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Where strategic and evolutionary stability depart – a study of minimal diversity games

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Abstract

A minimal diversity game is an n player strategic form game in which each player has m pure strategies at his disposal. The payoff to each player is always 1, unless all players select the same pure strategy, in which case all players receive zero payoff. Such a game has a unique isolated completely mixed Nash equilibrium in which each player plays each strategy with equal probability, and a connected component of Nash equilibria consisting of those strategy profiles in which each player receives payoff 1. The Pareto superior component is shown to be asymptotically stable under a wide class of evolutionary dynamics, while the isolated equilibrium is not. On the other hand, the isolated equilibrium is strategically stable, while the strategic stability of the Pareto efficient component depends on the dimension of the component, and hence on the number of players, and the number of pure strategies.

JEL Codes. C72, D44.

Keywords. Strategic form games, strategic stability, evolutionary stability.

1 Introduction

Imagine a castle that is being guarded by n soldiers. The castle has two gates, east and west, and the castle is currently under attack from enemy troops. If each gate is guarded by at least one soldier, the guards can lock the gates and the enemies won't enter. However, if one gate is left unattended then the castle will be taken and all soldiers will perish [1]. The question is:

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[1] This is an example of a binary minimal diversity game. In a binary minimal diversity game each player has two (pure) strategies and all players win if there is a minimum of diversity in the players choices, i.e. at least two players choose to do something different.

How should the soldiers choose to defend the castle? Clearly, all that is needed is that not all soldiers choose to defend the same gate.

The game thus defined has two types of Nash equilibria. In one there is complete lack of coordination, and each soldier chooses to defend each gate with equal probability. Hence, in this equilibrium there is a positive probability that all soldiers show up at the same gate, in which case the castle is taken by the enemy. In the second type of equilibrium the coordination problem is solved. Two soldiers choose different gates, one east and one west, while the remaining soldiers can, and may, do whatever they like.

The two types of Nash equilibria are very different. Can one convincingly argue that one is more reasonable than the other? In this paper we compare two approaches to answer this question. We show that, from the evolutionary perspective, the equilibria where the coordination problem is solved are most natural. We show that they form a set that is asymptotically stable under a wide class of evolutionary dynamics, while the equilibrium where the coordination problem is not solved is always unstable. We then take a look at the above choice problem from the perspective of strategic stability, notably at the more demanding notions of strategic stability: essentiality as defined by Wu Wen-Tsün and Jiang Jia-He ([13],[23]), best response stability as defined by Hillas ([8],[9]), and strategic stability in the sense of Mertens ([16],[17]). Here it turns out that the number of soldiers in the castle starts to matter. Surprisingly, the question is not whether the number of soldiers is large or small, but whether it is even or odd. If the number of players is even then the set of Nash equilibria where the coordination problem is solved contains a strategically stable set, otherwise it does not. The Nash equilibrium where the coordination problem is not solved is always strategically stable, in virtually any sense.

The main purpose of the paper is to substantiate these claims. We think these findings are interesting for at least two reasons. First of all, there is a whole class of very simple games where the two approaches of evolutionary stability and strategic stability make mutually exclusive predictions, namely whenever the number of players is odd. This contrasts with earlier, more positive findings, notably by Swinkels [20] and Demichelis and Ritzberger [3]. Secondly, the alternating behavior for strategic stability solutions is puzzling. Although we show that continuity arguments, based on behavior induced by inconsistent beliefs, do provide a basic intuition for our findings, continuity is explicitly ruled out by Kohlberg and Mertens [14] as a reasonable axiom for solution concepts. The question therefore remains what game theoretic arguments would justify the alternating behavior we find for the more demanding notions of strategic stability.

Refinement theory became popular when, with the increased use of game theoretic methods in Economics in the 1970's and early 1980's, it became clear that the notion of a Nash equilibrium is too weak as a solution concept to analyze relevant economic models. Refinements such as perfect, proper and sequential equilibrium were introduced and widely used. However, as it turned out, also refinements of Nash equilibrium allowed for many possible solutions, including very implausible ones. Moreover, refinements were often only motivated and introduced on an ad-hoc, example-driven basis. This encouraged game theorists to try to find a more systematic and consistent way to further refine among Nash equilibria. One attempt in this direction is made in the book Harsanyi and Selten [6], which has the ambitious aim to select a unique Nash equilibrium for every game.

Another influential approach is due to Kohlberg and Mertens [14]. In contrast to Harsanyi and Selten, Kohlberg and Mertens [14], henceforth called K&M, argue that any satisfactory selection criterion cannot select a single equilibrium, but forces us to select *sets* of closely related equilibria. Their second major contribution to the ongoing discussion was to produce a list of basic criteria such refined sets of equilibria should satisfy. K&M's point of view was that, while a full-fledged axiomatic approach may be out of reach, the search for a satisfactory solution concept should be guided at least by a list of properties that are desirable from a game theoretical perspective. Their own initial notion of strategically stable sets of equilibria fails on some of these criteria. However, Mertens [16] proposes a notion of strategic stability that satisfies all the requirements made by K&M and that passes several additional plausibility tests.

On the other side of the spectrum experimental research, e.g. by Güth et al. [4] on the ultimatum game, made it clear that classical game theory, in particular Nash equilibrium, is not always a good predictor of human behavior. These results inspired a new line of research in game theory that focused on models of bounded rationality, in an attempt to avoid the traditional game-theoretic approach to take the assumption that players are rational to its ultimate conclusions. One such attempt uses learning dynamics to model boundedly rational behavior, in line with earlier work by evolutionary biologists on animal behavior.

In their seminal work Maynard Smith and Price [15] showed that under certain conditions such learning dynamics converge to Nash equilibrium. Nash equilibria that are selected via learning dynamics tend to be rather "refined" Nash equilibria. This raises the question whether the outcomes predicted by the learning approach assuming only bounded rationality [2] might coincide with the outcomes given by the refinement approach, based on the extreme emphasis on rationality.

Indeed, Demichelis and Ritzberger [3] (henceforth called D&R) showed that for a wide class of evolutionary dynamics any asymptotically stable Nash equilibrium is automatically strategically stable in the sense of Mertens. [3]

However, as K&M emphasized, in general one should look for *sets* of Nash equilibria. Swinkels [20] noted that the analysis for sets of Nash equilibria is complicated by the fact that topological properties of the sets start to matter. He showed for a wide class of dynamics that asymptotically stable sets of the dynamics contain a strategically stable (even a hyper-stable) set as defined by K&M, *provided* that the set is contractible. D&R considerably sharpen this result. They show for a wide class of dynamics that sets that are asymptotically stable under the dynamics contain a stable set of Nash equilibria in the sense of Mertens, provided the set has non-zero Euler characteristic.

While these results show a strong connection between evolutionary and strategic stability under certain topological restrictions on the solution set, it does not rule out the existence of convincing examples where the two types of stability are not related. The aim of this paper is precisely to provide such examples.

More concretely, we present a natural class of coordination games, called minimal diversity games. Minimal diversity games are coordination games with a slight twist. In classical coordination games, all players have to choose the same action in order to coordinate. Here the task faced by the players is precisely to *avoid* choosing the same action. Each game within this class has a set of Nash equilibria that consists of one isolated completely mixed Nash equilibrium and one connected component of Pareto dominant Nash equilibria [4]. For two families of games within the class of minimal diversity games [5] we show that any sufficiently strong notion of strategic stability exclusively selects the isolated completely mixed equilibrium, while evolutionary stability exclusively selects the component of Pareto dominant Nash equilibria. Thus, minimal diversity games provide such examples of games where the predictions made by

[2] In the literature on bounded rationality evolutionary models are re-interpreted as learning models.

[3] As Swinkels [20] pointed out, traditional game theory and evolutionary game theory cannot be compared directly because of the additional symmetry assumptions made in the latter approach. The results in Ritzberger and Demichelis [3] and related literature are obtained once the assumption of symmetry in evolutionary game theory is dropped, and conflicts are considered to be inter-species rather than within the same species.

[4] The connected component is in fact a strict equilibrium set, as is shown in Balkenborg and Schlag [1].

[5] In one family there are two players with an odd number of strategies, and in the other there are two strategies and an odd number of players.

evolutionary stability and strategic stability are different, even in the strong sense that they may be mutually exclusive.

We conjecture that our results are not merely "accidental", but true in the following more general sense. In section 3 we show for any minimal diversity game with n players and m actions that the set of efficient Nash equilibria is a topological sphere of dimension $d = (n-1)(m-1)-1$. Our results strongly indicate that strategic stability and evolutionary stability make mutually exclusive predictions precisely when the dimension d of the set of efficient Nash equilibria is odd. This would be entirely in line with the conjecture of Swinkels that the topology of the component of Nash equilibria in question, specifically its Euler characteristic, is the decisive factor in the connection (and the distinction!) between the two types of stability.

In relation to this last observation it is interesting to see that, in those cases where the efficient set is not strategically stable, we can also show that it is not essential in the sense of Wu Wen-Tsun and Jiang Jia-He [13], [23]. This implies that many evolutionary dynamics never converge to any equilibrium [6]. In this sense strategic stability matters for evolutionary stability even when at first sight it seems to be irrelevant.

K&M already stressed the necessity to discard the focus on generic classes of strategic form games, but rather study natural subclasses within the class of strategic form games. K&M made this observation in the context of extensive form games, where the class of games considered was defined by fixing the game tree of an extensive form game, and by varying the payoffs of players at the terminal nodes of the game tree. In a similar vein our class of minimal diversity games is a natural class of potential games, very much in the spirit of games such as for example Colonel Blotto games [7]. In the class of potential games minimal diversity games are characterized only by the requirement that players, similar to the task of players in Colonel Blotto games, need to "avoid coordination" in order to maximize their joint payoff.

The paper is organized as follows. After some preliminaries we introduce the class of minimal diversity games. We show that the set of Nash equilibria of each minimal diversity game consists of an isolated completely mixed equilibrium and a connected component G of efficient equilibria. The latter is shown to be a topological sphere of dimension $d = (n-1)\cdot(m-1)-1$, where n is the number of players and m the number of actions. We then show for a wide class of dynamics (including the replicator dynamics as well as the class of Nash dynamics defined by

[6] This finding strengthens the observation in D&M that these sets of Nash equilibria are not robustly evolutionary stable. Notice, however, that there are still asymptotically stable attractors not containing fixed points in the nearby games

[7] See among others Hart [7], and Monderer and Shapley [18] for motivation and explanation of these games.

D&R) that the set G is asymptotically stable, while the completely mixed equilibrium is not. Thereafter we turn to strategic stability where we first observe that all equilibrium components of these games are strategically stable in the sense of Kohlberg and Mertens. We then derive the results indicated above, first for games with two players and arbitrary number of players, and next for games with two strategies and an arbitrary number of players. The Appendix is entirely devoted to the proof of Theorem 7.2.

2 Preliminaries

This paper concerns the class of minimal diversity games. However, many concepts applied here are defined for general normal form games. It will hence be useful to have some basic notation and terminology available for these games.

A finite normal form game consists of a finite set of players $N = \{1, \cdots, n\}$, and for each player $i \in N$ a finite pure strategy set S_i and a payoff function $u_i \colon S \to \mathbb{R}$ on the set $S := \prod_{i \in N} S_i$ of pure strategy profiles. We denote the game by (N, u), where $u = (u_i)_{i \in N}$ is the vector of payoff functions. A mixed strategy σ_i of player i is a vector $(\sigma_i(s_i))_{s_i \in S_i}$ that assigns a probability $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$. We denote the set of mixed strategies of player i by Σ_i. The set of all profiles $\sigma = (\sigma_i)_{i \in N}$ of mixed strategies is denoted by Σ. The support of a mixed strategy σ_i is the set of all pure strategies s_i with $\sigma_i(s_i) > 0$. The multilinear extension of the payoff function u_i of player i to the set Σ of all strategy profiles is given by the formula

$$u_i(\sigma) = \sum_{s \in S} \prod_{j \in N} \sigma_j(s_j) u_i(s).$$

By $u_i(\sigma \mid s_i)$ we denote the payoff to player i when player i plays pure strategy $s_i \in S_i$ while his opponents adhere to the mixed strategy profile σ. A strategy profile $\sigma \in \Sigma$ is a Nash equilibrium when $u_i(\sigma) \geq u_i(\sigma \mid s_i)$ holds for every player i and every pure strategy s_i of player i.

Minimal diversity games A minimal diversity game is a normal form game (N, u) such that $S_i = M = \{1, \ldots, m\}$ for every player $i \in N$, and

$$u_i(s_1, \cdots, s_n) = \begin{cases} 0 & \text{if } s_1 = s_2 = \ldots = s_n \\ 1 & \text{else.} \end{cases}$$

To simplify notation for minimal diversity games, for player i and pure strategy k, we denote the probability $\sigma_i(k)$ that player i assigns to pure strategy k in strategy profile σ by σ_{ik}.

Strategic stability In this paper we use several different notions of strategic stability, notably regularity, defined by Harsanyi [5], essentiality, defined by Wu Wen-Tsun and Jiang

Jia-He [13], [23], KM stability, hyperstability and full stability, defined by K&M, strategic stability in the sense of Mertens, defined by Mertens [16], and best response stability, defined by Hillas [8]. KM stability was simply called stability in K&M, and the same holds for best response stability in Hillas [8]. We use the terms KM stability and best response stability in this paper to avoid confusion.

We do not define most of these notions, because we rely on the results of D&R and Hillas et al. [9] for most of our conclusions. We explicitly use the definitions of essentiality and regularity though, and for that reason we state here their formal definitions. For two games (N, u) and (N, v) with the same player set, we write

$$\|u - v\| = \max\left\{|u_i(s) - v_i(s)| \mid s \in S, i \in N\right\}.$$

A closed set $C \subset \Sigma$ of Nash equilibria of the game (N, u) is called essential when for every open set $U \subset \Sigma$ containing C there is a $\varepsilon > 0$ such that every game (N, v) with $\|u - v\| < \varepsilon$ has a Nash equilibrium in U.

A Nash equilibrium σ of (N, u) is called regular if there exist open sets $U \ni u$ and $V \ni \sigma$ and a continuously differentiable function $g: U \to V$ such that for any $v \in U$ and $\tau \in V$ we have $g(v) = \tau$ precisely when τ is a Nash equilibrium of (N, v).

Notice that the definition automatically implies $g(u) = \sigma$. It is known that a regular equilibrium is strategically stable in virtually any sense: it is perfect, proper, essential, best response stable, and even strategically stable in the sense of Mertens.

Best response stability is, in an alternative formulation called CKM-stability, defined in the Appendix. For information on the remaining notions of strategic stability we refer the interested reader to the original papers.

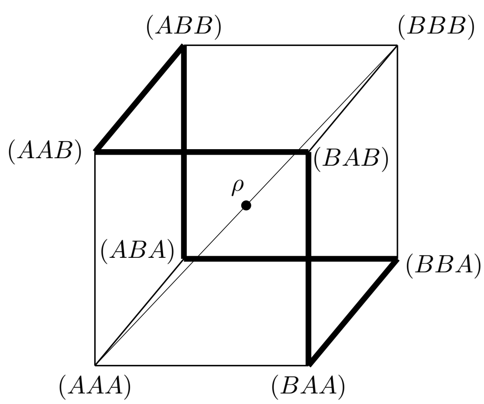
3 The Nash equilibria of a minimal diversity game

In this section we show that the set of Nash equilibria of a minimal diversity game consists of two components. One component consists of a single isolated completely mixed Nash equilibrium. The other component consists of all strategy profiles in which at least two players play a different pure strategy. We show that this second component is homeomorphic to a sphere of dimension $d = (n - 1) \cdot (m - 1) - 1$.

It is easy to verify that the completely mixed strategy profile in which each player randomizes with equal probability $\frac{1}{m}$ between all his pure strategies is a Nash equilibrium with an expected payoff of $1 - \left(\frac{1}{m}\right)^{n-1}$ for each player. We denote this Nash equilibrium by $\rho = (\rho_i)_{i \in N}$.

Secondly, every strategy profile where one (and hence every) player gets the maximal expected payoff one is clearly a Nash equilibrium. We denote the set of all such Nash equilibria by G. Note that G is in fact the set of all Pareto-efficient strategy profiles in the game.

To illustrate the set G we consider one of the simplest minimal diversity games, namely the game with three players and two strategies each. In this case each player's mixed strategy space is a line segment, so that the set of strategy profiles can be identified with a cube, as in the following graph.



The pure strategy profiles are the corners of the cube. The Nash equilibrium ρ in completely mixed strategies is in the center of the cube. The set G of Pareto efficient Nash equilibria is the cycle on the boundary consisting of six line segments. Along each line segment each pure strategy is not used by one of the players.

More generally, for each minimal diversity games the set G is a union of faces of Σ. The faces of G that have maximal dimension $d = (n-1) \times (m-1) - 1$ are characterized as follows. For each pure strategy k, let $J(k)$ be a player. This defines a map $J\colon M \to N$. Write

$$G_J = \{\sigma \mid \sigma_{J(k)}(k) = 0 \text{ for all } k \in M\}.$$

The set G_J is a face of Σ of dimension d as soon as $J^{-1}(i) \neq M$ for each player i. Further, G is the union of these faces G_J. We show the following two facts.

(1) Apart from ρ and the strategy profiles in G a minimal diversity game has no other Nash equilibria.

(2) G is a topological sphere of dimension $d = (n-1) \times (m-1) - 1$. This means that G is homeomorphic [8] to

$$\left\{ x \in \mathbb{R}^{d+1} \mid \sum_{j=1}^{d+1} x_j^2 = 1 \right\}.$$

Consequently, ρ is an isolated Nash equilibrium and G is a connected component of Nash equilibria [9].

Proposition 3.1 *The Nash equilibria of a minimal diversity game are precisely the strategy profiles in G, together with the completely mixed strategy profile ρ.*

Proof. Let σ be a Nash equilibrium of the game in completely mixed strategies. For each $k \in M$, write $\sigma_k = \prod_{i \in N} \sigma_{ik}$. Since in a completely mixed equilibrium each player is indifferent between all his strategies in M we have

$$1 - \frac{\sigma_k}{\sigma_{ik}} = 1 - \frac{\sigma_l}{\sigma_{il}}$$

for all players $i \in N$ and pure strategies $k, l \in M$. Therefore, for two given players i and j we have

$$\frac{\sigma_{il}}{\sigma_{ik}} = \frac{\sigma_l}{\sigma_k} = \frac{\sigma_{jl}}{\sigma_{jk}}.$$

Hence $\sigma_{ik} = \frac{\sigma_{il}}{\sigma_{jl}} \sigma_{jk}$ for all $k, l \in M$. Summing over k we obtain $1 = \frac{\sigma_{il}}{\sigma_{jl}}$. Hence $\sigma_{il} = \sigma_{jl}$ for every $l \in M$, which shows that all players use the same strategy. Therefore $(\sigma_{ik})^n = \prod_{j \in N} \sigma_{jk} = \sigma_k$ for every $i \in N$ and $k \in M$. Hence, since $\frac{\sigma_l}{\sigma_{il}} = \frac{\sigma_k}{\sigma_{ik}}$, it follows that $\sigma_{il} = \sigma_{ik}$ for every $i \in N$ and $k \in M$. Thus each player uses each pure strategy in M with equal probability and we conclude that $\sigma = \rho$.

Suppose next that σ is a Nash equilibrium, and that $\sigma_{ik} = 0$ for some $i \in N$ and $k \in M$. Then every player $j \neq i$ can ensure himself the maximal payoff 1 by playing pure strategy k. So, since σ is Nash equilibrium, every player $j \neq i$, and then also i, receives payoff 1 under σ. Hence $\sigma \in G$ by definition. ∎

3.1 G is a topological sphere

In this section we show that the connected component G of Pareto efficient Nash equilibria is homeomorphic to a sphere of dimension $d = (n-1) \cdot (m-1) - 1$.

We briefly sketch the outline of the proof. Let D be the "diagonal" consisting of all strategy profiles where all players play the same strategy. Clearly, D is a simplex of dimension $m - 1$.

[8] Although not diffeomorphic.

[9] Except when $m = n = 2$. In this case G consists of two isolated Nash equilibria, and hence it is homeomorphic to the two endpoints of the unit interval.

Within the affine hull of Σ, consider the affine space L through the center point ρ orthogonal to D. So, L has dimension

$$n \cdot (m-1) - (m-1) = (n-1) \cdot (m-1).$$

Moreover, the orthogonal projection of Σ onto L is a convex polyhedron. We show that this polyhedron is of full dimension, and that the set G of Pareto-efficient Nash equilibria is mapped one-to-one onto the boundary of the polyhedron P. Hence, G is homeomorphic to a sphere of dimension $d = (n-1) \cdot (m-1) - 1$.

Proposition 3.2 *For any minimal diversity game the set G of efficient Nash equilibria is homeomorphic to a sphere of dimension $d = (n-1) \cdot (m-1) - 1$.*

Proof. It suffices to show that G is homeomorphic to the boundary of a compact and convex set of dimension $(n-1) \cdot (m-1)$.

A mixed strategy profile $\sigma = (\sigma_i(k))_{i \in N, k \in M}$ of the minimal diversity game is a point in the Euclidean vectorspace $\mathbb{R}^{mn}$ endowed with the usual scalar product $\langle \sigma, \tau \rangle = \sum_{i \in N} \sum_{k \in M} \sigma_{ik} \tau_{ik}$ for σ, $\tau \in \mathbb{R}^{mn}$. For each $k \in M$, let d_k be the strategy profile where each player plays pure strategy k with probability one, and let $\delta_k = d_k - \rho$. For $i \in N$, let e_i be the vector in $\mathbb{R}^{mn}$ which assigns 1 to each coordinate corresponding to a pure strategy of player i and 0 to all other coordinates.

Claim 1. The vectors δ_k together with the vectors e_i span a linear subspace of dimension $m + n - 1$.

Proof of Claim 1. Since $\sum_{k \in M} \delta_k = 0$ this linear subspace has a dimension of at most $m + n - 1$. It has at least this dimension since, for a given pure strategy $l \in M$, the vectors δ_k for $k \neq l$ together with the vectors e_i are linearly independent. This can be seen as follows. Take a fixed $l \in M$. Consider a linear combination

$$0 = \sum_{k \neq l} \alpha_k \delta_k + \sum_{i \in N} \beta_i e_i = \sum_{k \neq l} \alpha_k d_k - \left(\sum_{k \neq l} \alpha_k \right) \rho + \sum_{i \in N} \beta_i e_i.$$

Since each coordinate (i, l) of this vector must be zero we obtain $\frac{1}{m} \sum_{k \neq l} \alpha_k = \beta_i$ for each i. Hence

$$\sum_{i \in N} \beta_i e_i = \left(\frac{1}{m} \sum_{k \neq l} \alpha_k \right) \sum_{i \in N} e_i = \left(\sum_{k \neq l} \alpha_k \right) \rho.$$

and the linear combination is equal to $\sum_{k \neq l} \alpha_k d_k$. Since each coordinate (i, k) must be zero it follows $\alpha_k = 0$ for all $k \neq l$ and hence also $\beta_i = 0$ for all $i \in N$.

We conclude that the linear subspace L consisting of the vectors orthogonal to all vectors δ_k and e_i has dimension $(m-1)\cdot(n-1)$.

Claim 2. L is the image of $\mathbb{R}^{mn}$ under the orthogonal projection

$$P(\sigma) = \sigma - \frac{1}{n}\sum_{k\in M}\langle\sigma,\delta_k\rangle\delta_k - \frac{1}{m}\sum_{i\in N}\langle\sigma,e_i\rangle e_i \qquad \text{with } \sigma\in\mathbb{R}^{mn}.$$

Proof of Claim 2. Clearly, P equals the identity map on L and hence L is contained in the image space of P. Further, for all indices i,j and k,l, we have $\langle\delta_k,e_i\rangle = 0$,

$$\langle\delta_k,\delta_l\rangle = \begin{cases} n-\frac{n}{m} & \text{for } k=l \\ -\frac{n}{m} & \text{for } k\neq l, \end{cases} \qquad \text{and} \qquad \langle e_i,e_j\rangle = \begin{cases} m & \text{for } i=j \\ 0 & \text{for } i\neq j. \end{cases}$$

Therefore $\langle P(\sigma),e_j\rangle = 0$ holds for any j, and for l we also obtain

$$\begin{aligned} \langle P(\sigma),\delta_l\rangle &= \langle\sigma,\delta_l\rangle - \frac{1}{n}\sum_{k\in M}\langle\sigma,\delta_k\rangle\langle\delta_k,\delta_l\rangle \\ &= \langle\sigma,\delta_l\rangle - \langle\sigma,\delta_l\rangle + \frac{1}{m}\sum_{k\in M}\langle\sigma,\delta_k\rangle = \frac{1}{m}\langle\sigma,\sum_{k\in M}\delta_k\rangle = 0. \end{aligned}$$

Thus the projection maps into L, which proves the claim.

The mixed strategy space Σ of the game is a convex polyhedron which is mapped under the projection P onto a convex polyhedron $P(\Sigma)$.

Claim 3. The convex polyhedron $P(\Sigma)$ spans the linear subspace L and is hence a compact and convex set of dimension $(m-1)\cdot(n-1)$.

Proof of Claim 3. For each $x\in L$ choose a sufficiently small scalar $\lambda>0$ such that $\lambda x+\rho$ has only non-negative coordinates. Since $\langle\lambda x+\rho,e_i\rangle = 1$, $\lambda x+\rho$ is a strategy profile. Thus, since $P(\rho)=0$, we have $P(\lambda x+\rho) = \lambda P(x)+P(\rho) = \lambda x$. It follows that $P(\Sigma)$ spans L, which proves claim 3.

We show next that the projection maps the set of Pareto efficient Nash equilibria G homeomorphically onto the boundary of $P(\Sigma)$ in L. Let $J\colon M\to N$ be a map with $J^{-1}(i)\neq M$ for all $i\in N$. The set

$$G_J = \left\{\sigma\in\Sigma \mid \sigma_{J(k)}(k) = 0 \text{ for all } k\in M\right\}$$

is a face of G of maximal dimension. For any $\sigma\in\Sigma$ we have by a straightforward calculation that

$$P(\sigma) = \sigma - \frac{1}{n}\sum_{k\in M}\langle\sigma,d_k\rangle d_k.$$

Suppose that two strategy profiles $\sigma, \tau \in G_J$ satisfy $P(\sigma) = P(\tau)$. Then by the previous observation

$$\sigma - \tau = \frac{1}{n} \sum_{k \in M} \langle \sigma - \tau, d_k \rangle d_k.$$

Further, for any l we have $\sigma_{J(l)}(l) = \tau_{J(l)}(l) = 0$. Hence, when we write $e_{i,l}$ for the vector in $\mathbb{R}^{mn}$ that has a one in coordinate (i, l) and zeroes otherwise, we have

$$0 = \langle \sigma - \tau, e_{J(l),l} \rangle = \langle \frac{1}{n} \sum_{k \in M} \langle \sigma - \tau, d_k \rangle d_k, e_{J(l),l} \rangle = \frac{1}{n} \langle \sigma - \tau, d_l \rangle$$

Hence, $\langle \sigma - \tau, d_l \rangle = 0$ for every l. It follows that $\sigma = \tau$. Consequently, the projection is one-to-one on each face G_J.

Note that the vectors e_{ik} in fact constitute the standard basis of $\mathbb{R}^{mn}$. Let $\mathcal{J}$ be the collection of all maps $J : M \to N$ with $J^{-1}(i) \neq M$ for all $i \in N$. Given $J \in \mathcal{J}$, define the linear functional $l(J) : \mathbb{R}^{mn} \to \mathbb{R}$ by $l(J)(e_{i,k}) = 1$ when $i \neq J(k)$ and $l(J)(e_{J(k),k}) = 0$.

A mixed strategy σ_i of player i can be identified with the vector $\sum_{k \in M} \sigma_i(k) e_{ik} \in \mathbb{R}^{mn}$. We have $l(J)(\sigma_i) = \sum_{k : i \neq J(k)} \sigma_i(k) \leq 1$ whereby equality holds if and only if $\sigma_i(k) = 0$ whenever $i = J(k)$. For any strategy profile $\sigma \in \Sigma$ we have hence $l(J)(\sigma) = \sum_{i \in N} l(J)(\sigma_i) \leq n$ whereby equality holds if and only $\sigma \in G_J$. Furthermore we obtain

$$l(J)(P(\sigma)) = l(J)(\sigma)$$

for all $\sigma \in \Sigma$, from the observation that

$$l(J)(d_k) = \sum_{i \in N} l(J)(e_{i,k}) = n - 1 = \frac{1}{m} \sum_{k \in M} l(J)(d_k) = l(J)(\rho)$$

so that $l(J)(\delta_k) = l(J)(d_k) - l(J)(\rho) = 0$ and hence

$$l(J)(P(\sigma)) = l(J)(\sigma) - \frac{1}{n} \sum_{k \in M} \langle \sigma, \delta_k \rangle l(J)(\delta_k) = l(J)(\sigma)$$

for all $\sigma \in \Sigma$. Hence $l(J)(P(\sigma)) \leq n$ for all $\sigma \in \Sigma$ whereby equality holds if any only if $\sigma \in G_J$.

(i) P is one-to-one on G into the boundary of $P(\Sigma)$. We already know that P is one-to-one on G_J. Now consider strategy profiles $\sigma, \tau \in \Sigma$ with $\sigma \in G_J$ and $\tau \notin G_J$. Then by the previous observations, $l(J)(P(\sigma)) = n$ while $l(J)(P(\tau)) < n$. Hence, $P(\sigma) \neq P(\tau)$. Thus, P is one-to-one on G.

ii) $P(G)$ equals the relative boundary of $P(\Sigma)$. $P(\Sigma \setminus G)$ equals the relative interior of $P(\Sigma)$. Consider the linear map $l : L \to \mathbb{R}^{\mathcal{J}}$ defined by $l(x) = (l(J)(x))_{j \in \mathcal{J}}$. From the previous discussion we conclude that $P(G) = l^{-1}(B)$ and that $P(\Sigma \setminus G) = l^{-1}(U)$, where $B = \{y \in \mathbb{R}^{\mathcal{J}} \mid y_J = n \text{ for some } J \in \mathcal{J}\}$ and $U = \{y \in \mathbb{R}^{\mathcal{J}} \mid y_J < n \text{ for all } J \in \mathcal{J}\}$.

We have shown that the map P is a homeomorphism of G onto the boundary of the compact convex set $P(\Sigma)$, which has dimension $(m-1)\cdot(n-1)$. This proves our claim. ∎

4 Evolutionary dynamics

In this section we show for the large class of evolutionary dymamics called strongly payoff consistent selection dynamics that G is asymptotically stable, but ρ is not.

As in D&R we define a *payoff consistent selection dynamics* to be a Lipschitz continuous vector field $f = (f_i)_{i \in N}$ on Σ that does not point outwards on Σ and that satisfies for all players i and all $x \in \Sigma$

$$\langle f_i(\sigma), \nabla_{\sigma_i} U_i(\sigma) \rangle \geq 0. \tag{1}$$

We call the selection dynamics *strongly payoff consistent* if, in addition, all Nash equilibria are fixed points and if the above inequality is strict for every strategy profile σ and every player i who has a pure strategy s_i in the support of σ_i with $u_i(\sigma) < u_i(\sigma \mid s_i)$. The notion of a strongly payoff consistent selection dynamics weakens the notion of a Nash dynamic in D&R just enough so that the replicator dynamics is captured as well as a special case.

Proposition 4.1 *Suppose we have a strongly payoff consistent selection dynamics in a minimal diversity game. Then the only asymptotically stable set of restpoints is the set G of efficient Nash equilibria. The completely mixed Nash equilibrium ρ is not asymptotically stable.*

Proof. Since all players have the same utility function and the dynamic is strongly payoff consistent, the chain rule implies that utility is non-decreasing along a trajectory and strictly increasing in any point where there is a better reply in the support. Now consider a strategy profile σ with $u(\sigma) < 1$ in which no player i has a better reply in the support of σ_i. Then all players must play mixed strategies which have the same support, which we denote by T.

To see this, note that otherwise one player would have a pure strategy s_i in the support of his strategy σ_i which is not in the support of the strategy σ_j of another player. By using only s_i player i can then increase everybody's payoff to 1. Contradiction.

Then the same argument as in the proof of Proposition 3.1, which showed that in a Nash equilibrium in completely mixed strategies every player must play each pure strategy with equal probability $1/m$, establishes here that in the strategy profile σ each player must use each strategy in T with equal probability. It follows that the dynamics can only have a finite number of points outside G for which (1) holds with equality for all i.

To see that no rest point σ with $u(\sigma) < 1$ is asymptotically stable under the dynamics, notice that we can find strategy profiles τ with $u(\sigma) < u(\tau)$ arbitrarily near to σ. Because utility is increasing on the trajectory starting in τ, the trajectory cannot move towards σ.

Now let c be such that $u(\sigma) < c$ for all the rest points not in G and consider the neighborhood U of G consisting of all τ with $u(\tau) \geq c$. Since utility is strictly increasing along any trajectory starting in $\tau \in U \setminus G$, Theorem 2.6.1 of Hofbauer and Sigmund [11] implies that any ω-limit point σ of the trajectory must satisfy $\dot{u}(\sigma) = 0$. So, inequality (1) holds for all i. Therefore, σ is an element of G. Hence, G is asymptotically stable. ■

Somewhat lengthy calculations (available from the authors on request) show that the finitely many rest points of the replicator dynamics outside G are unstable hyperbolic rest points. The stable manifold of such a rest point σ with its common support T can be shown to consist of the strategy profiles where all players use identical mixed strategies with support T. (It is straightforward to verify that σ is the unique strategy profile maximizing utility within this set, which is clearly forward invariant under the replicator dynamics. Dimensional arguments then imply that it is indeed the stable manifold.) Thus the stable manifold of each of these restpoints is of lower dimension than Σ. Hence there is an open and dense subset of Σ such that all trajectories starting in it converge to G. In this sense G is "almost" globally asymptotically stable. Since G is a strict equilibrium set in the sense of Balkenborg and Schlag [1], Theorem 6 of their paper implies that G is an asymptotically stable set of stable rest points of the replicator dynamics.

5 Strategic stability

Strategic stability is concerned with sets of Nash equilibria that satisfy necessary conditions for a solution to be acceptable for rational players. Most definitions of strategic stability require robustness of a set of Nash equilibria with respect to certain perturbations of the original game. In this section we are primarily concerned with essentiality (Wu Wen-Tsun and Jiang Jia-He [13], [23]), KM stability and full stability (K&M) and strategic stability in the sense of Mertens [16] and Hillas [8].

We study which of the Nash equilibrium components of a minimal diversity game, the set of Pareto efficient Nash equilibria G or the set $\{\rho\}$ consisting of the Nash equilibrium in completely mixed strategies, contains a strategically stable set of Nash equilibria according to any of the notions mentioned.

From the perspective of strategic stability, the analysis for the completely mixed Nash equilibrium ρ is straightforward. According to the next Theorem, the Nash equilibrium ρ is regular, and hence essential and stable in the sense of Mertens. Then it follows from K&M and Hillas et al. [9] that ρ is also strategically stable according to any of the other notions mentioned above.

Theorem 5.1 *The equilibrium ρ is regular in the sense of Harsanyi. Consequently ρ is also essential, and the set $\{\rho\}$ is stable in the sense of Mertens.*

Proof. Let (N, u) be a minimal diversity game, and let ρ be its completely mixed equilibrium. We show that ρ is regular. For player i, define the function $f_{i1}\colon \Sigma \to \mathbb{R}$ by

$$f_{i1}(\sigma) = \sum_k \sigma_{ik} - 1.$$

For player i and pure strategy $k \geq 2$, define the function $f_{ik}\colon \Sigma \to \mathbb{R}$ by

$$f_{ik}(\sigma) = u(\sigma \mid k) - u(\sigma \mid 1).$$

Since ρ is completely mixed, it is by the Implicit Function Theorem sufficient to show that the Jacobian matrix $\frac{\partial f}{\partial \sigma}(\rho)$ has non-zero determinant. Straightforward calculations show that, for $k \geq 2$,

$$f_{ik}(\sigma) = \prod_{j \neq i} \sigma_{j1} - \prod_{j \neq i} \sigma_{jk}.$$

So, it is easy to check that

$$\frac{\partial f_{ik}}{\partial \sigma_{jl}}(\rho) = \begin{cases} 0 & \text{if } k = 1 \text{ and } j \neq i \\ 1 & \text{if } k = 1 \text{ and } j = i \\ 0 & \text{if } k \geq 2 \text{ and } j = i \\ \left(\frac{1}{m}\right)^{n-2} & \text{if } k \geq 2,\ j \neq i \text{ and } l = 1 \\ -\left(\frac{1}{m}\right)^{n-2} & \text{if } k \geq 2,\ j \neq i \text{ and } l \neq 1. \end{cases}$$

It is now straightforward to show that the resulting Jacobian matrix $\frac{\partial f}{\partial \sigma}(\rho)$ has full rank. Hence, ρ is regular, and therefore also essential and stable in the sense of Mertens. ■

Note that the latter notion of strategic stability also implies many other types of strategic stability such as full stability, KM stability, and best response stability.

Next we study the strategic stability of the set G of efficient Nash equilibria. Consider a minimal diversity game, and let G be its set of efficient Nash equilibria. Any game equivalent to this game and any perturbation defined by restriction of the strategy space of it is a game with identical interests, that is, all players have the same payoff function. Then there are Nash equilibria nearby to strategy profiles equivalent to G in every perturbation close to an equivalent game. These can be found among the strategy profiles maximizing the utility function. The definitions of strategic stability in K&M hence imply the following [10].

Proposition 5.2 *The set G of efficient Nash equilibria of a minimal diversity game contains a fully stable set, and hence a KM-stable set.*

Thus, G is strategically stable under the milder notions of strategic stability defined in K&M. We proceed to show that the picture starts to change when we turn to the more demanding notions of strategic stability such as essentiality, best response stability, and strategic stability in the sense of Mertens.

Theorem 5.3 *If both the number of players and the number of pure strategies is even, then G is essential.*

Proof. By Theorem 4.1, G is an asymptotically stable set of rest points under the replicator dynamics. Consider the extension of this dynamics to a manifold with boundary containing Σ in its interior as described in D&R. Because the replicator dynamics leaves faces invariant, it can easily be shown that every trajectory $\sigma(t)$ starting outside of Σ gets projected orthogonally onto a trajectory $v(t)$ of the replicator dynamics on a face of Σ [11]. In particular it has the same ω-limit points. It follows that G is an asymptotically stable set of rest points under the extended dynamics. So, by Theorem 1 in D&R, the index of G is equal to its Euler characteristic, and hence not zero. Theorem 4 in Ritzberger [19] then implies that G is essential. ∎

Theorem 5.4 *If both the number of players and the number of pure strategies is even, then G contains a strategically stable set in the sense of Mertens.*

Proof. We can apply the arguments as in the proof of Theorem 2 in D&R to the dynamics constructed in the previous theorem. It is not important that the dynamics in question is not a Nash dynamics, because the arguments in D&R are purely local, and G is a component of rest points of the dynamics. Hence, G contains a stable set in the sense of Mertens. ∎

[10] The result extends to the set of strategy profiles maximizing the potential in any weighted potential game as defined in Monderer and Shapley [18].

[11] In a suitably chosen coordinate system, the differential equation for points outside σ can be rewritten as $(\dot{x}, \dot{y}) = g(x) - y$ where $y = 0$ corresponds to the face and $\dot{x} = g(x)$ to the replicator dynamics on that face.

Where strategic and evolutionary stability depart

We now come to the main results of the paper. We conjecture that for all minimal diversity games the following result holds. Suppose that the set G of efficient Nash equilibria has odd dimension. Then it does *not* contain a strategically stable set in the sense of Mertens or Hillas, and it also does not contain an essential set.

A fairly intuitive proof [12], using a generalization of the rock-scissors-paper game, is given for bimatrix games. For binary minimal diversity games (that is, minimal diversity games in which players only have two pure actions) we have an elementary proof showing that G does not contain an essential set. Further, specifically for binary minimal diversity games we develop a technique to linearize the Nash equilibrium correspondence on the class of KM-perturbed games. This then allows us to prove the above conjecture in that case.

Whether any of these techniques can be adapted to more general types of minimal diversity games is not known to us. At least we have proofs for examples with an arbitrarily large number of players, and an arbitrarily large number of pure actions. These results give us some confidence that the conjecture might be true in its full generality.

6 Bimatrix games

In this section we give the proof of the conjecture for two-player games. Notice that if the number of pure strategies is m in a two-player minimal diversity game, the dimension of the sphere G is $m-2$. So we obtain one example for each possible dimension of the sphere. Here we are interested in the case where m is odd.

Consider then a two-player minimal diversity games with an odd number of strategies $m \geq 3$. We show in two steps that G does not contain a best response stable set. It does therefore also not contain a strategically stable set in the sense of Mertens. Moreover, we show that G does not contain an essential set.

First, we introduce a weak notion of strategic stability called independent t-stability. We show that every best response stable set as well as every essential set must contain a t-stable set. Secondly we show that G does not contain an independent t-stable set.

Definition An independent t-perturbation of size ε of a game (N, u) is a collection $(t_i)_{i \in N}$ of

[12] Although the proof uses rather sophisticated tools from Hillas et al. [9].

maps $t_i \colon S_i \to \Sigma_i$ such that $\|t_i(s_i) - s_i\| < \varepsilon$ for every pure strategy $s_i \in S_i$. Each independent t-perturbation of a game defines a t-perturbed game (N, u^t) in normal form, with the same strategy sets S_i as the original game, but new utility functions $u_i^t \colon S \to \mathbb{R}$ given by

$$u_i^t(s) = u_i\left(t(s) \mid s_i\right),$$

where
$$t(s) = t((s_i)_{i\in N}) = (t_i(s_i))_{i\in N}.$$

A t-perturbed game obviously defines a particular payoff-perturbed game. The t-perturbations are, however, more general than trembling hand perturbations because the trembles of a player are correlated with his intended pure strategy choice. Because of this correlation it matters now for the choice of an optimal strategy (and hence for the Nash equilibria of the perturbed game) whether a player ignores his own trembles or not. In our definition of a t-perturbed game a player ignores his own trembles. As a consequence, a t-perturbation of a game with identical payoff does not have to have identical interests and therefore arguments as in Proposition 5.2 do not apply.

Definition A non-empty connected closed set C of Nash equilibria is an independent t-set if there is a Nash equilibrium close to C in every sufficiently small t-perturbation of the game.

Proposition 6.1 *Let $m \geq 3$ be odd. Then the set G of the two player minimal diversity game with m pure strategies is not an independent t-set.*

Proof. Let $m \geq 3$ be odd. Let $\varepsilon > 0$. For the two-player minimal diversity game with m strategies we construct an independent t-perturbation t of size ε such that the unique Nash equilibrium of the t-perturbed game is ρ. Set

$$t_{ik} = \sum_{l=1}^{m} \frac{\varepsilon^{(l-k) \bmod m}}{\sum_{l=0}^{m-1} \varepsilon^l} \cdot s_{il}$$

where we denote by s_{il} player i's l-th pure strategy. If we multiply the payoffs by the common factor $f = \sum_{l=0}^{m-1} \varepsilon^l$, the rescaled payoffs in the t-perturbed game are given by

$$f \cdot u_1^t(l,k) = f \cdot u_2^t(k,l) = f - \varepsilon^{(l-k) \bmod m}.$$

Consider a Nash equilibrium σ of the t-perturbed game. Assume that $\sigma \neq \rho$. We derive a contradiction. Assume without loss that $\sigma_2 \neq \rho_2$. We first show the following claim.

Claim. Let $1 \leq k \leq m$ be such that

$$\sigma_{2k} = \min_{1 \leq l \leq m} \sigma_{2l}.$$

Then $\sigma_{1,k-1} = 0$ (where $\sigma_{1,k-1} = \sigma_{1,m}$ if $k = 1$).

Proof of claim. We assume without loss that $k = m$. We compute that

$$f \cdot u_1^t(m, \sigma_2) = f - \sum_{l=1}^{m} \varepsilon^{m-l} \sigma_{2l} = f - \sum_{l=1}^{m-1} \varepsilon^{m-l} \sigma_{2l} - \sigma_{2m},$$

while

$$f \cdot u_1^t(m-1, \sigma_2) = f - \sum_{l=1}^{m} \varepsilon^{(m-1-l) \bmod m} \sigma_{2l} = f - \sum_{l=1}^{m-1} \varepsilon^{m-1-l} \sigma_{2l} - \varepsilon^{m-1} \sigma_{2m}.$$

We obtain

$$\begin{aligned} f \cdot u_1^t(m, \sigma_2) - f \cdot u_1^t(m-1, \sigma_2) &= \sum_{l=1}^{m-1} \varepsilon^{m-1-l}(1-\varepsilon)\sigma_{2l} - (1-\varepsilon^{m-1})\sigma_{2m} \\ &= \sum_{l=1}^{m-1} \varepsilon^{m-1-l}(1-\varepsilon)\sigma_{2l} - \sum_{l=1}^{m-1} \varepsilon^{m-1-l}(1-\varepsilon)\sigma_{2m} \\ &= \sum_{l=1}^{m-1} \varepsilon^{m-1-l}(1-\varepsilon)(\sigma_{2l} - \sigma_{2m}) > 0. \end{aligned}$$

Strict inequality holds because $\sigma_{2m} \leq \sigma_{2l}$ for all l and $\sigma_{2m} < \sigma_{2l}$ for at least one l. This concludes the proof of the claim.

Now we can proceed as follows. From the assumptions that $\sigma_2 \neq \rho_2$ and $\sigma_{2m} = \min_{1 \leq l \leq m} \sigma_{2l}$ we concluded that $\sigma_{1,m-1} = 0$. So, $\sigma_1 \neq \rho_1$ and $\sigma_{1,m-1} = \min_{1 \leq l \leq m} \sigma_{1l}$. Thus the claim implies that $\sigma_{2,m-2} = 0$. Iterating the argument yields $\sigma_{1,m-3} = 0$, then $\sigma_{2,m-4} = 0$ and so on, with the player index alternating between 2 and 1.

Thus, since m is odd, we see that $\sigma_{2l} = 0$ for l odd and $\sigma_{1l} = 0$ for l even. We obtain in particular after $m - 1$ steps that $\sigma_{21} = 0$. Now the claim yields $\sigma_{1m} = 0$. Iteration of the argument yields $\sigma_{1l} = 0$ for l odd and $\sigma_{2l} = 0$ for l even. We have shown that $\sigma_{1l} = \sigma_{2l} = 0$ for all l. Contradiction. This completes the proof. ■

Now we can show the main result of this section.

Theorem 6.2 *Let $m \geq 3$ be odd. Then the set G of the two player minimal diversity game with m pure strategies is not essential. Also it does not contain a best response stable set.*

Proof. Let (N, u) be a game. We argue that a closed set C of Nash equilibria of the game (N, u) that is essential or best response stable stable necessarily contains an independent t-set. The result then immediately follows from Proposition 6.1.

For essentiality this follows from the observation that, when the size of a t-perturbation is small, also $\|u - u^t\|$ is small by definition of u^t and the continuity of the payoff function u.

Assume that C is best response stable. Then, according to Hillas et al. [9], the set C must contain a so-called CT set [13]. This means in particular that, for every sufficiently small independent t-perturbation, the correspondence BR^t defined by

$$BR^t(\sigma) = \text{convex hull } \{t(s) \mid s \in PB(\sigma)\},$$

where PB is the pure best reply correspondence, has a fixed point close to C. Let $\sigma \in BR^t(\sigma)$ be such a fixed point. Then, because t is an independent t-perturbation, each σ_i is in the convex hull of the strategies $t_i(s_i)$ with $s_i \in PB_i(\sigma)$. Hence, for each player i there is a vector $\phi(\sigma, t)_i = (\phi(\sigma, t)_i(s_i))_{s_i \in S_i}$ in Σ_i such that

$$\sigma_i = \sum_{s_i \in S_i} \phi(\sigma, t)_i(s_i) \cdot t_i(s_i),$$

while moreover $\phi(\sigma, t)_i(s_i) > 0$ implies that $s_i \in PB_i(\sigma)$. Write $\phi(\sigma, t) = (\phi(\sigma, t)_i)_{i \in N}$. Using the definition of the payoff function u_i^t and the multilinearity of the payoff function u_i it is straightforward to check that

$$u_i^t(\phi(\sigma, t) \mid s_i) = u_i(\sigma \mid s_i)$$

for all pure strategies s_i of player i. Now suppose that $\phi(\sigma, t)_i(s_i) > 0$. Then, as noted before, $s_i \in PB_i(\sigma)$. Thus, according to the above displayed equality, s_i is a pure best reply to $\phi(\sigma, t)$ in the game (N, u^t). Therefore ϕ_i is a best reply to ϕ in the t-perturbed game (N, u^t) and hence $\phi(\sigma, t)$ is a Nash equilibrium of the t-perturbed game (N, u^t).

It remains to show that, for sufficiently small t, $\phi(\sigma, t)$ is close to C whenever σ is close to C. This follows readily once we observe that, for small t, $t_i(s_i)$ is close to s_i for all s_i. Hence, $\phi(\sigma, t)_i(s_i)$ is close to $\sigma_i(s_i)$ and $\phi(\sigma, t)$ is close to σ. This concludes the proof. ∎

7 Binary minimal diversity games

The second class of minimal diversity games for which we prove the conjecture that evolutionary and strategic stability make mutually exclusive choices is the class of binary minimal diversity games. A binary minimal diversity game is a game in strategic form with player set $N = \{1, \ldots, n\}$ in which each player has two pure strategies A and B at his disposal. We assume that n is odd and $n \geq 3$. Let S denote the set of pure strategy profiles $s = (s_i)_{i \in N}$ where

[13] See Subsection 5.1 of their paper for the definition.

$s_i \in \{A, B\}$ for all $i \in N$. Each player has the same payoff function $u_i = u$, where $u\colon S \to \mathbb{R}$ is defined by

$$u(s) := \begin{cases} 0 & \text{when } s_1 = \cdots = s_n \\ 1 & \text{else.} \end{cases}$$

A typical mixed strategy is denoted by $\sigma = (\sigma_{iA}, \sigma_{iB})_{i \in N}$, where σ_{iA} (σ_{iB}) denotes the probability with which player i plays pure strategy A (B). Obviously $\sigma_{iA} \geq 0$, $\sigma_{iB} \geq 0$, and $\sigma_{iA} + \sigma_{iB} = 1$. Alternatively we write $\sigma = (\sigma_i, 1 - \sigma_i)_{i \in N}$ for a generic strategy profile. The space of mixed strategy profiles is denoted by Σ.

7.1 Essentiality

In this section we show, for odd n, that G is not essential in the sense of Wu Wen-Tsun and Jiang Jia-He [13], [23]. Let (N, u) denote the binary minimal diversity game. It suffices to construct a small perturbation of the game (N, u) that does not have Nash equilibria close to G. Let (N, v) denote the strategic form game with player set N and payoff functions $v = (v_i)_{i \in N}$, where v_i is defined by

$$v_i(s) = \begin{cases} 1 & \text{if } s_i \neq s_{i-1} \\ 0 & \text{else,} \end{cases}$$

with the convention that player 0 equals player n. Take $\varepsilon > 0$. Consider the game $v(\varepsilon) = u + \varepsilon v$. We show that the strategy profile in which each player plays both his pure strategies with weight $\frac{1}{2}$ is the unique Nash equilibrium of the game $v(\varepsilon)$.

Theorem 7.1 *The unique Nash equilibrium of the game $v(\varepsilon)$ is the strategy profile $\big(\sigma_i, 1 - \sigma_i\big)_{i \in N}$ with $\sigma_i = \frac{1}{2}$ for all $i \in N$.*

Proof. The proof is in three steps.

A. First notice that the strategy profile in which each player plays both his pure strategies with weight $\frac{1}{2}$ is a Nash equilibrium of both the game u and the game v. Therefore it is also a Nash equilibrium of the game $v(\varepsilon)$. We show that $v(\varepsilon)$ does not have any other Nash equilibria.

B. Let $\big(\sigma_i, 1 - \sigma_i\big)_{i \in N}$ be a Nash equilibrium of the game $v(\varepsilon)$. Suppose that it is an element of the boundary of the space of strategy profiles. Suppose w.l.o.g. that $\sigma_1 = 0$. Then necessarily $\sigma_2 = 1$. Because n is odd, iterating this argument leads to $\sigma_1 = 1$. Contradiction. Hence the game $v(\varepsilon)$ only has completely mixed Nash equilibria.

C. Suppose there exists a completely mixed Nash equilibrium $\big(\sigma_i, 1 - \sigma_i\big)_{i \in N}$ of the game $v(\varepsilon)$ for which $\sigma_i \neq \frac{1}{2}$ for at least one $i \in N$. Assume w.l.o.g. that $\sigma_i > \frac{1}{2}$. From the equilibrium condition

$$1 - \prod_{j \neq i+1} \sigma_j + \varepsilon(1 - \sigma_i) = 1 - \prod_{j \neq i+1} (1 - \sigma_j) + \varepsilon \sigma_i$$

for player $i+1$ we can deduce that

$$\frac{\sigma_i}{1 - \sigma_i} = \frac{\prod_{j \neq i, i+1}(1 - \sigma_j) + \varepsilon}{\prod_{j \neq i, i+1} \sigma_j + \varepsilon}.$$

Define $\alpha_j = \frac{\sigma_j}{1-\sigma_j}$ for all $j \in N$. Note that $\alpha_i > 1$, because we assumed that $\sigma_i > \frac{1}{2}$. From $\alpha_i > 1$ we obtain the strict inequality

$$\alpha_i = \frac{\sigma_i}{1 - \sigma_i} = \frac{\prod_{j \neq i, i+1}(1 - \sigma_j) + \varepsilon}{\prod_{j \neq i, i+1} \sigma_j + \varepsilon} < \frac{\prod_{j \neq i, i+1}(1 - \sigma_j)}{\prod_{j \neq i, i+1} \sigma_j} = \prod_{j \neq i, i+1} \frac{1}{\alpha_j}.$$

Then we know that $\alpha_j < 1$ for at least one $j \in N$. So, there must be a player k with $\alpha_k > 1$ and $\alpha_{k-1} \leq 1$. Write $\alpha = \prod_{j \in N} \alpha_j$. From the equilibrium equality for player k we deduce that

$$\alpha_{k-1} \geq \prod_{j \neq k-1, k} \frac{1}{\alpha_j}$$

which can be rewritten to $\alpha \geq \alpha_k$. Hence, $\alpha \geq \alpha_k > 1$. However, since there also exists a player j for which $\alpha_j < 1$, the same line of reasoning yields $\alpha < 1$. Contradiction. ■

The above construction has a very intriguing interpretation in terms of the illustrative story in the introduction. Recall that in this story n soldiers are in charge of the defense of a castle with two gates, east and west. Both gates need to be guarded by at least one soldier to ensure safety of the soldiers. When $n = 2$, the defense tactics are evident: each soldier guards one gate, and this is a strategically stable solution. However, consider the case where there are three soldiers, let us say Edmund, Baldrick, and George. Suppose they agree that Edmund will guard the east gate, while Baldrick is assigned to the west gate, and George is free. However, all three of them are prone to make slight mistakes in the fulfilment of their duties. The assessments of each of them regarding the probability of the others to make a mistake are as follows: George trusts Baldrick more than Edmund, Baldrick trusts Edmund more than George, and Edmund trusts George more than Baldrick. Since George trusts Baldrick more than Edmund, his best option now is to help Edmund at the east gate. This relieves Edmund of his duties. Since he trusts George more than Baldrick, he moves over to the west gate, thus relieving Baldrick of his duties. Now, Baldrick trusts Edmund more than George, he in turn moves to the east gate, setting free George. However, compared to the initial situation, Baldrick and Edmund have now switched places. Hence, George will now move to the west gate. And as you can see, our brave soldiers will run around for ever. This effect occurs exactly when the number of soldiers is odd. With an even number of soldiers the dynamics will eventually settle in a situation where the number of soldiers present at each gate is equal. Notice it is crucial that the assessments of the soldiers regarding each others' reliability is not consistent.

In fact the perturbed games are generalized Shapley games as in Hofbauer and Swinkels [12]. This means that in these games any (sufficiently small) open neighborhood of G contains an attractor of *any* strongly payoff consistent dynamics, while it (in the perturbed game!) does not contain any Nash equilibria.

7.2 Strategic stability, definitions of Mertens and Hillas

In this section we argue that for odd n, G is not best response stable in the sense of Hillas [8]. The results of Hillas et al. [9] then imply that G is also not homotopy stable, and hence also not stable in the sense of Mertens [16]. In fact, we argue that, when n is odd, G is not a CKM set as it is defined in Hillas et al. [9]. The above claims are then implied by the results of that paper.

Let $\Gamma = (N, u)$ be a binary minimal diversity game. A KM perturbation of this game is a vector $\eta = (\eta_{iA}, \eta_{iB})_{i\in N}$ of non-negative numbers η_{iA} and η_{iB}. In the η-perturbed game player i is forced to play pure strategy A (B) with a minimal probability of η_{iA} (η_{iB}). We denote the η-perturbed game by $\Gamma(\eta)$. The size of KM perturbation η is $\|\eta\|_\infty$.

The set of KM perturbations is denoted by $\mathcal{K}$. Let $\mathcal{E}$ be the set of pairs (η, σ) in $\mathcal{K}\times\Sigma$ for which the mixed strategy profile σ is a Nash equilibrium of the game $\Gamma(\eta)$. A CKM perturbation is a continuous function ε from the space Σ of strategy profiles to the space $\mathcal{K}$ of KM perturbations. The size of CKM perturbation ε is

$$\|\varepsilon\| = \max\{\|\varepsilon(\sigma)\|_\infty \mid \sigma \in \Sigma\}.$$

The graph of ε is the set of pairs $(\varepsilon(\sigma), \sigma)$ for $\sigma \in \Sigma$. It is denoted by graph$[\varepsilon]$.

Definition. A closed set $S \subset \Sigma$ is called a CKM set if for every neighborhood U of S there exists $\zeta > 0$ such that for every CKM perturbation ε with $\|\varepsilon\| < \zeta$ there exists a point $(\eta, \sigma) \in$ graph$[\varepsilon] \cap \mathcal{E}$ with $\sigma \in U$.

Theorem 7.2 *Let $n \geq 3$ be odd. Then the set G of the binary minimal diversity game with n players does not contain a best response stable set.*

The proof is deferred to the Appendix. Note that an immediate consequence of the above

Theorem is that, for n odd, G is also neither homotopy stable, nor stable in the sense of Mertens.

8 Discussion and conclusion

In this paper we presented the class of minimal diversity games, a subclass of the class of potential games. The set of Nash equilibria of each minimal diversity game is shown to consist of a symmetric isolated completely mixed equilibrium ρ and a connected component G of strategy profiles that maximize the common payoff function.

The completely mixed equilibrium ρ is strategically stable in virtually any sense: it is perfect, proper, regular, essential, best response stable, and stable in the sense of Mertens. However, ρ is not asymptotically stable.

The connected component G of common payoff maximizers is shown to be asymptotically stable. It also contains a stable set in the sense of Kohlberg and Mertens. Moreover, we show that G is homeomorphic to a sphere of dimension $d = (n-1) \cdot (m-1) - 1$, where n is the number of players, and m is the number of pure strategies. Therefore, when d is even, the Euler characteristic of G is $+2$, and zero when d is odd. Thus, by D&R, when d is even, G is essential and stable in the sense of Mertens.

However, when d is odd, we show for the case where $n = 2$ and m is odd as well as the case where $m = 2$ and n is odd, that G is not essential, not best best response stable, and hence also not stable in the sense of Mertens. Thus, these cases provide examples of games where evolutionary stability and strategic stability make mutually exclusive predictions.

One may wonder whether there are game theoretic considerations that conclusively explain why G ought to be stable, or whether on the contrary there are arguments that explain why G should not be stable. However, no matter what one's stance is in these matters, what we want to emphasize here in this conclusion is that it is hard to think of any conclusive purely game theoretic argument why in the n even case the cycle should be stable, while in the n odd case it should not be stable. This inconsistency in treatment of G (depending on the dimension of G, and ultimately on its Euler characteristic) is exactly what the stronger versions of strategic stability, in particular stability in homology, best response stability, and essentiality, advocate in these examples.

The only reasonably convincing explanation we have found thus far is the allegorical story on the castle, where for odd dimensions the instability of G is due to inconsistent assessments of the players regarding each others' reliability. Still, such an explanation is yet far from any, positive or normative, game theoretic argument that is free from such detailed descriptions of players' behavior as the formation of inconsistent beliefs.

A final interesting observation in this context follows from a known result that any stability concept that satisfies the separate worlds axiom and that is at least as strong as homotopy stability must necessarily be homology type. So, any concept that is at least as strong as homotopy stability either violates the separate worlds axiom, or agrees with homotopy stability on treatment of G: G is stable precisely when its dimension is even. This may indicate that homotopy stability is too strong a requirement for strategic stability. The alternative in the defense of the stronger notions of strategic stability would be to find convincing game theoretic arguments for the relevance of dimensionality, and more specifically the relevance of topological invariants such as the Euler characteristic, for strategic stability. The correct interpretation of these results is still far from settled, and further research is definitely called for.

Appendix: Proof of Theorem 7.2

This Appendix is entirely devoted to the proof of Theorem 7.2. The proof of Theorem 7.2 consists of several intermediate steps, which we shall first briefly discuss.

In the next section, Section 9, we give a full description of the graph of the equilibrium correspondence over KM perturbations. In order to facilitate computations, we do not work directly with the original graph of the equilibrium correspondence. Rather, we apply a transformation to the space $\mathcal{K} \times \Sigma$ of perturbation-strategy profile pairs (η, σ) to obtain a completely linear description of the graph.

In Section 10 we show that, for $n \geq 3$ odd, the set G is not a CKM set. First we analyse what this means in the linearized environment. Then we construct a CKM perturbation that does not have fixed points close to G. This construction is again subdivided in several steps.

First we construct a KM perturbation, called the initial perturbation, having exactly $2n - 1$ equilibria. One is the completely mixed equilibrium ρ. Then there are $2(n - 2)$ equilibria in which the first k players play the same pure strategy, while the others play the same mixed strategy [14]. The exact probabilities used by the mixing players is precisely determined by the

[14]In a perturbed game it is not really possible to play a pure strategy due to the restrictions imposed by the

perturbation. Finally, there are 2 equilibria where the first $n-1$ players all play the same pure strategy, while the last player plays the other pure strategy. The completely mixed equilibrium does not concern us, we focus on the $2n-2$ remaining equilibria.

For example, for $n=5$, we have the following equilibria. First there is the completely mixed equilibrium (Z,Z,Z,Z,Z). Then there are the 6 equilibria

$$\begin{array}{ccc} (A,Z,Z,Z,Z) & & (B,Z,Z,Z,Z) \\ (A,A,Z,Z,Z) & \text{and} & (B,B,Z,Z,Z) \\ (A,A,A,Z,Z) & & (B,B,B,Z,Z) \end{array}$$

and finally the two equilibria (A,A,A,B) and (B,B,B,B,A). In this notation an A (or a B) in position i stands for "player i plays B (or A) with minimum probability", while a Z in position i means that player i plays a mixed strategy (in the sense that he plays both strategies with strictly more than minimum probability). Only the latter 8 equilibria are of interest to us because these are all close to the sphere G, while (Z,Z,Z,Z,Z) is not.

The next step in the construction is to show that these equilibria are pairwise linked via paths in the space of perturbations. In the above example, (A,Z,Z,Z,Z) gets linked to (A,A,Z,Z,Z), while, via another path, (A,A,A,Z,Z) gets linked to (A,A,A,A,B). In the same way (B,Z,Z,Z,Z) gets linked to (B,B,Z,Z,Z), and (B,B,B,Z,Z) gets linked to (B,B,B,B,A).

These paths are then used to construct a CKM perturbation that has no equilibria close to G. The idea behind the construction is to create a CKM perturbation that avoids an intersection with the paths connecting the equilibria of the initial perturbation.

9 Geometry of the equilibrium graph

In this section we give a description of the graph $\mathcal{E}$ of the equilibrium correspondence over KM perturbations.

9.1 Perturbed equilibria

Let $\eta = (\eta_{iA}, \eta_{iB})_{i \in N}$ be a KM perturbation of size at most ζ and let $\sigma = (\sigma_{iA}, \sigma_{iB})_{i \in N}$ be a strategy profile with $\sigma_{iA} \geq \eta_{iA}$ and $\sigma_{iB} \geq \eta_{iB}$. Throughout this section we assume that the perturbation is completely mixed. We derive the conditions under which σ is a Nash equilibrium of the perturbed game $\Gamma(\eta)$.

perturbations. When we write that a player plays a pure strategy, in this context we simply mean that he plays the other strategy with minimum weight.

Define the sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$ by

$$\mathcal{A} := \{i \in N \mid \sigma_{iB} = \eta_{iB}\}$$

$$\mathcal{B} := \{i \in N \mid \sigma_{iA} = \eta_{iA}\}$$

$$\mathcal{Z} := \{i \in N \mid \sigma_{iB} > \eta_{iB} \text{ and } \sigma_{iA} > \eta_{iA}\}.$$

Players in $\mathcal{A}$ are those players that under σ play pure strategy A with maximal probability given η. Similarly players in $\mathcal{B}$ are those players that under σ play pure strategy B with maximal probability given η. The remaining players are in $\mathcal{Z}$. Thus sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Z}$ partition the player set N. Dependence of sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$ on σ and η is suppressed in the notation, but should be kept in mind throughout.

First we derive the general conditions that state precisely when strategy profile σ is a Nash equilibrium of the perturbed game $\Gamma(\eta)$.

Lemma 9.1 *Suppose that σ is a Nash equilibrium of $\Gamma(\eta)$. Then there exists a real number $c \in (0,1)$ such that $\sigma_{iA} = c$ for all $i \in \mathcal{Z}$.*

Proof. Take two players i and j in $\mathcal{Z}$. Since both players are indifferent between playing A and B we have that

$$\prod_{k \neq i} \sigma_{kA} = \prod_{k \neq i} \sigma_{kB} \quad \text{and} \quad \prod_{k \neq j} \sigma_{kA} = \prod_{k \neq j} \sigma_{kB}.$$

So,

$$\frac{\sigma_{iA}}{1-\sigma_{iA}} = \frac{\sigma_{iA}}{\sigma_{iB}} = \frac{\prod_{k \neq i,j} \sigma_{kB}}{\prod_{k \neq i,j} \sigma_{kA}} = \frac{\sigma_{jA}}{\sigma_{jB}} = \frac{\sigma_{jA}}{1-\sigma_{jA}}.$$

Hence, since the function $\sigma \mapsto \frac{\sigma}{1-\sigma}$ is strictly increasing on the open interval $(0,1)$, it follows that $\sigma_{iA} = \sigma_{jA}$. ■

Write $z = |\mathcal{Z}|$. The general equilibrium conditions are as follows. We already have the feasibility conditions

(a) For all $i \in \mathcal{A}$ $\qquad \sigma_{iA} \geq \eta_{iA} \quad \text{and} \quad \sigma_{iB} = \eta_{iB}$

(b) For all $i \in \mathcal{B}$ $\qquad \sigma_{iA} = \eta_{iA} \quad \text{and} \quad \sigma_{iB} \geq \eta_{iB}$

(c) For all $i \in \mathcal{Z}$ $\qquad \sigma_{iA} > \eta_{iA} \quad \text{and} \quad \sigma_{iB} > \eta_{iB}$

Furthermore, there is a $c \in (0,1)$ such that

(1) For all $i \in \mathcal{Z}$ $$\sigma_{iA} = c$$

(2) For all $i \in \mathcal{Z}$ $$\prod_{j\in\mathcal{A}}(1-\eta_{jB})\cdot c^{z-1}\cdot\prod_{j\in\mathcal{B}}\eta_{jA} = \prod_{j\in\mathcal{A}}\eta_{jB}\cdot(1-c)^{z-1}\cdot\prod_{j\in\mathcal{B}}(1-\eta_{jA})$$

(3) For all $i \in \mathcal{A}$ $$\prod_{\substack{j\in\mathcal{A}\\ j\neq i}}(1-\eta_{jB})\cdot c^{z}\cdot\prod_{j\in\mathcal{B}}\eta_{jA} \leq \prod_{\substack{j\in\mathcal{A}\\ j\neq i}}\eta_{jB}\cdot(1-c)^{z}\cdot\prod_{j\in\mathcal{B}}(1-\eta_{jA})$$

(4) For all $i \in \mathcal{B}$ $$\prod_{j\in\mathcal{A}}(1-\eta_{jB})\cdot c^{z}\cdot\prod_{\substack{j\in\mathcal{B}\\ j\neq i}}\eta_{jA} \geq \prod_{j\in\mathcal{A}}\eta_{jB}\cdot(1-c)^{z}\cdot\prod_{\substack{j\in\mathcal{B}\\ j\neq i}}(1-\eta_{jA})$$

Throughout requirements (2), (3), and (4), the left-hand side of the (in)equalities represents the probability that players not equal to i itself all play pure strategy A, the right-hand side being the probability that they all play B. Due to the structure of a minimal diversity game, a player plays optimally precisely when he selects the pure strategy that has the lowest probability.

We treat three exhaustive and non-overlapping cases in which the general conditions reduce to a simpler system of (in)equalities.

CASE I. $z = 0$ In case $z = 0$, the conditions (1) and (2) are empty, and the system reduces to

$(I.3)$ For all $i \in \mathcal{A}$ $$\prod_{j\in\mathcal{B}}\frac{\eta_{jA}}{1-\eta_{jA}} \leq \prod_{\substack{j\in\mathcal{A}\\ j\neq i}}\frac{\eta_{jB}}{1-\eta_{jB}}$$

$(I.4)$ For all $i \in \mathcal{B}$ $$\prod_{\substack{j\in\mathcal{B}\\ j\neq i}}\frac{\eta_{jA}}{1-\eta_{jA}} \geq \prod_{j\in\mathcal{A}}\frac{\eta_{jB}}{1-\eta_{jB}}$$

CASE II. $z = 1$ We write $\mathcal{Z} = \{k\}$. From equation (2) of the general equilibrium conditions it follows that
$$\prod_{j\in\mathcal{B}}\frac{\eta_{jA}}{1-\eta_{jA}} = \prod_{j\in\mathcal{A}}\frac{\eta_{jB}}{1-\eta_{jB}}$$
This equality can be used to rewrite inequalities (3) and (4) to

For all $i \in \mathcal{A}$ $$\frac{c}{1-c} \leq \frac{1-\eta_{iB}}{\eta_{iB}}$$

For all $i \in \mathcal{B}$ $$\frac{c}{1-c} \geq \frac{\eta_{iA}}{1-\eta_{iA}}$$

Putting these observations together, and using the fact that the function $x \mapsto \frac{x}{1-x}$ is strictly increasing on the interval $(0,1)$ we obtain the system

$(II.1)$ $$\sigma_{kA} = c$$

$(II.2)$
$$\prod_{j\in\mathcal{B}}\frac{\eta_{jA}}{1-\eta_{jA}}=\prod_{j\in\mathcal{A}}\frac{\eta_{jB}}{1-\eta_{jB}}$$

$(II.3)$ For all $i\in\mathcal{A}$
$$\eta_{iB}\leq 1-c$$

$(II.4)$ For all $i\in\mathcal{B}$
$$\eta_{iA}\leq c$$

CASE III. $z\geq 2$ Using similar manipulations of the (in)equalities as in the previous case, one can reduce the general system of equilibrium conditions to

$(III.1)$ For all $i\in\mathcal{Z}$
$$\sigma_{iA}=c$$

$(III.2)$ For all $i\in\mathcal{Z}$
$$\prod_{j\in\mathcal{A}}(1-\eta_{jB})\cdot c^{z-1}\cdot\prod_{j\in\mathcal{B}}\eta_{jA}=\prod_{j\in\mathcal{A}}\eta_{jB}\cdot(1-c)^{z-1}\cdot\prod_{j\in\mathcal{B}}(1-\eta_{jA})$$

$(III.3)$ For all $i\in\mathcal{A}$
$$\eta_{iB}\leq 1-c$$

$(III.4)$ For all $i\in\mathcal{B}$
$$\eta_{iA}\leq c$$

This enables us to describe the geometry of the equilibrium correspondence. For sufficiently small size ζ we give a fully parameterized description of the graph $\mathcal{N}(\zeta)$ of the perturbed equilibrium correspondence.

TYPE I EQUILIBRIA Take a partition $\mathcal{A}$ and $\mathcal{B}$ of the player set N. For each perturbation $\eta=(\eta_{iA},\eta_{iB})_{i\in N}$ that satisfies conditions $(I.3)$ and $(I.4)$ we have an equilibrium $\sigma=(\sigma_{iA},\sigma_{iB})_{i\in N}$ defined by $\sigma_{iA}=\eta_{iA}$ for all $i\in\mathcal{B}$ and $\sigma_{iA}=1-\eta_{iB}$ for all $i\in\mathcal{A}$. Automatically, both sets $\mathcal{A}$ and $\mathcal{B}$ are not empty.

TYPE II EQUILIBRIA Take a partition $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}=\{k\}$ of the player set N. For each perturbation $\eta=(\eta_{iA},\eta_{iB})_{i\in N}$ that satisfies condition $(II.2)$ we have the following line segment of equilibria. For any c with $\eta_{kA}\leq c\leq 1-\eta_{kB}$,

$$\eta_{iA}\leq c \text{ for all } i\in\mathcal{B} \quad\text{and}\quad c\leq 1-\eta_{iB} \text{ for all } i\in\mathcal{A}$$

the strategy profile $\sigma=(\sigma_{iA},\sigma_{iB})_{i\in N}$ defined by $\sigma_{iA}=\eta_{iA}$ for all $i\in\mathcal{B}$, $\sigma_{iA}=1-\eta_{iB}$ for all $i\in\mathcal{A}$, and $\sigma_{kA}=c$, is an equilibrium. Automatically, since $n\geq 3$ and $z=1$, the sets $\mathcal{A}$ and $\mathcal{B}$ are not empty by $(II.2)$.

type iii equilibria Take a partition $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$ of the player set N, $\mathcal{A}$ or $\mathcal{B}$ possibly empty. For a perturbation $\eta = (\eta_{iA}, \eta_{iB})_{i\in N}$, define

$$c := \left(1 + \left(\prod_{j\in\mathcal{A}} \frac{1-\eta_{jB}}{\eta_{jB}} \cdot \prod_{j\in\mathcal{B}} \frac{\eta_{jA}}{1-\eta_{jA}}\right)^{\frac{1}{z-1}}\right)^{-1}$$

Suppose that η satisfies

For all $i \in \mathcal{A} \cup \mathcal{Z}$ $\qquad \eta_{iB} \le 1 - c$

For all $i \in \mathcal{B} \cup \mathcal{Z}$ $\qquad \eta_{iA} \le c.$

Then the strategy profile $\sigma = (\sigma_{iA}, \sigma_{iB})_{i\in N}$ defined by $\sigma_{iA} = \eta_{iA}$ for all $i \in \mathcal{B}$, $\sigma_{iA} = 1 - \eta_{iB}$ for all $i \in \mathcal{A}$, and $\sigma_{iA} = c$ for all $i \in \mathcal{Z}$ is an equilibrium of the game $\Gamma(\eta)$. Note that the isolated equilibrium ρ is a type III equilibrium, corresponding to the choice $\mathcal{A} = \mathcal{B} = \phi$ and $\mathcal{Z} = N$.

9.2 Linearization of the equilibrium graph

We can simplify the previous description a bit further. We use the map $h\colon (0,1) \to \mathbb{R}$ defined by

$$h(x) = \log x - \log(1-x)$$

to linearize the graph of the equilibrium correspondence and to give a full account of the combinatorial structure of the graph of the equilibrium correspondence.

Clearly h is strictly increasing, and a homeomorphism from the open interval $(0,1)$ to $\mathbb{R}$. Also note that $h(1-x) = -h(x)$ and that, for small values of x, the function value $h(x)$ is a large negative number. We use this map to give a piecewise linear description of the graph of the equilibrium correspondence as follows. Write $y_{iA} = h(\eta_{iA})$, $y_{iB} = h(\eta_{iB})$, $x_{iA} = h(\sigma_{iA})$ and $x_{iB} = h(\sigma_{iB})$. Notice that $\sigma_{iA} + \sigma_{iB} = 1$ holds precisely when $x_{iA} = -x_{iB}$.

type i equilibria Take a partition $\mathcal{A}$ and $\mathcal{B}$ of the player set N such that both sets $\mathcal{A}$ and $\mathcal{B}$ are not empty. For a perturbation $\eta = (\eta_{iA}, \eta_{iB})_{i\in N}$, define the variables y_{iA} and y_{iB} by $y_{iA} := h(\eta_{iA})$ and $y_{iB} := h(\eta_{iB})$. The conditions for η to have a type I equilibrium can now be written as

$(I.3^*)$ For all $i \in \mathcal{A}$ $\qquad \sum_{j\in\mathcal{B}} y_{jA} \le \sum_{\substack{j\in\mathcal{A}\\ j\neq i}} y_{jB}$

$(I.4^*)$ For all $i \in \mathcal{B}$ $$\sum_{\substack{j\in\mathcal{B}\\ j\neq i}} y_{jA} \geq \sum_{j\in\mathcal{A}} y_{jB}.$$

In which case we have the linear equilibrium $x = (x_{iA}, x_{iB})_{i\in N}$ of type I defined by $x_{iA} = y_{iA}$ for all $i \in \mathcal{B}$, $x_{iA} = -y_{iB}$ for all $i \in \mathcal{A}$, and $x_{iA} = -x_{iB}$ for all $i \in N$.

TYPE II EQUILIBRIA Take a partition $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z} = \{k\}$ of the player set N such that sets $\mathcal{A}$ and $\mathcal{B}$ are not empty. Take a perturbation $\eta = (\eta_{iA}, \eta_{iB})_{i\in N}$ that satisfies condition

$(II.2^*)$ $$\sum_{j\in\mathcal{B}} y_{jA} = \sum_{j\in\mathcal{A}} y_{jB}.$$

We have the following line segment of linear equilibria. For any γ with

For all $i \in \mathcal{A} \cup \{k\}$ $$y_{iB} \leq -\gamma$$

For all $i \in \mathcal{B} \cup \{k\}$ $$y_{iA} \leq \gamma.$$

the strategy profile $x = (x_{iA}, x_{iB})_{i\in N}$ defined by $x_{iA} = y_{iA}$ for all $i \in \mathcal{B}$, $x_{iA} = -y_{iB}$ for all $i \in \mathcal{A}$, and $x_{kA} = \gamma$, is a linear equilibrium of type II.

TYPE III EQUILIBRIUM Take a partition $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$ of the player set N, $\mathcal{A}$ or $\mathcal{B}$ possibly empty, and $|\mathcal{Z}| = z \geq 2$. For a perturbation $\eta = (\eta_{iA}, \eta_{iB})_{i\in N}$, let c be such that

$$(z-1)h(c) = \sum_{j\in\mathcal{A}} y_{jB} - \sum_{j\in\mathcal{B}} y_{jA}.$$

Note that c is uniquely defined this way. Suppose that η satisfies

For all $i \in \mathcal{A} \cup \mathcal{Z}$ $$\eta_{iB} \leq 1 - c$$

For all $i \in \mathcal{B} \cup \mathcal{Z}$ $$\eta_{iA} \leq c.$$

When we write

$$\gamma = h(c) = \frac{1}{z-1}\left[\sum_{j\in\mathcal{A}} y_{jB} - \sum_{j\in\mathcal{B}} y_{jA}\right],$$

these inequalities can be rewritten to

For all $i \in \mathcal{A} \cup \mathcal{Z}$ $$y_{iB} \leq -\gamma$$

For all $i \in \mathcal{B} \cup \mathcal{Z}$ $$y_{iA} \leq \gamma.$$

Then the strategy profile $x = (x_{iA}, x_{iB})_{i\in N}$ defined by $x_{iA} = y_{iA}$ for all $i \in \mathcal{B}$, $x_{iA} = -y_{iB}$ for all $i \in \mathcal{A}$, and $x_{iA} = \gamma$ for all $i \in \mathcal{Z}$ is a linear equilibrium of type III. Note that the choice $x_{iA} = x_{iB} = 0$ is a type III linear equilibrium, corresponding to to the Nash equilibrium ρ in the perturbed game.

10 The CKM perturbation

In this section we construct a CKM perturbation that does not intersect the equilibrium correspondence.

10.1 Translation to the linearized equilibrium correspondence

Let X_i denote the space of vectors (x_{iA}, x_{iB}) with $x_{iA} = -x_{iB}$. Write $X = \prod_{i\in N} X_i$. Let Y_i denote the space of vectors (y_{iA}, y_{iB}) and write $Y = \prod_{i\in N} Y_i$. A vector $(x_{iA}, x_{iB})_{i\in N}$ in X is said to be a linear equilibrium of the (linear) perturbation $(y_{iA}, y_{iB})_{i\in N}$ when it satisfies the conditions for type I, type II, or type III linear equilibrium of the previous section. We also sometimes write that the tuple $(x_{iA}, x_{iB}, y_{iA}, y_{iB})_{i\in N}$ is a linear equilibrium.

Lemma 10.1 *Suppose for every $K < 0$ that there exists an $L < 0$, a point $y^* \in Y$, and a continuous function $f\colon X \to Y$ with the following four properties.*

(i) $f(x)_{iA} \le K$ and $f(x)_{iB} \le K$ for all $x \in X$ and all coordinates (i, A) and (i, B)

(ii) for every $x \in X$, when (x, y^*) is a linear equilibrium, then $x_{iA} > L$ and $x_{iB} > L$ for all $i \in N$

(iii) for every $x \in X$ with $x_{iA} \le L$ or $x_{iB} \le L$ for at least one $i \in N$, $f(x) = y^*$

(iv) for every $x \in X$ with $x_{iA} > L$ and $x_{iB} > L$ for all $i \in N$, if $(x, f(x))$ is a linear equilibrium, then $x = 0$.

Then the sphere G of Pareto efficient Nash equilibria of the binary minimal diversity game is not a CKM set.

Proof. We want to show that G is not a CKM set. So, we have to construct a neighborhood U of the sphere G such that for every $\zeta > 0$ there is a CKM perturbation ε with $\|\varepsilon\| \le \zeta$ and $\operatorname{graph}[\varepsilon] \cap \mathcal{E} \cap U$ is empty.

Let U be the set of strategy profiles $\sigma \in \Sigma$ with $\sigma_{iA} < \frac{1}{4}$ or $\sigma_{iB} < \frac{1}{4}$ for at least one player $i \in N$. Clearly U is a neighborhood of G. Take $\zeta < 0$. Take

$$K = h(\zeta) = \log(\zeta) - \log(1 - \zeta).$$

Let L, y^*, and f satisfy the four conditions of the Lemma given this choice of K. For a vector $(z_{iA}, z_{iB})_{i\in N}$ with $0 < z_{iA} < 1$ and $0 < z_{iB} < 1$ for all $i \in N$ we define

$$H\colon (z_{iA}, z_{iB})_{i\in N} \mapsto (h(z_{iA}), h(z_{iB}))_{i\in N}.$$

It is easily checked that H is a homeomorphism from $(0,1)^N$ to $\mathbb{R}^N$. Therefore its inverse H^{-1} exists and is continuous. We define the CKM perturbation $\varepsilon\colon \Sigma \to \mathcal{K}$ by, for every $\sigma \in \Sigma$,

$$\varepsilon(\sigma) = \begin{cases} (H^{-1} \circ f \circ H)(\sigma) & \text{when } \sigma \text{ is completely mixed} \\ H^{-1}(y^*) & \text{otherwise.} \end{cases}$$

First notice that this is indeed a sound definition because, for every completely mixed strategy profile σ, $H(\sigma)$ is indeed an element of X.

A. The function ε is continuous. Suppose that the sequence $(\sigma^k)_{k=1}^{\infty}$ of strategy profiles converges to the strategy profile σ. When σ is completely mixed, we have $\varepsilon(\sigma^k) \to \varepsilon(\sigma)$ by the continuity of H, f, and H^{-1}. Suppose that σ is not completely mixed. Without loss we assume that there is a pair (i, A) with $\sigma_{iA} = 0$. So, $\sigma_{iA}^k \to 0$ as $k \to \infty$. Then $h(\sigma_{iA}^k) \to -\infty$ as $k \to \infty$. So, $h(\sigma_{iA}^k) \leq L$ for large k. Therefore, by (iii), $f(H(\sigma^k)) = y^*$ for large k. Hence, $(H^{-1} \circ f \circ H)(\sigma^k) \to H^{-1}(y^*)$ as $k \to \infty$.

B. We check that $\|\varepsilon(\sigma)\|_\infty \leq \zeta$ for all $\sigma \in \Sigma$. Take $\sigma \in \Sigma$. Then $H(\sigma) \in X$. So, by (i), $f(H(\sigma))_{iA} \leq K$ and $f(H(\sigma))_{iB} \leq K$ for all (i, A) and (i, B). Hence, by the definition of K, $h^{-1}(f(H(\sigma))_{iA}) \leq \zeta$ and $h^{-1}(f(H(\sigma))_{iB}) \leq \zeta$ for all (i, A) and (i, B), which implies that $\|\varepsilon(\sigma)\|_\infty \leq \zeta$.

C. We check that $\mathrm{graph}[\varepsilon] \cap \mathcal{E} \cap U$ is empty. Suppose that (η, σ) is an element of $\mathrm{graph}[\varepsilon] \cap \mathcal{E} \cap U$. Then $\eta = \varepsilon(\sigma)$. So, η is completely mixed by the definition of ε. Therefore, since σ is an equilibrium of the η-perturbed game, also σ is completely mixed. So, again by the definition of ε, $\eta = (H^{-1} \circ f \circ H)(\sigma)$, which implies that $H(\eta) = (f \circ H)(\sigma)$. Consequently, $(H(\sigma), f(H(\sigma))) = (H(\sigma), H(\eta))$ is a linear equilibrium.

If $H(\sigma)_{iA} > L$ and $H(\sigma)_{iB} > L$ for all $i \in N$, then $H(\sigma) = 0$ by (iv), which implies that $\sigma = \rho \notin U$. Therefore $H(\sigma)_{iA} \leq L$ or $H(\sigma)_{iB} \leq L$ for at least one $i \in N$. So, by (iii), $f(H(\sigma)) = y^*$, and $(H(\sigma), y^*) = (H(\sigma), f(H(\sigma)))$ is a linear equilibrium. Then, by (ii), $H(\sigma)_{iA} > L$ and $H(\sigma)_{iB} > L$ for all $i \in N$. This contradicts the earlier conclusion that $H(\sigma)_{iA} \leq L$ or $H(\sigma)_{iB} \leq L$ for at least one $i \in N$. ■

The remainder of this paper is devoted to the proof that, given $K < 0$, there do indeed exist an $L < 0$, a point $y^* \in Y$, and a continuous function $f\colon X \to Y$ that satisfy the four properties of Lemma 10.1.

10.2 The initial perturbation

Take $K < 0$ fixed. Define the initial perturbation y^* by

$$y^* = (y_{jA}, y_{jB})_{j\in N} = (K \cdot n^j \cdot 4^j, K \cdot n^j \cdot 4^j)_{j\in N}.$$

Write $y_j = K \cdot n^j \cdot 4^j$. Note that $y_j = y_{jA} = y_{jB} = K \cdot n^j \cdot 4^j$. Moreover, for every $k = 2, \dots, n$,

$$y_k < n \cdot \sum_{j=1}^{k-1} y_j \tag{$*$}$$

We show that the only linear equilibria of y^* are the ones associated with the partitions

$$\mathcal{A} = \{1, \dots, k\} \quad \text{and} \quad \mathcal{Z} = \{k+1, \dots n\}$$

for $k \leq n-2$, and

$$\mathcal{A} = \{1, \dots, n-1\} \quad \text{and} \quad \mathcal{B} = \{n\}$$

together with their symmetric counterparts

$$\mathcal{B} = \{1, \dots, k\} \quad \text{and} \quad \mathcal{Z} = \{k+1, \dots n\}$$

for $k \leq n-2$, and

$$\mathcal{B} = \{1, \dots, n-1\} \quad \text{and} \quad \mathcal{A} = \{n\}.$$

It is straightforward to check that these partitions indeed generate linear equilibria for y^*. We show that no other partitions generate linear equilibria for this particular perturbation.

First we show that there are no other linear equilibria of type I beyond the two mentioned above. Assume w.l.o.g. that $n \in \mathcal{B}$. Suppose that there is a $k \neq n$ with $k \in \mathcal{B}$. Then, by the above inequality $(*)$,

$$\sum_{\substack{j\in\mathcal{B}\\ j\neq k}} y_j \leq y_n < \sum_{j=1}^{n-1} y_j \leq \sum_{j\in\mathcal{A}} y_j.$$

This violates inequality $(I.4^*)$.

Next, we show that there are no linear equilibria of type II. Assume w.l.o.g. that there is a $k \in \mathcal{B}$ with $k > j$ for all $j \in \mathcal{A}$. Then, by $(*)$,

$$\sum_{j\in\mathcal{B}} y_j \leq y_k < \sum_{j=1}^{k-1} y_j \leq \sum_{j\in\mathcal{A}} y_j,$$

which violates equation $(II.2^*)$.

Finally we show that there are no linear equilibria of type III that have both $\mathcal{A} \neq \phi$ and $\mathcal{B} \neq \phi$. Suppose both $\mathcal{A} \neq \phi$ and $\mathcal{B} \neq \phi$. Assume w.l.o.g. that there is a $k \in \mathcal{B}$ with $k > j$ for all $j \in \mathcal{A}$. Then $\gamma > 0$. So, by inequality $(*)$, for every $i \in \mathcal{A}$,

$$\begin{aligned}(n-1)\cdot\gamma \geq (z-1)\cdot\gamma &= \sum_{j\in\mathcal{A}} y_j - \sum_{j\in\mathcal{B}} y_j > \sum_{j\in\mathcal{A}} y_j - y_k > \sum_{j\in\mathcal{A}} y_j - n\cdot\sum_{j=1}^{k-1} y_j \\ &\geq \sum_{j\in\mathcal{A}} y_j - n\cdot\sum_{j\in\mathcal{A}} y_j = -(n-1)\cdot\sum_{j\in\mathcal{A}} y_j \\ &\geq -(n-1)y_i.\end{aligned}$$

This violates the requirement $y_{iB} \leq -\gamma$ for $i \in \mathcal{A}$.

10.3 Linking linear equilibria

Suppose that $k \leq n-4$ (and hence necessarily $n \geq 5$). We focus on the equilibrium that is generated by the partition

$$\mathcal{A} = \{1,\ldots,k\} \quad \text{and} \quad \mathcal{Z} = \{k+1,\ldots n\}.$$

We indicate this equilibrium by (A,Z,Z), the coordinates in positions $(k,k+1,k+2)$. We wish to construct a path in the linear equilibrium correspondence that connects this linear equilibrium to linear equilibrium (A,A,Z) generated by partition

$$\mathcal{A} = \{1,\ldots,k+1\} \quad \text{and} \quad \mathcal{Z} = \{k+2,\ldots n\}$$

via the linear equilibria (A,Z,B) and (A,A,B) generated by the partitions

$$\mathcal{A} = \{1,\ldots,k\} \quad \text{and} \quad \mathcal{B} = \{k+2\} \quad \text{and} \quad \mathcal{Z} = \{k+1\}\cup\{k+3,\ldots n\}$$

and

$$\mathcal{A} = \{1,\ldots,k+1\} \quad \text{and} \quad \mathcal{B} = \{k+2\} \quad \text{and} \quad \mathcal{Z} = \{k+3,\ldots n\}$$

respectively.

Take $K < 0$ fixed. Define $R = 5n\cdot K$. Take $\lambda \geq 1$. The parameter λ is our parametrization for the line segment of perturbations above which we consider the equilibrium correspondence. The constant R is chosen in such a way that the $(1,B)$ coordinate of the resulting perturbation is smaller (in the sense of "more negative") than the $(1,B)$ coordinate of y^*. We define the perturbation

$$y^*(k,\lambda) = (y_{iA}, y_{iB})_{i\in N}$$

by [15]

$$y_{iA} = \begin{cases} \lambda \cdot R & \text{when } i = 1, \ldots, k \\ 3nR & \text{when } i = k+1 \\ 2nR & \text{when } i = k+2 \\ 4nR & \text{when } i = k+3, \ldots, n-1 \\ (4n)^n R & \text{when } i = n \end{cases} \quad \text{and} \quad y_{iB} = \begin{cases} \lambda \cdot R & \text{when } i = 1, \ldots, k \\ R & \text{when } i = k+1 \\ R & \text{when } i = k+2 \\ 4nR & \text{when } i = k+3, \ldots, n-1 \\ (4n)^n R & \text{when } i = n \end{cases}$$

Notice that $y_{iA} = \lambda \cdot R$ for $i = 1, \ldots, k$. Thus, this value varies as λ changes. The other values are fixed throughout the construction. We show that the equilibrium correspondence over the path of perturbations for $\lambda \geq 1$, when restricted to the equilibria (A, Z, Z), (A, Z, B), (A, A, B), and (A, A, Z), looks as follows.

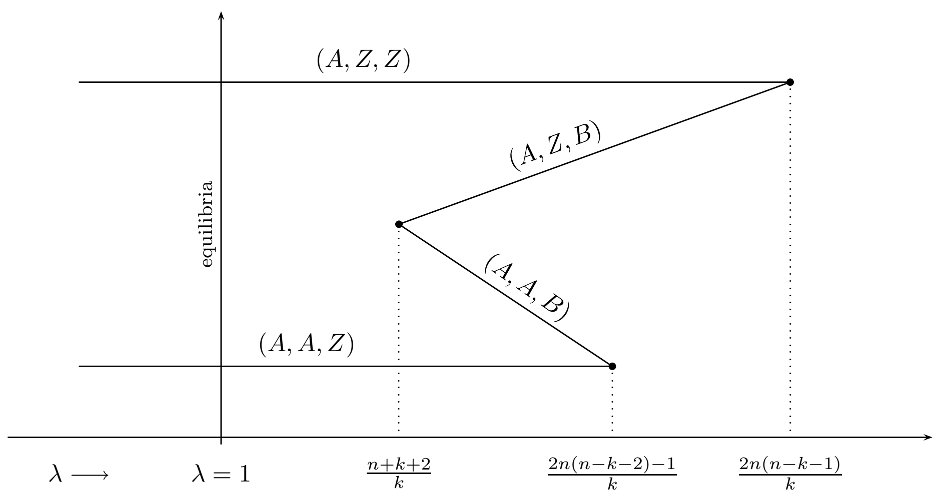


FIGURE: the graph of the equilibrium correspondence for $k \leq n-4$

Notice that indeed

$$1 < \frac{n+k+2}{k} < \frac{2n(n-k-2)-1}{k} < \frac{2n(n-k-1)}{k}.$$

The second inequality follows from the assumption that $k \leq n-4$ (and hence $n-k-2 \geq 2$).

Claim 1. The (A, Z, Z) equilibrium exists precisely when $\lambda \leq \frac{2n(n-k-1)}{k}$. Given the partition, we know that

$$\gamma = \frac{1}{z-1} \sum_{j \in \mathcal{A}} y_{jB} = \frac{k\lambda R}{n-k-1} < 0.$$

[15]Formally y_{iA} and y_{iB} depend on both k and λ. For simplicity we suppress this dependence in the notation though and simply keep the dependence in mind in the calculations.

Thus, the equilibrium conditions reduce to $y_{iA} \leq \gamma$ for all $i \in \mathcal{Z}$. So, since $y_{k+2,A} > y_{iA}$ for all $i \in \mathcal{Z}$, $i \neq k+2$, the condition for this equilibrium to exist is

$$2nR \leq \frac{k\lambda R}{n-k-1} \quad \Leftrightarrow \quad \lambda \leq \frac{2n(n-k-1)}{k}.$$

Claim 2. The (A, A, Z) equilibrium exists precisely when $\lambda \leq \frac{2n(n-k-2)-1}{k}$. Given the partition, we know that

$$\gamma = \frac{1}{z-1} \sum_{j \in \mathcal{A}} y_{jB} = \frac{k\lambda R + R}{n-k-2} < 0.$$

Thus, the equilibrium conditions reduce to $y_{iA} \leq \gamma$ for all $i \in \mathcal{Z}$. So, since $y_{k+2,A} > y_{iA}$ for all $i \in \mathcal{Z}$, $i \neq k+2$, the condition for this equilibrium to exist is

$$2nR \leq \frac{k\lambda R + R}{n-k-2} \quad \Leftrightarrow \quad 2n \geq \frac{k\lambda + 1}{n-k-2} \quad \Leftrightarrow \quad \frac{2n(n-k-2)-1}{k} \geq \lambda.$$

Claim 3. The (A, Z, B) equilibrium exists precisely when $\frac{n+k+2}{k} \leq \lambda \leq \frac{2n(n-k-1)}{k}$. Given the partition, we know that

$$\gamma = \frac{1}{z-1} \left[\sum_{j \in \mathcal{A}} y_{jB} - \sum_{j \in \mathcal{B}} y_{jA} \right] = \frac{1}{n-k-2} \left[k \cdot \lambda \cdot R - 2nR \right]$$

while the equilibrium conditions reduce to $y_{k+1,B} \leq -\gamma$ and $y_{k+2,A} \leq \gamma$. This yields

$$2nR \leq \frac{k\lambda R - 2nR}{n-k-2} \leq -R$$

which can be rewritten to

$$2n \geq \frac{k\lambda - 2n}{n-k-2} \geq -1$$

and

$$2n + 2n(n-k-2) \geq k\lambda \geq 2n - (n-k-2)$$

and hence

$$\frac{n+k+2}{k} \leq \lambda \leq \frac{2n(n-k-1)}{k}.$$

Claim 4. The (A, A, B) equilibrium exists precisely when $\frac{n+k+2}{k} \leq \lambda \leq \frac{2n(n-k-2)-1}{k}$. Given the partition, we know that

$$\gamma = \frac{1}{z-1} \left[\sum_{j \in \mathcal{A}} y_{jB} - \sum_{j \in \mathcal{B}} y_{jA} \right] = \frac{k\lambda R + R - 2nR}{n-k-3}$$

while the equilibrium conditions reduce to $y_{k+1,B} \leq -\gamma$ and $y_{k+2,A} \leq \gamma$. This yields

$$2nR \leq \frac{k\lambda R + R - 2nR}{n-k-3} \leq -R$$

which can be rewritten to

$$2n + 2n(n-k-3) - 1 \geq k\lambda \geq 2n - (n-k-3) - 1$$

and further to

$$\frac{n+k+2}{k} \leq \lambda \leq \frac{2n(n-k-2)-1}{k}.$$

Again the interval for λ is not degenerate because we assume that $k \leq n-4$.

Now suppose that $k = n-2$, so that

$$\mathcal{A} = \{1, \ldots, n-2\} \quad \text{and} \quad \mathcal{Z} = \{n-1, \ldots n\}.$$

We connect the (A, Z, Z) equilibrium [16] generated by this partition to the (A, A, B) equilibrium generated by the partition

$$\mathcal{A} = \{1, \ldots, n-1\} \quad \text{and} \quad \mathcal{B} = \{n\}.$$

In order to do this, again take $R < 0$ fixed. For $\lambda \geq 1$ we define the perturbation

$$y^*(n-2, \lambda) = (y_{iA}, y_{iB})_{i \in N}$$

by [17]

$$y_{iA} = \begin{cases} \lambda \cdot R & \text{when } i = 1, \ldots, n-2 \\ 3nR & \text{when } i = n-1 \\ 2nR & \text{when } i = n \end{cases} \qquad \text{and} \quad y_{iB} = \begin{cases} \lambda \cdot R & \text{when } i = 1, \ldots, n-2 \\ R & \text{when } i = n-1 \\ R & \text{when } i = n \end{cases}$$

We show that the equilibrium correspondence over the path of perturbations for $\lambda \geq 1$, when restricted to the equilibria (A, Z, Z), (A, Z, B), and (A, A, B), looks as follows. Notice that indeed $1 < \frac{2n}{n-2}$.

[16] The letter coding here refers to the coordinates $(n-2, n-1, n)$.
[17] For simplicity we again suppress the dependence of y_{iA} and y_{iB} on λ.

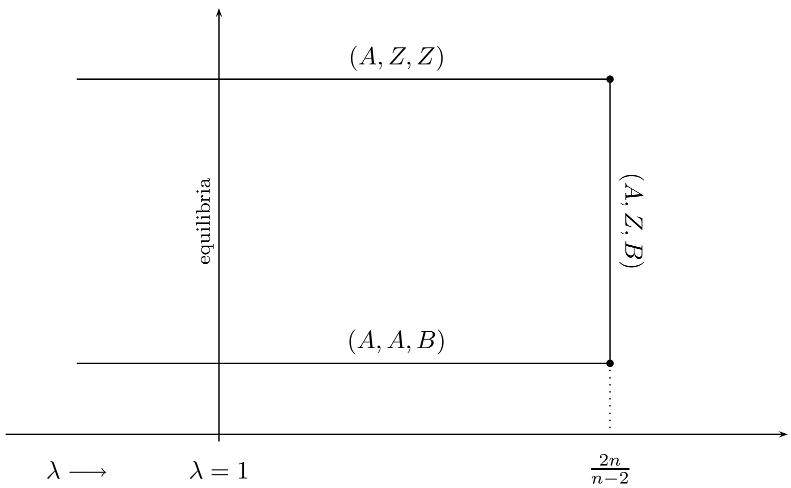


FIGURE: the graph of the equilibrium correspondence for $k = n-2$

Claim 1. The (A,Z,Z) equilibrium exists precisely when $\lambda \leq \frac{2n}{n-2}$. Given the partition, we know that

$$\gamma = \frac{1}{z-1} \sum_{j \in \mathcal{A}} y_{jB} = (n-2)\lambda R < 0.$$

Thus, the equilibrium conditions reduce to $y_{iA} \leq \gamma$ for $i = n-1, n$. So, since $y_{n,A} = 2nR > 3nR = y_{n-1,A}$, the condition for this equilibrium to exist is

$$2nR \leq (n-2)\lambda R \quad \Leftrightarrow \quad \lambda \leq \frac{2n}{n-2}.$$

Claim 2. The (A,A,B) equilibrium exists precisely when $\lambda \leq \frac{2n}{n-2}$. Given the partition, we know that $|B| = 1$. So, since $y_{n-1,B} \geq y_{jB}$ for all $j = 1, \ldots, n-2$, the equilibrium conditions reduce to

$$y_{nA} \leq \sum_{j=1}^{n-2} \lambda y_{jB} \quad \Leftrightarrow \quad 2nR \leq (n-2)\lambda R \quad \Leftrightarrow \quad \lambda \leq \frac{2n}{n-2}.$$

Claim 3. The (A,Z,B) equilibrium exists precisely when $\lambda = \frac{2n}{n-2}$. This type II equilibrium exists precisely when

$$\sum_{j \in \mathcal{B}} y_{jA} = \sum_{j \in \mathcal{A}} y_{jB}.$$

On the line segment of perturbations we consider this reduces to the equation

$$y_{nA} = \sum_{j=1}^{n-2} y_{jB} \quad \Leftrightarrow \quad 2nR = (n-2)\lambda R \quad \Leftrightarrow \quad \lambda = \frac{2n}{n-2}.$$

For the perturbation where this equality holds we have the following line segment of linear equilibria. For any γ with

for $i = 1, \ldots, n-1$ $\qquad y_{iB} \leq -\gamma$

for $i = n-1, n$ $\qquad y_{iA} \leq \gamma$

the vector $x = (x_{iA}, x_{iB})_{i \in N}$ defined by $x_{iA} = y_{iA}$ for all $i \in \mathcal{B}$, $x_{iA} = -y_{iB}$ for all $i \in \mathcal{A}$, and $x_{kA} = \gamma$, is a linear equilibrium. For this particular perturbation these conditions reduce to

$$y_{n-1,B} = y \leq -\gamma \quad \text{and} \quad y_{nA} = 2nR \leq \gamma.$$

On the side of the line segment of linear equilibria where the inequality $y_{n-1,B} \leq -\gamma$ becomes binding, the line segment connects to the line segment of (A, A, B) equilibria. On the side where the inequality $y_{nA} \leq \gamma$ becomes binding it connects to the line segment of (A, Z, Z) linear equilibria.

10.4 Construction of f

Take an arbitrary $K < 0$. In this final section we show that there exist an $L < 0$, a point $y^* \in Y$, and a continuous function $f \colon X \to Y$ that satisfy the four properties of Lemma 10.1. Note that the constructions in the previous two sections only depend on K (and on $R = 5nK$).

Take $k \leq n-2$ and k odd. Further let $0 \leq \mu \leq 1$. Define

$$z(k, \mu) = \begin{cases} (1-2\mu)y^* + 2\mu y^*(k, 1) & \text{when } \mu \leq \frac{1}{2} \\ y^*\left(k, \frac{4n(n-k-1)}{k} \cdot \mu + 1 - \frac{2n(n-k-1)}{k}\right) & \text{when } \mu \geq \frac{1}{2}. \end{cases}$$

Define

$$Z_k = \{z(k, \mu) \mid 0 \leq \mu \leq 1\}.$$

For k odd, $k \leq n-2$, define sets E_k, N_k and S_k as follows. Let E_k be the set of pairs $(z(k, \mu), x)$ for which $z(k, \mu) \in Z_k$ and x is a linear equilibrium of $z(k, \mu)$. Let N_k be the set of pairs $(z(k, \mu), x)$ for which $z(k, \mu) \in Z_k$ and x is a linear equilibrium of $z(k, \mu)$ of one of the forms (A, Z, Z), (A, Z, B), (A, A, B), or (A, A, Z). Let $S_k \subset X$ be the collection of points $x \in X$ for which there is a $z(k, \mu) \in Z_k$ such that x is a linear equilibrium of the perturbation $z(k, \mu)$ of one of the forms (A, Z, Z), (A, Z, B), (A, A, B), or (A, A, Z). Note that, for $k = n-2$, type (A, A, Z) equilibria do not occur.

Before we proceed we first need to state and prove four claims concerning the sets S_k, N_k and E_k. The proofs are at times somewhat lengthy, and can be skipped in a first reading of this part of the paper.

Claim 1. Each N_k is compact. Hence, also each S_k is compact.

Proof of claim 1. The claim immediately follows from the observation that S_k is in fact the projection of the closed and bounded set N_k. ∎

Claim 2. N_k is isolated in E_k. So, there is an open neighborhood U_k of N_k such that for every $(z, x) \in U_k$ with $(z, x) \in E_k$ we automatically have $(z, x) \in N_k$.

Proof of claim 2. Take sequences $x^t \to x$ and $\mu^t \to \mu$ as $t \to \infty$ such that $(z(k, \mu^t), x^t) \in E_k$ for all t and $(z(k, \mu), x) \in N_k$. We show that $(z(k, \mu^t), x^t) \in N_k$ for large t. We assume without loss that all linear equilibria $(z(k, \mu^t), x^t)$ have the same partition $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Z}$. Then we know that $\mathcal{A} \subset \mathcal{A}(\mu)$, $\mathcal{B} \subset \mathcal{B}(\mu)$, and $\mathcal{Z}(\mu) \subset \mathcal{Z}$. Moreover, $i \in \mathcal{Z}$ and $i \in \mathcal{A}(\mu)$ can only happen when $\gamma(\mu) = x_{iA} > 0$, and similarly $i \in \mathcal{Z}$ and $i \in \mathcal{B}(\mu)$ can only happen when $\gamma(\mu) = x_{iA} < 0$.

A. Suppose x is of type (A, Z, Z) under $z(k, \mu)$. Then $\gamma(\mu) < 0$. So, $\mathcal{A} = \mathcal{A}(\mu) = \{1, \ldots, k\}$. Hence, since $\{k+1, \ldots, n\} = \mathcal{Z}(\mu) \subset \mathcal{Z}$, x^t is of type (A, Z, Z) under $z(k, \mu^t)$. The same argument is valid when x is of type (A, A, Z) under $z(k, \mu)$.

Suppose x is of type (A, Z, B) under $z(k, \mu)$. Then $\mu > \frac{1}{2}$. So, we may assume that $\mu^t > \frac{1}{2}$ for all t. If $\gamma(\mu) \leq 0$, then x^t is either of type (A, Z, B) or of type (A, Z, Z) for large t, because $\mathcal{A} = \mathcal{A}(\mu) = \{1, \ldots, k\}$ and $\{k+1\} \cup \{k+3, \ldots, n\} = \mathcal{Z}(\mu) \subset \mathcal{Z}$. In either case $(z(k, \mu^t), x^t)$ in N_k for large t. If $\gamma(\mu) > 0$. Then $\mathcal{B} = \{k+2\}$. Moreover, since $k+1 \in \mathcal{Z}(\mu)$, $x_{k+1,A} = \gamma(\mu) < -R$. Now suppose that $i \in \mathcal{Z}$ for some $i \in \{1, \ldots, k\}$. Then $x^t_{iA} = \gamma(\mu^t)$ for all t. However, since $i \leq k$, we know that $i \in \mathcal{A}(\mu)$, so that $x_{iA} = -\lambda R \geq -R$. Thus, since $x^t_{iA} \to x_{iA}$ and $\gamma(\mu^t) \to \gamma(\mu)$, $\gamma(\mu) \geq -R$. Contradiction. Hence, $i \in \mathcal{A}$ for all $i \in \{1, \ldots, k\}$, and $\mathcal{A} = \mathcal{A}(\mu)$.

Suppose x is of type (A, A, B) under $z(k, \mu)$. Then $\mu > \frac{1}{2}$. So, we may assume that $\mu^t > \frac{1}{2}$ for all t. If $\gamma(\mu) \leq 0$, then x^t is either of type (A, A, B) or of type (A, A, Z) for large t. In either case $(z(k, \mu^t), x^t)$ in N_k for large t. If $\gamma(\mu) > 0$. Then $\mathcal{B} = \{k+2\}$. Further, $k+1 \in \mathcal{A}(\mu)$, so either $k+1 \in \mathcal{A}$ or $k+1 \in \mathcal{Z}$. In either case, $\gamma(\mu^t) \leq -R$. Now take $i \in \{1, \ldots, k\}$, and assume that $i \in \mathcal{Z}$. Then $x^t_{iA} = \gamma(\mu^t) \leq -R$ for all t. So, also $x_{iA} \leq -R < -\lambda R$. This implies that $i \notin \mathcal{A}(\mu)$. Contradiction. Hence, $\{1, \ldots, k\} \subset \mathcal{A}$, and x^t is either of type (A, A, B) or of type (A, Z, B). In either case, $(z(k, \mu^t), x^t) \in N_k$. This completes the proof of Claim 2. ∎

Claim 3. N_k only consists of the line segments indicated in the pictures in the previous section. That is, for $\mu < \frac{1}{2}$, linear equilibria of type (A, Z, B) do not exist, and for $\mu < \frac{1}{2}$ and $k \leq n-4$, also linear equilibria of type (A, A, B) do not exist.

Proof of claim 3. Suppose that $(z(k,\mu), x) \in E_k$ and $\mu < \frac{1}{2}$. We show that it is not of one of the types (A, Z, B) or (A, A, B).

A. Suppose that $(z(k,\mu), x)$ is of type (A, Z, B). When $k \leq n-4$. Then $|\mathcal{Z}| \geq 2$. We show that $\gamma > -y_{k+1,B}$, which contradicts the assumption that $k+1 \in \mathcal{Z}$.

For $\mu = 0$, writing $y_{iA} = y_{iB} = y_i$, we know that

$$y_k < n \cdot \sum_{j=1}^{k-1} y_j.$$

Thus,

$$\gamma = \frac{1}{z-1}\left[\sum_{j=1}^{k} y_j - y_{k+2}\right] > \frac{1}{n-1}\left[\sum_{j=1}^{k} y_j - n \cdot \sum_{j=1}^{k+1} y_j\right] \geq \frac{-n}{n-1} y_{k+1} > -y_{k+1}.$$

For $\mu = \frac{1}{2}$,

$$\gamma = \frac{1}{z-1}\left[\sum_{j=1}^{k} y_{jB} - y_{k+2,A}\right] = \frac{1}{z-1}\left[kR - 2nR\right] > \frac{-n}{n-1} R > -R = -y_{k+1,B}.$$

Thus, the inequality $\gamma > -y_{k+1,B}$ holds for both $\mu = 0$ and $\mu = \frac{1}{2}$. Then by linearity it also holds for all $\mu \in [0, \frac{1}{2}]$.

For $k = n-2$. In this case a type (A, Z, B) linear equilibrium can only exist when

$$\sum_{j \in \mathcal{A}} y_{jB} = \sum_{j \in \mathcal{B}} y_{jA} \qquad \Leftrightarrow \qquad \sum_{j=1}^{n-2} y_{jB} = y_{nA}.$$

However, for both $\mu = 0$ and $\mu = \frac{1}{2}$ it can easily be seen that

$$\sum_{j=1}^{n-2} y_{jB} > y_{nA},$$

so that this strict inequality also holds for all $\mu \in [0, \frac{1}{2}]$ by linearity.

B. Suppose that $k \leq n-4$ and that $(z(k,\mu), x)$ is of type (A, A, B). We show that $\gamma > -y_{1B}$, which violates the assumption that $1 \in \mathcal{A}$. For $\mu = 0$ we have

$$\gamma = \frac{1}{n-k-3}\left[\sum_{j=1}^{k+1} y_j - y_{k+2}\right] > \frac{-(n-1)}{n-k-3} \cdot \sum_{j=1}^{k+1} y_j > -y_1.$$

For $\mu = \frac{1}{2}$ we have

$$\gamma = \frac{1}{n-k-3}\left[\sum_{j=1}^{k+1} y_{jB} - y_{k+2,A}\right] = \frac{(k+1)\cdot R - 2nR}{n-k-3} > \frac{-nR}{n-k-3} > -R = y_{1B}.$$

Hence, by linearity we find that $\gamma > -y_{1B}$ for all $\mu \in [0, \frac{1}{2}]$. ■

Claim 4. The sets $S_1, S_3, \ldots, S_{n-2}$ are mutually disjoint.

Proof of claim 4. Take $k \leq n-2$, k odd, and $l \leq n-2$, l odd, with $k \neq l$. Without loss, $k < l$. Then $l \geq k+2$. Take an $x \in S_k$. Then, since $l \geq k+2$, $x_{lA} < 0$. However, for every $y \in S_l$ we have $y_{lA} > 0$. Hence, $x \notin S_l$, and S_k and S_l are disjoint. ■

These three claims already enable us to prove that G is not stable under the homotopy definition. We however wish to prove a somewhat stronger statement, namely that G is not a CKM-set. For this we need the following improvement on claim 2.

Claim 5. There is an open neighborhood O_k of S_k such that for every $x \in O_k$ and every $z \in Z_k$ with $(z, x) \in E_k$ we automatically have $(z, x) \in N_k$.

Proof of claim 5. The proof is in several steps.

A. First take $x \in S_k$ and $z(k,\mu) \in Z_k$ with $(z(k,\mu), x) \in E_k$. We show that $(z(k,\mu), x) \in N_k$. Since $x \in S_k$ there exists a $z(k,\nu) \in Z_k$ such that $(z(k,\nu), x) \in N_k$. If both $\mu \geq \frac{1}{2}$ and $\nu \geq \frac{1}{2}$, or if both $\mu \leq \frac{1}{2}$ and $\nu \leq \frac{1}{2}$, then by linearity of the linear equilibrium correspondence, the line segment between $(z(k,\mu), x)$ and $(z(k,\nu), x)$ is a subset of E_k. Hence, since $(z(k,\nu), x) \in N_k$, also $(z(k,\mu), x) \in N_k$ in these two cases in view of claims 2 and 3.

Thus, we only need to consider the other two cases, namely (1) $\mu > \frac{1}{2}$ and $\nu < \frac{1}{2}$, and (2) $\mu < \frac{1}{2}$ and $\nu > \frac{1}{2}$. Notice that $x_{1A} > 0$, since $x \in S_k$. So, either $1 \in \mathcal{A}(\mu)$ or $1 \in \mathcal{Z}(\mu)$. Suppose that $1 \in \mathcal{Z}(\mu)$. We derive a contradiction.

A1a. For $k \leq n-4$, k odd. Suppose the linear equilibrium x is of type (A, Z, Z) under $z(k,\nu)$. So, $\mathcal{A}(\nu) = \{1, \ldots, k\}$ and $\mathcal{Z}(\nu) = \{k+1, \ldots, n\}$. Then $x_{1A} > 0 > x_{nA}$. So, since $1 \in \mathcal{Z}(\mu)$, we necessarily have $n \in \mathcal{B}(\mu)$. Now notice that, for both $z = y^*$ and $z = y^*(k,1)$ we have the strict inequality

$$z_{nA} = z_{nB} < n \cdot \sum_{j=1}^{n-1} z_{jB}.$$

Therefore this strict inequality also holds for every $z(k,\mu)$. So, writing $z = z(k,\mu)$, and taking into account that $\gamma(\mu) = x_{1A} > 0$, we get

$$\begin{aligned} -\gamma(\mu) &= \frac{1}{z-1}\left[\sum_{j\in\mathcal{B}} z_{jA} - \sum_{j\in\mathcal{A}} z_{jB}\right] \\ &\leq \frac{1}{n-1}\left[\sum_{j\in\mathcal{B}} z_{jA} - \sum_{j\in\mathcal{A}} z_{jB}\right] \\ &\leq \frac{1}{n-1}\left[z_{nA} - \sum_{j=1}^{n-1} z_{jB}\right] \\ &< \sum_{j=1}^{n-1} z_{jB} \leq z_{1B}. \end{aligned}$$

The conclusion $-\gamma(\mu) < z_{1B}$ contradicts the equilibrium condition when $1 \in \mathcal{Z}(\mu)$. The analogous suitably adjusted argument (using $k+1$ instead of k) can be used for the (A, A, Z) equilibrium.

A1b. For $k \leq n-4$ and k odd. Suppose the linear equilibrium x is of type (A, Z, B) or of type (A, A, B) under $z(k, \nu)$. So, $1 \in \mathcal{A}(\nu)$ and $k+2 \in \mathcal{B}(\nu)$. Then $x_{1A} > 0$ and $x_{k+2,A} < 0$. Thus, because we assumed that $1 \in \mathcal{Z}(\mu)$, necessarily $k+2 \in \mathcal{B}(\mu)$. We derive a contradiction. Note that, by A1, $\nu \geq \frac{1}{2}$, so that $z(k, \nu) = y^*(k, \lambda)$ for

$$\lambda = \frac{4n(n-k-1)}{k} \cdot \nu + 1 - \frac{2n(n-k-1)}{k}.$$

Hence, $x_{k+2,A} = y^*(k, \lambda)_{k+2,A} = 2nR$. We only need to consider the case where $\mu < \frac{1}{2}$. Then

$$\begin{aligned} z(k, \mu)_{k+2,A} &= (1-2\mu)y^*_{k+2,A} + 2\mu y^*(k, 1)_{k+2,A} \\ &= (1-2\mu)(4n)^{k+2}K + 2\mu 2nR \\ &\leq (1-2\mu)(4n)^3K + 2\mu 10n^2K \\ &< 10n^2K = 2nR = x_{k+2,A}, \end{aligned}$$

which implies that $k+2 \notin \mathcal{B}(\mu)$. Contradiction.

A2. When $k = n-2$. Then x is of one of three types (A, Z, Z), (A, Z, B), or (A, A, B) under $z(n-2, \nu)$.

A2a. When x is of type (A, Z, Z) under $z(n-2, \nu)$, and $\nu > \frac{1}{2}$. Then $x_{iA} = -\lambda R > 0$ for all $i \leq n-2$, and $x_{iA} = \lambda(n-2)R < 0$ for $i = n-1, n$. So, by the assumptions that $\mu < \frac{1}{2}$ and $1 \in \mathcal{Z}(\mu)$, we necessarily have $\mathcal{Z}(\mu) = \{1, \ldots, n-2\}$ and $\mathcal{B}(\mu) = \{n-1, n\}$. Then however, $x_{nA} = z(n-2, \mu)_{nA}$. This, because $\mu < \frac{1}{2}$, yields

$$\lambda(n-2)R = 2\mu 2nR + (1-2\mu)(4n)^nR$$

which, substituting $t = 2\mu$, can be rewritten to

$$(n-2)\lambda = 2nt + (1-t)(4n)^n.$$

Then however

$$\lambda = \frac{2nt + (1-t)(4n)^n}{n-2} > \frac{2n}{n-2}$$

because $t = 2\mu < 1$. This contradicts the assumption that x is a linear equilibrium of type (A, Z, Z) under $z(n-2, \nu)$.

A2b. When x is of type (A, Z, Z) under $z(n-2, \nu)$ and $\nu < \frac{1}{2}$. Then $x_{iA} > x_{i-1,A} > 0$ for all $2 \leq i \leq n-2$, and $x_{n-1,A} = x_{nA} < 0$ for $i = n-1, n$. So, when $1 \in Z(\mu)$, then necessarily $\mathcal{A}(\mu) = \{2, \ldots, n-2\}$ $\mathcal{B}(\mu) = \{n-1, n\}$, and $\mathcal{Z}(\mu) = \{1\}$. Thus, x is a type II linear equilibrium under $z(n-2, \mu)$, and the equality

$$\sum_{j \in \mathcal{B}(\mu)} z(n-2, \mu)_{jA} = \sum_{j \in \mathcal{A}(\mu)} z(n-2, \mu)_{jB}$$

should hold. Now notice that for both $z = y^*$ and $z = y^*(n-2, \lambda)$ with $\lambda \leq \frac{2n}{n-2} + 1$ we have

$$z_{n-1,A} + z_{nA} < \sum_{j=2}^{n-2} z_{jB}$$

so that

$$z(n-2, \mu)_{n-1,A} + z(n-2, \mu)_{nA} < \sum_{j=2}^{n-2} z(n-2, \mu)_{jB}$$

holds for every $0 \leq \mu \leq 1$. Contradiction.

A2c. When x is of type (A, Z, B) under $z(n-2, \nu)$. Then automatically $\nu > \frac{1}{2}$. So, we may assume that $\mu < \frac{1}{2}$. Since $\nu > \frac{1}{2}$, we know that $x_{iA} = -\lambda R > 0$ for all $i \leq n-2$, and $x_{nA} = 2nR < 0$. So, since $\mu < \frac{1}{2}$ and $1 \in \mathcal{Z}(\mu)$, necessarily $\{2, \ldots, n-2\} \subset \mathcal{Z}(\mu)$. Further, since $n-1 \in \mathcal{Z}(\nu)$, we have

$$x_{n-1,A} < -z(n-1, \nu)_{n-1,B} \leq -z(n-1, \nu)_{1B} = x_{1A}.$$

Thus, $x_{n-1,A} < x_{1A}$, and then necessarily $n-1 \in \mathcal{B}(\mu)$. If $x_{n-1,A} < 0$. So, $\{1, \ldots, n-2\} = \mathcal{Z}(\mu)$ and $\{n-1, n\} = \mathcal{B}(\mu)$. This however means that

$$-\lambda R = x_{1A} = \gamma(\mu) = -\frac{z(n-2, \mu)_{n-1,A} + z(n-2, \mu)_{nA}}{n-3}.$$

However, since $\mu < \frac{1}{2}$, we can deduce that $z(n-2, \mu)_{n-1,A} < 3nR$ and $z(n-2, \mu)_{n-1,A} < 2nR$. Then, since $\lambda \leq \frac{2n}{n-2}$, the above equality implies that

$$\frac{2n(n-3)}{n-2} \geq 5n$$

which contradicts the assumption that $n \geq 3$.

A2d. When x is of type (A, A, B) under $z(k, \nu)$, $\nu < \frac{1}{2}$ and $\mu > \frac{1}{2}$. Then, since $\nu < \frac{1}{2}$, we have

$$x_{n-1,B} < x_{n-2,B} < \cdots < x_{1B} < 0.$$

Then the assumption $1 \in \mathcal{Z}(\mu)$ implies that $\mathcal{Z}(\mu) = \{1\}$, $\mathcal{A}(\mu) = \{2, \ldots, n-1\}$ and $\mathcal{B}(\mu) = \{n\}$. However, since $z(n-2, \mu)_{iB} = \lambda \cdot R$ for all $i = 2, \ldots, n-2$, $\mathcal{A}(\mu) = \{2, \ldots, n-1\}$ implies that $n \geq 5$ is not possible. Therefore necessarily $n = 3$. In this case, $\mathcal{Z}(\mu) = \{1\}$, $\mathcal{A}(\mu) = \{2\}$ and $\mathcal{B}(\mu) = \{3\}$. So, x is a type II linear equilibrium under $z(n-2, \mu)$. Type II equilibria however only occur when we have the equality

$$\sum_{j \in \mathcal{B}(\mu)} z(n-2, \mu)_{jA} = \sum_{j \in \mathcal{A}(\mu)} z(n-2, \mu)_{jB}$$

which for $n = 3$ reduces to the requirement $z(n-2, \mu)_{3A} = z(n-2, \mu)_{2B}$. However, since $\mu > \frac{1}{2}$, we have $z(n-2, \mu)_{3A} = 2nR = 6R$ and $z(n-2, \mu)_{2B} = R$. Contradiction.

A2e. When x is of type (A, A, B) under $z(k, \nu)$, $\nu > \frac{1}{2}$ and $\mu < \frac{1}{2}$. Since $\nu > \frac{1}{2}$, and x is of type (A, A, B) under $z(n-2, \nu)$, it follows that $x_{1B} = \lambda \cdot R$ and $x_{n-1,B} = R$. Hence

$$x_{1B} < x_{n-1,B} < 0$$

which immediately contradicts the assumption $1 \in \mathcal{Z}(\mu)$.

So, the assumption $1 \in \mathcal{Z}(\mu)$ led to a contradiction in all cases. Hence, $1 \in \mathcal{A}(\mu)$. This implies that $x_{1A} = -z(k, \mu)_{1B}$. We continue the proof that $z(k, \mu), x) \in N_k$.

A3. First we show that the function $\mu \mapsto z(k, \mu)_{1B}$ is strictly decreasing, so that the value of x_{1A} uniquely determines the value of μ. For $\mu \geq \frac{1}{2}$, $z(k, \mu)_{1B} = y^*(k, \lambda)_{1B} = \lambda \cdot K$, where

$$\lambda = \frac{4n(n-k-1)}{k} \cdot \mu + 1 - \frac{2n(n-k-1)}{k}.$$

Obviously λ is strictly increasing in μ. Hence, because $K < 0$, $\lambda \cdot K$ is strictly decreasing in μ.

For $\mu \leq \frac{1}{2}$,

$$z(k, \mu)_{1B} = (1 - 2\mu) y^*_{1B} + 2\mu y^*(k, 1)_{1B} = (1 - 2\mu) \cdot 4nK + 2\mu \cdot R.$$

Since $R = 5n \cdot K < 4nK$, this expression is indeed strictly decreasing in μ.

Now we can finish the proof of A. Since, $\mu \mapsto z(k, \mu)_{1B}$ is strictly decreasing, and both $1 \in \mathcal{A}(\nu)$ and $1 \in \mathcal{A}(\mu)$, we find that $\mu = \nu$. Then however also $z(k, \mu) = z(k, \nu)$. Hence, $z(k, \mu), x) = z(k, \nu) \in N_k$. This completes the proof of part A.

B. Now take $x \in S_k$ and $z \in Z_k$ with $(z, x) \notin N_k$. Then in view of A also $(z, x) \notin E_k$. So, there exists a neighborhood $U_{(z,x)}$ of (z, x) such that $U_{(z,x)} \cap E_k$ is empty. Now let V_k be the union of sets U_k and $U_{(z,x)}$, where U_k is chosen as in claim 2. Then V_k is a neighborhood of the compact set $S_k \times Z_k$ such that $V_k \cap E_k = N_k$. By compactness of $S_k \times Z_k$ there are open neighborhoods O_k of S_k and W_k of Z_k such that $O_k \times W_k$ is a subset of V_k. This concludes the proof of claim 4. ■

We continue with the construction of y^*, L, and f as specified in Lemma 10.1. Let y^* be the initial perturbation defined in Section 10.2. Further, due to Claims 1 and 4 we can find bounded and open sets V_k in X such that $S_k \subset V_k \subset O_k$ for each k, and such that the respective closures $V_1, V_3, \ldots, V_{2n-3}$ are mutually disjoint [18]. Since each V_k is bounded, we can choose an $L < 0$ such that for each V_k and each $x \in V_k$ we have $x_{iA} > L$ and $x_{iB} > L$.

Next, by the Lemma of Urysohn, for each k there exists a continuous function

$$g_k \colon X \to [0, 1]$$

such that $g_k = 1$ on S_k and $g_k = 0$ outside V_k. Define the function $f \colon X \to Y$ by

$$f(x) = \begin{cases} z(k, g_k(x)) & \text{if } x \in V_k \text{ for some } k = 1, \ldots, n-2,\ k \text{ odd} \\ y^* & \text{else} \end{cases}$$

We show that y^*, L, and f satisfy the conditions of Lemma 10.1.

A. We show that f is continuous.

Proof of A. Take a sequence $(x^t)_{t=1}^{\infty}$ converging to x. We show that $f(x^t) \to f(x)$ as $t \to \infty$. The only slightly non-trivial case is where $x^t \in V_k$ for some k, and $x \notin V_k$. In that case $f(x) = y^*$, while by the continuity of g_k we have $g_k(x^t) \to 0$ as $t \to \infty$. Hence,

$$f(x^t) = z(k, g_k(x^t)) \to z(k, 0) = y^* = f(x)$$

as $t \to \infty$. ■

B. We show that $f(x)_{iA} \le K$ and $f(x)_{iB} \le K$ for all $x \in X$ and all coordinates (i, A) and (i, B).

Proof of B. Trivial once we observe that this holds for y^* as well as for all $y^*(k, \lambda)$. ■

C. Suppose that (x, y^*) is a linear equilibrium. We show that $x_{iA} > L$ and $x_{iB} > L$ for all $i \in N$.

[18] Recall that we also need to take care of the equilibria of type (B, Z, Z), (B, B, Z), and (B, B, A). The corresponding $\frac{n-1}{2}$ sets are indexed by indices $n, n+2, \ldots, 2n-3$.

Proof of C. Also easy once we observe that, according to Section 10.2, the only linear equilibrium of the form (x, y^*) for which x is not an element of some S_k is when $x_{iA} = x_{iB} = 0$ for all i, in which case the inequalities also hold. ■

D. For every $x \in X$ with $x_{iA} \leq L$ or $x_{iB} \leq L$ for at least one $i \in N$ we have $f(x) = y^*$.

Proof of D. Immediate from the definition of f, once we observe that the condition $x_{iA} \leq L$ or $x_{iB} \leq L$ for at least one $i \in N$ implies that x is not an element of any V_k. ■

E. Suppose $x \in X$ with $x_{iA} > L$ and $x_{iB} > L$ for all $i \in N$. Suppose that $(x, f(x))$ is a linear equilibrium. Then $x = 0$.

Proof of E. Suppose that $x \in V_k$ for some k. Then, since $(x, f(x)) \in E_k$ by construction, we know from Claim 4 that $(x, f(x)) \in N_k$. This implies that $x \in S_k$. So, $g_k(x) = 1$ by construction. Hence,

$$f(x) = z(k, 1) = y^* \left(k, \frac{2n(n-k-1)}{k} + 1 \right).$$

However, from Section 10.3 we know that for $\lambda > \frac{2n(n-k-1)}{k}$ the perturbation $y^*(k, \lambda)$ does not have any linear equilibria in S_k. Thus, $x \notin V_k$ for all k. Then $f(x) = y^*$. According to Section 10.2, $x = 0$ is the only linear equilibrium of y^* outside the sets $S_k \subset V_k$. ■

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