u + t_0, q = 1) \right) + \sum_h x^h \left(w^h, p_0^u + t_0, 1 \right) \\
&= \sum_h \phi^h \left(x_0^h (p_0^u + t_0, 1) \right) + \sum_h \omega^h - c y_0^u \\
&= \sum_h \frac{(2b^h H - b - Ht_0 - Hc)(2b^h H + b + Ht_0 + Hc)}{16H^2} + \omega - \frac{c[b - Hc - Ht_0]}{4},
\end{aligned} \tag{2.10}$$

which is a constant, so that $\mathcal{U}^u(t_0)$ is linear.

For any level of the unit tax t_0 and as in any partial equilibrium welfare analysis, call the expression

$$M^u(t_0) = \sum_h \phi^h \left(x_0^h (p_0^u + t_0, 1) \right) - c y_0^u \tag{2.11}$$

the Marshallian surplus. We can rewrite $M^u(t_0)$ as

$$\begin{aligned}
M^u(t_0) &= \sum_h \phi^h \left(x_0^h (p_0^u + t_0, 1) \right) - c x_0 (p_0^u + t_0, 1) + q_0 x_0 (p_0^u + t_0, 1) - q_0 x_0 (p_0^u + t_0, 1) \\
&= [\sum_h \phi^h \left(x_0^h (p_0^u + t_0, 1) \right) - q_0 x_0 (p_0^u + t_0, 1)] \; + \; [p_0^u x_0 (p_0^u + t_0, 1) - c x_0 (p_0^u + t_0, 1)] \\
&\quad + \; t_0 x_0 (p_0^u + t_0, 1) \\
&= [\sum_h \phi^h \left(x_0^h (p_0^u + t_0, 1) \right) - q_0 x_0 (p_0^u + t_0, 1)] \; + \; \Pi^{mu}(t_0, 1) \; + \; t_0 x_0 (p_0^u + t_0, 1) .
\end{aligned} \tag{2.12}$$

Given the quasi-linear structure of preferences, the first term in the last equation of (2.12) is the consumer surplus resulting from the monopolist's profit maximizing choice. Thus, the Marshallian surplus $M^u(t_0)$ is the sum of consumer surplus, the government's tax revenue, and the profit of the monopolist when the monopoly is subject to a unit tax.

2.3. The case of ad valorem taxation and personalized lump-sum transfers.

Now suppose that the monopoly commodity is taxed in an ad valorem manner, with the ad valorem tax denoted by τ_0, so that the price faced by the monopolist is $p_0^a = \frac{q_0}{1+\tau_0}$. The problem of the monopolist is

$$\Pi^{ma}(\tau_0, q) := \max_{p_0^a} x_0\left(p_0^a(1+\tau_0), q\right)\left[p_0^a - cq\right] \tag{2.13}$$

yielding the solution $p_0^a = P_0^a\left(\tau_0, q\right) = \frac{b+Hcq(1+\tau_0)}{2H(1+\tau_0)}$.[9] Note that for the profit maximizing price and quantity to be non-negative, we require that

$$\frac{b - cqH}{cqH} \geq \tau_0 \geq \frac{-(b + 2cqH^2)}{2cqH^2}. \tag{2.14}$$

Consider an exactly same economy as in the previous section with government being able to do personalized lump-sum transfers, but with the monopolist facing an ad valorem tax on his good. Let the incomes consumer receive in this ad valorem economy be denoted by $\langle R_h \rangle$. An equilibrium can be defined exactly as in the unit tax case. A harmless normalization $q = 1$ can be adopted and the set of equilibria are fully parametrized by the H variables τ_0, $R_1, \ldots, R_{H-1}$. Exactly as in the above section, we can also define $\mathcal{U}^a\left(\tau_0\right)$, the frontier of equilibrium utility profiles of this economy that are made possible when τ_0 is fixed and,

$$M^a\left(\tau_0\right) = [\sum_h \phi^h\left(x_0^h\left(p_0^a(1+\tau_0), 1\right)\right) - q_0 x_0] \; + \; \Pi^{ma}\left(\tau_0, 1\right) \; + \; p_0^a \tau_0 x_0\left(p_0^a(1+\tau_0), 1\right), \tag{2.15}$$

the Marshallian surplus corresponding to τ_0.

2.4. Unit vs ad valorem: the case of personalized lump-sum transfers.

As demonstrated out by Suits and Musgrave [1955], for every unit tax t_0 there exists an equivalent ad valorem tax that results in the monopolist choosing the same level of the output (and conversely). This is obtained by solving the following for τ_0:

$$x_0(P_0^u(t_0, q) + t_0, q) = x_0(P_0^a(\tau_0, q)(1+\tau_0), q) \tag{2.16}$$

With $q = 1$ this implies

$$\frac{b - Hc - Ht_0}{4} = \frac{b - Hc(1+\tau_0)}{4} \tag{2.17}$$

and we obtain the equivalent ad valorem tax rate as

$$\tau_0 = \frac{t_0}{c}. \tag{2.18}$$

[9] It is easy to check that the problem yields the profit maximizing quantity and profits of the monopolist as well as the consumer price as $y_0^a = x_0\left(q_0, q\right) = \frac{b-Hcq(1+\tau_0)}{4}$, $\Pi^{ma}\left(t_0, q\right) = \frac{(b-Hcq(1+\tau_0))^2}{8H(1+\tau_0)}$, and $q_0 = \frac{b+Hcq(1+\tau_0)}{2H}$.

Since the profit maximizing outputs of the monopolist are the same under both t_0 and its equivalent ad valorem tax, the demands for the numeraire by the monopolist as an input are also equal in the two scenarios. This means that the amounts of the numeraire left that can be potentially distributed to consumers are also same under t_0 and the equivalent ad valorem tax.

Clearly: *when the government can do unrestricted transfers of the numeraire between consumers, each equilibrium allocation under a unit tax is also attainable as an equilibrium allocation under the equivalent ad valorem tax and*

$$\mathcal{U}^u(t_0) = \mathcal{U}^a\left(\frac{t_0}{c}\right). \tag{2.19}$$

Further, Blackorby and Murty [2007] result can be easily verified in this example:

$$\Pi^{mu}(t_0,1) - \Pi^{ma}(\frac{t_0}{c},1) = -[t_0 y_0 - \frac{t_0}{c} p_0^a y_0] = \frac{[b - Hc - Ht_0]^2 t_0}{8H(c+t_0)} \begin{bmatrix} > \\ < \\ = \end{bmatrix} 0 \Leftrightarrow t_0 \begin{bmatrix} > \\ < \\ = \end{bmatrix} 0, \tag{2.20}$$

where y_0 is the output level chosen by the monopolist when faced with t_0 and p_0^a is the profit maximizing price for the equivalent ad valorem tax rate $\frac{t_0}{c}$.

(2.20) says that: *the sums of monopoly profit and tax revenue are the same when the monopolist faces a unit or an equivalent ad valorem tax, although for a positive tax on the monopolist, the monopoly profit under unit taxation is larger than under an equivalent ad valorem tax, while the tax revenue under unit taxation is smaller than under an equivalent ad valorem tax. The reverse is true if t_0 is negative.*

Since consumer surplus is the same under t_0 and its equivalent ad valorem tax, (2.20), (2.12), and (2.15) imply that: *the Marshallian surpluses under unit and ad valorem taxation are equal:*

$$M^u(t_0) = M^a(\frac{t_0}{c}). \tag{2.21}$$

This demonstrates the welfare equivalence of unit and ad valorem taxation of a monopoly in the case when government can do personalized lump-sum transfers.

2.5. The optimal tax on monopoly with personalized lump-sum transfers.

The optimal tax on the monopoly good is obtained by choosing the tax rate that maximizes the Marshallian surplus:

$$\begin{gathered} \max_{t_0, p_0^u, y_0^u} \sum_h \phi^h(x_0^h(p_0^u + t_0, 1)) - c y_0^u \\ \text{subject to} \\ x_0(p_0^u + t_0, 1) = y_0^u \quad \text{and} \quad p_0^u = P_0^u(t_0, 1). \end{gathered} \tag{2.22}$$

From (2.10) this is equivalent to

$$\max_{t_0} \sum_h \frac{(2b^h H - b - Ht_0 - Hc)(2b^h H + b + Ht_0 + Hc)}{16H^2} - \frac{c[b - Hc - Ht_0]}{4}. \tag{2.23}$$

The solution of this problem is

$$\overset{*}{t}_0 = \frac{cH - b}{H}. \tag{2.24}$$

Given (2.5) and (2.4), we find that: *the Pareto optimal unit tax on a monopolist when the government can implement personalized lumpsum transfers is* $\overset{*}{t}_0 < 0$*. Further, at this tax rate, the consumer price* $\overset{*}{q}_0$ *of the monopoly good is equal to the marginal cost* c *of the monopolist.*

Thus, as in Guesnerie and Laffont [1978], the optimal tax on the monopolist when the government can implement personalized lump-sum transfers is a subsidy that leads him to choose a level of monopoly output that corresponds to the level in a perfectly competitive economy: in this economy, the distortion created by a monopoly can be fully corrected by subsidizing the monopolist.

The first-best utility possibility frontier is

$$\mathcal{U}^{FB} \equiv \mathcal{U}^u(\overset{*}{t}_0)\,. \tag{2.25}$$

Since the ad valorem tax that is equivalent to $\overset{*}{t}_0$ is $\frac{\overset{*}{t}_0}{c}$, we also have

$$\mathcal{U}^{FB} = \mathcal{U}^a(\frac{\overset{*}{t}_0}{c}). \tag{2.26}$$

2.6. Reconciliation with Skeath and Trandel [1994].

Skeath and Trandel [1994] argue that for every unit tax on a monopolist there exists an ad valorem tax that yields a higher consumer surplus, a higher monopoly profits, and also a higher tax revenue to the government. Note the following with respect to their argument:

(1) Their proof for the existence of such an ad valorem tax rate holds only if the government can implement positive rates of taxation. The revealed preference argument employed will fail if the tax rate is negative.

(2.) The discussion above reveals that if for every unit tax $t_0 > 0$ there exists an ad valorem tax, say $\tau_0 > 0$, that yields a higher Marshallian surplus, then there is also a unit tax $\bar{t}_0 = c\tau_0 > 0$ that is equivalent to τ_0 that yields a higher Marshallian surplus than t_0. We can continue the argument further and establish that there exists also a unit tax and an equivalent ad valorem that yield a higher Marshallian surplus than $\bar{t}_0$. This argument can go on for ever if we confine ourselves to positive rates of taxation. This is because in this partial equilibrium framework based on Marshallian surplus, the optimal rate of taxation on the monopolist is negative. Thus, every positive unit tax will always be Pareto dominated by another positive unit and an equivalent positive ad valorem tax.

(3) At the Pareto optimal unit tax (which is negative) there exists no ad valorem tax that can yield a higher Marshallian surplus. There however exists an equivalent ad valorem tax that yields the same Marshallian surplus.

Thus, we reconcile the arguments of Skeath and Trandel [1994] with our arguments above on the equivalence of unit and ad valorem taxation of a monopoly with respect to the Marshallian surplus.

2.7. The problem with private ownership with no personalized lump-sum transfers.

Now consider economies with taxation of the monopoly good, where consumers receive a share of the monopoly profits, the aggregate endowment, ω, of the numeraire commodity is held by the consumers according to a distribution $\langle \omega^h \rangle$, and where government redistributes its tax revenue as *uniform* lumpsum transfers (That is, each consumer receives $1/H$ of the government deficit or surplus.). Suppose the share of consumer h in the monopoly profit is $\theta_h \in [0,1]$ with $\sum_h \theta_h = 1$. Consumer h's income is composed of his profit income, the lump-sum transfer from the government, and his endowment income. A unit tax equilibrium in this private ownership economy with profit shares $\theta = \langle \theta_h \rangle$ is given by

$$\begin{aligned} x_0(p_0^u + t_0, q) = y_0^u, \qquad p_0^u = P_0^u(t_0, q) \\ w_h = \theta_h[p_0^u y_0^u - c y_0^u] + \frac{t_0 y_0^u}{H} + q\omega^h, \ \forall\ h. \end{aligned} \tag{2.27}$$

Note that, from Walras law, (2.27) implies that the market for the non-produced input clears:

$$\sum_h x^h(p_0^u + t_0, q, w_h) = \omega - c y_0^u. \tag{2.28}$$

(2.27) is a system of $2+H$ equations in $4+H$ unknowns $p_0^u, y_0^u, q, t_0, w_1, \ldots, w_H$. The system is homogeneous of degree zero in $p_0^u, q, t_0, w_1, \ldots, w_H$ and so admits the normalization $q = 1$. Hence, effectively, there is one degree of freedom in choosing equilibria, that is the set of equilibria of this θ ownership economy can be parametrized by the variable t_0 such that (2.5) holds. The equilibrium values of $p_0^u, y_0^u, w_1, \ldots, w_H$ are (uniquely) determined once t_0 is fixed. Precisely, they are

$$\begin{aligned} p_0^u = \frac{b + Hc - Ht_0}{2H}, \qquad y_0^u = \frac{b - Hc - Ht_0}{2H}, \text{ and} \\ w_h = \frac{(b - Hc - Ht_0)}{8H}[\theta^h(b - Hc - Ht_0) + 2t_0] + \omega^h \ \forall\ h. \end{aligned} \tag{2.29}$$

Similarly too we can define an ad valorem tax equilibrium, where τ_0 is the ad valorem tax rate and $p_0^a = q_0/(1+\tau_0)$ is the producer price faced by the monopolist under the ad valorem tax. The income of consumer h under the ad valorem tax is

$$R_h = \theta_h[p_0^a y_0^a - c\,(y_0^a)\,q] + \frac{1}{H}\tau_0 p_0^a y_0^a + \omega^h, \tag{2.30}$$

with $p_0^a = P_0^a(\tau_0, 1)$ and $y_0^a = x_0^a\big(p_0^a(1+\tau_0), 1\big)$.

As in Blackorby and Murty [2007], (2.20) and (2.18) can be used to show that, in an economy where monopoly profit is taxed at 100% and rebated back to the consumers as uniform lumpsum transfers, a unit tax equilibrium has an equivalent ad valorem tax representation and vice-versa.[10] Hence, the set of equilibrium allocations are the same under both the tax systems, and this implies that the two taxes are equivalent in terms of individual well-being.

However, when monopoly profits are not taxed and there is private ownership of the monopoly, then the switch from unit to an equivalent ad valorem tax (or vice-versa) implies that the incomes of the consumers, in general, change because of the difference in the composition of profit income and the uniform lumpsum transfer from the government[11]: if $t_0 \neq 0$ and $\theta_h \neq \frac{1}{H}$ for all h, then we have

$$\begin{aligned} w_h &= \theta_h \Pi^{mu} + \frac{1}{H} t_0 y_0 + \omega^h \\ &\neq \\ R_h &= \theta_h \Pi^{ma} + \frac{1}{H} \tau_0 p_0^a y_0 + \omega^h, \end{aligned} \tag{2.31}$$

even though

$$\sum_h w_h = \sum_h R_h. \tag{2.32}$$

In terms of consumer demands (given quasi-linear preferences which are linear in the numeraire good),

$$x^h(q_0, 1, w_h) \neq x^h(q_0, 1, R_h), \ \forall h. \tag{2.33}$$

even though

$$\sum_h x^h(q_0, 1, w_h) = \sum_h x^h(q_0, 1, R_h) \tag{2.34}$$

and the consumer demands for the monopoly good, which are not subject to income effects, also remain the same in this switch. The above implies that although the aggregate demands remain unchanged in the switch from unit to the equivalent ad valorem tax, the individual demands for the numeraire good and, hence, the utilities of consumers change as do the set of equilibrium allocations.[12] Thus, the issue of dominance cannot be studied directly.

[10] The income distribution achieved at a unit-tax equilibrium of such an economy is also achieved by implementing the equivalent ad valorem tax. In particular, the income distribution achieved is the same as the one achieved in a private ownership economy with a demogrant, where the share θ_h of *any* consumer h is $\frac{1}{H}$. (In such private ownership economies, at a unit tax and its equivalent ad valorem tax equilibria, we have $\langle w_h \rangle = \langle R_h \rangle$.)

[11] Unless $\theta_h = \frac{1}{H}$ for all h. This case is theoretically equivalent to the case of 100% taxation of profits.

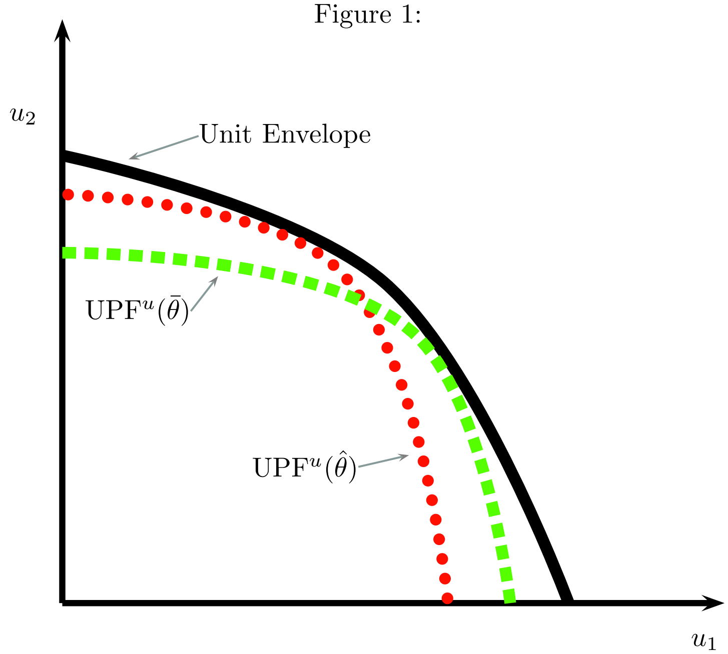
[12] For more general preference structures, the switch may not even result in an equilibrium allocation.

2.8. A sketch of the solution for the case of private ownership with no personalized lump-sum transfers.

In order to be able to make a comparison of the two tax regimes in the general case (as well as for the example above) we proceed in an indirect manner which ultimately yields results. Consider the move from unit-taxation to ad valorem taxation as the reverse is more or less the same. At every unit-tax equilibrium of a given private ownership economy, the equivalent ad valorem tax leads to the same production decision by the monopolist. However, as discussed above, under this ad valorem tax, the given allocation of profit shares results in different distributions of consumer incomes and hence different consumption decisions.

We proceed in the following manner. First, for each private ownership economy $\theta = \langle \theta_h \rangle$ with H consumers (that is, for each possible allocation of shares to the consumers), we define the unit-tax utility possibility set ($UPS^u(\theta)$) as the set of all utility profiles corresponding to all possible unit-tax equilibria of the given private ownership economy. We construct the utility possibility frontier ($UPF^u(\theta)$) by maximizing the utility of one consumer holding the utilities of all other consumers fixed and subject to equilibrium conditions for unit-taxation in the given private-ownership economy. Next we construct the outer envelope of these utility possibility frontiers. That is, for each feasible fixed level of utilities for persons 2 through H, we maximize, by choosing the allocation of private shares, the utility of consumer one. (See Figure 1, which illustrates this for $H = 2$).

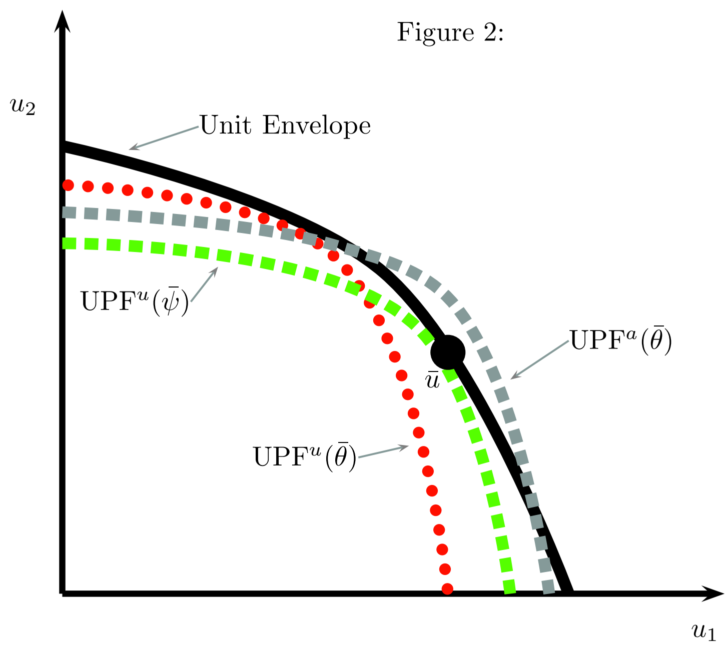
Figure 1:



The unit-tax envelope

Picking a particular fixed set of shares, say $\bar{\theta} = \langle \bar{\theta}_h \rangle$, we then search along this unit-tax envelope to see if there is a point on it that is also supported as an equilibrium of $\bar{\theta}$ private-ownership ad valorem economy. Under one set of regularity conditions we show such a point (a vector of consumers' utilities), say $\bar{u} = \langle \bar{u}_h \rangle$, exists by a fixed-point argument (see Figure 2.) Since $\bar{u}$ lies on the unit envelope, there exists a share profile, say $\bar{\psi} = \langle \bar{\psi}_h \rangle$, such that the Pareto frontier of the corresponding unit-tax economy is tangent to the unit envelope at $\bar{u}$. We show that under our regularity conditions, at $\bar{u}$, the consumer incomes and equilibrium prices and quantities in the ad valorem and unit economies are the same. However, we find that $\bar{\psi}$ is not equal to $\bar{\theta}$ and that $\bar{u}$ never belongs to the utility possibility set of the $\bar{\theta}$ ownership unit economy *unless* the shares in $\bar{\theta}$ were all equal to $1/H$ (and hence equivalent to one hundred per cent profit taxation problem that was solved in Blackorby and Murty [2007]) or the optimal tax on the monopolist happened to be equal to zero. In this way, we obtain a point in the utility possibility set of a $\bar{\theta}$ private-ownership economy with ad valorem taxes which is not present in the utility possibility set of a $\bar{\theta}$ private-ownership economy with unit taxes, demonstrating that unit taxation does not dominate ad valorem when the monopoly is privately owned. (Figure 2 makes this clear by indicating both the utility possibility sets.)

Figure 2:



$\bar{u} \in \mathrm{UPS}^a(\bar{\theta})$, $\bar{u} \in \mathrm{UPF}^u(\bar{\psi})$, $\bar{u} \notin \mathrm{UPS}^u(\bar{\theta})$

The converse is proved in a similar way under another similar set of regularity constraints by searching for a unit-tax equilibrium along the ad valorem-tax envelope for a given allocation of shares. If both our sets of regularity constraints hold simultaneously then, taken together, these results substantiate the claim that neither tax system Pareto-dominates the other.

2.9. Implementing the solution in the case of quasi-linear preferences and constant marginal cost.

For the quasi-linear example studied in Section 2.7, it was found that set of equilibria corresponding to any $\theta = \langle \theta_h \rangle$ private ownership economy with unit taxation of the monopoly good and uniform lump-sum transfer is fully parametrized by t_0, where t_0 satisfies (2.5). Precisely, the parametrization was derived in (2.29). This means that there is a tax equilibrium in this economy that corresponds to the first-best optimal tax $\overset{*}{t}_0 = \frac{cH-b}{H}$. The independence of the demands for the monopoly good from incomes of consumers under quasi-linear preference structures implies that the demands for the monopoly good as well as the demand by the monopolist for the numeraire commodity as input at the private-ownership tax equilibrium will also be the demands at the first-best. This implies that the tax equilibrium allocation of the θ private-ownership economy corresponding to $\overset{*}{t}_0$ is also first-best Pareto optimal, in other words, there is a point on the utility possibility frontier $UPF^u(\theta)$ that is tangent to $\mathcal{U}^{FB}$. This is true for *every* θ private ownership economy, where $\theta_h \in [0,1]$ for all h and $\sum_h \theta_h = 1$. Clearly, from this argument it follows that: *in the example with quasi-linear preferences, the unit-tax envelope is a subset of* $\mathcal{U}^{FB}$. Using exactly the same argument, we also claim that: *in the example with quasi-linear preferences, the ad valorem-tax envelope is a subset of* $\mathcal{U}^{FB}$.

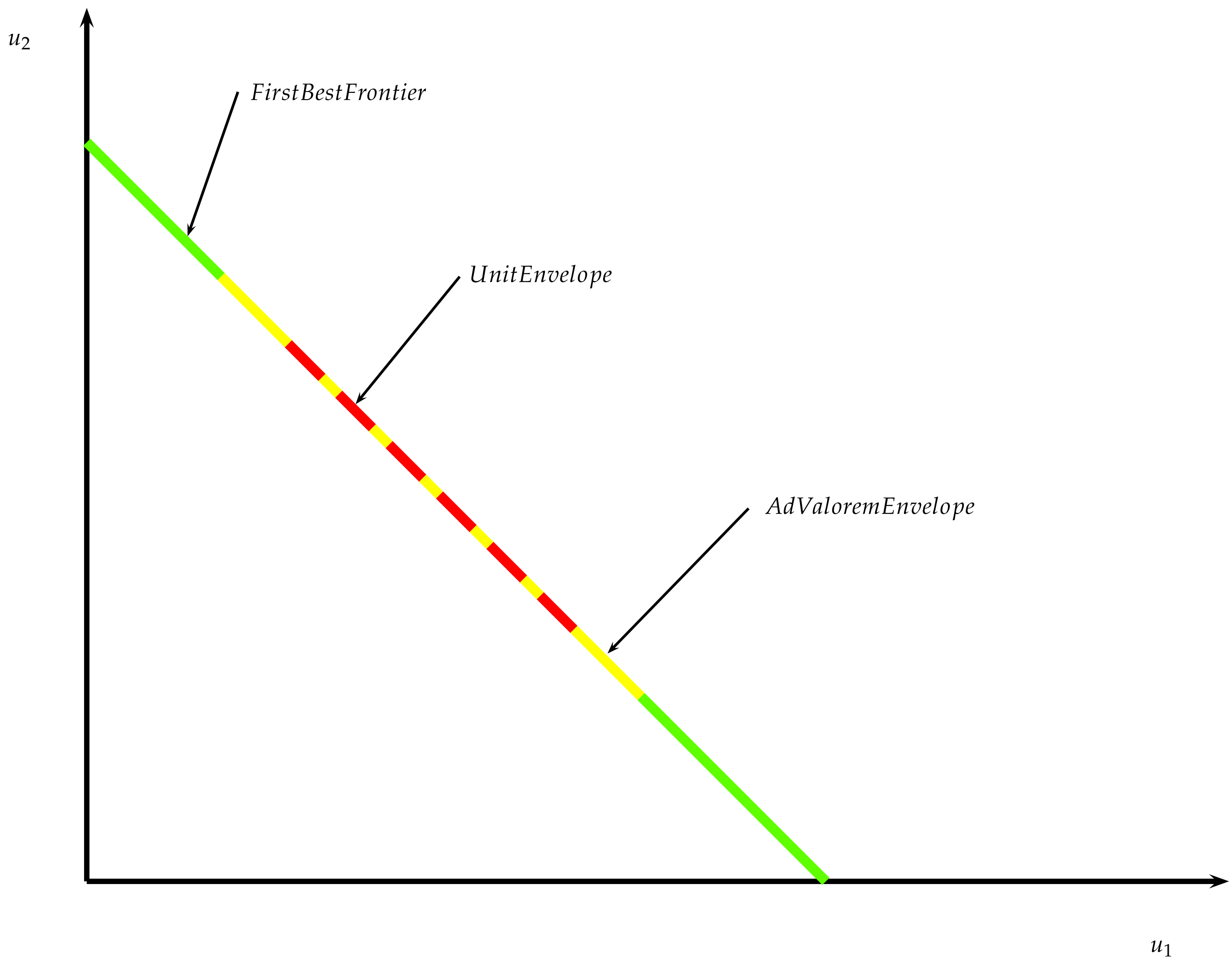
Further, we also find that: *quasi-linear preferences imply that the unit-tax envelope is a subset of the ad valorem-tax envelope.* This can be verified by looking at the ranges of values that the utility of each consumer h takes along the unit-tax envelope and the ad-valorem tax envelope. Consider the unit-tax envelope. Since the demands for the monopoly good, the input demand for the numeraire by the monopolist, and the unit tax are fixed at the first-best levels at every point on it, the monopoly profits and the demogrant are equal along all these points. The utility level of any consumer h varies along this envelope precisely because his income varies–as his profit share varies from zero to one–along this envelope.

For every h, let the range of utility levels along the ad-valorem envelope be denoted by $u_h \in [\underline{u}^h_u, \bar{u}^h_u]$.[13] Similarly, for every h, let the range of utility levels along the ad-valorem envelope be denoted by $u_h \in [\underline{u}^h_a, \bar{u}^h_a]$.[14] The upper-bounds correspond to the case where h's profit share is one, while the lower bounds correspond to the case where h's profit share is zero. The precise relation between the first-best frontier and the unit-tax and the ad valorem-tax envelopes is shown in Figure 3.

[13] Straightforward calculations yield $\underline{u}^h_u = \frac{(2b^hH-b-H\overset{*}{t}_0-Hc)(2b^hH+b+H\overset{*}{t}_0+Hc)}{16H^2} + \frac{(b-Hc-H\overset{*}{t}_0)}{4H}\overset{*}{t}_0 + \omega^h - b^h\frac{(b+H\overset{*}{t}_0+Hc)}{4H} + \frac{(b+H\overset{*}{t}_0+Hc)^2}{8H^2}$ and $\bar{u}^h_u = \frac{(2b^hH-b-H\overset{*}{t}_0-Hc)(2b^hH+b+H\overset{*}{t}_0+Hc)}{16H^2} + \frac{(b-Hc-H\overset{*}{t}_0)}{4H}[\frac{(b-Hc-H\overset{*}{t}_0)}{2} + \overset{*}{t}_0] + \omega^h - b^h\frac{(b+H\overset{*}{t}_0+Hc)}{4H} + \frac{(b+H\overset{*}{t}_0+Hc)^2}{8H^2}$.

[14] Straightforward calculations yield $\underline{u}^h_a = \frac{(2b^hH-b-Hc(1+\tau_0))(2b^hH+b+Hc(1+\tau_0))}{16H^2} + \frac{(b-Hc(1+\tau_0))}{8H(1+\tau_0)}[\frac{\tau_0(b+Hc(1+\tau_0))}{H}] + \omega^h - b^h\frac{(b+Hc(1+\tau_0))}{4H} + \frac{(b+Hc(1+\tau_0))^2}{8H^2}]$ and $\bar{u}^h_a = \frac{(2b^hH-b-Hc(1+\tau_0))(2b^hH+b+Hc(1+\tau_0))}{16H^2} + \frac{(b-Hc(1+\tau_0))}{8H(1+\tau_0)}[\frac{(b-Hc(1+\tau_0))}{2} + \frac{\tau_0(b+Hc(1+\tau_0))}{H}] + \omega^h - b^h\frac{(b+Hc(1+\tau_0))}{4H} + \frac{(b+Hc(1+\tau_0))^2}{8H^2}$, where $\tau_0 = \frac{\overset{*}{t}_0}{c}$.

Figure 3:



The case of quasi linear preferences: the first-best frontier and the unit and ad valorem envelopes

If we now look at the differences between the lower and upper bounds of the intervals $[\underline{u}_u^h, \bar{u}_u^h]$ and $[\underline{u}_a^h, \bar{u}_a^h]$, then we find that[15]

$$\bar{u}_u^h - \bar{u}_a^h = \frac{\overset{*}{t}_0(H-1)}{16H^3(c+\overset{*}{t}_0)}(b - Hc - H\overset{*}{t}_0)^2 < 0 \tag{2.35}$$

and

$$\underline{u}_u^h - \underline{u}_a^h = -\frac{\overset{*}{t}_0}{8H^2(c+\overset{*}{t}_0)}(b - Hc - H\overset{*}{t}_0)^2 > 0 \tag{2.36}$$

implying that $[\underline{u}_u^h, \bar{u}_u^h] \subset [\underline{u}_a^h, \bar{u}_a^h]$ for all h.

Intuitively, this is true because $\overset{*}{t}_0 < 0$, so that from (2.20) it follows that the demogrant under ad valorem taxation is smaller than under unit taxation, while the profits are larger under ad valorem taxation. Thus, for $\theta_h = 0$ (respectively, $\theta_h = 1$), when the consumer receives only demogrant (respectively, profit) income, apart, of course, from his endowment income, his income and hence utility is larger (respectively, smaller) under the unit tax than under the equivalent ad valorem tax. Precisely, this implies that the regularity condition (mentioned in the previous subsection and called Assumption 6 later in the general case), which is required to prove that ad valorem taxation does not dominate unit taxation, holds for this example.[16] The solution outlined in the previous section for welfare comparison of unit and ad valorem taxation under private ownership can now be employed in this example. Pick any profile of valid profit share-profile θ. A search along the ad valorem envelope will yield a utility profile $\bar{u}$ that also corresponds to an equilibrium in a θ private ownership economy with unit taxation. This is precisely because the unit envelope is a subset of the ad valorem envelope. Arguments made in the previous section while explaining the solution in the general case follow to show that ad valorem taxation does not dominate unit taxation in private ownership economies.

Further, a search along the unit envelope for a utility profile that corresponds to an equilibrium in an ad valorem private ownership economy with share-profile θ may or may not be successful as the unit envelope is a subset of the ad valorem envelope. If successful, the arguments outlined in the solution provided in the previous section apply. If not, then too the utility profile that lies on the ad valorem envelope and corresponds to a private ownership economy with share-profile θ is not attainable in a unit private ownership economy with share-profile θ unless $\theta_h = \frac{1}{H}$ for all h. This demonstrates that unit taxation does not dominate ad valorem taxation in private ownership economies.

[15] To sign these expressions, note that (2.18) implies $1+\tau_0 = \frac{c+t_0}{c}$, which in turn implies that $c+t_0 > 0$.

[16] For every consumer h and given any share of profit $\theta_h \in [0,1]$, the income associated with an equivalent unit tax at any point on the ad valorem envelope lies between the incomes h receives under ad valorem taxation at the two end points of the ad valorem frontier corresponding to shares zero and one.

2.10. Issues with extension to more general economies.

The example studied above restricted focus to the case of quasi-linear preferences and did not allow for taxation of the numeraire good. In more general economies, the relations between the first-best frontier and the unit-tax and ad valorem-tax envelopes may be more general. With more general preferences, the demands for the monopoly good will not be independent of consumer incomes and hence will vary along the first-best frontier. This implies that, even if there are common instruments that parametrize tax equilibria in both economies with personalized lump-sum transfers and economies with private-ownership[17], the utility profiles obtained at a private-ownership tax equilibrium may be different from the one obtained at the first-best for the same set of values of these instruments. This is because the income profiles associated with these instrument values may differ in the two equilibria in more general economies.[18] This may imply that a second-best utility possibility frontier corresponding to a private-ownership economy with taxation of the monopoly good may not be tangent to the first-best frontier, *i.e.,* the unit-tax (or the ad valorem-tax) envelope may not be a subset of the first-best frontier. Further, it could also be the case that neither of the two envelopes is a subset of the other.[19]

3. The general case: Description of the economy.

Consider an economy where $\mathcal{H}$ is the index set of consumers who are indexed by h. The cardinality of $\mathcal{H}$ is H. There are $N+1$ goods, of which the good indexed by 0 is the monopoly good. The remaining goods are produced by competitive firms.

The aggregate technology of the competitive sector is Y^c,[20] the technology of the monopolist is $Y^0 = \{(y_0, y^m) | y_0 \leq F^m(y^m)\}$, where $y^m \in \mathbf{R}^N_+$ is its vector of input demands and the technology of the public sector for producing g units of a public good is $Y^g(g) = \{y^g \in \mathbf{R}^N_+ |\ F(y^g) \geq g\}$. For all $h \in \mathcal{H}$, the gross consumption set is $X^h \subseteq \mathbf{R}^{N+1}$. The aggregate endowment is denoted by $(\omega_0, \omega) \in \mathbf{R}^{N+1}_{++}$ and is distributed among consumers as $\langle \omega^h_0, \omega^h \rangle$.[21] For all $h \in \mathcal{H}$, a gross consumption bundle is denoted by (x^h_0, x^h), and u^h denotes the utility function defined over the gross consumption set. The production bundle of the competitive sector is denoted by y^c, of the public sector by y^g, and of the monopolist by (y_0, y^m).

17 As we saw in the quasi-linear case, where the tax on the monopoly good, t_0, was the common instrument.

18 For example, in a two-good case with general preferences, unit-for-unit transfers between consumers of the amount of the non-monopoly good available at a first-best that corresponds to these instrument values may not be feasible due to income effects such transfers generate. This is unlike in the example with quasi-linear preferences.

19 See a working paper version of this paper, Blackorby and Murty [2008], for the relative positions of the first-best frontier and the unit and ad valorem envelopes for the general case.

20 Aggregate profit maximization in this sector is consistent with individual profit maximization by many different firms, as we assume away production externalities.

21 Any H dimensional vector of variables pertaining to all H consumers such as $(u_1, \ldots, u_H)$ is denoted by $\langle u^h \rangle$.

The economy is summarized by $E = (\langle \omega_0^h, \omega^h \rangle, \langle X^h, u^h \rangle, Y^0, Y^c, Y^g)$. An allocation in this economy is denoted by $z = \left(\langle x_0^h, x^h \rangle, y_0, y^m, y^c, y^g \right)$. A private ownership economy is one where the consumers own shares in the profits of both the competitive and monopoly firms. A profile of consumer shares in aggregate profits is given by $\langle \theta_h \rangle \in \Delta_{H-1}$.[22] The consumer price of the monopoly good is $q_0 \in \mathbf{R}_{++}$, $q \in \mathbf{R}_{++}^N$ is the vector of consumer prices of the competitively supplied goods. The wealth of consumer h is given by w_h. The producer price of the monopoly good is $p_0 \in \mathbf{R}_{++}$, $p \in \mathbf{R}_{+}^N$ is the vector of producer prices of the competitively supplied goods. The individual and aggregate consumer demands for the monopoly good are given by

$$x_0(q_0, q, \langle w_h \rangle) = \sum_h x_0^h(q_0, q, w_h), \tag{3.1}$$

and the individual and aggregate consumer demand vectors for the competitively supplied commodities are given by

$$x(q_0, q, \langle w_h \rangle) = \sum_h x^h(q_0, q, w_h). \tag{3.2}$$

The indirect utility function of consumer h is denoted by $V^h(q_0, q, w^h)$.[23] We assume that the monopolist is naive, in the sense that it does not take into account the effect of its decision on consumer incomes.[24] Its cost and input demand functions are denoted by $C(y_0, p)$ and $y^m(y_0, p)$, respectively. The aggregate competitive profit and supply functions are denoted by $\Pi^c(p)$ and $y^c(p)$, respectively. We use the following general assumptions on preferences and technologies in our analysis.

Assumption 1: For all $h \in \mathcal{H}$, the gross consumption set is $X^h = \mathbf{R}_+^{N+1}$, the utility function u^h is increasing, strictly quasi-concave, and twice continuously differentiable in the interior of its domain X^h. This, in turn, implies that the indirect utility function V^h is twice continuously differentiable.[25] We also assume that the demand functions $(x_0^h(), x^h())$ are twice continuously differentiable on the interior of their domain.

Assumption 2: The technologies Y^0, Y^c, and $Y^g(g)$ are closed, convex, satisfy free disposability. Y^0 and Y^c contain the origin. The public good production function F is strictly concave and twice continuously differentiable on the interior of its domain.

Assumption 3: The profit function of the competitive sector, Π^c, is assumed to be differentially strongly convex and the cost function $C(y_0, p)$ of the monopolist is assumed

[22] Δ_{H-1} is the $H-1$-dimensional unit simplex. Assuming that consumers have the same shares of monopoly and competitive sectors' profits makes the notation considerably simpler without any loss of generality.

[23] There is also a public good g but, as it remains constant throughout the analysis, it is suppressed in the utility function.

[24] Likewise we assume that consumers are naive; they do not anticipate changes in theirs incomes due to change in the profits of the monopolist.

[25] See Blackorby and Diewert [1979].

to be differentially strongly concave in prices and increasing and convex in output.[26] The competitive supply $y^c(p)$ is given by Hotelling's Lemma as $\nabla_p \Pi^c(p)$ and the input demands of the monopolist are given by $y^m(y_0, p) = \nabla_p C(y_0, p)$. The marginal cost $\nabla_{y_0} C(y_0, p)$ is positive on the interior of the domain of C.

3.1. A unit-tax private-ownership equilibrium.

The monopolist's optimization problem, when facing a unit tax $t_0 \in \mathbf{R}$ and when the vector of unit taxes on the competitive goods is $t \in \mathbf{R}^N$, is

$$P_0^u\left(p, t_0, t, \langle w_h \rangle\right) := \operatorname{argmax}_{p_0^u} \left\{ p_0^u \cdot x_0\left(p_0^u + t_0, p + t, \langle w_h \rangle\right) - C\left(x_0(p_0^u + t_0, p + t, \langle w_h \rangle), p\right) \right\}. \tag{3.3}$$

As discussed in detail in Guesnerie and Laffont [1978] the profit function of the monopolist (the function over which it optimizes) is not in general concave. Following them we assume that the solution to monopolist's profit maximization problem is locally unique and smooth. Under Assumptions 1, 2, and 3, the first-order condition for this problem is

$$\nabla_{q_0} x_0\left(p_0^u + t_0, p + t, \langle w_h \rangle\right) \ \left[p_0^u - \nabla_{y_0} C(y_0, p)\right] + x_0(p_0^u + t_0, p + t, \langle w_h \rangle) = 0 \tag{3.4}$$

which implicitly defines the solution $p_0^u = P_0^u(p, t_0, t, \langle w_h \rangle)$.

Assumption 4: P_0^u is single-valued and twice continuously differentiable function such that

$$\nabla_{t_0} P_0^u(p, t_0, t, \langle w_h \rangle) \neq -1. \tag{3.5}$$

As discussed in Guesnerie and Laffont [1978], $\nabla_{t_0} P_0^u \neq -1$ implies that the government can control q_0 by controlling t_0. Since consumer demands are homogeneous of degree zero in consumer prices and incomes, $\nabla_{q_0} x_0$ is homogeneous of degree minus one in these variables. Also, the cost function C is homogeneous of degree one in p. Hence, it follows that the left side of (3.4) is homogeneous of degree zero in $p_0^u, p, t_0, t,$ and $\langle w_h \rangle$. This implies that the function $P_0^u(p, t_0, t, \langle w_h \rangle)$ is homogeneous of degree one in $p, t_0, t,$ and $\langle w_h \rangle$.

A unit-tax equilibrium in private-ownership economy with shares $\langle \theta_h \rangle \in \Delta_{H-1}$ is given by[27]

$$-x\left(q_0, q, \langle w_h \rangle\right) + y^c(p) - y^m(y_0^u, p) - y^g + \omega \geq 0, \tag{3.6}$$

$$-x_0\left(q_0, q, \langle w_h \rangle\right) + y_0^u + \omega_0 \geq 0, \tag{3.7}$$

$$p_0^u - P_0^u\left(p, t_0, t, \langle w_h \rangle\right) = 0, \tag{3.8}$$

$$\begin{aligned} w_h = \theta_h \left[p_0^u y_0^u - C(y_0^u, p) + \Pi^c(p)\right] + \frac{1}{H} \left[t^T(y^c - y^m - y^g) + t_0 y_0^u - p y^g\right] \\ + q^T \omega^h + q_0 \omega_0^h, \ \forall h \in \mathcal{H} \end{aligned} \tag{3.9}$$

and

$$F(y^g) - g \geq 0, \ p_0^u \geq 0, \ p \geq 0_N, \ q_0 = p_0^u + t_0 \geq 0, q = p + t \geq 0_N. \tag{3.10}$$

[26] See Avriel, Diewert, Schaible, and Zang [1988].

[27] The superscript notation T stands for transpose. A matrix with subscript N is a square matrix with dimension N.

3.2. An ad valorem-tax private-ownership equilibrium.

The monopolist's profit maximization problem, when confronted with ad valorem taxes (τ_0, τ) is[28]

$$
\begin{aligned}
&P_0^a(p, \tau_0, \tau, \langle R_h \rangle) := \\
&\operatorname{argmax}_{p_0^a} \Big\{ p_0^a x_0 \Big(p_0^a(1+\tau_0), p^T(I_N + \boldsymbol{\tau}), \langle R_h \rangle \Big) \\
&\qquad - C\Big(x_0(p_0^a(1+\tau_0), p^T(I_N + \boldsymbol{\tau}), \langle R_h \rangle), p \Big) \Big\},
\end{aligned}
\tag{3.11}
$$

Assume that P_0^a is a single valued twice continuously differentiable function; Assumption 4 then implies that $(1+\tau_0)\nabla_{\tau_0} P_0^a \neq -P_0^a$.

A monopoly ad-valorem tax equilibrium in a private ownership economy with shares $\langle \theta_h \rangle \in \Delta_{H-1}$ satisfies

$$
-x\left(q_0, q, \langle R_h \rangle\right) + y^c(p) - y^m(p, y_0^a) - y^g + \omega \geq 0, \tag{3.12}
$$

$$
-x_0\left(q_0, q, \langle R_h \rangle\right) + y_0^a + \omega_0 \geq 0, \tag{3.13}
$$

$$
p_0^a = P_0^a\left(p, \tau_0, \tau, \langle R_h \rangle\right), \tag{3.14}
$$

$$
\begin{aligned}
R_h = \theta_h \left[p_0^a y_0^a - C(y_0^a, p) + \Pi^c(p)\right] + &\frac{1}{H}\left[\tau_0 p_0^a y_0^a + p^T \boldsymbol{\tau}\left[y^c - y^g - y^m\right] - p^T y^g\right] \\
&+ q^T \omega^h + q_0 \omega_0^h \ \forall h \in \mathcal{H},
\end{aligned}
\tag{3.15}
$$

$$
F(y^g) - g \geq 0, \tag{3.16}
$$

and

$$
p_0^a \geq 0, \ p \geq 0_N, \ q_0 = p_0^a(1+\tau_0) \geq 0, \ q = (I_N + \boldsymbol{\tau})p \geq 0_N. \tag{3.17}
$$

As in the unit-tax case, the function P_0^a is homogeneous of degree one in its arguments.

4. Unit versus ad valorem taxes in private ownership economies.

This section consists of two subsections. In the first we set out the assumptions and notation that we need and then show that the set of unit-tax Pareto optima for a given private-ownership economy does not contain the set of ad-valorem tax Pareto optima for the same private ownership economy. Hence, unit taxation does not dominate ad valorem taxation. The following subsection proves the converse.

[28] Symbols in bold face such as $\boldsymbol{\tau}$ and $\mathbf{p}$ stand for diagonal matrices with diagonal elements being the elements of vectors τ and p, respectively.

4.1. An ad valorem-tax private-ownership equilibrium on the envelope of unit-tax utility possibility frontiers

For each possible profile of profit shares, $\langle\theta_h\rangle \in \Delta_{H-1}$, we obtain a unit-tax Pareto frontier by solving the following problem for all utility profiles $(u_2, \ldots, u_H)$ for which solution exists:

$$\begin{aligned} \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle) := \max_{p_0^u, p, t_0, t, \langle w_h\rangle, q_0, q} & V^1(q_0, q, w_1) \\ \text{subject to} & \\ & V^h(q_0, q, w_h) \geq u_h, \text{ for } h = 2, \ldots, H, \\ & \text{and (3.6) to (3.10).} \end{aligned} \tag{4.1}$$

The envelope for the Pareto manifolds of all possible private ownership economies with unit taxes (which we will call the unit envelope) is obtained by solving the following problem for all utility profiles $(u_2, \ldots, u_H)$ for which solutions exist:

$$\begin{aligned} \hat{\mathcal{U}}^u(u_2, \ldots, u_H) := & \max_{\langle\theta_h\rangle} \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle) \\ & \text{subject to} \\ & \sum_h \theta_h = 1, \text{ and } \theta_h \in [0,1], \forall\, h \in \mathcal{H}. \end{aligned} \tag{4.2}$$

Denote the solution to this problem by

$$\langle \overset{*}{\theta}{}^u_h \rangle = \langle \overset{*}{\theta}{}^u_h(u_2, ..., u_h) \rangle. \tag{4.3}$$

That is, for given utility levels, $(u_2, \ldots, u_H)$, $\langle \overset{*}{\theta}{}^u_h \rangle$ is the vector of shares that maximizes the utility of consumer 1.

Next we generate an algorithm that identifies the ad valorem tax-equilibria that lie on the unit envelope.

Let A^u be the set of all allocations corresponding to the utility profiles on the unit envelope $\hat{\mathcal{U}}^u(u_2, ..., u_h)$. Define the identity mapping $I : A^u \to A^u$ with image $(\langle x_0^h(z), x^h(z)\rangle, y_0(z), y^m(z), y^c(z), y^g(z)) = z$.[29]

Let $\rho^u : A^u \to \mathbf{R}^H$ with image $\rho^u(z) = \langle u^h(x_0^h(z), x^h(z))\rangle$ be a utility map of the allocations in A^u. That is, for every $z \in A^u$, the set of utility levels enjoyed by consumers at that allocation is $\rho^u(z)$.

With some abuse of notation, let $\theta_h^u(z) = \overset{*}{\theta}{}^u_h(\rho^u(z))$ for $h \in \mathcal{H}$ be the solution of the problem (4.2) at the allocation z.

Our strategy is based on a fixed point argument and requires the restriction that all prices and taxes belong to a compact and convex set. A natural way to do so is to adopt

[29] That is, for every $z = \big(\langle x_0^h, x^h\rangle, y_0, y^m, y^c, y^g\big) \in A^u$, the mapping I assigns $\langle x_0^h(z), x^h(z)\rangle = \langle x_0^h, x^h\rangle$, $y^c(z) = y^c, y^g(z) = y^g, y_0(z) = y_0$, and $y^m(z) = y^m$.

a price normalization rule, which the equilibrium system allows as it is homogeneous of degree zero in the variables.[30] Let b be such a normalization rule such that the set

$$\mathcal{S}_b^u := \{(p_0, t_0, p, t) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}_+^N \times \mathbf{R}^N | \;\; b(p_0, t_0, p, t, \langle w_h \rangle) = 0 \text{ for any } \langle w_h \rangle \;\} \quad (4.4)$$

is compact. Let $\langle \bar{p}_0^u, \bar{t}_0, \bar{p}, \bar{t} \rangle$ and $\langle \underline{p}_0^u, \underline{t}_0, \underline{p}, \underline{t} \rangle$ be the vectors of maximum and minimum values attained by p_0^u, t_0, p, and t in $\mathcal{S}_b^u$. For example, $\bar{p}_0^u$ solves

$$\max \{p_0^u \in \mathbf{R}_+ | \; \exists t_0, p, t, \langle w_h \rangle \text{ such that } b(p_0^u, p, t_0, t, \langle w_h \rangle) = 0\}. \quad (4.5)$$

Define the mapping $\psi^u : A^u \to \mathcal{S}_b^u$ as $\psi^u(z) = (\psi_p^u(z), \psi_w^u(z))$, where $\psi_p^u(z) = (p_0^u(z), t_0(z), p(z), t(z))$ is the vector of unit taxes and producer prices associated with allocation z (a unit tax equilibrium), while $\psi_w^u(z) = \langle w^h(z) \rangle$ is the profile of consumer incomes associated with allocation z. Under Assumptions 1 to 4, ψ^u is continuous. For every $z \in A^u$, $q_0(z) = p_0^u(z) + t_0(z)$ and $q(z) = p(z) + t(z)$. Since $\mathcal{S}_b^u$ is compact, there exist $\langle \bar{q}_0, \bar{q} \rangle$ and $\langle \underline{q}_0, \underline{q} \rangle$ which denote the vector of maximum and minimum possible consumer prices that can be attained under the adopted price normalization rule.

Given (i) an appropriate normalization rule and (ii) the fact that, for a monopolist, $p_0^u(z) \geq \nabla_{y_0} C(y_0(z), p(z))$, we have, for every $z \in A^u$,

$$p(z) \in [\underline{p}, \bar{p}], \;\; p_0^u(z) \in [\underline{p}_0^u, \bar{p}_0^u], \;\; t(z) \in [\underline{t}, \bar{t}], \;\; t_0(z) \in [\underline{t}_0, \bar{t}_0], \;\; q_0(z) \in [\underline{q}_0, \bar{q}_0], \;\; q(z) \in [\underline{q}, \bar{q}], \quad (4.6)$$

and[31]

$$\nabla_{y_0} C(y_0(z), p(z)) \in [\underline{p}_0^u, \bar{p}_0^u,]. \quad (4.7)$$

Next, we use Suits and Musgrave [1953] argument to show that we can separate $q(z)$ and $q_0(z)$ into (equivalent) ad valorem taxes and producer prices defined by functions $(\tau_0(z), \tau(z))$ and $(p_0^a(z), p(z))$, which ensure that $(y_0(z), y(z))$ and $(p_0^a(z), p(z))$ are the profit maximizing outputs and prices in the monopoly and the competitive sector when the ad valorem taxes are $(\tau_0(z), \tau(z))$, that is, $(\tau_0(z), \tau(z))$ and $(p_0^a(z), p(z))$ solve

$$\begin{aligned} q_0(z) &= p_0^a(z)(1 + \tau_0(z)) \text{ and} \\ p_0^a(z) &= P_0^a \left(\tau_0(z), p(z), t(z), \langle w^h(z) \rangle \right) > 0 \\ q(z) &= (\boldsymbol{\tau}(z) + I_N) p(z). \end{aligned} \quad (4.8)$$

From an argument in Suits and Musgrave [1953][32] , for every $z \in A^u$, if

$$\frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))} > -1 \quad (4.9)$$

[30] The rationale for this including a discussion of valid normalization rules and a proof that their choice does not affect the solution (Lemma B3) is in Appendix B (Blackorby and Murty [2008]) of the working paper version.

[31] Note, normalization rules such as the unit hemisphere ensure that $\nabla_{y_0} C(y_0(z), p(z))$ lies in a compact set when the monopolist optimizes. This is because, under such a normalizaton, $\underline{p}_0^u = 0$ and hence, $\nabla_{y_0} C \geq \underline{p}_0^u = 0$.

[32] See also Blackorby and Murty [2007].

then choosing

$$\tau_0(z) = \frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))}, \tag{4.10}$$

$$\tau(z) = \mathbf{p}(z)^{-1} t(z) \tag{4.11}$$

$$p_0^a(z) = \frac{q_0(z)}{1+\tau_0(z)} = \frac{q_0(z)\ \nabla_{y_0} C(y_0(z), p(z))}{\nabla_{y_0} C(y_0(z), p(z))\ +\ t_0(z)} > 0. \tag{4.12}$$

does the job[33]

Define the mapping $\psi_p^a(z) := (p_0^a(z), \tau_0(z), p(z), \tau(z))$. ψ_p^a identifies the (equivalent) ad valorem taxes and prices associated with an allocation $z \in A^u$ that results in the same output decisions as in the unit tax equilibrium.

Since $\tau_0(z)$ and $p_0^a(z)$ are continuous functions, (4.10)–(4.12) imply that there exist compact intervals $[\underline{\tau}_0, \bar{\tau}_0]$, $[\underline{\tau}, \bar{\tau}]$, and $[\underline{p}_0^a, \bar{p}_0^a]$ such that for every $z \in A^u$, we have

$$\tau_0(z) \in [\underline{\tau}_0, \bar{\tau}_0]\,, \tag{4.13}$$

$$\tau(z) \in [\underline{\tau}, \bar{\tau}] \tag{4.14}$$

and

$$p_0^a(z) \in \left[\underline{p}_0^a, \bar{p}_0^a\right]. \tag{4.15}$$

We define

$$\mathbf{S}_b^u = \left[\underline{p}_0^a, \bar{p}_0^a\right] \times [\underline{\tau}_0, \bar{\tau}_0] \times \left[\underline{p}, \bar{p}\right] \times [\underline{\tau}, \bar{\tau}]\,, \tag{4.16}$$

and for every $z \in A^u$, we have $(p_0(z), \tau_0(z), p(z), \tau(z)) \in \mathbf{S}_b^u$, which is a compact and convex set.

For each allocation $z \in A^u$ we need to be able to identify the incomes of the consumers. Define an income map for consumer h as the map $r^{uh} : A^u \times \mathcal{S}_b^u \times [0,1] \to \mathbf{R}$, which for every $z \in A^u$, $\pi = (p_0^u, t_0, p, t) \in \mathcal{S}_b^u$, and $\theta_h \in [0,1]$ has image

$$\begin{aligned} r^{uh}(z, \pi, \theta_h) =& \theta_h \left[p_0^u y_0(z) - C(y_0(z), p) + \Pi^c(p)\right] + \\ & \frac{1}{H}\left[t_0 y_0(z) + t^T[y^c(z) - y^m(z) - y^g(z))] - p^T y^g(z)\right] \\ & + [p+t]^T \omega^h + [p_0^u + t_0]\omega_0^h \end{aligned} \tag{4.17}$$

[33] If (4.9) is not satisfied then there is no ad valorem tax that that yields the same profit-maximizing output as the given unit tax $t_0(z)$, for as seen in the equation immediately below, violation of (4.9) would imply that $p_0^a(z)$ is either less than zero or does not exist.

and so[34]

$$\sum_h r^{uh}\left(z,\psi_p^u(z),\theta_h^u(z)\right) = \Big[p_0^u(z)y_0(z) - p(z)^T y^m(z) + p^T(z)y^c(z)\Big]\sum_h \theta_h^u(z)$$
$$+ \Big[t_0(z)y_0(z) + t^T(z)[y^c(z) - y^m(z) - y^g(z))] - p^T(z)y^g(z)\Big]$$
$$+ [p(z)+t(z)]^T\omega + [p_0^u(z)+t_0(z)]\omega_0$$
$$= q_0(z)y_0(z) - q(z)^T y^m(z) + q(z)^T y^c(z) - q^T(z)y^g(z) + q^T(z)\omega + q_0(z)\omega_0 \tag{4.18}$$

For all $h \in \mathcal{H}$ let $r^{ah} : A^u \times \mathbf{S}_b^u \times [0,1] \to \mathbf{R}$ be defined so that

$$r^{ah}\left(z,\psi_p^a(z),\theta_h\right) = \theta_h\Big[p_0^a(z)y_0(z) - p(z)^T y^m(z) + p^T(z)y^c(z)\Big]$$
$$+ \frac{1}{H}\Big[\tau_0(z)p_0^a(z)y_0(z) + \tau^T(z)\mathbf{p}(z)\left[y^c(z) - y^m(z) - y^g(z))\right] - p^T(z)y^g(z)\Big] \tag{4.19}$$
$$+ (\boldsymbol{\tau}(z) + I_N)p(z)\omega^h + p_0^a(z)(1+\tau_0(z))\omega_0^h.$$

The maps $\langle r^{ah}\rangle$ generate the incomes of consumers at any allocation $z \in A^u$ using the equivalent ad valorem price-tax configuration and arbitrary ownership shares $\langle\theta_h\rangle$. Note that since $p_0^a(z)(1+\tau_0(z)) = p_0^u(z)+t_0(z) = q_0(z)$ and $p^a(z)(1+\tau(z)) = p(z)+t(z) = q(z)$, we have from (4.18) and (4.19)

$$\sum_h r^{ah}\left(z,\psi_p^a(z),\theta_h\right) = \Big[p_0^a(z)y_0(z) - p^T(z)y^m(z) + p^T(z)y^c(z)\Big]\sum_h \theta_h$$
$$+ \Big[\tau_0(z)p_0^a(z)y_0(z) + \tau^T(z)\mathbf{p}(z)\left[y^c(z) - y^m(z) - y^g(z)\right] - p^T(z)y^g(z)\Big]$$
$$+ (\boldsymbol{\tau}(z) + I_N)p(z)\omega + p_0^a(z)(1+\tau_0(z))\omega_0 \tag{4.20}$$
$$= q_0(z)y_0(z) - q^T(z)y^m(z) + q^T(z)y^c(z) - q^T(z)y^g(z) + q^T(z)\omega + q_0(z)\omega_0$$
$$= \sum_h r^{uh}\left(z,\psi_p^u(z),\theta_h\right) = \sum_h w^h(z).$$

This demonstrates that the aggregate income at allocation z under unit-taxation and income rule $\langle\theta_h^u(z)\rangle$ is the same as the aggregate income at z with equivalent ad valorem taxes and any income rule $\langle\theta_h\rangle$.

For all $z \in A^u$, denote the monopoly and competitive profits and government revenues under unit and ad valorem taxes as

$$\Pi^{mu}(z) = p_0^u(z)y_0^u(z) - C(y_0^u(z), p(z)),$$
$$\Pi^{ma}(z) = p_0^a(z)y_0^a(z) - C(y_0^a(z), p(z)),$$
$$\Pi^c(z) = p(z)y(z), \tag{4.21}$$
$$G^u(z) = t_0(z)y_0(z) + t^T(z)[y^c(z) - y^m(z) - y^g(z)] - p^T(z)y^g(z), \text{ and}$$
$$G^a(z) = p_0^a(z)\tau_0(z)y_0(z) + p^T(z)\boldsymbol{\tau}(z)[y^c(z) - y^m(z) - y^g(z)] - p^T(z)y^g(z).$$

[34] Recall that $q_0(z) = p_0^u(z) + t_0(z)$ and $q(z) = p(z) + t(z)$.

From (4.20), we note that, at $z \in A^u$, the sums of profit and government revenue are the same under the unit and ad valorem systems, that is,

$$\begin{aligned} &\Pi^{mu}(z) + \Pi^c(z) + G^u(z) = \Pi^{ma}(z) + \Pi^c(z) + G^a(z) \\ \Leftrightarrow &- [\Pi^{ma}(z) - \Pi^{mu}(z)] = G^a(z) - G^u(z). \end{aligned} \tag{4.22}$$

We calculate the following difference, recalling the Suits and Musgrave relation (4.10)

$$\begin{aligned} G^u(z) - G^a(z) =& [t_0(z) - \tau_0(z)p_0^a(z)]y_0(z) \\ =& [t_0(z) - \frac{t_0(z)p_0^a(z)}{\nabla_{y_0}C(z)}]y_0(z) \\ =& [\frac{\nabla_{y_0}C(z) - p_0^a(z)}{\nabla_{y_0}C(z)}]t_0(z)y_0(z). \end{aligned} \tag{4.23}$$

Remark 1: Since, under monopoly, $\nabla_{y_0}C(z) - p_0^a(z) < 0$, from (4.23) it follows that (i) the unit demogrant is bigger than (smaller than, equal to) the ad valorem demogrant iff $t_0(z) < 0$ ($t_0(z) > 0$, $t_0(z) = 0$) and (ii) the monopoly profit under unit taxation is bigger than (smaller than, equal to) the monopoly profit under ad valorem taxation if and only if $t_0(z) > 0$ ($t_0(z) < 0$, $t_0(z) = 0$) .

Assumption 5: For every $\langle\theta_h\rangle \in \Delta_{H-1}$ and for every $h \in \mathcal{H}$, there exist $\bar{z}^h$ and $\underline{z}^h$ in A^u such that $r^{uh}(\bar{z}^h, \psi_p^u(\bar{z}^h), \theta_h^u(\bar{z}^h)) - r^{ah}(\bar{z}^h, \psi_p^a(\bar{z}^h), \theta_h) > 0$ and $r^{uh}(\underline{z}^h, \psi_p^u(\underline{z}^h), \theta_h^u(\underline{z}^h)) - r^{ah}(\underline{z}^h, \psi_p^a(\underline{z}^h), \theta_h) < 0$.

The continuity of the function $r^{uh}(z^h, \psi_p^u(z^h), \theta_h^u(z^h)) - r^{ah}(z^h, \psi_p^a(z^h), \theta_h)$ in z and the intermediate value theorem imply that, for every h, there is a point in A^u where its income from unit taxation and the equivalent ad valorem taxation is the same.

Lemma 1: *Let the mapping* $\overset{*}{\theta}{}^u : \rho^u(A^u) \to \Delta_{H-1}$ *be a surjective function and Assumption 5 be true. Fix* $\langle\theta_h\rangle \in \Delta_{H-1}$*. For every* $h \in \mathcal{H}$*, there exists* $z^h \in A^u$ *such that* $r^{uh}(z^h, \psi_p^u(z^h), \theta_h^u(z^h)) - r^{ah}(z^h, \psi_p^a(z^h), \theta_h) = 0$.

The following theorem proves the existence of an ad valorem equilibrium of a given private ownership economy on the unit envelope. Fixing a vector of ownership shares, say $\langle\theta_h\rangle$, we search along the unit-tax envelope to see if there is a point on it that is also supported as an equilibrium of a $\langle\theta_h\rangle$ ownership ad valorem economy. The search is facilitated by the fact that at each point $\langle u_h\rangle$ on the unit-tax envelope, there is an equivalent vector of ad valorem taxes and prices that results in the same output decisions. Under Assumption 5, the continuity of the income maps $\langle r^{uh}\rangle$ and $\langle r^{ah}\rangle$ implies that there is a point on the unit-tax envelope where the corresponding unit taxes and the equivalent ad valorem taxes result also in the same incomes to all consumers, and hence to the same consumer demands.

Theorem 1: *Let $E = (\langle (X^h, u^h)\rangle, Y^0, Y^c, Y^g, \langle (\omega_0^h, \omega^h)\rangle)$ be an economy. Fix the profit shares as $\langle \theta_h \rangle$ and suppose the following are true:*
(*i*) *Assumptions 1 through 5 hold;*
(*ii*) *the mapping $\rho^u : A^u \to \rho^u(A^u)$ is bijective;*
(*iii*) *b is a normalization rule such that $\mathcal{S}_b^u$ is compact, and the mapping $\psi_p^a : A^u \to \mathbf{S}_b^u$ is a continuous function;*
(*iv*) *A^u is compact and $\rho^u(A^u)$ is a $H-1$ dimensional manifold;*
(*v*) *for every $z \in A^u$*

$$\frac{t_0(z)}{\nabla_{y_0} C(y_0(z), p(z))} > -1; \tag{4.24}$$

(*vi*) *the mapping $\overset{*}{\theta}{}^u : \rho^u(A^u) \to \Delta_{H-1}$ is a surjective function.*
Then
(*a*) *there exists a $\overset{*}{z} \in A^u$ such that $r^{uh}(\overset{*}{z}, \psi_p^u(\overset{*}{z}), \theta_h^u(\overset{*}{z})) = r^{ah}(\overset{*}{z}, \psi_p^a(\overset{*}{z}), \theta_h)$ for $h \in \mathcal{H}$;*
(*b*) *$\overset{*}{z}$ is also an allocation underlying an ad valorem tax equilibrium of the private ownership economy with shares $\langle \theta_h \rangle$;*
(*c*) *$\theta_h^u(\overset{*}{z}) = \theta_h$ for all $h \in \mathcal{H}$ if and only if $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$;*
(*d*) *$\rho^u(\overset{*}{z}) \in U^u(\langle \theta_h \rangle) := \left\{ \langle u_h \rangle \in \mathbf{R}^H | u_1 \le \mathcal{U}^u(u_2, \ldots, u_H, \langle \theta_h \rangle) \right\}$ if and only if $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$.*

Conclusions (a) and (b) of the above theorem imply that given a ownership profile $\langle \theta_h \rangle$ there exists an allocation $\overset{*}{z}$ such that $\rho^u(\overset{*}{z})$ lies on the unit envelope and $\overset{*}{z}$ is also supported as an equilibrium of the ad valorem $\langle \theta_h \rangle$ economy. (c) says that unless $\theta_h = \frac{1}{H}$ for all h, the private ownership unit economy that is tangent to the unit envelope at $\overset{*}{u} = \rho^u(\overset{*}{z})$, is not the same as the $\langle \theta_h \rangle$ unit economy. (d) says that unless $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$, the utility imputation $\overset{*}{u}$ never belongs to the utility possibility frontier corresponding to the $\langle \theta_h \rangle$ unit economy. All these conclusions imply that (unless $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$) though $\overset{*}{u}$ belongs to the utility possibility set corresponding to the $\langle \theta_h \rangle$ *ad valorem* economy, it does not belong to the utility possibility set corresponding to the $\langle \theta_h \rangle$ unit economy.

4.2. A unit-tax private-ownership equilibrium on the envelope of ad valorem-tax utility possibility frontiers.

Arguments for proving that, for any private ownership economy, the unit utility possibility set is not a subset of the ad valorem utility possibility set, are similar to the ones in the previous section. The Pareto manifold for a private ownership economy with ad valorem taxes can be derived in a manner similar to (4.1). We denote its image by $u_1 = \mathcal{U}^a(u_2, \ldots, u_H, \langle \theta_h \rangle)$ for shares $\langle \theta_h \rangle \in \Delta_{H-1}$. An envelope for all Pareto manifolds of private ownership economies with ad valorem taxes (which we call the ad valorem envelope) is obtained by solving the following problem, where we choose the shares $\langle\theta_h\rangle \in \Delta_{H-1}$ to solve

$$\hat{\mathcal{U}}^a(u_2,\ldots,u_H) := \max_{\langle\theta_h\rangle} \mathcal{U}^a(u_2,\ldots,u_H,\langle\theta_h\rangle)$$
$$\text{subject to} \tag{4.25}$$
$$\sum_h \theta_h = 1 \text{ and } \theta_h \in [0,1], \forall h.$$

We denote the solution to this problem by $\langle\overset{*}{\theta}{}^a_h\rangle = \langle\overset{*}{\theta}{}^a_h(u_2,...,u_h)\rangle$.

Under a set of assumptions analogous to the ones in the previous subsection, a theorem analogous to Theorem 1 can be proved to show that, for every allocation of shares $\langle\theta_h\rangle \in \Delta_{H-1}$, there exists a unit-tax equilibrium of a $\langle\theta_h\rangle$ ownership economy on the ad valorem-tax envelope, and utility profile corresponding to it will, generally (unless $\theta_h = \frac{1}{H}$ for all h or $\tau_0 = 0$), not belong to the utility possibility set corresponding to a $\langle\theta_h\rangle$ ownership ad valorem economy. In particular, Assumption 6 below (which, as was discussed in Section 2.9, is satisfied by our earlier quasi-linear example) is analogous to Assumption 5.[35]

Assumption 6: For every $\langle\theta_h\rangle \in \Delta_{H-1}$ and for every $h \in \mathcal{H}$, there exist $\bar{z}^h$ and $\underline{z}^h$ in A^a such that $\mathbf{r}^{ah}(\bar{z}^h, \boldsymbol{\psi}^a_p(\bar{z}^h), \theta^a_h(\bar{z}^h)) - \mathbf{r}^{uh}(\bar{z}^h, \boldsymbol{\psi}^u_p(\bar{z}^h), \theta_h) > 0$ and $\mathbf{r}^{ah}(\underline{z}^h, \boldsymbol{\psi}^a_p(\underline{z}^h), \theta^a_h(\underline{z}^h)) - \mathbf{r}^{uh}(\underline{z}^h, \boldsymbol{\psi}^u_p(\underline{z}^h), \theta_h) < 0$.

5. Concluding Remarks

We show that the Marshallian social surpluses that result from unit and ad valorem taxation of a monopoly, where the government can implement unrestricted transfers, are identical.

If the government's redistributive ability is constrained so that it can only implement a uniform lump-sum transfer and there is private ownership of the monopoly then, we show that under one set of regularity conditions unit taxation does not dominate ad valorem taxation, while under another similar set of regularity conditions the reverse holds. In the case of quasi-linear preferences, the latter regularity holds. In the case of more general preferences, the possibility of both sets of regularity conditions holding simultaneously is not ruled out, and so it could be the case that the set of second-best Pareto optima in a unit-tax economy neither dominates the set of second-best ad valorem-tax Pareto optima nor is dominated by it. In particular, if the shares in the private sector profits are equal for all consumers (which is equivalent to the case of one hundred per cent profit taxation in Blackorby and Murty [2007]) then the two sets of Pareto optima coincide. This conclusion is at odds with most of the existing literature comparing unit and ad valorem taxation.

Earlier claims that equilibria in unit-tax economies are dominated by equilibria in ad valorem-tax economies did not deal with the fact that the monopoly profits must be redistributed to consumers either via government taxation and lump-sum transfers or via

[35] The functions $\mathbf{r}^{ah}$, $\mathbf{r}^{uh}$, $\boldsymbol{\psi}^a_p$, and $\boldsymbol{\psi}^u_p$ are analogous to the functions r^{ah}, r^{uh}, ψ^a_p, and ψ^u_p in Assumption 5. The set A^a is analogous to the set A^u.

the private ownership of firms. Nevertheless, the move from a unit-tax equilibrium to an ad valorem one is not simply an accounting identity as it is in a competitive economy.

6. Appendix A

This appendix contain the proofs of all of the theorems in the paper. Appendix B (Blackorby and Murty [2008]) in the working paper version of this paper rationalizes some of the assumptions in Theorems 1 and 2 in terms of the underlying primitives of the problem in so far as possible.

Proof of Lemma 1: The proof follows from Assumption 5 and the intermediate value theorem given the continuity of the function $r^{uh}(z^h, \psi^u_p(z^h), \theta^u_h(z^h)) - r^{ah}(z^h, \psi^a_p(z^h), \theta_h)$ in z. ■

Proof of Theorem 1:

Under the maintained assumptions, conclusions of Lemma 2 follow. For every h, pick z^h as defined in Lemma 2.[36] Renormalize the utility function u^h such that $u^h(x^h_0(z^h), x^h(z^h)) = 0$.

The mapping $\kappa : \rho^u(A^u) \to \kappa(\rho^u(A^u))$ with image

$$\kappa(u) = \frac{u}{||u||}, \tag{6.1}$$

for every $u = (u_1, \dots, u_H) \in \rho^u(A^u)$ is a homeomorphism.[37] Denote $\kappa(\rho^u(A^u))$ by Ω. Define the inverse of κ as $\mathcal{K} : \Omega \to \rho^u(A^u)$.[38]

Define the correspondence $T : A^u \times \mathbf{S}^u_b \to \Omega$ as

$$\begin{aligned} T(z, \pi) = \{\beta \in \Omega | \beta_h = 0 \text{ if there exists } h \text{ such that} \\ r^{uh}(z, \pi, \theta^u_h(z)) \neq r^{ah}(z, \pi, \theta_h)\}. \end{aligned} \tag{6.3}$$

We claim that T is non-empty, compact, convex valued, and upper-hemi continuous. It is trivial to prove that T is nonempty and convex valued. We now show that it is upper-hemi continuous, which implies that it is compact valued, given Assumptions (iii) and (iv). Suppose $(z^v, \pi^v) \to (z, \pi) \in A^u \times \mathbf{S}^u_b$ and $\beta^v \to \beta$ such that $\beta^v \in T(z^v, \pi^v)$ for all v. We need to show that $\beta \in T(z, \pi)$. If there exists h such that $r^{uh}(z, \pi, \theta^u_h(z)) - r^{ah}(z, \pi, \theta_h) \neq 0$ then, by the definition of the mapping T, we have $\beta^h = 0$. Without loss of generality, assume $r^{uh}(z, \pi, \theta^u_h(z)) - r^{ah}(z, \pi, \theta_h) > 0$. Since the functions $r^{uh}(z, \pi, \theta^u_h(z)) - r^{ah}(z, \pi, \theta_h)$ are continuous for all h in z and π, there exists v' such that for all $v \geq v'$, we have

[36] If this is not unique, choose one such z^h.

[37] See also Quinzii [1992], p. 51.

[38] Its image is

$$\mathcal{K}(\alpha) = \lambda(\alpha)\alpha \tag{6.2}$$

where $\lambda(\alpha) = \max\{\lambda \geq 0 | \lambda\alpha \in \rho^u(A^u)\}$.

$r^{uh}(z^v, \pi^v, \theta^u_h(z^v)) - r^{ah}(z^v, \pi^v, \theta^h) > 0$. Hence $\beta^{hv} = 0$ for all $v \geq v'$. Therefore $\beta^v \to \beta$ implies that $\beta^h = 0$.
Define the correspondence $K : \Omega \times \mathbf{S}^u_b \to \Omega \times \mathbf{S}^u_b$, as[39]

$$K(\alpha, \pi) = (T(\rho^{u-1}(\mathcal{K}(\alpha)), \pi),\ \Psi^a_p(\rho^{u-1}(\mathcal{K}(\alpha)))). \tag{6.4}$$

Under the maintained assumptions of this theorem, this correspondence is convex valued and upper-hemi continuous. The Kakutani's fixed point theorem implies that there is a fixed point $(\overset{*}{\alpha}, \overset{*}{\pi}\rangle$ such that $\overset{*}{\alpha} \in T(\rho^{u-1}(\mathcal{K}(\overset{*}{\alpha})), \overset{*}{\pi})$ and $\overset{*}{\pi} \in \Psi^a_p(\rho^{u-1}(\mathcal{K}(\overset{*}{\alpha})))$.
Let $\overset{*}{z} := \rho^{u-1}(\mathcal{K}(\overset{*}{\alpha}))$. Hence, $\overset{*}{z} \in A^u$ and it is unique (as ρ^u and $\mathcal{K}$ are bijective).
We now prove that

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) = r^{uh}(\overset{*}{z}, \overset{*}{\pi}, \theta^u_h(\overset{*}{z})),\ \forall h \in \mathcal{H}. \tag{6.5}$$

If there exists h such that

$$r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) \neq r^{uh}(\overset{*}{z}, \overset{*}{\pi}, \theta^u_h(\overset{*}{z})), \tag{6.6}$$

then by the definition of the correspondence T, we have $\overset{*}{\alpha}_h = 0$. By the definition of the homeomorphism $\mathcal{K}$, this would imply $\overset{*}{u}^h = 0$. As ρ^u is bijective, our utility normalization implies that $\overset{*}{z} = z^h$ and hence $r^{ah}(\overset{*}{z}, \overset{*}{\pi}, \theta_h) = r^{uh}(\overset{*}{z}, \overset{*}{\pi}, \theta^u_h(\overset{*}{z}))$, which is a contradiction. This proves (a).

The price and the ad valorem tax configuration $\psi^a_p(\overset{*}{z}) = (p^a_0(\overset{*}{z}), \tau_0(\overset{*}{z}), p(\overset{*}{z}), \tau(\overset{*}{z}))$ and the income configuration $\langle r^{ah}(\overset{*}{z}, \psi^a_p(\overset{*}{z}), \theta_h)\rangle$ define an ad valorem tax equilibrium of the private ownership economy $\langle\theta_h\rangle$, the underlying equilibrium allocation is $\overset{*}{z}$ and the consumer prices are $(q_0(\overset{*}{z}), q^T(\overset{*}{z})) = (p^a_0(\overset{*}{z})[1 + \tau_0(\overset{*}{z})], p^T(\overset{*}{z})[I_N + \boldsymbol{\tau}(\overset{*}{z})]) = (p^u_0(\overset{*}{z}) + t^u_0(\overset{*}{z}), p(\overset{*}{z}) + t(\overset{*}{z}))$. This proves (b).
At $\overset{*}{z}$ we know, from (4.20), that the sums of profits and government revenue are the same under the unit and ad valorem systems, that is

$$\begin{aligned} &\Pi^u(\overset{*}{z}) + G^u(\overset{*}{z}) = \Pi^a(\overset{*}{z}) + G^a(\overset{*}{z}) \\ \Leftrightarrow &- [\Pi^a(\overset{*}{z}) - \Pi^u(\overset{*}{z})] = G^a(\overset{*}{z}) - G^u(\overset{*}{z}). \end{aligned} \tag{6.7}$$

From conclusion (a) we have for all $h \in \mathcal{H}$

$$\begin{aligned} w^h(\overset{*}{z}) = \theta^u_h(\overset{*}{z})[\Pi^u(\overset{*}{z}) + \Pi^c(\overset{*}{z})] + \frac{1}{H}G^u(\overset{*}{z}) &= \theta_h[\Pi^a(\overset{*}{z}) + \Pi^c(\overset{*}{z})] + \frac{1}{H}G^a(\overset{*}{z}) \\ \Rightarrow \theta^u_h(\overset{*}{z})\Pi^u(\overset{*}{z}) - \theta_h\Pi^a(\overset{*}{z}) &= \frac{1}{H}[G^a(\overset{*}{z}) - G^u(\overset{*}{z})] \\ \Rightarrow \theta^u_h(\overset{*}{z})\Pi^u(\overset{*}{z}) - \theta_h\Pi^a(\overset{*}{z}) &= \frac{1}{H}[\Pi^u(\overset{*}{z}) - \Pi^a(\overset{*}{z})]. \end{aligned} \tag{6.8}$$

39 Note, ρ^{u-1} is the inverse mapping corresponding to ρ^u.

The last equality follows from (6.7). Hence, (6.8) implies that $\theta_h^u(\overset{*}{z}) = \theta_h$ for all $h \in \mathcal{H}$ iff $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $\Pi^u(\overset{*}{z}) - \Pi^a(\overset{*}{z}) = 0$. The latter is true when $t_0(\overset{*}{z}) = 0$. Thus, (c) is true.
We now prove (d). Let $\overset{*}{u} := \rho^u(\overset{*}{z})$.
If $\overset{*}{u} \in U^u(\langle\theta_h\rangle) := \{\langle u_h\rangle \in \mathbf{R}^H | u_1 \leq \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle)\}$, then since $\overset{*}{u} \in \rho^u(A)$, we have, because of Assumption (vi), the unique solution to (4.2) as

$$\overset{*}{\theta}{}^u(\overset{*}{u}) = \langle\theta_h\rangle. \tag{6.9}$$

From (c) this is true iff $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$.
If $\theta_h = \frac{1}{H}$ for all $h \in \mathcal{H}$ or $t_0(\overset{*}{z}) = 0$, then again (6.9) follows from (c), and we have $\overset{*}{u} \in U^u(\langle\theta_h\rangle) := \{\langle u_h\rangle \in \mathcal{H} | u_1 \leq \mathcal{U}^u(u_2, \ldots, u_H, \langle\theta_h\rangle)\}$. ■

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