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Influence of aggregation and measurement scale on ranking a compromise alternative in AHP

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Analytic Hierarchy Process (AHP) is one of the most popular multi-attribute decision aid method. However, results depend on the preference measurement scale and the aggregation technique used. In this paper we describe a decision problem with an inherent trade-off between two criteria. A decision maker has to choose among three alternatives: two extremes and one superior "compromise". Six different measurement scales described previously in the literature and the new proposed logarithmic scale are considered for applying the additive and the multiplicative AHP. The results are compared with the standard consumer choice theory. The geometric and power scales offer no chance (for the additive AHP) and very few chances (for the multiplicative AHP) for a compromise to be selected. The logarithmic scale used with the multiplicative AHP is the most in agreement with the consumer choice theory.

Keywords: Decision Analysis, Multiple criteria analysis, Utility theory, Additive AHP, Multiplicative AHP, Logarithmic scale

1. Introduction

Analytic Hierarchy Process (AHP) (Saaty, 1977; Saaty, 1980) is a multi-criteria decision method applied in numerous situations with impressive results. The *Journal of the Operational Research Society* has recently related several successful applications in different areas: Information Systems (Ahn and Choi, 2008), Supply Chain Management (Sha and Che, 2006; Akarte et al, 2001), Public services (Mingers et al, 2007; Fukuyama and Weber, 2002), Health (Lee1 and Kwak, 1999; Li and al., 2008), Strategy (Leung and al., 2006), E-learning (Tavana, 2006), Defence (Wheeler 2006) and Manufacturing (Bañuelas and Antony, 2007). Several papers have compiled the AHP success stories (Forman and Gass, 2001; Golden et al., 1989; Ho, 2008; Kumar and Vaidya, 2006; Liberatore and Nydick, 2008; Omkarprasad and Sushil, 2006; Vargas, 1990; Zahedi, 1986). The heart of the method are the comparison matrices $\mathbf{A} = (a_{i,j})$, $i,j = 1,\ldots, n$, where $a_{i,j}$ are pairwise comparisons from alternatives/criteria given by the decision maker on a verbal scale of nine levels (table 1). Local priorities $\mathbf{l_l} = l_i$, $i = 1,\ldots, n$ are then calculated from these comparison matrices. Finally, the local priorities are weighted with the criterion priority and aggregated to give the global priority p_i of the alternatives

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Although AHP is widely used, there are differences on some fine points of the method. The literature has proposed different variants

- to derive priorities from the comparison matrix,
- to aggregate the local priorities, and
- to convert verbal comparisons into quantifiable values.

Levels	Definitions
A	**Equal importance**
B	Equal - weak importance
C	**Weak importance**
D	Weak - strong
E	**Strong importance**
F	Strong - very strong importance
G	**Very strong importance**
H	Very strong - absolute importance
I	**Absolute importance**

Table 1: The nine levels of the comparison scale, including the intermediate levels B, D, F, and H.

An AHP matrix is said perfectly consistent if for all comparison $a_{i,j}$ respect the following transitivity (1) and reciprocity (2) rules:

$$a_{i,j} = a_{i,k} \cdot a_{k,j} \quad \text{where } i, j \text{ and } k \text{ are any alternatives of the matrix} \tag{1}$$

$$a_{i,j} = \frac{1}{a_{j,i}} \tag{2}$$

However, AHP accepts some inconsistencies in the entries, which reflect the real world. The calculated priorities are plausible only for slightly inconsistent matrices, i.e consistency ratio (C.R.) < 0.1 (cf. e.g (Saaty, 1980) for the calculations).

The principal right eigenvalue method (Saaty 1977, 1980) is the first introduced method for the calculation of priorities. Others methods like the principal left eigenvalue (Johnson, et al., 1979), the geometric mean (Crawford and Williams, 1985), the least squares (Jensen, 1984), the weighted least squares (Blankmeyer, 1987; Chu, et al., 1979), and the logarithmic least absolute values (Cook and Kress, 1988) have been proposed. A discussion has arisen over the "best" method (Barzilai, et al., 1987; Barzilai and Golany, 1990; Barzilai, 1997; Barzilai, 2001; Harker and Vargas, 1987; Saaty and Vargas, 1984a, 1984b; Saaty, 2001; Saaty, 2003,). In contrast experimental studies (Budescu, et al., 1986; Choo and Wedley, 2004; Golany and Kress, 1993; Ishizaka and Lusti, 2006; Jones and Mardle, 2004; Stam and Duarte Silva, 2003) show a high agreement between these methods. In our paper, we will focus on the other two debated points: the measurement scale and the aggregation of local priorities.

The calculation of global priorities p_i results from the aggregation of the local priorities l_{ij} and the weight w_j of the criterion j. Saaty (1977, 1980) has proposed an additive approach (3). This method has been attacked by Belton and Gear (1983) and

Holder (1990; 1991) because the introduction of a copy of an alternative or a near copy (Dyer, 1990) would change the ranking. This phenomenon is called in the literature "rank reversal". Saaty (1990), Harker and Vargas (1990) have defended the method saying that it is legitimate that the introduction of new information (even a copy of the existing one) is able to change the ranking.

Barzilai and Golany (1994), Barzilai (1997), Triantaphyllou (2001) argued that the rank reversal problem in the AHP is due to an erroneous use of the additive aggregation method. Instead the multiplicative method (4) (Lootsma et al., 1990; Lootsma, 1993; Leskinen and Kangas, 2005) should be used. Contrary to the additive AHP, where the sum of the criteria is equal to the unity $\sum_j w_j = 1$, the multiplicative AHP does not require this normalisation.

$$p_i = \sum_j w_j \cdot l_{i,j} \tag{3}$$

$$p_i = \prod_j l_{i,j}^{\ w_j} \tag{4}$$

where p_i: global priority of the alternative i
$l_{i,j}$: local priority of the alternative i
w_j: weight of the criterion j

In response, Vargas (1997) gives an example where the exact weight of objects can be retrieved only by an additive aggregation of the local comparisons. Due to its rank reversal preservation and its non-linear properties (Triantaphyllou and Baig, 2005), the multiplicative AHP (MAHP) seems to receive a growing attention. In particular Stam and Duarte Silva (2003) notice that the additive AHP tends to overweight extreme alternatives, which seems not to be the case for the MAHP. He suggests that further research should be done to confirm these observations. It is the aim of this paper.

In this paper we describe a decision problem with an inherent trade-off between two criteria. We compare AHP and its variant MAHP with the utility theory to evaluate the choice among three alternatives: two extremes and one superior "compromise". The utility theory has a normative approach and AHP a descriptive or a practical orientation (Winkler, 1990). In this paper, we aim to demonstrate that the aggregation method of local priorities and the measurement scale in AHP has a strong influence on the selection of the superior compromise and therefore on the degree of concordance with utility theory.

One of AHP's strengths is the possibility to evaluate quantitative and also qualitative criteria and alternatives on the same preference scale, namely a verbal scale. The use of verbal responses is intuitively appealing, user-friendly and more common in our everyday lives than numbers. It may also allow some ambiguity in non-trivial comparisons. To derive priorities, the verbal comparisons must be converted into numerical ones. In Saaty's AHP the verbal statements are converted into integers from one to nine. Theoretically there is no reason to be restricted to these numbers. Therefore other scales have been proposed (table 2). Harker and Vargas (1987) have evaluated a quadratic and a root square scale in only one simple example and argued in favour of Saaty's 1 to 9 scale. However, one example seems not enough to conclude the superiority of the 1-9 linear scale. The entered comparisons are not unique: they depend on the decision-maker. Lootsma (1989) argued that the geometric scale is preferable to the 1-9 linear scale. Salo and Hämäläinen (1997) point out that the integers from one to nine yield local weights, which are unevenly dispersed, so that there is lack of sensitivity when comparing elements, which are preferentially close to each other. Based on this observation, they propose a balanced scale where the local weights are evenly dispersed over the weight range [0.1, 0.9]. Earlier, Ma and Zheng (1991) had calculated an inverse linear scale, which also gives more uniformly distributed priorities than the 1-9 scale. For our study, we will also propose a logarithmic scale, which is smoother for high values. Figures 1 and 2 show the used scales in the study.

Scale	Definition	Parameters
Linear (Saaty, 1977)	$c = a \cdot x$	$a > 0$; $x = \{1, 2, \ldots, 9\}$
Power (Harker and Vargas, 1987)	$c = x^a$	$a > 1$; $x = \{1, 2, \ldots, 9\}$
Geometric (Lootsma, 1989)	$c = a^{x-1}$	$a > 1$; $x = \{1, 2, \ldots, 9\}$
Logarithmic	$c = \log_a(x+1)$	$a > 1$; $x = \{1, 2, \ldots, 9\}$
Root square (Harker and Vargas, 1987)	$c = \sqrt[a]{x}$	$a > 1$; $x = \{1, 2, \ldots, 9\}$
Inverse linear (Ma and Zheng, 1991)	$c = 9/(10-x)$	$x = \{1, 2, \ldots, 9\}$
Balanced (Salo and Hämäläien, 1997)	$c = w/(1-w)$	$w = \{0.5, 0.55, 0.6, \ldots, 0.9\}$

Table 2: Different scales for comparing two alternatives (for the comparison of A and B, $c=1$ indicates $A \approx B$; $c>1$ indicates $A>B$; when $A<B$, the reciprocal values $1/c$ are used).

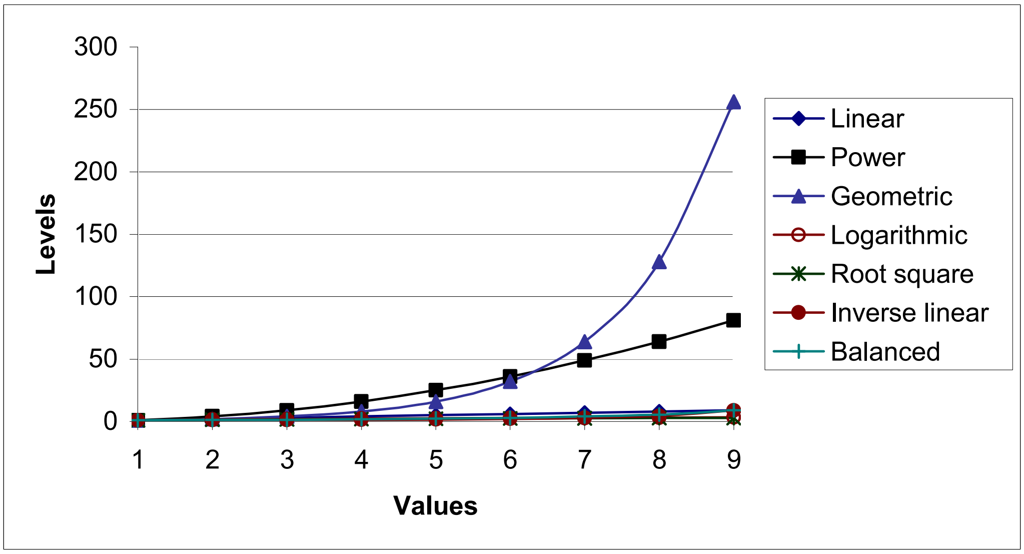

Figure 1: Graph of the judgement scales used in the study.

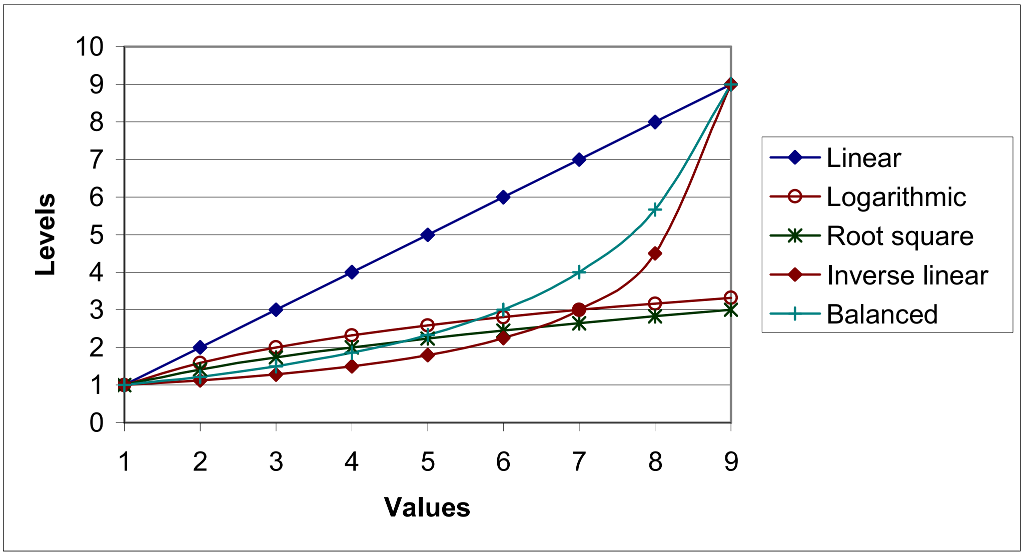

Figure 2: Graph of the judgement scales without the geometric and power scales.

Among all the proposed scales, the linear scale with the integers one to nine and their reciprocals have been used by far the most often in applications. Saaty (1980; 1991) advocates it as the best scale to represent weight ratios. Combined with cluster techniques the upper limit scale problem can be avoided (Saaty 1991; Ishizaka 2004a, 2004b). However, the cited examples deal with objective measurable alternatives like the areas of figures, whereas AHP treats mainly decision processes on subjective issues. We understand the difficulty of verifying the effectiveness of scales through subjective issues. Salo and Hämäläinen (1997) demonstrate the superiority of the balanced scale when comparing two elements. The choice of the "best" scale is a very heated debate. Some scientists agree that the choice depends on the person and the decision problem (Harker and Vargas, 1987; Pöyhönen and al., 1997). Our paper aims to shed some light on the choice of the appropriate scale and aggregation technique. We will run a complete enumeration with the different type of scales for the additive and multiplicative AHP and then draw a parallel with the consumer choice theory.

2. Theory of the consumer choice

2.1 Introduction

For our study, we choose a simple multi-criteria decision problem. A simple problem is not only easier to treat but it also captures the essence of a choice that AHP and MAHP must take. If it cannot hand a simple one, it is doubtful that it will be able to handle a difficult one.

A company has to hire a new sales engineer. The position requires both engineering and sales skills. Three candidates with very different profiles are available (see table 3).

Candidates	Technical skills	Sales skills
A	High	Low
B	Medium	Medium
C	Low	High

Table 3: Candidates A, B, C with their respective knowledge.

Which candidate will be selected? The consumer choice theory sets three main axioms about the preferences of the consumer:

- *Rationality*: The consumer preferences are complete (no preferences are undefined) and consistent (satisfying equations (1) and (2)).[2]
- *Monotonicity*: The consumer prefers to hire a candidate who has more skill than less, where skill is being considered as a normal aptitude or attribute. For example, if two candidates have the same skill in sales but the first has more ability in engineering than the second one, the first candidate will be preferred. The hypothesis implies that the indifference curves have a negative slope like in the figure 3.
- *Convexity*: Simply, the consumer prefers a mix to the extremes. For instance, if a consumer is indifferent to either 10 apples or 10 oranges, then the consumer prefers 5 apples and 5 oranges to either of these options. The hypothesis is discussed in introductory economic textbooks as the "law of diminishing marginal rates of substitution". It implies that the indifference curves are upward bowed (all points on a line between two points on an indifference curve must be on a higher indifference curve) like in the figure 3.

Indifference curves connect all alternatives (represented by vectors of attributes) which leave the consumer indifferent. In figure 3, the set of curves U_1, U_2 and U_3 have three different inclinations. They represent the utility of three different people. Curves U_1 are the indifference curves for a person having symmetric preferences between the importance of criteria skill in sales and skill in engineering for the position to fill. Curves U_2 correspond to a big importance in the criterion skill in engineering and curves U_3 indicate a preference for the skill in sales.

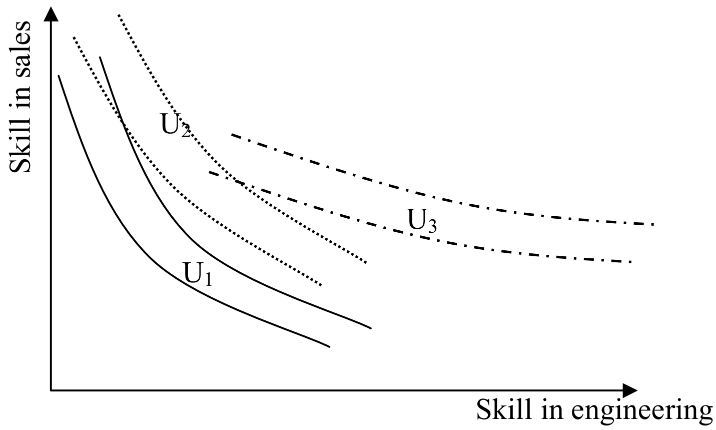

Figure 3: Three types of indifference curves.

The employer will choose the candidate on the highest indifference curve. For example, in figure 4, the candidate B is preferred because he lies on a higher indifference curve than A and C.

[2] AHP only partially requires this hypothesis

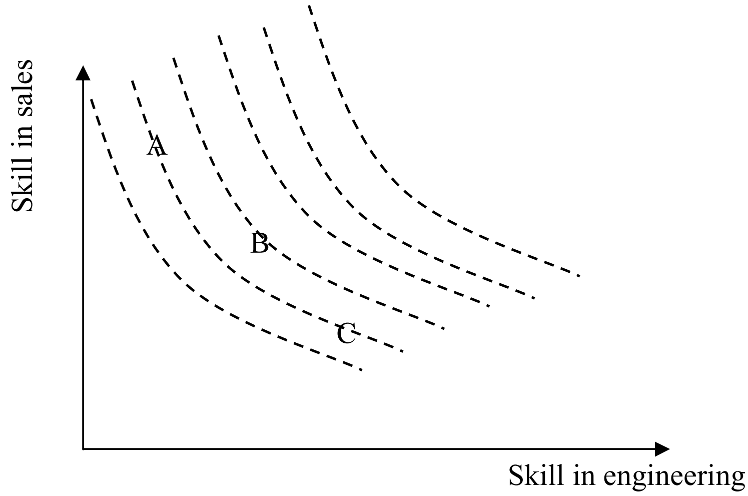

Figure 4: The candidate B is preferred because he lies on a higher utility curve of the employer.

2.2 Decision with the consumer choice theory

We consider the problem of table 3, where a decision maker has to decide between three candidates A, B, and C for a position to fulfil. They have different skills in engineering and sales. Let s_I^{eng} and s_I^{sales} be the objectives measures of their engineering and sales skills for I, one of the three candidates. From table 3:

$$s_A^{eng} > s_B^{eng} > s_C^{eng} \tag{5}$$

$$s_A^{sales} < s_B^{sales} < s_C^{sales} \tag{6}$$

We assume the decision maker to have a standard utility function u (s_I^{eng}, s_I^{sales}) which satisfies the assumptions of table 3 and depends only on the two engineering and sales skills. The candidate I with the highest utility u (s_I^{eng}, s_I^{sales}) will be preferred.

In order to use AHP, the decision maker has to estimate the relative skill

$$c_{I,J}^{x} = \frac{s_I^x}{s_J^x}$$

of candidate *I* in comparison to candidate *J* with respect to his ability in skill x. We are aware that the $c_{I,J}^{x}$ are subjective and potentially inaccurate estimates which must, moreover fit into a measurement scale, for example the 1/9, 1/8, …1/2, 1, 2, …,8 ,9 Saaty scale. The inequalities (5) and (6) for the absolute skill measures imply:

$$c_{A,C}^{eng} > c_{B,C}^{eng} > 1,\ c_{A,B}^{eng} > 1 \tag{7}$$

$$c_{A,C}^{sales} < c_{B,C}^{sales} < 1,\ c_{A,B}^{sales} < 1 \tag{8}$$

Since (1), the transitivity rule $c_{A,C}^{eng} = c_{A,B}^{eng} \cdot c_{B,C}^{eng}$ holds for known skills and since both factors in the product are larger than one, we obtain

$$c_{A,C}^{eng} > \max\{c_{A,B}^{eng}; c_{B,C}^{eng}\} \tag{9}$$

As the transitivity rule is too rigid in our inconsistent world, we will throughout impose the weak consistency requirement (9).

Identically for the criterion skill in sales, we can derivate the weak consistency requirement:

$$c_{A,C}^{sales} < \min\{ c_{A,B}^{sales} \; ; \; c_{B,C}^{sales} \} \tag{10}$$

For our study, we assume that the skill in engineering and in sales have the same utility. In AHP this means that the criteria skill in sales and skill in engineering are of equal weight.

The comparison $c_{I,J}^{x}$ between candidate I and J as regards to the criterion x can take seventeen values (table 1). If we evaluate three alternatives with AHP, 17^3 = 4913 different matrices are possible. This result must then be squared because we have two criteria. However, most of these matrices would be highly inconsistent and not even describe the problem of the table 3. To respect our design, we include the conditions (7), (8), (9), (10) and we consider only acceptable inconsistent matrices (consistency ratio C.R.< 0.1). Moreover, the number of cases where the compromise B is selected depends on the utility function. In our study we describe four special cases: the perfect complements, the geometric mean, perfect substitutes and the Cobb-Douglas utility function.

In accordance with standard consumer theory, we would expect that in most cases the consumer would prefer the compromise alternative B.

a) Perfect complements

Perfect complements are goods that are always consumed together in fixed proportions. For example, we buy a left and a right shoe. The indifference curves are L-shaped. The utility function describing perfect complement preferences is given by:

$$u(x_1, x_2) = \min[a \cdot x_1, b \cdot x_2], \tag{11}$$

where a and b are positive numbers that indicate the proportions in which the goods are consumed.

Under the assumption made on the skills the compromise candidate B will always be preferred by the decision maker if

$$\min[s_B^{eng}, s_B^{sales}] > \max[\min[s_A^{eng}, s_A^{sales}], \min[s_C^{eng}, s_C^{sales}]], \tag{12}$$

The condition (12) can be easily separate in two conditions, one for each skill:

$$\min[s_B^{eng}, s_B^{sales}] > \min[s_A^{eng}, s_A^{sales}] \text{ AND } \min[s_B^{eng}, s_B^{sales}] > \min[s_C^{eng}, s_C^{sales}] \tag{13}$$

The condition (13) is weaker than the conditions (5) and (6). In fact, if $s_B^{eng} > s_B^{sales}$, the right part of (13) implies $s_B^{sales} > s_A^{sales}$, which respects (6). However if $s_B^{sales} > s_B^{eng}$ the right part of (13) implies $s_B^{eng} > s_A^{eng}$, which contradicts (5). We therefore use solely the (7-10) and the consistency (C.R. < 0.1) conditions as requirement for the selection of candidate B (figure 5). With the perfect complements condition, the candidate B is selected **12650**.

```
winB = 0                                          // counter B wins
FOR c^{eng}_{A,B} = 2 TO 8                        // c^{eng}_{A,B} > 1
   FOR c^{eng}_{B,C} = 2 TO 8                     // c^{eng}_{B,C} > 1
      FOR c^{eng}_{A,C} = 3 TO 9
         IF c^{eng}_{A,C} > c^{eng}_{A,B} AND c^{eng}_{A,C} > c^{eng}_{B,C} THEN   // c^{eng}_{A,C} > max{c^{eng}_{A,B}; c^{eng}_{B,C}}
            IF CR< 0.1 THEN
              winB = winB + 1
            END IF
         END IF
      END FOR
   END FOR
END FOR
```

Figure 5: Pseudo code for the calculation of the maximum numbers of B wins.

b) Geometric mean

In the problem of table 3, if the advantage of candidate A over B in engineering is small compared to that of B over C is, we can write the inequality:

$$c^{eng}_{A,B} < c^{eng}_{B,C} \tag{14}$$

If the corresponding statement holds with respect to sales skill (15), then B is a very attractive compromise candidate.

$$c^{sales}_{C,B} < c^{sales}_{B,A} \tag{15}$$

In terms of absolute skills we get:

$$\frac{s^{eng}_A}{s^{eng}_B} < \frac{s^{eng}_B}{s^{eng}_C}$$

or

$$s^{eng}_B > \sqrt{s^{eng}_A \cdot s^{eng}_C}\text{, where } s^{eng}_A > s^{eng}_C$$

Similary for the sales skill,

$$s^{sales}_B > \sqrt{s^{sales}_A \cdot s^{sales}_C}\text{, where } s^{sales}_C > s^{sales}_A$$

Thus B’s skill must be better than the geometric mean of the skills of the others.

With the conditions (7-10), (14), (15) and the consistency condition, B is selected in **2115** cases (figure 6 for the pseudo code).

```
winB = 0                                        // counter B wins
FOR c_{A,B}^{eng} = 2 TO 8                      // c_{A,B}^{eng} > 1
 FOR c_{B,C}^{eng} = 2 TO 8                     // c_{B,C}^{eng} > 1
  FOR c_{A,C}^{eng} = 3 TO 9
   FOR c_{C,B}^{sales} =2 TO 8                  // c_{C,B}^{sales} > 1
    FOR c_{B,A}^{sales} = 2 TO 8                // c_{B,A}^{sales} > 1
     FOR c_{C,A}^{sales} =3 To 9
              IF c_{A,B}^{eng} < c_{B,C}^{eng} AND c_{C,B}^{sales} < c_{B,A}^{sales} AND c_{A,C}^{eng} >max{ c_{A,B}^{eng} ; c_{B,C}^{eng} }AND
                 c_{C,A}^{sales} > max{ c_{B,A}^{sales} ; c_{C,B}^{sales} } AND CR< 0.1 THEN
                   winB = winB+1
              END IF
           END FOR
         END FOR
       END FOR
END FOR
```

Figure 6: Pseudo code for the calculation of the number of times that B is selected with the geometric mean condition.

c) Perfect substitutes

As both skills have the same importance, indifference curves of two perfect substitutes goods are all parallel straight lines with slope of -1. He is preferred if and only if

$$s_B^{sales} + s_B^{eng} > \max\left[s_A^{sales} + s_A^{eng} ; s_C^{sales} + s_C^{eng}\right] \tag{16}$$

In figure 7, the compromise candidate would be selected if the slope $\overline{AB}$ is higher than $\overline{AC}$. It is the case for B''. Candidate B has the same preference than A and C, while B' is rejected.

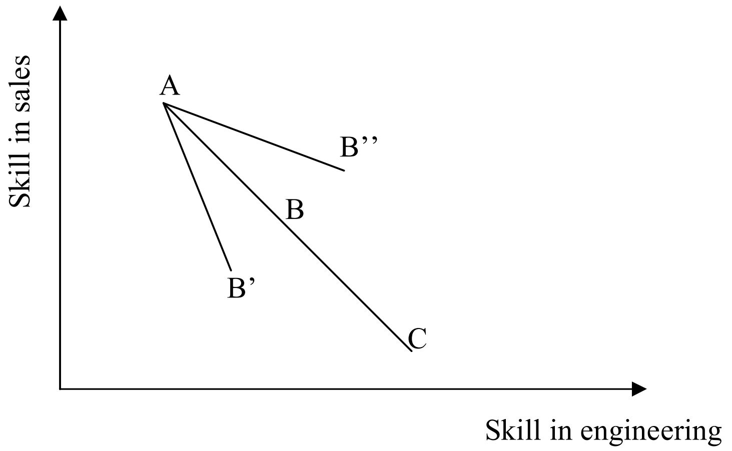

Figure 7: Compromise candidate is selected if he is above the line through A and C.

Mathematically, the slope $\overline{AB}$ is higher than $\overline{AC}$ when:

$$\frac{s_B^{sales} - s_A^{sales}}{s_B^{eng} - s_A^{eng}} > \frac{s_C^{sales} - s_A^{sales}}{s_C^{eng} - s_A^{eng}}$$

and expressed with ratios:

$$\frac{\frac{s_B^{sales}}{s_A^{sales}} - \frac{s_A^{sales}}{s_A^{sales}}}{\frac{s_B^{eng}}{s_C^{eng}} - \frac{s_A^{eng}}{s_C^{eng}}} > \frac{\frac{s_C^{sales}}{s_A^{sales}} - \frac{s_A^{sales}}{s_A^{sales}}}{\frac{s_C^{eng}}{s_C^{eng}} - \frac{s_A^{eng}}{s_C^{eng}}}$$

$$\frac{c_{B,A}^{sales} - 1}{c_{B,C}^{eng} - c_{A,C}^{eng}} > \frac{c_{C,A}^{sales} - 1}{1 - c_{A,C}^{eng}} \tag{17}$$

With the conditions (7-10), (17), and the consistency condition, B is selected in **7739** cases (pseudo code similar than figure 6).

d) Cobb-Douglas

Cobb-Douglas indifference curves look just like convex monotonic curves. The utility function is given by

$$u(x_1, x_2) = x_1^c x_2^d, \tag{18}$$

where *c* and *d* are positive number that describe the preference of the consumer.

In our study, we assume that the skill in engineering and in sales have the same preference, then $c = d = 1/2$.

Under this assumption, the compromise candidate B will always be preferred by the decision-maker if:

$$\left[s_B^{eng}\right]^{\frac{1}{2}}\left[s_B^{sales}\right]^{\frac{1}{2}} \succ \max\left[\left[s_A^{eng}\right]^{\frac{1}{2}}\left[s_A^{sales}\right]^{\frac{1}{2}};\left[s_C^{eng}\right]^{\frac{1}{2}}\left[s_C^{sales}\right]^{\frac{1}{2}}\right] \tag{19}$$

The condition (19) can be decomposed in two equations, one for each candidate:

$$\left[s_B^{eng}\right]^{\frac{1}{2}}\left[s_B^{sales}\right]^{\frac{1}{2}} \succ \left[s_A^{eng}\right]^{\frac{1}{2}}\left[s_A^{sales}\right]^{\frac{1}{2}} \text{ AND } \left[s_B^{eng}\right]^{\frac{1}{2}}\left[s_B^{sales}\right]^{\frac{1}{2}} \succ \left[s_C^{eng}\right]^{\frac{1}{2}}\left[s_C^{sales}\right]^{\frac{1}{2}}$$

and by grouping the same skills, we obtain:

$$\left[\frac{s_B^{sales}}{s_A^{sales}}\right]^{\frac{1}{2}} \succ \left[\frac{s_A^{eng}}{s_B^{eng}}\right]^{\frac{1}{2}} \text{ AND } \left[\frac{s_B^{eng}}{s_C^{eng}}\right]^{\frac{1}{2}} \succ \left[\frac{s_C^{sales}}{s_B^{sales}}\right]^{\frac{1}{2}}$$

with the definition of the comparison $c_{12} = s_1/s_2$ and squaring:

$$c_{B,A}^{sales} \rangle c_{A,B}^{eng} \text{ AND } c_{B,C}^{eng} \rangle c_{C,B}^{sales} \tag{20}$$

Under the conditions (7-10) and (20), the compromise B would be selected **2379** (pseudo code similar to figure 6).

3. Decision with AHP and MAHP

3.1 Introduction

In this paragraph, we describe and discuss the hiring decision problem from table 3 solved with AHP and MAHP. All the possible matrices combinations with an acceptable consistency are used with each preference scale. For the MAHP four different weights normalisations are applied. Then, we compare the results of the AHP and MAHP with the consumer choice theory. The final position of the compromise candidate, B, is our particular interest.

3.2 Description

All matrices modelling our problem (i.e. respecting conditions (7), (8), (9), (10) and C.R.<0.1) are considered with the seven measurement scales (table 2) and the two different aggregation methods. Tables 4 indicate the parameters used in our study.

Scale type	Parameters
Linear	a = 1
Geometric	a = 2
Power	a = 2
Logarithmic	a = 2
Root square	a = 2

Table 4: Parameters used with the different scales.

The priorities are calculated with the normalised geometric mean, namely,

$$l_i = \sqrt[n]{\prod_{j=1}^{n} c_{i,j}} \Big/ \sum_{j=1}^{n} l_j \qquad (21)$$

where l_i is the priority of the alternative i
$c_{i,j}$ is the comparison between i and j
n is the dimension of the matrix

This calculation provides similar results as the eigenvalue method for matrices of dimension three (Saaty, 1984b; Ishizaka, 2004b, 2006).

3.3 Results

We have seen with the standard consumer theory in section 2.2, that we would expect that in many cases the consumer would prefer the compromise alternative B. The choice of a power or geometric scale excludes definitely (for AHP) or almost definitely (for MAHP) the compromise alternative (see table 5). These scales are too extremist. With the other scales, the MAHP captures the obvious cases where B should win (higher scores than the geometric mean and the Cobb-Douglas). However it is still below the result of the perfect substitutes. The normalisation of the criteria weights has little impact on the final result. The selection of B with the traditional AHP is much more difficult.

Scale type	Additive AHP		Multiplicative AHP $\sum w_j = 0.5$	
	# of times B wins	% of times B wins	# of times B wins	% of times B wins
Geometric	0	0	1	0
Power	0	0	129	1
Linear 1-9	109	1	4904	39
Logarithmic	444	4	6745	53
Root square	845	7	6197	49
Inverse linear	1179	9	4021	32
Balanced	1213	10	5828	46
Geometric mean	2115	17	2115	17
Cobb-Douglas	2379	19	2379	19
Perfect substitutes	7739	61	7739	61
Perfect complements	12650	100	12650	100

Scale type	Multiplicative AHP $\sum w_j = 1$		Multiplicative AHP $\sum w_j = 2$		Multiplicative AHP $\sum w_j = 4$	
	# of times B wins	% of times B wins	# of times B wins	% of times B wins	# of times B wins	% of times B wins
Geometric	0	0	0	0	0	0
Power	130	1	128	1	128	1
Inverse linear	4021	32	4040	32	4039	32
Linear 1-9	4918	39	4877	39	4908	39
Balanced	5871	46	5888	47	5954	47
Root square	6227	49	6242	49	6260	49
Logarithmic	6750	53	6760	53	6772	54

Table 5: Number of combinations where the compromise alternative is selected under AHP.

Saaty's linear scale 1 to 9 gives few chance for the alternative B to be selected. Only 109 possibilities out of 12650 would yield B, which appears to be an unreasonable result. Furthermore, in all the 109 cases, B yield a special configuration with the necessary but non sufficient condition $c_{A,B}^{eng}$ **= 2** and $c_{B,C}^{sales}$ **= 1/2** (see section 3.4).

The root square scale (845 selections for B), the balanced scale (1213) and the inverse linear scale (1179) offer more possibilities that the compromise alternative will be selected but still under the geometric mean criteria given by the consumer choice theory.

3.4 Example of compromise selection with Saaty's scale

The compromise will be selected only in a few cases with the linear scale 1 to 9 and under the non sufficient condition that $c_{A,B}^{eng}$ **= 2** and $c_{B,C}^{sales}$ **= 1/2**. Figure 8 gives one example of them.

	A	B	C	priorities
A	1	**2**	9	0.606
B	½	1	6	0.333
C	1/9	1/6	1	0.061

Matrix for the engineering skill
(C.R. = 0.01)

	A	B	C	priorities
A	1	1/7	1/8	0.061
B	7	1	**1/2**	0.353
C	8	2	1	0.586

Matrix for sales skill
(C.R. = 0.03)

Figure 8: Example where the compromise candidate B is the best alternative (A = 0.329; B = 0.343; C = 0.328). The consistency ratios (C.R.) are acceptable.

The global priorities are the average of the priorities' skills:

A = 1/2·0.606 + 1/2·0.061 = 0.329

B = 1/2·0.333 + 1/2·0.353 = 0.343 (winner)

C = 1/2·0.061 + 1/2·0.585 = 0.328

3.5 Surprising example of compromise rejection with Saaty's scale

AHP with Saaty's linear scale 1 to 9 prefers the extremes even if a compromise offers a better solution. We have three candidates:

- A is very good in sales but very poor in engineering
- C is very good in engineering but very poor in sales
- B is very good in sales but not as good as A and he is very good in engineering but not as good as C.

With the theory of consumer choice (see Section 2), the candidate B offers a clearly higher utility (under convex or linear preferences) and should be chosen.

One plausible representation in matrix comparisons could be given by figure 9.

	A	B	C	priorities
A	1	3	9	0.655
B	1/3	1	7	0.290
C	1/9	1/8	1	0.055

Matrix for the engineering skill
(C.R. = 0.08)

	A	B	C	priorities
A	1	1/8	1/9	0.054
B	8	1	1/2	0.357
C	9	2	1	0.589

Matrix for sales skill
(C.R. = 0.04)

Figure 9: Example where the compromise candidate B should be the best alternative but is the worst classified (A=0.339; B=0.325; C=0.336). The consistency ratios (C.R.) are acceptable.

The global priorities of this example are:

1. A with 0.339 (winner)
2. C with 0.336
3. B with 0.325

AHP does not classify the candidate B in first place but in last place!

4. Conclusion

In our paper we have shown that the additive AHP will overrate alternatives with extreme ratings and penalize balanced ones. It may be mathematically impossible for a non-dominated compromise to achieve the highest overall ratings, which make little sense from a practical point of view. The use of the power or the geometric scale would give no chance (for AHP) or very few (for MAHP) for the compromise to be selected, which is problematic from the perspective of consumer choice theory.

The impact of the preferences scales is different with the AHP and the MAHP. We first discuss the impact in the AHP. The linear scale 1 to 9 offers very few possibilities for the compromise to be selected. These scales should be avoided unless we face a high concave utility function.

The logarithmic scale, root square, the inverse linear and the balanced scales offer more possibilities for the compromise alternative to be selected, albeit they may ignore some superior compromise.

We do not suggest any fixed scale as a standard tool for AHP. This is because the interpretation of verbal expressions varies from one person to another. However our observations confirm the work of Pöyhönen and al. (1997), which do not support the Saaty's scale and prefer a more balanced scale like the balanced scale or the inverse linear scale.

The multiplicative AHP, independently of the measurement scale (apart the geometric and the power scale) and the normalisation of the weights criteria, ensures due consideration to the compromise alternative when compared with alternatives extremely attractive with respect to one criterion and extremely unattractive with respect to the other. This observation is particularly true for the new proposed logarithmic scale.

The examples presented here are typical of decision problems and must hence be taken very seriously. From the perspective of economists, decision making is almost always about making compromises. Trying to reach a better outcome in one dimension is often at the expense of achieving a worse outcome in another dimension. For instance, the production cost of a firm can often only be lowered at the expense of producing lower quality output. A good decision-maker will typically have to correctly trade-off one dimension for another. If a decision aid like the additive AHP tends to recommend extremes, which are good in only one respect, it will fail in its purpose. Thus, in addition to his property of rank reversal preservation, there appear to be important motivations for using the multiplicative AHP.

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