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BOUQUETS OF GEOMETRIC LATTICES: SOME ALGEBRAIC AND TOPOLOGICAL ASPECTS

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Introduction

Matroid theory is in the center of Combinatorics, Finite Geometry, Lattice theory and Combinatorial Optimization. During the last decades, extensive search was done in order to find a good degree of generality which still preserves the validity of deep results known for matroids. One of such generalizations is the concept of bouquet of matroids introduced in 1983 by Deza, Frankl and Laurent and studied in a dozen papers (cf. [7, 11, 14, 17] and references mentioned there). The following matroidal features were extended in a satisfactory way till now:

—classical axiomatizations and their equivalence (axiomatizations through flats, independent sets, circuits, rank function, closure operator) (cf. [11, 17])

—operations and extremal theorems for perfect matroid design case (cf. [11, 12, 6])

—diagram representation and geometrical aspects (cf. [14, 17])

—algorithmic and polyhedral aspects (cf. [8, 9])

—orientation (cf. [13]).

This paper is a follow-up work in the above series of articles on bouquets and it deals especially with the following features: other operations (contraction, restriction and cuts), strong maps and mapping cylinders, representability, topological aspects and, in particular, shellability of various simplicial complexes associated with bouquets and relation with connectivity properties.

On the other hand, the starting point of this paper was the important paper of Wachs and Walker [23]. We realized that their principal concepts and results (strong map, mapping cylinder, realization theorem) stated for geometric semilattices could be naturally extended for the broader framework of bouquets.

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We also give new examples of geometric semilattices; actually, our transversal geometries include all examples of [23].

The paper is organized as follows. Sections 1 to 3 recall briefly generalities on bouquets of matroids: main axiomatizations (through flats in Section 1 and through independent sets and circuits in Section 3), central examples of transversal geometries and d-injection geometries in Section 2, structure of the semilattice $\mathcal{L}(\mathcal{J})$ of all bouquets with given independence system $\mathcal{J}$ in Section 4. In Section 5, we introduce bouquets of geometric lattices as the lattice representation of bouquets of matroids. In Section 6, we consider operations on bouquets: contraction, restriction and cuts and we study their effect on the independence system of the bouquet. In Section 7, we study strong maps on bouquets; we give two new examples of strong maps coming from the closure operator between comparable bouquets having the same independence system (Theorem 7.2) and from the projection map for transversal matroid designs (Theorem 7.6). Then, using the mapping cylinder construction, we prove a realization theorem (Corollary 7.20) which essentially says that every bouquet with M branches can be obtained from a "better" bouquet having only $m \leq M$ branches by deleting one upper interval. In Section 8, we study the shellability of bouquets of matroids; we prove that the connectivity of the basis graph is a necessary condition for shellability and that this condition is, in fact, sufficient for the class of bouquets of matroids with the 2-union property, i.e. of bouquets whose independence system can be written as "union" of two matroids (Theorem 8.19). We also show that the Hirsch conjecture holds for bouquets of matroids with the 2-union property if and only if they are shellable.

1. Flat axioms for bouquets of matroids

We first define bouquets of matroids through their flat axioms which are a direct relaxation of the matroidal axioms.

Definition 1.1. Axiomatization through flats.
Let X be a finite set and $X_1, \ldots, X_m$ be subsets of X forming a clutter, i.e. $X_i \not\subseteq X_j$ for all $i \neq j$. Let $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_s$ be pairwise disjoint families of subsets of X and $\mathcal{G} = \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_s$. Then, the family $\mathcal{G}$ is called *a bouquet of matroids* on X of *rank* s with *roofs* $X_1, \ldots, X_m$ if:

($F1$) $\mathcal{G} \subseteq \bigcup_{i=1}^m 2^{X_i}$ and $X_1, \ldots, X_m \in \mathcal{G}$.

($F2$) $\mathcal{G}$ is stable under intersection, i.e. $G \cap G' \in \mathcal{G}$ for all $G, G' \in \mathcal{G}$.

($F3$) if $G \in \mathcal{G}_i$, $G' \in \mathcal{G}_j$ and $G \subsetneq G'$, then $i < j$

($F4$) if $G \in \mathcal{G}_r$ for $0 \leq r \leq s-1$, $x \in X - G$ and $G \cup x \in \bigcup_{i=1}^m 2^{X_i}$, then there exists (a unique) $G' \in \mathcal{G}_{r+1}$ such that $G \cup x \subseteq G'$.

Elements of $\mathcal{G}$ are called *flats* or *closed sets,* elements of $\mathcal{G}_r$ are called *r-flats* or *flats of rank r*, for $0 \leq r \leq s$. The *roofs* of the geometry $\mathcal{G}$ are the maximal (for set inclusion) flats. Clearly, for each $i \in [1, m]$, the interval $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$ is the set of flats of a matroid on X_i and $\mathcal{G} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_m$ is therefore the "bouquet" of the m matroids $\mathcal{M}_i$, its rank s being the maximum value of the ranks of the matroids $\mathcal{M}_i$. We will sometimes refer to the matroids $\mathcal{M}_i$ composing the bouquet $\mathcal{G}$ as its *branches* or *flowers*. The above observation yields naturally the following equivalent definition for bouquets which essentially says that a union (in the set of theoretical sense) of matroids is a bouquet if and only if it is stable under intersection.

Definition 1.2. Axiomatization through flats.
Let X be a finite set and $X_1, \ldots, X_m$ be a clutter of subsets of X. A family $\mathcal{G}$ of subsets of X is the set of flats of a bouquet of matroids on X with roofs $X_1, \ldots, X_m$ if:

($F1$) $\mathcal{G} \subseteq \bigcup_{i=1}^{m} 2^{X_i}$ and $X_1, \ldots, X_m \in \mathcal{G}$

($F2$) $\mathcal{G}$ is stable under intersection

($F3'$) $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$ is the set of flats of a matroid on X_i for each $i \in [1, m]$.

We recall some more definitions for bouquets. Let $\mathcal{G}$ be a bouquet of matroids of rank s on X with roofs $X_1, \ldots, X_m$. When every subset of a roof is a flat, one says that $\mathcal{G}$ is *free*. $\mathcal{G}$ is called *well-cut* when all roofs have the same rank s, i.e. when the set of roofs coincides with the set $\mathcal{G}_s$ of s-flats. Obviously, there is a unique flat of rank 0 and one can assume w.l.o.g. that it is $\emptyset$. The bouquet $\mathcal{G}$ is called *simple* when all 1-flats have cardinality 1. When, for each $r \in [0, s]$, all r-flats have the same cardinality l_r, the bouquet is called a *design with parameters* $(l_0, \ldots, l_s)$. An *epimorphism* between two bouquets of matroids $\mathcal{G}, \mathcal{G}'$ is a surjective mapping from $\mathcal{G}$ onto $\mathcal{G}'$ which preserves rank and incidence; if, furthermore, it is one-to-one, then it is called an *isomorphism*. Given a matroid $\mathcal{M}$, the bouquet $\mathcal{G}$ is called *$\mathcal{M}$-unisupported* if, for each $i \in [1, m]$, the matroids $\mathcal{M}$ and $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$ are isomorphic. Clearly, if $\mathcal{G}$ is unisupported, then $\mathcal{G}$ is well-cut and all roofs have same cardinality. As we shall see, transversal matroid designs represent an important class of unisupported bouquets.

2. Examples of bouquets: Transversal and injection geometries

Bouquets of matroids are, in fact, a special case of the more general concept of *$\mathcal{F}$-squashed geometries* introduced by Deza and Frankl in [11, 12]. In brief, $\mathcal{F}$ being a clutter of subsets of a finite set X, $\mathcal{F}$-squashed geometries are a generalization of the matroidal structure in which the flats, in addition to satisfying some axioms similar to axioms ($F1$)–($F4$) from Definition 1.1, have to be contained in some element of $\mathcal{F}$; this amounts to replace in Definition 1.1 the clutter of the roofs by the "covering" clutter $\mathcal{F}$ (i.e. each roof X_i is contained in some $F \in \mathcal{F}$). By specifying the clutter $\mathcal{F}$, one obtains various classes of squashed geometries, such as transversal geometries, permutation geometries [6, 7], injection geometries [11] and more generally, d-transversal geometries [14, 17]. We recall now precisely the classes of transversal geometries and d-injection geometries that we will especially consider in this paper.

Definition 2.1. Let $N_1, \ldots, N_d$ be d ($d \geq 2$) finite sets. For $\alpha \in [1, d]$, a set A, $A \subseteq N_1 \times \cdots \times N_d$, is called *injective by* N_α if, for all distinct elements $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$ of A, $a_\alpha \neq b_\alpha$ holds. Then, a set $A \subseteq N_1 \times \cdots \times N_d$ is called *d-injective* if A is injective by N_α for all $\alpha \in [1, d]$ and a set $A \subseteq N_1 \times N_2$ is called *transversal* if A is injective by N_1.

One denotes by $\mathcal{T}(N_1, N_2)$ the family of all transversal subsets of $N_1 \times N_2$ and by $\mathcal{J}(N_1, \ldots, N_d)$ the family of all d-injective subsets of $N_1 \times \cdots \times N_d$.

Definition 2.2. Let $\mathcal{G}_0, \ldots, \mathcal{G}_s$ be pairwise disjoint families of subsets of $X = N_1 \times N_2$ (resp. $X = N_1 \times \cdots \times N_d$) and $\mathcal{G} = \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_s$. Then $\mathcal{G}$ is called a *transversal geometry* (resp. *d-injection geometry*) on X of rank s if:

($G1$) *each set* $G \in \mathcal{G}$ is transversal (resp. d-injectif)

($G2$) $\mathcal{G}$ is stable under intersection

($G3$) if $G \in \mathcal{G}_i$, $G' \in \mathcal{G}_j$ and $G \subsetneq G'$, then $i < j$

($G4$) if $G \in \mathcal{G}_r$ for $0 \leq r \leq s-1$, $x \in X - G$ and $G \cup x$ is transversal (resp. d-injectif), then there exists (a unique) $G' \in \mathcal{G}_{r+1}$ such that $G \cup x \subseteq G'$.

When the geometry $\mathcal{G}$ is a design with parameters $(l_0, \ldots, l_s)$, then $\mathcal{G}$ is called a *transversal matroid design* (resp. *d-injection design*). In this case, one can easily compute the number of r-flats for $0 \leq r \leq s$ (cf. [12]); in particular, for a transversal matroid design $\mathcal{G}$ of rank s on $[1, n] \times [1, m]$, one has: $|\mathcal{G}_s| = m^s$ and for a d-injection design $\mathcal{G}$ of rank s on $\prod_{i=1}^d [1, n_i]$, one has: $|\mathcal{G}_s| = \prod_{j=0}^s \prod_{i=1}^d (n_i - l_j)/(l_s - l_j)$.

As noted in [12], transversal matroid designs arise as extremal intersecting families of transversal sets; more precisely, if $\mathcal{A}$ is a family of transversal subsets of $[1, n] \times [1, m]$ such that $|A \cap A'| \in \{l_1, \ldots, l_s\}$ for $A \neq A' \in \mathcal{A}$, then, for n big enough, $|\mathcal{A}| \leq m^s$ and equality holds if and only if $\mathcal{A}$ is the set of roofs of a transversal matroid design. This result can be rephrased in coding theory terminology; for this, see that any transversal subset of $[1, n] \times [1, m]$ of cardinality n can be represented as an n-tuple of $[1, m]^n$ and thus transversal matroid designs correspond to extremal codes of length n over the alphabet with m letters and with a prescribed number of distances.

Similarly, d-injection designs correspond to extremal intersecting families of d-injective sets. Notice that any d-injective subset of $[1, n]^d$ of cardinality n can be written as $\{(i, a_2^i, \ldots, a_d^i): i \in [1, n]\}$ and thus be viewed as a set $(\sigma_2, \ldots, \sigma_d)$ of $d-1$ permutations of $[1, n]$ with $\sigma_j(i) = a_j^i$ for $i \in [1, n]$, $j \in [2, d]$. Hence the set of roofs of a d-injection design on $[1, n]^d$ with $l_s = n$ can be seen as a subset of the group $(\mathscr{S}_n)^{d-1}$, $\mathscr{S}_n$ denoting the symmetric group of order n. The case when it is a subgroup is particularly interesting and we refer to [7] for the case $d=2$ and to [17] for some results in general case.

Transversal and d-injection geometries are highly structured objects; so, most of them are unisupported. For this, let p_i denote the ith projection from the product set $N_1 \times \cdots \times N_d$ onto N_i. Let $\mathscr{G}$ be a transversal geometry on $N_1 \times N_2$ or a d-injection geometry on $N_1 \times \cdots \times N_d$; then, for each roof X_i of $\mathscr{G}$, the matroids $\mathscr{M}_i = \mathscr{G} \cap [\emptyset, X_i]$ and $p_1(\mathscr{M}_i)$ are isomorphic and, if $\mathscr{G}$ is a design, then $p_1(\mathscr{M}_i)$ is a perfect matroid design (PMD) with the same parameters. In some cases, the matroid $p_1(\mathscr{M}_i)$ does not depend on the choice of the roof X_i, i.e. the matroids $\mathscr{M}_i$ are pairwise isomorphic for all roofs X_i.

Proposition 2.3 [16]. *Let $\mathscr{G}$ be a transversal matroid design on $[1, n] \times [1, m]$ with $l_s = n$. Then, $\mathscr{M} = p_1(\mathscr{M}_i)$ is a fixed* PMD *on $[1, n]$ for all roofs X_i of $\mathscr{G}$ and the projection p_1 is an epimorphism from $\mathscr{G}$ onto $\mathscr{M}$.*

Proposition 2.4 [17]. *Let $\mathscr{G}$ be a d-injection design of rank s on $\prod_{i=1}^{d} [1, n_i]$ with $l_s = n_1$. Assume that one of the following conditions holds:*

(i) *$\mathscr{G}$ is concentrated, i.e. for all $G, G' \in \mathscr{G}$, there exists $G'' \in \mathscr{G}$ such that $p_1(G) \cap p_1(G') = p_1(G \cap G'')$*

(ii) *$n_1 = \cdots = n_d = n$ and the set of roofs of $\mathscr{G}$ forms a subgroup of $(\mathscr{S}_n)^{d-1}$. Then, $\mathscr{M} = p_1(\mathscr{M}_i)$ is a fixed* PMD *on $[1, n]$ and the projection p_1 is an epimorphism from $\mathscr{G}$ onto $\mathscr{M}$.*

We will see in Section 7 that, in transversal case, the projection p_1 is an example of strong map.

We now survey some of the known examples of transversal geometries.

Example 2.5. $\mathscr{T}(N_1, N_2)$ is a (full) transversal matroid design. Clearly, $\mathscr{T}(N_1, N_2)$ can also be defined as the set: $L(N_1, N_2) = \{(A, f) : A \subseteq N_1$ and $f : A \to N_2$ mapping$\}$ (this example is due to Delsarte, [10]).

Example 2.6. Let V_1, V_2 be finite dimension vector spaces over the finite field GF(q). The set $\mathscr{T}_V(V_1, V_2) = \{W \leqslant V_1 \times V_2 : \dim(p_1(W)) = \dim W\}$ is a (linear) transversal matroid design on $V_1 \times V_2$. It is easy to see that $\mathscr{T}_V(V_1, V_2)$ is isomorphic to the set: $L_q(V_1, V_2) = \{(W, f) : W \leqslant V_1$ and $f \in \mathrm{Lin}(W, V_2)\}$ (this example was considered by Stanton [21] who calls it the semilattice of bilinear forms). One defines similarly the affine analogue of the above set.

Example 2.7. A transversal matroid design on $[1, n] \times \mathrm{GF}(m)$ with $l_s = n$ is said to be *linear* if its set of roofs – when viewed as a subset of $\mathrm{GF}(m)^n$ – forms a vector subspace. We refer to [7] for many examples of linear transversal matroid designs and for the exposition of a sufficient and necessary condition for their existence (Prop. 4.4 in [7]).

Example 2.8. Let E be a set of mappings from $[1, n]$ to $[1, m]$ which is sharply t-transitive (i.e. for all distinct elements $x_1, \ldots, x_t \in [1, n]$ and all elements $y_1, \ldots, y_t \in [1, m]$, there exists a unique $f \in E$ such that $f(x_i) = y_i$ for $i \in [1, t]$). Then, the meet semilattice generated by the sets $\{(x, f(x) : x \in [1, n]\}$ is a transversal matroid design with parameters $(0, 1, 2, \ldots, t-1, n)$ (cf. Prop. 3.8 in [14]). Note that sharply t-transitive sets of mappings are well known objects; so they correspond, in fact, to transversal t-designs (from Hanani, [15]), or, equivalently, to orthogonal arrays of strength t (precisely to $\mathrm{OA}(m, n; 1)$ with order m, index 1 and degree n) and also, for m prime power, to MDS-codes (cf. [19]).

Example 2.9. Let V be a finite dimension vector space over $\mathrm{GF}(q)$, $A(V)$ denote the family of affine subspaces of V and $H \subseteq A(V)$ the family of affine hyperplanes. Any affine subspace S can be identified with the set $H(S)$ of hyperplanes containing S. If one considers the partition of H into the parallelism classes, then the collection: $\mathcal{A}(V)^* = \{H(S) : S \neq \emptyset \text{ and } S \in A(V)\}$ is a transversal matroid design on H (this example is taken from [23] where the poset $A(V)^* - \{\emptyset\}$ ordered by the reverse inclusion is considered instead).

There are many examples of d-injection geometries (cf. [11, 12]); let us simply mention that examples 2.6, 2.7, 2.8 have analogues for the injective case and we recall the following:

Example 2.10. $\mathcal{J}(N_1, \ldots, N_d)$ is a (full) d-injection design.

3. Other axiomatizations for bouquets of matroids

It is a well known fact that a matroid can be equivalently defined through the axioms of its flats, independent sets, circuits (or stigmes), rank function, closure operator (cf. [22]). The same holds for bouquets of matroids for which we recall the main axioms that we will need throughout the paper; we refer to [8, 17] for an extensive treatment of various axiomatizations of bouquets.

Let $\mathcal{G}$ be the set of flats of a bouquet of matroids of rank s on X with roofs $X_1, \ldots, X_m$ and, for $i \in [1, m]$, $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$ be the matroid determined on X_i. For each $i \in [1, m]$, let us denote by r_i, σ_i, $\mathcal{J}_i$, $\mathcal{S}_i$ the rank function, the closure operator, the family of independent sets, the family of stigmes, respectively, of the matroid $\mathcal{M}_i$. Then, one is naturally led to define the rank function r, the closure operator σ, the family $\mathcal{J}$ of independent sets, the family $\mathcal{D}$ of circuits of $\mathcal{G}$ as follows:

—the family of *independent sets* is: $\mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_m$

—the family of *circuits* is the family $\mathcal{D}$ of all minimal dependent sets, i.e. $D \in \mathcal{D}$ if and only if $D \notin \mathcal{J}$ and $D - x \in \mathcal{J}$ for all $x \in D$.

At this point, let us note that the family $\mathcal{J}$ of independent sets is an *independence system* (IS, for short) on X, i.e. it satisfies:

(I0) if $I \in \mathcal{J}$ and $J \subseteq I$, then $J \in \mathcal{J}$

and the family $\mathcal{D}$ is a clutter, i.e. it satisfies:

$(D1)$ if $D, D' \in \mathcal{D}$ and $D \subseteq D'$, then $D = D'$.

Furthermore, the family $\mathcal{D}$ can be partitioned into $\mathcal{D} = \mathcal{S} \cup \mathcal{C}$ where $\mathcal{S} = \mathcal{D} \cap (\bigcup_{i=1}^{m} 2^{X_i})$ and $\mathcal{C} = \mathcal{D} - \mathcal{S}$, with $\mathcal{S}_i = \mathcal{D} \cap 2^{X_i}$ being the collection of stigmes of $\mathcal{M}_i$. Elements of $\mathcal{S}$ are called *stigmes* – they correspond to the "matroidal" part of $\mathcal{D}$ – and elements of $\mathcal{C}$ are called *critical sets* – they correspond to the "non matroidal" part of $\mathcal{D}$. In fact, the IS $\mathcal{J}$ is completely determined by the clutter $\mathcal{D}$ of circuits and conversely. Actually, the additional information that the bouquet $\mathcal{G}$ is providing, is, respectively, the decomposition of $\mathcal{J}$ as the union of the m matroidal IS: $\mathcal{J}_1, \ldots, \mathcal{J}_m$ and the decomposition of $\mathcal{D}$ into the stigmes $\mathcal{S}$ and the critical sets $\mathcal{C}$.

The *rank function* r and the *closure operator* σ of the bouquet $\mathcal{G}$ are defined as follows:

—for a set $A \subseteq X_i$ for some $i \in [1, m]$, $r(A) = r_i(A)$ and $\sigma(A) = \sigma_i(A)$

—for a set $A \notin \bigcup_{i=1}^{m} 2^{X_i}$, $r(A) = \infty$ and $\sigma(A) = X \cup \infty$ where ∞ is an "infinity" point.

In other words, one considers the rank and the closure only for sets that are contained in some roof of $\mathcal{G}$. Note, that, from the flat axioms, the above definition is consistent, i.e. $r_i(A) = r_j(A)$ and $\sigma_i(A) = \sigma_j(A)$ for $A \subseteq X_i \cap X_j$. Moreover, for $A \in \bigcup_{i=1}^{m} 2^{X_i}$, one has:

$\sigma(A) = A \cup \{x \notin A\colon$ there exists $S \in \mathcal{S}$ such that $x \in S$ and $S \subseteq A \cup x\}$ and $r(A) = \max(|I| : I \in \mathcal{J}$ and $I \subseteq A)$; i.e. $r(.)$ coincides with the rank function of the IS $\mathcal{J}$ on subsets of roofs. We recall the axioms for circuits and independent sets since we will need them in the remaining of the paper.

Definition 3.1. Axiomatization through circuits.

A family $\mathcal{D}$ of subsets of X if the family of circuits of a bouquet of matroids on X if $\mathcal{D}$ can be partitioned into two subfamilies $\mathcal{S}$, $\mathcal{C}$ satisfying:

$(D1)$ $D \not\subseteq D'$ for all distinct $D, D' \in \mathcal{D}$

$(D2)$ if $S, S' \in \mathcal{S}$, $S \neq S'$ and $x \in S \cap S'$, then there exists $D \in \mathcal{D}$ such that $D \subseteq S \cup S' - x$

$(D3)$ if $S \in \mathcal{S}$, $C \in \mathcal{C}$ and $x \in S \cap C$, then there exists $C' \in \mathcal{C}$ such that $C' \subseteq S \cup C - x$.

Then the roofs of the bouquet are the maximal subsets of X that do not contain any $C \in \mathcal{C}$.

Definition 3.2. Axiomatization through independent sets.

Given a clutter $X_1, \ldots, X_m$ of subsets of X, a family $\mathcal{J}$ of subsets of X is the family of independent sets of a bouquet of matroids on X with roofs $X_1, \ldots, X_m$

if:

($I1$) $\mathcal{J}_i = \mathcal{J} \cap 2^{X_i}$ is the family of independent sets of a matroid on X_i, for all $i \in [1, m]$

($I2$) $\mathcal{J} = \bigcup_{i=1}^{m} \mathcal{J}_i$

($I3$) if $I \in \mathcal{J}_i \cap \mathcal{J}_j$ and $x \in X_i - X_j$, then $I \cup x \in \mathcal{J}$.

Let $\mathcal{J}$ be an IS on X and $\mathcal{B}$ be its family of *bases*, i.e. $\mathcal{B}$ is formed by the maximal sets $I \in \mathcal{J}$. Recall that the IS $\mathcal{J}$ is the family of independent sets of a matroid on X (i.e. is a matroidal IS) if it satisfies the following *augmentation axiom:*

($I4$) if $I, J \in \mathcal{J}$ and $|I| < |J|$, then there exists an element $x \in J - I$ such that $I \cup x \in \mathcal{J}$

or equivalently, if $\mathcal{B}$ satisfies the following *basis exchange axiom:*

(B) for all $B, B' \in \mathcal{B}$ and $x \in B - B'$, there exists $x' \in B' - B$ such that $B - x + x' \in \mathcal{B}$.

In application, we recall how to construct bouquets from a matroid ([13], example 3.1). Take a matroid $\mathcal{M}$ on X, a clutter $X_1, \ldots, X_m$ of subsets of X such that $X_i \cap X_j$ is closed in $\mathcal{M}$; define $\mathcal{S}$ as the family of stigmes of $\mathcal{M}$ that are contained in some X_i, $\mathcal{C}$ as the family of minimal sets that are not contained in any X_i and $\mathcal{D} = \mathcal{S} \cup \mathcal{C}$. Then $\mathcal{D}$ is the family of circuits of a bouquet of matroids $\mathcal{G}$ with roofs $X_1, \ldots, X_m$; one says that $\mathcal{G}$ is *induced from the matroid* $\mathcal{M}$.

We now mention the related notion of representability for bouquets.

Definition 3.3. A bouquet of matroids $\mathcal{G}$ on X is called *representable over the field* F if there exists a vector space V over F and a mapping φ from X to V which preserves the rank, i.e. $r(\varphi(A)) = r(A)$ for all sets $A \subseteq X$ with $r(A) \neq \infty$ where $r(A)$ denotes the rank of A in $\mathcal{G}$ and $r(\varphi(A))$ the vectorial rank of $\varphi(A)$.

For instance, bouquets induced from a vectorial matroid and linear transversal matroid designs are representable. The above definition extends the notion of representability introduced in [13] for bouquets induced from matroids and coincides with it when the map φ is one-to-one. It also covers the definition of representability given in [11] for injection geometries (we point out an error in the formulation in [11] in Section 6: in the relation "$r(A) = r(\varphi(A))$ for all $A \subseteq X$", the condition $r(A) \neq \infty$ was omitted).

We finally introduce some definitions concerning bouquets of matroids whose IS have specific matroidal properties.

Definition 3.4. Let $\mathcal{J}$ be an IS on X and $\mathcal{G}$ be a bouquet of matroids on X with IS $\mathcal{J}$. Let $p \geq 1$ be an integer.

(i) if the IS $\mathcal{J}$ is matroidal, then the bouquet $\mathcal{G}$ is called a *geometric semilattice* (see Section 5 for more remarks concerning this terminology)

(ii) the IS $\mathcal{J}$ is said to have the *p-intersection property* if p is the least integer such that $\mathcal{J}$ can be written as the intersection of p matroids; in this case, one also says that the bouquet $\mathcal{G}$ has the *p-intersection property*

(iii) the IS $\mathcal{J}$ is said to have the *p-union property* if p is the least integer such that $\mathcal{J}$ can be decomposed as the bouquet of m matroidal IS (i.e. as the union of m matroidal IS satisfying axioms $(I1)$–$(I3)$); in this case, one also says that the bouquet $\mathcal{G}$ has the *m-union property*.

Theorem 3.5. *Any well-cut (i.e. all roofs have the same rank) transversal geometry is a geometric semilattice.*

Proof. Let $\mathcal{G}$ be well-cut transversal geometry on $[1, n] \times [1, m]$ with IS $\mathcal{J}$. We prove that $\mathcal{J}$ is matroidal by showing that the basis exchange axiom (B) holds. For this, let B, B' be two bases of $\mathcal{J}$ and (i, x) be an element of $B - B'$. We prove that there exists an element (i', x') of $B' - B$ such that the set $B'' = B - (i, x) + (i', x')$ is a base of $\mathcal{J}$; since $\mathcal{G}$ is well-cut, it is enough to verify that $B'' \in \mathcal{J}$. We first suppose that $i \in p_1(B')$. Hence B' contains an element (i, x') with $x \neq x'$. We prove that the set $B'' = B - (i, x) + (i, x')$ is independent. For this, let F be the $(s-1)$-flat of $\mathcal{G}$ containing $B - (i, x)$ and G be the s-flat containing B; thus, $F \subsetneq G$. Then the set $F \cup (i, x')$ is transversal, i.e. $i \notin p_1(F)$; else, there exists an element $(i, z) \in F$ and, since $F \subseteq G$, (i, z) and (i, x) are two elements of the transversal set G which implies that $z = x$ and thus $B \subseteq F$, yielding a contradiction. From axiom $(G4)$, there exists an s-flat G' containing $F \cup (i, x')$. Now, if $B'' \notin \mathcal{J}$, there exists a circuit D such that $(i, x') \in D$ and $D \subseteq B'' \subseteq G'$; therefore, D is a stigme and (i, x') belongs to the closure F of $B - (i, x)$, yielding a contradiction. We now suppose that $i \notin p_1(B')$. Consider again the $(s-1)$-flat F containing $B - (i, x)$. For rank considerations, $p_1(B') \not\subseteq p_1(F)$; hence, one can take elements $i' \in p_1(B') - p_1(F)$ and $(i', x') \in B'$. Then $(i', x') \notin B$; else, one would have $(i, x) = (i', x')$, contradicting the fact that $i \notin p_1(B')$. Hence the set $F \cup (i', x')$ is transversal and thus contained in an s-flat which, similarly as before, implies that the set $B'' = B - (i, x) + (i', x')$ is independent. □

Corollary 3.6. *The full injection geometry $\mathcal{J}(N_1, \ldots, N_d)$ has the d'-intersection property, for some $d' \leq d$.*

Proof. For $i \in [1, d]$, denote by $\mathcal{T}_i$ the family of all subsets of $N_1 \times \cdots \times N_d$ which are injective by N_i; then $\mathcal{T}_i$ is a full transversal geometry. Since $\mathcal{J}(N_1, \ldots, N_d) = \bigcap_{i=1}^{d} \mathcal{T}_i$ and each of the geometries involved is free, i.e. coincides with the its own IS, one deduces that $\mathcal{J}(N_1, \ldots, N_d)$ can be written as the intersection of d matroids. □

Problem 3.7. Is it the case that any well-cut d-injection geometry has the d'-intersection property for some $d' \leq d$?

4. The semilattice $\mathscr{L}(\mathscr{I})$

Let $\mathscr{I}$ be an IS on X and $\mathscr{D}$ be its family of circuits. In general, there exist several bouquets of matroids whose IS is $\mathscr{I}$ or, equivalently, whose family of circuits is $\mathscr{D}$; in other words, there exist several ways of decomposing $\mathscr{I}$ as a union of matroids satisfying the independent set axioms $(I1)$–$(I3)$. For instance, if $\mathscr{B}$ denotes the set of bases (maximal independent sets) of $\mathscr{I}$, then, for $B \in \mathscr{B}$, the family $\mathscr{I}_B = \{I \in \mathscr{I} : I \subseteq B\}$ is obviously a matroidal IS and $\mathscr{I} = \bigcup_{B \in \mathscr{B}} \mathscr{I}_B$ always provides a decomposition of $\mathscr{I}$ as a (free) bouquet of matroids.

Example 4.1. Let $\mathscr{I}$ be the IS on $[1,4]$ whose bases are: 12, 13, 23, 14. Then, $\mathscr{I} = \mathscr{I}_{12} \cup \mathscr{I}_{13} \cup \mathscr{I}_{23} \cup \mathscr{I}_{14}$ and $\mathscr{I} = \mathscr{I}_{\{12,13,23\}} \cup \mathscr{I}_{14}$ are two distinct ways of decomposing $\mathscr{I}$ as bouquet of matroids.

Therefore, we are naturally led to consider the collection $\mathscr{L}(\mathscr{I})$ of all bouquets of matroids on X whose IS is $\mathscr{I}$. The study of $\mathscr{L}(\mathscr{I})$ has been initiated in [8]; it was motivated by the fact that "best" bouquets in $\mathscr{L}(\mathscr{I})$ permit to find sharp estimations for the performance of the so-called greedy algorithm applied on $\mathscr{I}$ for searching maximum weight independent sets. Here, by "best" bouquet, we mean a bouquet composed of as few matroids as possible and, as we will see, they are maximal for some order relation on $\mathscr{L}(\mathscr{I})$. Note also that saying that the IS $\mathscr{I}$ has the p-union property amounts to saying that there exists a bouquet in $\mathscr{L}(\mathscr{I})$ composed of p matroids and all other bouquets are composed of at least p matroids.

Any bouquet of matroids $\mathscr{G}$ of $\mathscr{L}(\mathscr{I})$ admits $\mathscr{D}$ as family of circuits and is characterized by the partition of $\mathscr{D}$ into $\mathscr{S} \cup \mathscr{C}$; hence, one denotes $\mathscr{G}$ by $\mathscr{G}(\mathscr{S}, \mathscr{C})$ or simply by $\mathscr{G}(\mathscr{S})$, $\mathscr{S}$ being the set of stigmes, $\mathscr{C}$ the collection of critical sets and $(\mathscr{S}, \mathscr{C})$ satisfying axioms $(D2)$–$(D3)$. We define an *order relation* on $\mathscr{L}(\mathscr{I})$ as follows: $\mathscr{G}(\mathscr{S}_1, \mathscr{C}_1) \leqslant \mathscr{G}(\mathscr{S}_2, \mathscr{C}_2)$ if and only if $\mathscr{S}_1 \subseteq \mathscr{S}_2$ or, equivalently, $\mathscr{C}_2 \subseteq \mathscr{C}_1$. We state some properties of the poset $(\mathscr{L}(\mathscr{I}), \leqslant)$. First, notice that axioms $(D2)$–$(D3)$ are trivially verified for the partition of $\mathscr{D}$ into $\mathscr{S} = \emptyset$, $\mathscr{C} = \mathscr{D}$, implying that the bouquet $\mathscr{G}(\emptyset, \mathscr{D}) = \mathscr{I}$ is the least element of $\mathscr{L}(\mathscr{I})$. We consider the following family:

$$\mathscr{C}^* = \{D \in \mathscr{D} : \text{there exists } D' \in \mathscr{D},\ D' \neq D \text{ and } x \in D \cap D' \text{ such that } D \cup D' - x \in \mathscr{I}\} \qquad \mathbf{(4.2)}$$

and define $\mathscr{S}^* = \mathscr{D} - \mathscr{C}^*$. It follows easily from $(D2)$–$(D3)$ that, for any bouquet $\mathscr{G} \in \mathscr{L}(\mathscr{I})$, one has the inclusion $\mathscr{C}^* \subseteq \mathscr{C}$. Therefore, if axiom $(D3)$ holds for the pair $(\mathscr{S}^*, \mathscr{C}^*)$ (note that axiom $(D2)$ is always satisfied), then $\mathscr{G}(\mathscr{S}^*, \mathscr{C}^*) = \mathscr{G}^*$ is the greatest element of $\mathscr{L}(\mathscr{I})$. More precisely, one has the following result:

Theorem 4.3 (cf. [8], Prop. 2.1, 2.3). *The poset $\mathscr{L}(\mathscr{I})$ is a meet semilattice and the* *meet of any two elements* $\mathcal{G}_1(\mathcal{S}_1, \mathcal{C}_1)$, $\mathcal{G}_2(\mathcal{S}_2, \mathcal{C}_2)$ *is defined by*: $\mathcal{G}_1 \wedge \mathcal{G}_2 = \mathcal{G}(\mathcal{S}_1 \cap \mathcal{S}_2, \mathcal{C}_1 \cup \mathcal{C}_2)$

—*its least element is* $\mathcal{G}(\emptyset, \mathcal{D}) = \mathcal{J}$

—*its atoms are the bouquets* $\mathcal{G}(\{S\}, \mathcal{D} - \{S\})$ *for all* $S \in \mathcal{S}^*$

—$\mathcal{L}$ *is atomic, i.e. every element* $\mathcal{G}(\mathcal{S}, \mathcal{C})$ *is the join of atoms*:

$$\mathcal{G}(\mathcal{S}, \mathcal{C}) = \bigvee_{S \in \mathcal{S}} \mathcal{G}(\{S\}, \mathcal{D} - \{S\})$$

—$\mathcal{L}$ *is a lattice if and only if* $\mathcal{G}(\mathcal{S}^*, \mathcal{C}^*) \in \mathcal{L}(\mathcal{J})$, *i.e. axiom* $(D3)$ *holds for the pair* $(\mathcal{S}^*, \mathcal{C}^*)$, *and, in this case, the greatest element of* $\mathcal{L}(\mathcal{J})$ *is* $\mathcal{G}^* = \mathcal{G}(\mathcal{S}^*, \mathcal{C}^*)$.

We now give some classes of IS for which the poset $\mathcal{L}(\mathcal{J})$ is a lattice.

Theorem 4.4 (Theorem 2.4, [8]). *Suppose that the* IS $\mathcal{J}$ *is the family of stable sets of a graph, or, equivalently, that* $|D| = 2$ *for all* $D \in \mathcal{D}$. *Then, the poset* $\mathcal{L}(\mathcal{J})$ *is a lattice.*

When the IS $\mathcal{J}$ is matroidal with $\mathcal{M}$ as family of flats, then $\mathcal{C}^* = \emptyset$ and axiom $(D3)$ obviously holds for the pair $(\emptyset, \mathcal{D})$; the bouquet $\mathcal{G}^* = \mathcal{G}(\emptyset, \mathcal{D})$ coincides in fact with the matroid $\mathcal{M}$ and every bouquet $\mathcal{G}$ of $\mathcal{L}(\mathcal{J})$ is a geometric semilattice.

Theorem 4.5. *Suppose that* $\mathcal{J}$ *is a matroidal* IS *with* $\mathcal{M}$ *as family of flats. Then, the poset* $\mathcal{L}(\mathcal{J})$ *is a lattice with* $\mathcal{M}$ *as greatest element.*

Proposition 4.6. *Let* $\mathcal{G} = \mathcal{G}(\mathcal{S}, \mathcal{C})$ *be a bouquet of* 2 *matroids of* $\mathcal{L}(\mathcal{J})$ *with roofs* X_1, X_2 *and suppose that* $\mathcal{J}$ *is not matroidal. Then,* $\mathcal{G} = \mathcal{G}^*$, *i.e.* $\mathcal{S} = \mathcal{S}^*$, $\mathcal{C} = \mathcal{C}^*$ *holds.*

Proof. It is enough to show that $\mathcal{C} = \mathcal{C}^*$ holds. For this, suppose for contradiction that there exists a circuit $C \in \mathcal{C} - \mathcal{C}^*$. It is easy to see that all critical sets are of the form $\{x, y\}$ with $x \in X_1 - X_2$ and $y \in X_2 - X_1$. Thus, we have that $C = \{x, y\}$ with x, y as above. We prove that $r(X_1 - X_2) = r(X_2 - X_1) = 1$ holds. We can suppose that $|X_2 - X_1| \geq 2$. Take $z \in X_2 - X_1$ with $z \neq y$, then $C' = \{x, z\} \in \mathcal{C}$. From Definition 4.2 of $\mathcal{C}^*$, we deduce that $C \cup C' - x = \{y, z\} \notin \mathcal{J}$ and thus $\{y, z\} \in \mathcal{S}$. Similarly, for all $z' \in X_2 - X_1$ with $z' \neq y, z$, $\{y, z'\} \in \mathcal{S}$ which, together with axiom $(D2)$, implies that $\{z, z'\} \in \mathcal{S}$. This implies therefore that $r(X_2 - X_1) = 1$ and, similarly, $r(X_1 - X_2) = 1$. If r denotes the rank of $\mathcal{G}$, one obtains that $r(X_1) = r(X_2) = r$ and $r(X_1 \cap X_2) = r - 1$. Consequently, for any base B of $\mathcal{J}$, if $B \subseteq X_1$, then $|B \cap (X_1 - X_2)| = 1$ and the same for index 2. We now show that this implies that $\mathcal{J}$ is a matroidal IS, yielding therefore a contradiction. For this, we show that the basis exchange axiom (B) holds; i.e. for two distinct bases B, B' of $\mathcal{J}$ with $B \subseteq X_1$, $B' \subseteq X_2$ and an element $x \in B - B'$, there exists an element $y \in B' - B$ such that $B - x + y \in \mathcal{J}$. When $x \in X_1 - X_2$, then $B - x \subseteq X_2$;

from the augmentation axiom ($I4$) applied to the independent sets $B-x$, B' in matroid $\mathcal{J}\cap 2^{X_2}$, there exists an element $y\in B'-B$ such that $B-x+y\in\mathcal{J}$. When $x\in X_1\cap X_2$, then $x\in B\cap X_2-B'\cap X_1$; by applying again ($I4$) to the independent sets $(B-x)\cap X_2$ and $B'\cap X_1$, there exists an element $y\in B'\cap X_1-B\cap X_2$ such that the set $B\cap X_2-x+y$ is independent. Let a denote the unique element of $B-X_2$, then $B\cap X_2=B-a$. Since the independent set $B-\{a,x\}+y$ is contained in $X_1\cap X_2$ and $a\in X_1-X_2$, one deduces from axiom ($I3$) that the set $B-x+y$ is independent. □

Theorem 4.7. *Let $\mathcal{J}$ be an* IS *with the* 2*-union property. Then the poset $\mathcal{L}(\mathcal{J})$ is a lattice whose greatest element $\mathcal{G}^*$ is a bouquet of* 2 *matroids.*

Proof. Since $\mathcal{J}$ has the 2-union property, there exists $\mathcal{G}\in\mathcal{L}(\mathcal{J})$ which is a bouquet of 2 matroids. One deduces from Proposition 4.6 that $\mathcal{G}=\mathcal{G}^*$ and, from Theorem 4.3, that $\mathcal{L}(\mathcal{J})$ is a lattice. □

Theorem 4.7 does not extend to the case of IS having the m-union property for $m\geq 3$; we refer to [8] for an example of an IS $\mathcal{J}$ with the 3-union property for which $\mathcal{L}(\mathcal{J})$ is not a lattice.

Given a bouquet of matroids $\mathcal{G}$ of $\mathcal{L}(\mathcal{J})$, let m denote the number of roofs of $\mathcal{G}$, i.e. the number of matroids composing the bouquet $\mathcal{G}$. One may ask which are the "best" bouquets in $\mathcal{L}(\mathcal{J})$, i.e. the bouquets composed by the least possible number of matroids. For instance, the least element $\mathcal{J}=\mathcal{G}(\emptyset,\mathcal{D})$ is the "worst" bouquet since it involves as many matroids as the number of bases of $\mathcal{J}$. On the other hand, when $\mathcal{J}$ is a matroidal IS, then the greatest element of $\mathcal{L}(\mathcal{J})$ is the best possible since it is, in fact, a matroid. The following result shows that, generally, if $\mathcal{G}_1\preccurlyeq\mathcal{G}_2$, then $\mathcal{G}_2$ is better than $\mathcal{G}_1$, i.e. is composed by less matroids. Note that, if $\mathcal{G}_1\preccurlyeq\mathcal{G}_2$, then $\mathcal{C}_2\subseteq\mathcal{C}_1$ and thus no flat $G\in\mathcal{G}_1$ contains a critical set of $\mathcal{C}_2$ and the closure $\sigma_2(G)$ is well defined.

Theorem 4.8 (Proposition 2.6, [8]). *Let $\mathcal{G}_1$, $\mathcal{G}_2$ be two bouquets of matroids of $\mathcal{L}(\mathcal{J})$ whose respective numbers of roofs are m_1, m_2. If $\mathcal{G}_1\preccurlyeq\mathcal{G}_2$ holds in $\mathcal{L}(\mathcal{J})$, then the closure operator σ_2 of $\mathcal{G}_2$ induces an epimorphism from $\mathcal{G}_1$ onto $\mathcal{G}_2$ and $m_2\leq m_1$ holds.*

The above result can be rephrased as follows: if $\mathcal{G}_1\preccurlyeq\mathcal{G}_2$, then the bouquet $\mathcal{G}_2$ is obtained from $\mathcal{G}_1$ by *aggregation* of the branches of $\mathcal{G}_1$; a branch of $\mathcal{G}_2$ with roof X_2 results from the aggregation of all branches of $\mathcal{G}_1$ whose roofs are contained in X_2. The best bouquets, i.e. those having minimum number of branches, are among the maximal elements of $\mathcal{L}(\mathcal{J})$ and, when $\mathcal{L}(\mathcal{J})$ is a lattice, then the greatest element $\mathcal{G}^*$ of $\mathcal{L}(\mathcal{J})$ is the best bouquet. In fact, Theorem 4.8 can be strengthened; when $\mathcal{G}_1\preccurlyeq\mathcal{G}_2$, the closure operator σ_2 induces an epimorphism from FL($\mathcal{G}_1$) onto FL($\mathcal{G}_2$) where, for a bouquet $\mathcal{G}$, FL($\mathcal{G}$) denotes its chain complex formed by the chains of flats: $F_1\subsetneq F_2\subsetneq\cdots\subsetneq F_s$ of $\mathcal{G}$.

Proposition 4.9. *Let $\mathcal{G}_1 \leqslant \mathcal{G}_2$ be two bouquets of $\mathcal{L}(\mathcal{J})$. Let G, G' be flats of $\mathcal{G}_2$ with $G \subsetneqq G'$ and F be a flat of $\mathcal{G}_1$ such that $\sigma_2(F) = G$. Then there exists a flat F' of $\mathcal{G}_1$ such that $\sigma_2(F') = G'$ and $F \subsetneqq F'$.*

Proof. One can assume w.l.o.g. that $r(G') = r(G) + 1$. Then, $G' = \sigma_2(G \cup x)$ for some $x \in G' - G$. Since $G = \sigma_2(F)$, $x \notin F$ and, in fact, $F' = \sigma_1(F \cup x)$ exists. Else, if F' does not exist, there exists a critical set $C \in \mathscr{C}_1$ such that $x \in C$ and $C \subseteq F \cup x$; since $F \cup x \subseteq G'$, then $C \notin \mathscr{C}_2$ and thus $C \in \mathscr{S}_2$, implying that $x \in \sigma_2(F) = G$, which yields a contradiction. Observe now that $F' \subseteq G'$ holds; take $y \in F' = \sigma_1(F \cup x)$, then there exists $S \in \mathscr{S}_1$ such that $y \in S$ and $S \subseteq F \cup x \cup y$, but $S \in \mathscr{S}_2$ since $\mathscr{S}_1 \subseteq \mathscr{S}_2$, which implies therefore that $y \in \sigma_2(F \cup x) \subseteq G'$. In fact, $G' = \sigma_2(F')$ holds for rank considerations. □

Theorem 4.10. *If $\mathcal{G}_1 \leqslant \mathcal{G}_2$ in $\mathcal{L}(\mathcal{J})$, then the closure operator σ_2 of $\mathcal{G}_2$ induces an epimorphism from* FL($\mathcal{G}_1$) *onto* FL($\mathcal{G}_2$) *which, to a chain: $F_1 \subsetneqq F_2 \subsetneqq \cdots \subsetneqq F_s$ of flats of $\mathcal{G}_1$ associates the chain: $\sigma_2(F_1) \subsetneqq \sigma_2(F_2) \subsetneqq \cdots \subsetneqq \sigma_2(F_s)$ of flats of $\mathcal{G}_2$.*

The proof follows easily from Proposition 4.9. □

5. Bouquets of geometric lattices

In this section, we look in more detail at the family of flats of a bouquet of matroids viewed as a poset with inclusion as order relation. For the case of matroids, this is a classical approach. It is well known that the poset of flats of a matroid is a geometric lattice and, more precisely, that finite geometric lattices correspond bijectively to simple matroids. Similarly, bouquets of matroids correspond to what we call bouquets of geometric lattices.

Definition 5.1. A poset P is a bouquet of geometric lattices if P is a meet semilattice in which every interval is a geometric lattice.

Proposition 5.2. *The poset of flats of a bouquet of matroids is a bouquet of geometric lattices.*

The above result can be easily seen to hold. Conversely and similarly to the matroidal case, a simple bouquet of matroids can be derived from every bouquet of geometric lattices.

Let P be a bouquet of geometric lattices with maximal elements $z_1, \ldots, z_m$ and X as set of atoms; define X_i as the set of atoms under z_i, then $X_1, \ldots, X_m$ is a clutter of subsets of X. The following facts can be easily checked:

—P has a minimum element 0

—P is ranked with rank $r(.)$, i.e. every unrefinable chain from 0 to $x \in P$ has the same length $r(x)$

—define a set $I \subseteq X$ of atoms to be independent if $\vee I$ exists and $r(\vee I) = |I|$ and let $\mathcal{J}(P)$ be the family of independent sets of atoms

—$\mathcal{J}(P) = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_m$ where $\mathcal{J}_i = \{I \in \mathcal{J}(P) : \vee I \leq z_i\}$ is in fact the collection of independent sets of atoms of the geometric lattice $P \cap [0, z_i]$ and thus $\mathcal{J}_i$ is a matroidal IS on X_i

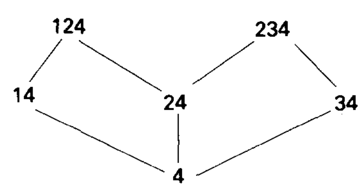
—the above decomposition is in fact a bouquet of matroids. For this, it suffices to verify that axiom ($I3$) holds. Take $I \in \mathcal{J}(P)$ with $\vee I \leq z_i \wedge z_j$ and an atom x with $x \leq z_i$ but $x \nleq z_j$. Hence, $\vee I \vee x \leq z_i$ and $\vee I \vee x \neq \vee I$ which implies that $r(\vee I \vee x) = |I| + 1$ and thus $I + x \in \mathcal{J}(P)$, stating axiom ($I3$)

—if x, y are distinct atoms such that $x \vee y$ exists, then one has $r(x \vee y) = 2$. Therefore, the bouquet of matroids $\mathcal{G}(P)$ on X with roofs $X_1, \ldots, X_m$ whose IS is $\mathcal{J}(P)$ is a simple bouquet of matroids. Hence, we have stated the following:

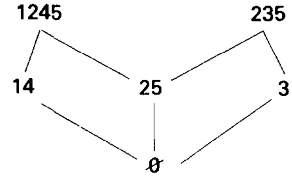
Proposition 5.3. *There is a bijective correspondence between bouquets of geometric lattices and simple bouquets of matroids.*

Similarly to what happens in the matroidal case, a bouquet of matroids is not completely specified by the bouquet of geometric lattices determined by its flat family. For instance, the bouquets $\mathcal{G}$, $\mathcal{G}'$ whose flat structure is shown below are distinct bouquets that are associated to the same bouquet of geometric lattices.

bouquet $\mathcal{G}$:



bouquet $\mathcal{G}'$:



Remark 5.4. The class of geometric semilattices which has been studied in [23] coincides with the class of bouquets of geometric lattices P for which $\mathcal{J}(P)$ is a matroidal IS. At this point, let us mention that this terminology "geometric semilattice" had been also used by Zaslavsky in [24] for denoting in fact the broader class of bouquets of geometric lattices as defined here. We saw in Theorem 3.5 that all well-cut transversal geometries are geometric semilattices. Actually, it turns out that the examples of geometric semilattices considered in [23] are, in fact, transversal geometries; they correspond to Examples 2.5, 2.6, 2.9.

6. Operations on bouquets of matroids

There are many known operations on matroids that preserve, in a way, the matroidal properties. Some of them operate on the lattice of flats of the matroid and, as such, are specifically poset operations; this is the case, for instance, for interval taking, direct product, truncation, etc. Some other ones, as restriction or contraction, are more easily described as operations on the family of independent sets of the matroid. Here, we define poset analogues of these operations for bouquets of matroids and we show which properties of the bouquet and, in particular, which matroidal properties of its independence system are carried out through the operations.

Let $\mathcal{G}$ be a bouquet of matroids on X of rank s with rank function $r(.)$, closure operator $\sigma(.)$ and IS $\mathcal{J}$. Given a subset T of X and an integer k, $0 \leq k \leq s$, there are several ways for constructing new bouquets from $\mathcal{G}$. We consider the following families:

(a) *upper interval*: $[T, \rightarrow) = \{G \in \mathcal{G}: G \supseteq T\}$

(b) *T-deletion*: $\mathcal{G} - T = \{G - T: G \in \mathcal{G}$ and $G \supseteq T\}$

(c) *T-contraction*: $\mathcal{G} \cdot T = \{G \in \mathcal{G}: r(G \cup T) \neq \infty$ and $r(G \cup T) = r(G) + r(T)\}$

(d) *T-restriction*: $\mathcal{G} \mid T = \{G \cap T: G \in \mathcal{G}\}$

(e) *k-truncation*: $\mathcal{G}^k = \{G \in \mathcal{G}: r(G) \leq k\}$.

For the operations of interval, deletion and contraction, we obviously suppose that $r(T) \neq \infty$. The families $[T, \rightarrow)$ and $\mathcal{G} - T$ are clearly isomorphic as posets; hence it is enough to study, for instance, the T-deletion operation. Before showing that the above families are all bouquets and studying their IS, we recall some preliminary results.

Claim 6.1. Let $\mathcal{G}$ be a bouquet of matroids and $\mathcal{G}' \subseteq \mathcal{G}$ be a lower order ideal of $\mathcal{G}$, i.e. if $F \in \mathcal{G}$, $G \in \mathcal{G}'$ and $F \subseteq G$, then $F \in \mathcal{G}'$. Then $\mathcal{G}'$ is a bouquet of matroids.

Proof. We use Definition 1.2 for proving that $\mathcal{G}'$ is a bouquet. It is clear that $\mathcal{G}'$ is stable under intersection and every interval of $\mathcal{G}'$, being also an interval of $\mathcal{G}$, is a matroid. □

Given an IS $\mathcal{J}$ on X and a subset T of X, the following families are obviously IS:

(a) *T-contraction*: $\mathcal{J} \cdot T = \{I \in \mathcal{J}: I \cup J \in \mathcal{J}$ for J maximal subset of $X - T$ with $J \in \mathcal{J}\}$

(b) *T-restriction*: $\mathcal{J} \mid T = \{I \in \mathcal{J}: I \subseteq T\}$

(c) *k-truncation*: $\mathcal{J}^k = \{I \in \mathcal{J}: |I| \leq k\}$

It will be clear from the context whether, in notation $\mathcal{J} \cdot T$, $\mathcal{J}$ is considered as IS or as flat family of a (free) bouquet.

Proposition 6.2 ([22], Chapter 4). *Let $\mathcal{M}$ be the family of flats of a matroid on X with $\mathcal{I}$ as* IS *and T be a subset of X. Then,*

(a) *the* IS $\mathcal{I} \cdot T$ *is a matroidal* IS *on T (no nice characterization of its flats being known)*

(b) *the T-restriction $\mathcal{M} \mid T$ is a matroid on T with* IS $\mathcal{I} \mid T$

(c) *the k-truncation $\mathcal{M}^k$ is a matroid on X with* IS $\mathcal{I}^k$.

In the following we study respectively, restriction, deletion, contraction, truncation and general cuts of bouquets; we analyse what is the effect of each of these operations on a bouquet having specific matroidal properties and, in particular, on geometric semilattices and bouquets with the 2-union property.

Theorem 6.3. *Let $\mathcal{G}$ be a bouquet of matroids on X with* IS $\mathcal{I}$ *and T be a subset of X. Then $\mathcal{G} \mid T$ is a bouquet of matroids on T with* IS $\mathcal{I} \mid T$. *Furthermore, if $\mathcal{G}$ is a bouquet of m matroids, then $\mathcal{G} \mid T$ is a bouquet of m' matroids for some m', $1 \leq m' \leq m$.*

Proof. We suppose that $\mathcal{G}$ is a bouquet of m matroids with roofs $X_1, \ldots, X_m$ and with IS $\mathcal{I}$. For $i \in [1, m]$, $\mathcal{I}_i = \{I \in \mathcal{I} : I \subseteq X_i\}$ is the IS of the matroid $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$. We denote by m' the number of maximal elements of the collection $\{X_1 \cap T, \ldots, X_m \cap T\}$; we can assume that these maximal sets are $X_i \cap T$ for $i \in \{1, \ldots, m'\}$. Using Definition 1.2, we prove that $\mathcal{G} \mid T$ is a bouquet of matroids with roofs $X_i \cap T$ for $i \in [1, m']$. Observe that axioms $(F1)$, $(F2)$ hold trivially. For $i \in [1, m']$, consider the interval $\mathcal{G} \mid T \cap [\emptyset, X_i] = \{G \cap T : G \in \mathcal{M}_i\}$. This interval is therefore the restriction of the matroid $\mathcal{M}_i$ to the set $X_i \cap T$; hence, from Proposition 6.2, it is a matroid on $X_i \cap T$ whose IS is $\mathcal{I}_i \mid X_i \cap T$. Thus, axiom $(F3')$ holds and the IS of $\mathcal{G} \mid T$ is given by: $\bigcup_{i=1}^{m'} \mathcal{I}_i \mid X_i \cap T = \mathcal{I} \mid T$. □

Corollary 6.4. *Let $\mathcal{G}$ be a bouquet of matroids on X and T be a subset of X. If $\mathcal{G}$ has the m-union property, then $\mathcal{G} \mid T$ has the m'-union property for some m', $0 \leq m' \leq m$. In particular, if $\mathcal{G}$ is a geometric semilattice, then so is $\mathcal{G} \mid T$.*

Proof. Since $\mathcal{G}$ has the m-union property, there exists a bouquet $\mathcal{G}' \in \mathcal{L}(\mathcal{I})$ which is composed by exactly m matroids. Since $\mathcal{G} \mid T$ and $\mathcal{G}' \mid T$ have the same IS $\mathcal{I} \mid T$, the proof follows from Theorem 6.3 applied to the bouquet $\mathcal{G}'$. □

Theorem 6.5. *Let $\mathcal{G}$ be a bouquet of matroids on X with* IS $\mathcal{I}$ *and T be a subset of X of finite rank. Then, $\mathcal{G} - T$ is a bouquet of matroids on $X - T$ with* IS $\mathcal{I} \cdot (X - T)$. *Furthermore, if $\mathcal{G}$ is a bouquet of m matroids, then $\mathcal{G} - T$ is a bouquet of m' matroids for some m', $0 \leq m' \leq m$.*

Proof. We suppose that $\mathcal{G}$ is a bouquet of m matroids with roofs $X_1, \ldots, X_m$. For $i \in [1, m]$, $\mathcal{I}_i = \{I \in \mathcal{I} : I \subseteq X_i\}$ is the IS of the matroid $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$. We denote by m' the number of roofs containing T, we can suppose that $T \subseteq X_1 \cap \cdots \cap X_{m'}$. One can easily verify that $\mathscr{G} - T$ is a bouquet with roofs $X_1 - T, \ldots, X_{m'} - T$ and that its rank function $\rho(.)$ is given by: $\rho(A) = r(A \cup T) - r(T)$ for all $A \in \bigcup_{i=1}^{m'} 2^{X_i - T}$. We now prove that the IS of $\mathscr{G} - T$ is $\mathscr{I} \cdot (X - T)$. Take first a set I which is independent for $\mathscr{G} - T$; hence, $I \subseteq X_i - T$ with $i \in [1, m']$ and $|I| = \rho(I) = r(I \cup T) - r(T)$, i.e. $r(I \cup T) = |I| + r(T)$. By applying the augmentation axiom ($I4$) in matroid $\mathscr{M}_i$, one can find a set $J \subseteq T$ such that $I \cup J$ is an independent subset of $I \cup T$ of cardinality $r(I \cup T)$; therefore, $|J| = r(T)$, implying that $I \in \mathscr{I} \cdot (X - T)$. Conversely, if $I \in \mathscr{I} \cdot (X - T)$, let $J \subseteq T$ such that $I \cup J \in \mathscr{I}$ and $r(T) = |J|$. Then, $I \cup J$ is an independent subset of $I \cup T$ of size $|I| + r(T)$, implying that $\rho(I) = |I|$, i.e. I is an independent set for $\mathscr{G} - T$. □

Corollary 6.6. *Let $\mathscr{G}$ be a bouquet of matroids on X with* IS *$\mathscr{I}$ and T be a subset of X of finite rank. Then*

(i) *assume that $\mathscr{L}(\mathscr{I})$ is a lattice and $\mathscr{G}$ has the m-union property, then $\mathscr{G} - T$ has the m'-union property for some m', $0 \leqslant m' \leqslant m$*

(ii) *if $\mathscr{G}$ is a geometric semilattice, then so is $\mathscr{G} - T$*

(iii) *if $\mathscr{G}$ has the 2-union property, then $\mathscr{G} - T$ has the 2-union property or is a geometric semilattice.*

Proof. The assertions (ii), (iii) follow from (i) and Theorems 4.5, 4.7. We now prove (i). By assumption, the greatest element $\mathscr{G}^*$ of $\mathscr{L}(\mathscr{I})$ is a bouquet of m matroids. Since $\mathscr{G} \leqslant \mathscr{G}^*$, it follows that the set T has also finite rank in $\mathscr{G}^*$ and, thus, we can apply Theorem 6.5 to the bouquet $\mathscr{G}^*$ and deduce that $\mathscr{G}^* - T$ is a bouquet of m' matroids for $m' \leqslant m$. Since $\mathscr{G}^* - T$ and $\mathscr{G} - T$ have the same IS, *we deduce that* $\mathscr{G} - T$ has the m''-union property for some $m'' \leqslant m' \leqslant m$. □

Note that the assertion (ii) of Corollary 6.6 is a restatement of Theroem 4.1 [23].

Theorem 6.7. *Let $\mathscr{G}$ be a bouquet of matroids on X with* IS *$\mathscr{I}$ and T be a subset of X of finite rank. Then, $\mathscr{G} \cdot T$ is a bouquet of matroids on $X - T$ with* IS *$\mathscr{I} \cdot (X - T)$.*

Proof. We suppose that $\mathscr{G}$ is a bouquet of m matroids with roofs $X_1, \ldots, X_m$ and with rank function $r(.)$. For $i \in [1, m]$, $\mathscr{I}_i$ denotes the IS of the matroid $\mathscr{M}_i = \mathscr{G} \cap [\emptyset, X_i]$. We denote by m' the number of roofs containing T and we suppose that $T \subseteq X_1 \cap \cdots \cap X_{m'}$.

Using Claim 6.1, we prove that $\mathscr{G} \cdot T$ is a bouquet of matroids on $X - T$ by showing that $\mathscr{G} \cdot T$ is a lower ideal of $\mathscr{G}$. For this, take $\mathscr{G} \in \mathscr{G} \cdot T$, $F \in \mathscr{G}$ with $F \subseteq G$; thus, $G \cup T \subseteq X_i$ for some $i \in [1, m']$, $r(G \cup T) = r(G) + r(T)$ and we can suppose w.l.o.g. that $r(G) = r(F) + 1$. Take $x \in G - F$; then, we have the relation: $r(G) + r(T) = r(G \cup T) = r(F \cup T \cup x) \leqslant r(F \cup T) + 1$ and, since $r(G) = r(F) + 1$, we deduce that: $r(F) + r(T) \leqslant r(F \cup T)$. In matroid $\mathscr{M}_i$, the reverse inequality holds, implying therefore the equality: $r(F \cup T) = r(F) + r(T)$, i.e. $F \in \mathcal{G} \cdot T$. Therefore, $\mathcal{G} \cdot T$ is a bouquet of matroids whose roofs are the maximal flats of $\mathcal{G} \cdot T$; one denotes them by $Y_1, \ldots, Y_{m''}$. Obviously, the rank function $\rho(.)$ of $\mathcal{G} \cdot T$ is given by: $\rho(A) = r(A)$ for all sets A contained in some roof Y_j for $j \in [1, m'']$.

We verify now that the IS of $\mathcal{G} \cdot T$ is $\mathcal{J} \cdot (X - T)$. For this, take an independent set I of $\mathcal{G} \cdot T$. Hence, $I \subseteq Y_j$ for some $j \in [1, m'']$ and its closure G in $\mathcal{G} \cdot T$ satisfies $r(G) = r(I) = |I|$ and $r(G \cup T) = r(G) + r(T) = |I| + r(T)$. By applying axiom $(I4)$, one finds a set J such that $J \subseteq T$, $I \cup J \in \mathcal{J}$ and $|I \cup J| = r(G \cup T)$, i.e. $|J| = r(T)$, implying that $I \in \mathcal{J} \cdot (X - T)$. Conversely, take $I \in \mathcal{J} \cdot (X - T)$ and let $J \subseteq T$ such that $I \cup J \in \mathcal{J}$, $|J| = r(T)$. Define the closure G of I in $\mathcal{G}$; then $I \cup J$ is an independent subset of $G \cup T$ of size $|I| + |J| = r(G) + r(T)$, implying the relation: $r(G \cup T) = r(G) + r(T)$ and thus that I is an independent set for $\mathcal{G} \cdot T$. □

We observe that T-contraction and T-deletion are two operations that yield distinct bouquets $\mathcal{G} \cdot T$ and $\mathcal{G} - T$ which have the same IS $\mathcal{J} \cdot (X - T)$. Hence, Corollary 6.6 remains valid when replacing $\mathcal{G} - T$ by $\mathcal{G} \cdot T$ and we do not repeat it; note that the assertion (ii) is then a restatement of Theorem 4.3 [23]. In fact, in the poset $\mathcal{L}(\mathcal{J} \cdot (X - T))$, the bouquet $\mathcal{G} - T$ is better than the bouquet $\mathcal{G} \cdot T$, i.e. $\mathcal{G} \cdot T \preccurlyeq \mathcal{G} - T$, or, in other words, $\mathcal{G} - T$ is obtained from $\mathcal{G} \cdot T$ by aggregation of its flowers.

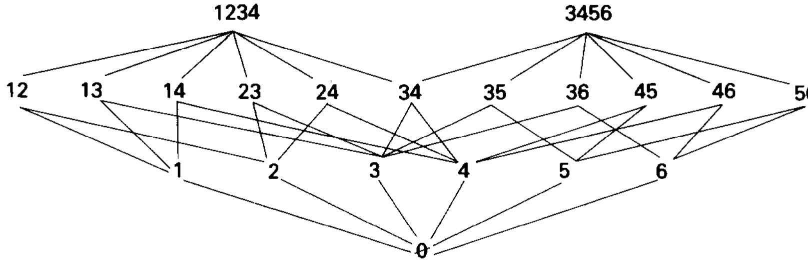
Proposition 6.8. *Let $\mathcal{G}$ be a bouquet of matroids on X with* IS *$\mathcal{J}$, closure operator $\sigma(.)$ and let T be a subset of X of finite rank. Then, $\mathcal{G} \cdot T \preccurlyeq \mathcal{G} - T$ holds in the poset $\mathcal{L}(\mathcal{J} \cdot (X - T))$. Furthermore, the mapping $\Theta: \mathcal{G} \cdot T \to \mathcal{G} - T$ that, to each flat $G \in \mathcal{G} \cdot T$ associates the flat $\Theta(G) = \sigma(G \cup T) - T$ of $\mathcal{G} - T$, is an epimorphism from $\mathcal{G} \cdot T$ onto $\mathcal{G} - T$.*

Proof. In order to show that $\mathcal{G} \cdot T \preccurlyeq \mathcal{G} - T$ holds, we have to verify that all stigmes of $\mathcal{G} \cdot T$ are stigmes of $\mathcal{G} - T$. Let S be a stigme of $\mathcal{G} \cdot T$, i.e. S is a circuit of the IS $\mathcal{J} \cdot (X - T)$ and S is contained in a flat G of $\mathcal{G} \cdot T$. Then, the set $\sigma(G \cup T) - T$ is a flat of $\mathcal{G} - T$ containing S, which implies that S is a stigme of $\mathcal{G} - T$. We observe that the mapping Θ coincides with the closure operator of the bouquet $\mathcal{G} - T$ and, therefore, Theorem 4.8 implies that Θ is an epimorphism from $\mathcal{G} \cdot T$ onto $\mathcal{G} - T$. □

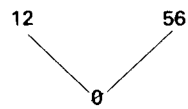
Example 6.9. Let $\mathcal{G}$ be the bouquet of matroids on $[1, 6]$ whose flat configuration is shown below; its IS $\mathcal{J}$ has bases: 123, 124, 134, 234, 345, 346, 356, 456. For the set $T = 34$, one defines the T-deletion $\mathcal{G} - T$ and the T-contraction $\mathcal{G} \cdot T$ whose flat configurations are shown below. Observe that their common IS is $\mathcal{J} \cdot (X - T)$ with bases: 1, 2, 5, 6. Observe also that $\mathcal{G} - T$ is a bouquet of 2 matroids while

$\mathscr{G} \cdot T$ is a bouquet of 4 matroids

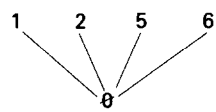
bouquet $\mathscr{G}$:



bouquet $\mathscr{G} - T$:



bouquet $\mathscr{G} \cdot T$:



We now consider the operation of deletion of intervals on bouquets. For a bouquet $\mathscr{G}$ and a set T, one defines the family: $\mathscr{G} - [T, \rightarrow) = \{G \in \mathscr{G}: G \not\supseteq T\}$ obtained by deleting the upper interval $[T, \rightarrow)$ from $\mathscr{G}$. One can obviously suppose that T is a flat of $\mathscr{G}$ and the following holds easily:

Proposition 6.10. *Let $\mathscr{G}$ be a bouquet of matroids on X with* IS $\mathscr{I}$ *and closure operator $\sigma(.)$ and let $T \in \mathscr{G}$. Then, the family $\mathscr{G} - [T, \rightarrow)$ is a bouquet of matroids whose* IS *is given by*: $\{I \in \mathscr{I}: \sigma(I) \not\supseteq T\}$.

An atom (1-flat) T of $\mathscr{G}$ is called *universal* if T is contained in all roofs of $\mathscr{G}$. Note that, for $T \in \mathscr{G}$, one has always the inclusion: $\mathscr{G} \cdot T \subseteq \mathscr{G} - [T, \rightarrow)$ and that equality holds if and only if T is a universal atom of $\mathscr{G}$. Hence, if T is a universal atom of $\mathscr{G}$, then $\mathscr{G} - [T, \rightarrow)$ has an IS the family $\mathscr{I} \cdot (X - T)$; therefore, Corollary 6.6 remains valid when replacing $\mathscr{G} - T$ by $\mathscr{G} - [T, \rightarrow)$ and the assertion (ii) then implies Corollary 4.5 [23].

For matroids, one has the following result:

Proposition 6.11 (Corollary 4.7, [23]). *Let $\mathcal{M}$ be a matroid and T be a flat of $\mathcal{M}$, then $\mathcal{M} - [T, \rightarrow)$ is a geometric semilattice.*

More generally, one can delete several intervals from a bouquet $\mathscr{G}$; so, if $T_1, \ldots, T_p$ are distinct flats of $\mathscr{G}$, then the family $\mathscr{G} - \bigcup_{i=1}^{p} [T_i, \rightarrow)$ is still a bouquet of matroids, also called *wounded bouquet*. Particularly interesting is the study of wounded matroids. So, we saw above that, when deleting one interval from a matroid, one obtains a geometric semilattice. A beautiful result from [23] shows that, conversely, any geometric semilattice can be realized as a matroid with one less interval. We will see in Section 7 that, more generally, any bouquet with the m-union property can be realized as a bouquet of m matroids with one less interval (under the condition that $\mathscr{L}(\mathscr{I})$ be a lattice). When one deletes several intervals from a matroid, one has the following result:

Propositon 6.12. *Let $\mathcal{M}$ be a matroid and $T_1, \ldots, T_p$ be p distinct flats of $\mathcal{M}$. Then, the family $\mathcal{M} - \bigcup_{i=1}^{p} [T_i, \rightarrow)$ is a bouquet of matroids having the p'-intersection property for some $p' \leqslant p$.*

Proof. Proposition 6.10 implies that $\mathcal{M}' = \mathcal{M} - \bigcup_{i=1}^{p} [T_i, \rightarrow)$ is a bouquet with IS: $\mathscr{I}' = \{I \in \mathscr{I}: \sigma(I) \not\supseteq T_i \text{ for } i = 1, \ldots, p\}$; hence $\mathscr{I}' = \bigcap_{i=1}^{p} \{I \in \mathscr{I}: \sigma(I) \not\supseteq T_i\}$, each of the families in the latter intersection being a matroidal IS from Proposition 6.11. □

The following question is of interest, at least for small values of p, for instance $p = 2$.

Problem 6.13. If $\mathscr{G}$ is a bouquet of matroids having the p-intersection property, can $\mathscr{G}$ be realized as a matroid with p deleted intervals?

We conclude this section by mentioning the related operation of cuts on bouquets of matroids. Following [13], an *elementary cut* consists of deleting exactly one roof from the bouquet and a *cut* is any sequence of elementary cuts. A cut is *uniform* when it consists of removing all roofs at once. For instance, deletion of intervals is a particular cut and iterated uniform cuts produce the truncation of bouquets.

Proposition 6.14. *Let $\mathscr{G}$ be a bouquet of matroids on X of rank s with* IS *$\mathscr{I}$. For k, $0 \leqslant k \leqslant s$, the k-truncation $\mathscr{G}^k$ is a bouquet of rank k on X with* IS *$\mathscr{I}^k$.*

Proof. Easy. □

Corollary 6.15 (Prop. 4.2, [23]). *If $\mathscr{G}$ is a geometric semilattice, then so is any truncation of $\mathscr{G}$.*

Proof. It follows from Proposition 6.2, 6.14. □

7. Strong maps and mapping cylinder operation

The notion of strong map on a geometric lattice is an important tool in matroid theory. It can be extended to bouquets of geometric lattices and, actually, the definition of strong map adopted by Wachs–Walker ([23]) for geometric semilattices turns out to be well adapted for the general class of bouquets, so we will consider the same notion. We refer to [18] for a further study of strong maps on bouquets.

Definition 7.1. Let P_1, P_2 be two bouquets of geometric lattices. A function $f: P_1 \to P_2$ is called a *strong map* if:

($S1$) f is rank reducing, i.e. $r_2(f(x)) \leq r_1(x)$ for $x \in P_1$

($S2$) for each atom $a \in P_1$ and $x \in P_1$, if $a \vee x$ exists in P_1, then $f(a) \vee f(x)$ exists in P_2 and $f(a) \vee f(x) = f(a \vee x)$

($S3$) for each atom $a \in P_1$ and $x \in P_1$, if $f(a) \vee f(x)$ exists in P_2 and $f(a) \nleq f(x)$, then $a \vee x$ exists in P_1.

Note that this definition reduces to the usual definition of strong maps on geometric lattices when P_1, P_2 are geometric lattices ([22], chap. 17). It can be verified that a strong map is *order preserving*, i.e. if y covers x in P_1, then $f(y)$ covers or is equal to $f(x)$ in P_2. Also, ($S2$) remains valid if one replaces the atom a by any element $y \in P_1$.

Most examples of strong maps on geometric lattices from [22] and all examples of strong maps on geometric semilattices from [23] extend easily to the case of general bouquets; we do not repeat them. We introduce two new examples of strong maps: the first one is coming from the closure operator between two comparable bouquets of $\mathscr{L}(\mathscr{I})$, the second one from the projection map for transversal matroid designs.

Theorem 7.2. *Let $\mathscr{I}$ be an* IS *on X and $\mathscr{G}_1, \mathscr{G}_2$ be two bouquets of matroids of $\mathscr{L}(\mathscr{I})$ such that $\mathscr{G}_1 \leq \mathscr{G}_2$. Then, the map from $\mathscr{G}_1$ onto $\mathscr{G}_2$ induced by the closure operator σ_2 of $\mathscr{G}_2$ is a surjective rank preserving strong map.*

Proof. We know from Theorem 4.8 that σ_2 induces an epimorphism from $\mathscr{G}_1$ onto $\mathscr{G}_2$, i.e. a surjective and rank preserving map. We show that σ_2 is a strong map, i.e. satisfies ($S2$),($S3$). Take a 1-flat $F = \sigma_1(a)$ of $\mathscr{G}_1$ and $G \in \mathscr{G}_1$ such that $\sigma_1(F \cup G) = H \in \mathscr{G}_1$ exists; we can suppose that $a \notin G$, else ($S2$) trivially holds. We have that $\sigma_2(F) \cup \sigma_2(G) \subseteq \sigma_2(H)$; if $a \notin \sigma_2(G)$, then, for rank considerations, $\sigma_2(H) = \sigma_2(\sigma_2(F) \cup \sigma_2(G))$, which states ($S2$). Suppose for contradiction that $a \in \sigma_2(G)$; then $\sigma_2(G) = \sigma_2(G \cup a)$ which, since σ_2 is a rank preserving map from $\mathscr{G}_1$ onto $\mathscr{G}_2$, implies that $r_1(G) = r_1(G \cup a)$, yielding a contradiction with the fact that $a \notin G$. We now verify that ($S3$) holds. Take a 1-flat $F = \sigma_1(a)$ of $\mathscr{G}_1$, $G \in \mathscr{G}_1$ such that $\sigma_2(F) \nsubseteq \sigma_2(G)$ and $\sigma_2(\sigma_2(F) \cup \sigma_2(G))$ exists. Suppose for contradiction that $\sigma_1(F \cup G)$ does not exist, hence $\sigma_1(G \cup a)$ does not exist. Thus, there exists a critical set C of $\mathscr{G}_1$ such that $a \in C$ and $C \in G \cup a$. Then C is also contained in the flat $\sigma_2(\sigma_2(F) \cup \sigma_2(G))$ which implies that C is a stigme of $\mathscr{G}_2$. We therefore deduce that $a \in \sigma_2(G)$, yielding a contradiction with the fact that $\sigma_2(F) \nsubseteq \sigma_2(G)$. □

In application of Theorem 7.2, we have the following examples of surjective rank preserving strong maps; they correspond, for the case of geometric semilattices, to examples 4, 5, 6 from [23].

Example 7.3. Given an IS $\mathcal{J}$ and a bouquet $\mathcal{G} \in \mathcal{L}(\mathcal{J})$ with closure operator $\sigma(.)$, the function: $\mathcal{J} \rightarrowtriangle \mathcal{G}$

$I \rightsquigarrow \sigma(I)$.

Example 7.4. Given a matroidal IS $\mathcal{J}$, $\mathcal{M}$ the associated matroid with closure operator $\sigma(.)$ and a bouquet $\mathcal{G} \in \mathcal{L}(\mathcal{J})$, the function: $\mathcal{G} \rightarrowtriangle \mathcal{M}$

$G \rightsquigarrow \sigma(G)$.

Example 7.5. Given a bouquet of matroids $\mathcal{G}$ with closure operator $\sigma(.)$ and T a set of finite rank, the function: $\mathcal{G} \cdot T \rightarrowtriangle \mathcal{G} - T$

$G \rightsquigarrow \sigma(G \cup T) - T$.

Recall that, if $\mathcal{G}$ is a transversal matroid design on $[1, n] \times [1, m]$ with parameters $(l_0, \ldots, l_{s-1}, l_s = n)$, then $\mathcal{G}$ is $\mathcal{M}$-unisupported where $\mathcal{M}$ is a PMD on $[1, n]$ with the same parameters, i.e. $p_1(\mathcal{M}_i) = \mathcal{M}$ with $\mathcal{M}_i = \mathcal{G} \cap [\emptyset, X_i]$ for all roofs X_i of $\mathcal{G}$, p_1 denoting the first projection from $[1, m] \times [1, n]$ onto $[1, n]$.

Theorem 7.6. *Let $\mathcal{G}$ be a transversal matroid design on $[1, n] \times [1, n]$ with $l_s = n$ and $\mathcal{M}$ be its* PMD *support. Then the map induced from $\mathcal{G}$ onto $\mathcal{M}$ by the projection p_1 is a surjective rank preserving strong map.*

Proof. We already know from Proposition 2.3 that p_1 induces an epimorphism from $\mathcal{G}$ onto $\mathcal{M}$. We show that it is a strong map. It is easy to see that axiom $(S2)$ holds. We now verify $(S3)$. Consider a 1-flat $F \in \mathcal{G}$, $G \in \mathcal{G}$ such that $p_1(F) \nsubseteq p_1(G)$. Since $p_1(F)$ has rank 1, one deduces that $p_1(F) \cap p_1(G) = \emptyset$ and thus that $F \cup G$ is a transversal set. Therefore, from axiom $(G4)$, there exists a flat $G' \in \mathcal{G}$ such that $G \cup F \subseteq G'$, which states $(S3)$. □

Remark 7.7. This result does not extend to unisupported d-injection designs for $d \geqslant 2$, i.e. the projection p_1 is not a strong map from $\mathcal{G}$ onto $\mathcal{M} = p_1(\mathcal{G})$. The reason for this being that, for $F, G \in \mathcal{G}$, the condition $p_1(F) \cap p_1(G) = \emptyset$ does not imply that $F \cup G$ is a d-injective set and thus $(S3)$ does not hold.

In the following, we show how to relate strong maps between bouquets of matroids to strong maps between their IS or other related bouquets. We first state a preliminary result.

Claim 7.8. Let $\mathcal{G}_i$ be a simple bouquet of matroids on X_i with IS $\mathcal{J}_i$ and family of circuits $\mathcal{D}_i$, for $i = 1, 2$. Suppose that $\mathcal{J}_i$ has the m_i-union property, for $i = 1, 2$, and let $f: \mathscr{G}_1 \multimap \mathscr{G}_2$ be a rank preserving strong map. Given a subset $A = \{a_1, \ldots, a_t\}$ of X_1, we denote by $f\{A\}$ the subset $\{f(a_1), \ldots, f(a_t)\}$. The following assertions hold:

(i) $A \in \mathscr{I}_1$ if and only if $f\{A\} \in \mathscr{I}_2$ and $|f\{A\}| = t$

(ii) if $A \in \mathscr{D}_1$ then, either $f(A) \in \mathscr{D}_2$ and $|f\{A\}| = t$; conversely, if $f\{A\} \in \mathscr{D}_2$ and $|f\{A\}| = t$, then $A \in \mathscr{D}_1$

(iii) Suppose that f is surjective and let $\mathscr{C}_1^*$, C_2^* be the families defined by relation (4.2); if $f\{A\} \in C_2^*$ and $|f\{A\}| = t$, then $A \in \mathscr{C}_1^*$

(iv) if f is surjective, then $m_1 \leq m_2$ holds.

Proof. (i), (ii) are easy to verify. We state (iii). If $f\{A\} \in C_2^*$, then, by definition of C_2^*, there exists a set $D \in \mathscr{D}_2$, $D \neq f\{A\}$, an element $x \in D \cap f\{A\}$ such that $f\{A\} \cup D - x \in \mathscr{I}_2$. Since f is surjective, we can find $B \in \mathscr{D}_1$ such that $D = f\{B\}$, $x = f(a)$ with $a \in A \cap B$ and $f\{A \cap B\} = f\{A\} \cap f\{B\}$; therefore $A \cup B - a \in \mathscr{I}_1$ from (i), thus implying that $A \in \mathscr{C}_1^*$. For proving (iv), consider a decomposition of $\mathscr{I}_2$ as a bouquet of matroids with m_2 roofs: $Y_1, \ldots, Y_{m_2}$ and define the sets: $Z_i = \{x \in X_1 : f(x) \in Y_i\}$ for $1 \leq i \leq m_2$; then it is a routine to verify that $\mathscr{I}_1$ can be decomposed as bouquet of m_2 matroids with roofs the sets Z_i's, therefore implying: $m_1 \leq m_2$. □

Theorem 7.9. *Let $\mathscr{G}_i$ be a simple bouquet of matroids with* IS *$\mathscr{I}_i$ and closure operator σ_i, for $i = 1, 2$, and $f: \mathscr{G}_1 \multimap \mathscr{G}_2$ be a rank preserving strong map. Then, there exists a unique strong map $\bar{f}: \mathscr{I}_1 \multimap \mathscr{I}_2$ such that the diagram below commutes. Furthermore, if f is surjective, then so is $\bar{f}$.*

$$\begin{array}{ccc} \mathscr{G}_1 & \xrightarrow{f} & \mathscr{G}_2 \\ \sigma_1 \uparrow & & \uparrow \sigma_2 \\ \mathscr{I}_1 & \xrightarrow[\bar{f}]{} & \mathscr{I}_2 \end{array}$$

Proof. It can be easily verified, using Claim 7.8(i), that the map $\bar{f}$ defined by: $\bar{f}(I) = \{f(a_1), \ldots, f(a_t)\}$ for $I = \{a_1, \ldots, a_t\} \in \mathscr{I}_1$, is the unique strong map satisfying Theorem 7.9. □

Theorem 7.10. *Let $\mathscr{G}_i$ be a simple bouquet of matroids with* IS *$\mathscr{I}_i$ and assume that $\mathscr{L}(\mathscr{I}_i)$ is a lattice with greatest element $\mathscr{G}_i^*$ whose closure operator is denoted by σ_i^*, for $i = 1, 2$. Let $f: \mathscr{G}_1 \multimap \mathscr{G}_2$ be a surjective rank preserving strong map. Then, there exists a unique map $f^*: \mathscr{G}_1^* \multimap \mathscr{G}_2^*$ such that the diagram below commutes; furthermore f^* is rank preserving and surjective strong map.*

$$\begin{array}{ccc} \mathscr{G}_1 & \xrightarrow{f} & \mathscr{G}_2 \\ \sigma_1^* \downarrow & & \downarrow \sigma_2^* \\ \mathscr{G}_1^* & \xrightarrow[f^*]{} & \mathscr{G}_2^* \end{array}$$

Proof. We denote by $\mathscr{D}_i$ the circuit family of $\mathscr{I}_i$, by $\mathscr{S}_i$ (resp. $\mathscr{S}_i^*$) the stigmes of $\mathscr{G}_i$ (resp. $\mathscr{G}_i^*$) and by $\mathscr{C}_i$ (resp. $\mathscr{C}_i^*$) the critical sets of $\mathscr{G}_i$ (resp. $\mathscr{G}_i^*$). Since $\mathscr{L}(\mathscr{I}_i)$ is a lattice, the family $\mathscr{C}_i^*$ is given by relation (4.2), for $i=1,2$. We first recall two relations that we will use in the proof: for all $I \in \mathscr{I}_i$, $\sigma_i^*(I) = \sigma_i^*(\sigma_i(I)$ for $i = 1, 2$ and for $I \in \mathscr{I}_1$, $f(\sigma_1(I)) = \sigma_2(f\{I\})$. Let $F^* \in \mathscr{G}_1^*$ and $I \in \mathscr{I}_1$ be a basis of F^*; necessarily, the map f^* must satisfy: $f^*(F^*) = f^*(\sigma_1^*(\sigma_1(I))) = \sigma_2^*(f(\sigma_1(I))) = \sigma_2^*(\sigma_2(f\{I\})) = \sigma_2^*(f\{I\})$. Hence, we are led to define f^* by: $f^*(F^*) = \sigma_2^*(f\{I\})$ where I is a basis of $F^* \in \mathscr{G}_1^*$. We first verify that f^* is well defined, i.e. if $\sigma_1^*(I) = \sigma_1^*(I')$, then $\sigma_2^*(f\{I\}) = \sigma_2^*(f\{I'\})$. It is enough to prove that $f(a) \in \sigma_2^*(f\{I'\})$ for all $a \in I$. If $a \in I$, then there exists $D \in \mathscr{S}_1^*$ such that $a \in D \subseteq I' \cup a$; hence $f(a) \in f\{D\} \subseteq f\{I'\} \cup f(a)$. From Claim 7.8(ii), we deduce that $f\{D\} \in \mathscr{D}_2$. Also, from Claim 7.8(iii), we have that $f\{D\} \in \mathscr{S}_2^*$, implying that $f(a) \in \sigma_2^*(f\{I'\})$.

Obviously f^* is rank preserving surjective and we leave it to the reader to verify that f^* is a strong map. □

In the case when $\mathscr{G}_1$, $\mathscr{G}_2$ are geometric semilattices, then Theorem 7.10 remains valid without the assumption that f be surjective and rank preserving, as stated in [23] (Theorem 5.1); actually, a slight modification of our proof also shows it.

Corollary 7.11. *With the notations of Theorem* 7.10, *suppose that $\mathscr{I}_i$ has the m_i union property for $i = 1, 2$, then $m_1 = m_2$ holds.*

Proof. From Theorem 7.10, f^* is a surjective rank preserving map from the bouquet $\mathscr{G}_1^*$ on the bouquet $\mathscr{G}_2^*$, hence f^* maps the m_1 roofs of $\mathscr{G}_1^*$ onto the m_2 roofs of $\mathscr{G}_2^*$ and thus $m_2 \leq m_1$ holds. The reverse inequality follows from Claim 7.8 (iv), hence implying that $m_1 = m_2$. □

We now present a poset operation on bouquets of geometric lattices that uses strong maps as essential tool; this is the operation of mapping cylinder which has been introduced in [23] for geometric semilattices. Again it turns out that bouquets of geometric lattices seem to offer the correct level of generality at which the mapping cylinder construction applies nicely.

Definition 7.12. Let P_1, P_2 be two bouquets of geometric lattices and $f: P_1 \to P_2$ be a strong map. The *mapping cylinder* $C(P_1, P_2, f)$ is the poset whose element set is $P_1 \cup P_2$ and whose order relation $<_c$ is defined as follows: for $x, y \in P_1 \cup P_2$, $x <_c y$ if one of the following holds:

(i) $x < y$ in P_1 when $x, y \in P_1$
(ii) $x < y$ in P_2 when $x, y \in P_2$
(iii) $f(x) \leq y$ when $x \in P_1$, $y \in P_2$.

Theorem 7.13. *Let P_1, P_2 be two bouquets of geometric lattices and $f: P_1 \to P_2$ be a surjective rank preserving strong map. Then, the mapping cylinder $C(P_1, P_2, f)$ is a bouquet of geometric lattices.*

Remark 7.14. This theorem is a companion to Theorem 6.1 from [23] which states that, when P_1, P_2 are geometric semilattices, then $C(P_1, P_2, f)$ is a geometric semilattice. If one looks carefully at the proof of Theorem 6.1 ([23]), one can notice that, in the first part of it, it is shown that $C(P_1, P_2, f)$ is a bouquet of geometric lattices, using only the assumption that P_1, P_2 are bouquets; this part therefore includes the proof of Theorem 7.13 and we do not repeat it. In the last part of the proof of Theorem 6.1 ([23]), using the additional information that the bouquets P_1, P_2 are geometric semilattices, it is deduced that the bouquet $C(P_1, P_2, f)$ too is a geometric semilattice; this result will also follow from the more general statement in Corollary 7.16.

Theorem 7.15. *With the notations of Theorem* 7.10, *the map* φ, φ: $C(\mathcal{G}_1, \mathcal{G}_2, f) \to C(\mathcal{G}_1^*, \mathcal{G}_2^*, f^*)$ *defined by* $\varphi(F) = \sigma_i^*(F)$ *for* $G \in \mathcal{G}_i$, $i = 1, 2$ *is a surjective rank preserving strong map.*

Proof. φ is obviously surjective and rank preserving. From Corollary 7.11 and the proof of Claim 7.8(iv), if $\mathcal{G}_2^*$ has roofs $Y_1, \ldots, Y_m$, then $\mathcal{G}_1^*$ has for roofs the sets $Z_i = \{a \in X_i : f(a) \in Y_i\}$ for $i = 1, \ldots, m$; this implies the relations:

(a) if $S \in \mathcal{S}_1^*$, then either $S = \{a, b\}$ with $f(a) = f(b)$, or $f\{S\} \in \mathcal{S}_2^*$

(b) if $f\{S\} \in \mathcal{S}_2^*$ and $|f\{S\}| = |S|$, then $S \in \mathcal{S}_1^*$.

Set $P = C(\mathcal{G}_1, \mathcal{G}_2, f)$ and $P^* = C(\mathcal{G}_1^*, \mathcal{G}_2^*, f^*)$. We show that φ is a strong map. We first prove that $(S2)$ holds. For this, take an atom $F \in P$, $G \in P$ such that $F \not\leq G$ and $F \vee G$ exists in P. If $F = \emptyset \in \mathcal{G}_2$ and $G \in \mathcal{G}_1$, then $F \vee G = f(G)$ and thus $\varphi(f(G)) = f^*(\varphi(G))$ dominates $\varphi(F), \varphi(G)$ and, in fact, $f^*(\varphi(G)) = \varphi(F) \vee \varphi(G)$. Now suppose (the other cases are easy) that $F = \sigma_1(a)$, $G = \sigma_1(I) \in \mathcal{G}_1$ and $F \vee G$ exists in $\mathcal{G}_2$, i.e. $\sigma_1(I \cup a)$ does not exist; then $F \vee G = \sigma_2(f\{I\} \cup f(a))$ and $\varphi(F \vee G) = \sigma_2^*(f\{I\} \cup f(a))$ dominates $f^*(\varphi(F)), f^*(\varphi(G))$ and thus $\varphi(F), \varphi(G)$, implying that $\varphi(F) \vee \varphi(G) \leq f^*(\varphi(F)) \vee f^*(\varphi(G)) \leq \varphi(F \vee G)$. Equality holds for rank considerations, after noticing that $\varphi(F) \not\leq \varphi(G)$; else, $a \in \sigma_1^*(I)$ which, from (a), implies that $f(a) \in \sigma_2^*(f\{I\})$, contradicting the fact that $F \not\leq G$.

We now prove that $(S3)$ holds. For this, take an atom $F \in P$, $G \in P$ such that $\varphi(F) \vee \varphi(G)$ exists in P^* and $\varphi(F) \not\leq \varphi(G)$. When $\varphi(F) \vee \varphi(G) \in \mathcal{G}_1^*$, then $F = \sigma_1(a)$, $G = \sigma_1(I)$ and $I \cup a \in \mathcal{I}_1$, so $F \vee G$ exists in $\mathcal{G}_1$. Suppose now that $\varphi(F) \vee \varphi(G) \in \mathcal{G}_2^*$. If $F = \emptyset \in \mathcal{G}_2$, $G \in \mathcal{G}_1$ then $f(G)$ dominates F, G and $F \vee G$ exists. If $F = \sigma_1(a) \in \mathcal{G}_1$, $G = \sigma_2(I) \in \mathcal{G}_2$, then $\varphi(F) \vee \varphi(G) = \sigma_2^*(I \cup f(a))$, implying that $I \cup f(a) \in \mathcal{I}_2$; hence $\sigma_2(I \cup f(a))$ dominates F, G and $F \vee G$ exists. Suppose now that $F = \sigma_1(a)$, $G = \sigma_1(I) \in \mathcal{G}_1$ and $\sigma_1^*(I \cup a)$ does not exist. Then $\varphi(F) \vee \varphi(G) = \sigma_2^*(f\{I\} \cup f(a))$ and, by computing the rank of both sides in P^*, we deduce that $f(a) \in \sigma_2^*(f(\{I\})$. If $f(a) \notin f\{I\}$, then there exists $D \in \mathcal{S}_2^*$ such that $f(a) \in D \subseteq f\{I\} \cup f(a)$; from (b), $D = f\{S\}$ with $S \in \mathcal{S}_1^*$ and $a \in S \subseteq I \cup a$, contradicting the fact that $\sigma_1^*(I \cup a)$ does not exist. Therefore $f(a) \in f\{I\}$; hence $\sigma_2(f\{I\})$ dominates $f(F)$, $f(G)$ and thus F, G, i.e. $F \vee G$ exists. □

Corollary 7.16. *With the notations of Theorem* 7.10, *and, following Corollary* 7.11, *let m be the integer such that* $\mathscr{G}_1$, $\mathscr{G}_2$ *have the m-union property. Then, the bouquet* $C(\mathscr{G}_1, \mathscr{G}_2, f)$ *has the* m'*-union property for some* m', $1 \leq m' \leq m$.

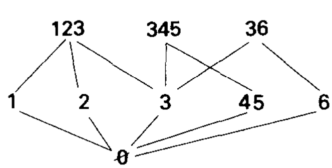
Proof. It follows from Claim 7.8(iv) applied to the strong map φ defined in Theorem 7.15 after noticing that $C(\mathscr{G}_1^*, \mathscr{G}_2^*, f^*)$ is a bouquet of m matroids. □

Let us describe in more detail the mapping cylinder operation. Let $P = C(P_1, P_2, f)$ be the mapping cylinder obtained from P_1, P_2, f as in Theorem 7.13. Suppose that P_2 is a bouquet of m geometric lattices of rank r, with maximal elements $z_1, \ldots, z_m$ and with least element 0_2. From the definition of the order relation $<_c$, P is also a bouquet of m geometric lattices with maximal elements $z_1, \ldots, z_m$; its rank is $r+1$, its atoms are 0_2 (which is in fact a universal atom of P) together with the atoms of P_1 and its least element is the least element 0_1 of P_1. Furthermore, if one deletes the upper interval $[0_2, \rightarrow)$ from P, one obtains exactly the poset P_1, i.e. $P_1 = P - [0_2, \rightarrow)$ can be realized as the bouquet P with one interval deleted. As a consequence, we have results 7.19, 7.20, 7.21. Next, we give some precisions on how to define the mapping cylinder as a bouquet of matroids, i.e. in set theoretical terminology.

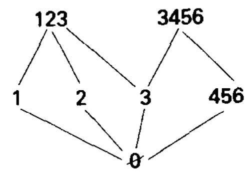
Remark 7.17. Let $\mathscr{J}$ be an IS on X and $\mathscr{G}_1, \mathscr{G}_2 \in \mathscr{L}(\mathscr{J})$ such that $\mathscr{G}_1 \preccurlyeq \mathscr{G}_2$. From Theorems 7.2, 7.13, the poset $P = C(\mathscr{G}_1, \mathscr{G}_2, \sigma_2)$ is a bouquet of geometric lattices. Let w be an arbitrary element that does not belong to X. Then one can define P as a bouquet of matroids on $X \cup w$ whose flats are exactly the sets $G \in \mathscr{G}_1$ or $G \cup w$ for $G \in \mathscr{G}_2$. Hence, assuming that the 0-flat of $\mathscr{G}_2$ is $\emptyset$, the set $F_w = \{w\}$ is a universal 1-flat of $C(\mathscr{G}_1, \mathscr{G}_2, \sigma_2)$; we keep these notations in the remaining of the section. Notice that this amounts to the embedding of the bouquet $\mathscr{G}_1$ of rank r on X in the bouquet $C(\mathscr{G}_1, \mathscr{G}_2, \sigma_2)$ of rank $r+1$ on $X \cup w$. Hence, the mapping cylinder operation is closely related to the notion of embedding of geometries and, also, as noted in [23], to the notion of single element extensions of matroids. We give for illustration an example.

Example 7.18. Let $\mathscr{G}_1, \mathscr{G}_2$ be the bouquets of matroids on $[1, 6]$ whose flat configurations are shown below. They have the same IS and $\mathscr{G}_1 \preccurlyeq \mathscr{G}_2$ holds. We picture below the bouquet of matroids $C(\mathscr{G}_1, \mathscr{G}_2, \sigma_2)$.

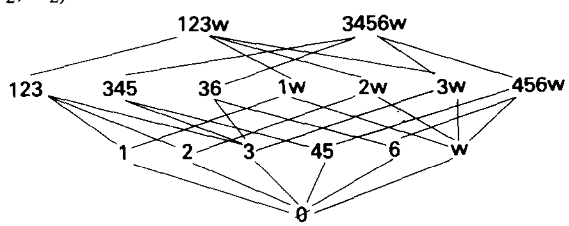
Bouquet $\mathscr{G}_1$:



Bouquet $\mathcal{G}_2$:



Bouquet $C(\mathcal{G}_1, \mathcal{G}_2, \sigma_2)$:



Proposition 7.19. *Let $\mathcal{J}$ be an IS on X and $\mathcal{G}_1 \leqslant \mathcal{G}_2 \in \mathcal{L}(\mathcal{J})$ be two bouquets of, respectively, m_1, m_2 matroids; hence $m_2 \leqslant m_1$. Then $C(\mathcal{G}_1, \mathcal{G}_2, \sigma_2)$ is a bouquet of m_2 matroids and $\mathcal{G}_1 = C(\mathcal{G}_1, \mathcal{G}_2, \sigma_2) - [F_w, \rightarrow)$.*

Corollary 7.20. *Let $\mathcal{J}$ be an IS having the m-union property and assume that $\mathcal{L}(\mathcal{J})$ is a lattice with greatest element $\mathcal{G}^*$. Then, for all $\mathcal{G} \in \mathcal{L}(\mathcal{J})$, $C(\mathcal{G}, \mathcal{G}^*, \sigma^*)$ is a bouquet of m matroids and $\mathcal{G} = C(\mathcal{G}, \mathcal{G}^*, \sigma^*) - (F_w, \rightarrow)$; i.e. any bouquet with the m-union property can be realized as a bouquet of m matroids with one upper interval deleted.*

We deduce in particular from Corollary 7.20 that any bouquet with the 2-union property can be realized as a bouquet of 2 matroids with one less upper interval. We also deduce that any geometric semilattice can be realized as a matroid with one interval deleted, thus restating the "realization" part of Theorem 3.2 [23].

Remark 7.21. We obtain an alternative proof for Theorem 3.5 in the design case: if $\mathcal{G}$ is a transversal matroid design with PMD support $\mathcal{M}$, then, since the projection p_1 is a surjective rank preserving strong map (Theorem 7.6), $C(\mathcal{G}, \mathcal{M}, p_1)$ is a matroid and, from Proposition 6.11, $\mathcal{G} = C(\mathcal{G}, \mathcal{M}, p_1) - [F_w, \rightarrow)$ is therefore a geometric semilattice.

8. On the shellability of bouquets of matroids

To any poset P, one can associate a simplicial complex $\Delta(P)$, called its *order complex*, whose simplices are the maximal chains $x_1 < x_2 < \cdots < x_r$ of elements of P. Recall that a simplicial complex is exactly an IS in which all singletons are independent sets, the simplices correspond then to the independent sets of the IS; the notation of simplicial complex being more specifically used in topological or geometrical context. For any simplex m of a simplicial complex Δ, one denotes by $\tilde{m}$ the subcomplex of Δ formed by all subsets of m; its dimension is one less than its cardinality. The dimension of Δ is the maximum dimension of its simplices, Δ is called *pure* when all maximum simplices have the same dimension. Similarly, an IS is *pure* when all its bases have the same cardinality which is then called the rank of the IS.

Let Δ be a pure d-dimensional simplicial complex with vertex set X, $|X| = n$. A *shelling* of Δ is a special ordering of the maximal simplices of Δ which is favourable for induction arguments. Then, Δ is said to be *shellable* if it admits a shelling order. The ordering: $m_1, m_2, \ldots, m_s$ of the maximum simplices of Δ is a shelling order if:

$$\text{for all } i, j,\ 1 \leq i < j \leq s, \text{ there exists } k,\ 1 \leq k < j, \text{ and } x \in m_j \text{ such that: } m_i \cap m_j \subseteq m_j \cap m_k = m_j - \{x\} \tag{8.1}$$

This amounts to saying that the subcomplex $\tilde{m}_j \cap \bigcup_{i=1}^{j-1} \tilde{m}_i$ is a pure complex of dimension $d-1$. The distance between two maximum simplices m, m' is the length k of a shortest simplicial path $m = m_0, m_1, \ldots, m_k = m'$ where the m_i's are maximum simplices such that $m_i \cap m_{i-1}$ is a $(d-1)$-simplex for $i = 1, \ldots, k$; if no such path exists, then the distance between m, m' is ∞. The *diameter* of Δ, diam Δ, is the maximum distance between any two maximal simplices of Δ. One says that Δ satisfies the *Hirsch conjecture* if diam $\Delta \leq n - d + 1$ holds.

It is well known that shellable complexes share many combinatorial and topological properties. For instance, an r-dimensional shellable complex has the homotopy type of a wedge of r-spheres (Theorem 1.3, [4]), its reduced homology is known: it vanishes in all dimensions other than r (Proposition 3.10, [2]) and some naturally associated commutative ring is Cohen–Macaulay (for more details, see [3, 4] and references mentioned there).

To any bouquet of geometric lattices P are naturally associated two simplicial complexes: its order complex $\Delta(P)$ and the complex $\mathcal{J}(P)$ of its independent sets of atoms. Similarly, for a bouquet of matroids $\mathcal{G}$, one considers respectively the complex of chains of flats of $\mathcal{G}$, also called its *flat complex* and denoted by FL($\mathcal{G}$); and its independence system $\mathcal{J}$, also called *independence complex*. When FL($\mathcal{G}$) is shellable, we also say that $\mathcal{G}$ is shellable. Note that, as was done by Björner for matroids ([2]), one may associate other complexes to a bouquet such as its broken circuit complex; this will be the object of further study in [18]. It is known that when P is a geometric lattice, then both $\Delta(P)$ and $\mathcal{J}(P)$ are shellable ([2, 20]); this result was extended to geometric semilattices in [23]. Therefore, it follows from Theorem 3.5 that all well cut transversal geometries are shellable. We now study the shellability of general bouquets. The results presented here come from [18] which will also contain other results of topological nature. Let us mention an application of shellable IS to the study of tight bounds for their reliability polynomials ([5]).

It is obviously not true that any bouquet of matroids is shellable; for a counterexample, consider a bouquet whose branches are matroids on disjoint groundsets. In fact, a shellable bouquet must satisfy strong connectivity properties; so, its basis graph must be connected. We will see that, for the case of bouquets of matroids with the 2-union property, this condition is indeed a sufficient condition for shellability. Note that a shellable bouquet must be well cut, which amounts to saying that its IS must be pure.

Proposition 8.2. *Let $\mathscr{G}$ be a well cut bouquet of m matroids of rank r with roofs $X_1, X_1, \ldots, X_m$. If its flat complex* FL($\mathscr{G}$) *is shellable, then there exists an ordering of the roofs, say $X_1, X_2, \ldots, X_m$, such that:*

$$\textit{for all } j \geq 2, \textit{ there exists } k,\ 1 \leq k < j, \textit{ such that } r(X_k \cap X_j) = r - 1 \qquad \mathbf{(8.3)}$$

Recall that a maximal chain m of FL($\mathscr{G}$) is of the form: $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r$ with F_i being an i-flat and F_r is some roof X_j of $\mathscr{G}$, also called the roof of the chain m; the length of the chain is: $|m| = r$. The following can be easily verified.

Claim 8.4. Let m, m' be two maximal chains of flats of $\mathscr{G}$ with distinct roofs X_i, X_j. Then, $|m \cap m'| = r - 1$ holds if and only if m, m' differ only by their roofs and, then, $r(X_i \cap X_j) = r - 1$ holds.

Proof of Proposition 8.2. Consider a shelling order of the maximal chains of FL($\mathscr{G}$): $m_1, \ldots, m_s$. We deduce from (8.1):

$$\text{for } j \geq 2, \text{ there exists } k,\ 1 \leq k < j, \text{ such that } |m_k \cap m_j| = |m_j| - 1 = r - 1 \qquad \mathbf{(8.5)}$$

Suppose, for instance, that the first chain m_1 has roof X_1. Let $i \geq 2$ be the first index such that m_i has a roof distinct from X_1, say m_i has roof X_2. Then, one deduces from (8.5) and Claim 8.4 that $r(X_1 \cap X_2) = r - 1$. Let $j \geq i + 1$ be the first index such that m_j has neither roof X_1 or X_2, say m_j has roof X_3. One deduces again from (8.5) and Claim 8.4 that $r(X_1 \cap X_3) = r - 1$ or $r(X_2 \cap X_3) = r - 1$. Clearly, after iteration of this process, one obtains an ordering of the roofs satisfying (8.3). □

Definition 8.6. Let $\mathscr{G}$ be a well cut bouquet of m matroids of rank r with roofs $X_1, \ldots, X_m$. Its *roof graph* G_R is the graph with vertex set $[1, m]$ and whose edges are defined as follows: two vertices $i, j \in [1, m]$ are adjacent if and only if $r(X_i \cap X_j) = r - 1$.

Definition 8.7. Let $\mathscr{I}$ be a pure IS of rank r and $\mathscr{B}$ its family of bases. Its *basis graph* G_B is the graph with vertex set $\mathscr{B}$ and whose edges are defined as follows: two bases B, B' are adjacent if and only if $|B \cap B'| = r - 1$.

Two bases B, B' are adjacent in G_B if and only if B' is obtained from B by pivoting (or shifting), i.e. by exchanging exactly one element of B by an element of B'. Hence, the basis graph is connected if and only if any basis can be obtained from any other by a finite sequence of pivots; let us simply recall that pivoting is a fundamental tool in the simplex algorithm for linear programming. For instance, G_B is connected when $\mathcal{J}$ is a matroidal IS. Observe that, for a bouquet $\mathcal{G} \in \mathcal{L}(\mathcal{J})$, its roof graph and its basis graph are closely related; so, for the free bouquet $\mathcal{G} = \mathcal{J}$, both graphs coincide and, in general, they are simultaneously connected, as shows Proposition 8.8. Observe also that the diameter of the IS $\mathcal{J}$ (as simplicial complex) coincides with the diameter, diam G_B, of its basis graph; therefore, for a pure IS of rank r on X, $|X| = n$, saying that it satisfies the Hirsch conjecture amounts to saying that diam $G_B \leqslant n - r$ holds. It is proven in [20] that the Hirsch conjecture holds for matroidal IS; we extend this result to IS with the 2-union property and with connected basis graph in Proposition 8.16.

Proposition 8.8. *Let $\mathcal{J}$ be a pure* IS. *The following assertions are equivalent*:
(i) *the basis graph G_B is connected*
(ii) *the roof graph G_R of any bouquet $\mathcal{G} \in \mathcal{L}(\mathcal{J})$ is connected.*

Proof. The implication (i)$\Rightarrow$(ii) follows from the fact that, if two bases B, B' contained in distinct roofs X_i, X_j are adjacent in G_B, then i, j are adjacent in G_R. Conversely, the implication (ii)$\Rightarrow$(i) follows from the fact that any two bases B, B' contained in roofs X_i, X_j with $r(X_i \cap X_j) = r - 1$ are connected; for this, take a maximal independent subset I of $X_i \cap X_j$, $|I| = r - 1$, $x \in X_i - X_j$ and $y \in X_j - X_i$. Then, from axiom ($I3$), the sets $B_i = I + x$ and $B_j = I + y$ are bases of $\mathcal{J}$ respectively contained in X_i, X_j and they are adjacent in G_B; now one can connect B to B_i in the matroid on X_i and, similarly, B' to B_j and thus B to B'. □

Proposition 8.9. *Let $\mathcal{G}$ be a well cut bouquet of matroids. If its flat complex* FL($\mathcal{G}$) *is shellable, then its roof graph is connected or, equivalently, its basis graph is connected.*

Proof. Let $X_1, \ldots, X_m$ be an ordering of the roofs of $\mathcal{G}$ satisfying (8.3); one verifies by induction on $i \geqslant 2$ that i is connected to 1 in G_R, henceforth implying that G_R is connected. □

Corollary 8.10. *The full d-injection geometry $\mathcal{J}(N_1, \ldots, N_d)$ with $|N_i| = \cdots = |N_d| = n \geqslant 2$ is not shellable for all $d \geqslant 2$.*

Proof. Observe that any two distinct bases B, B' of $\mathcal{J}(N_1, \ldots, N_d)$ are d-injective sets of size n satisfying $|B \cap B'| \leqslant n - 2$. Therefore the basis graph is totally disconnected. □

This fact was observed in [1], Fig. 7.1(b), for the case $d = 2$.

It turns out that, for bouquets of matroids having the 2-union property, the connectivity of the basis graph (or of the roof graph) is enough for ensuring shellability, i.e. the converse of Proposition 8.9 is true. For stating this result, we need another type of poset shellability, introduced in [1], which is favourable for induction proofs. Recall that the length of a poset is the maximum length of the chains of $\Delta(P)$.

Definition 8.11. Let P be a finite ranked poset. A *recursive atom ordering* of P is defined by induction on the length of P as follows:

—if P has length 1, then any atom ordering is a recursive atom ordering

—if P has length greater than 1, a recursive atom ordering of P is an ordering $a_1, a_2, \ldots, a_t$ of the atoms of P satisfying:

for $j \in [1, t]$, the poset $[a_j, \rightarrow)$ admits a recursive atom ordering which begins with the atoms that cover some a_i for $j < i$ **(8.12)**

for $j \in [2, t]$, there exists i, $1 \leq i < j$, such that $a_i \vee a_j$ exists. **(8.13)**

Note that (8.13) is slightly different from axiom (ii) in the original definition of [1], however both definitions coincide for the case of bouquets of geometric lattices that we consider, also it suffices to adjoin a top element to P for obtaining the original definition of [1] for bounded posets. It is proved in [1] that the existence of a recursive atom ordering of P is equivalent to chain lexicographical shellability which implies the shellability of $\Delta(P)$.

Proposition 8.14 (Theorem 7.2, [23]). *Let $\mathcal{G}$ be a geometric semilattice of rank r with* IS *$\mathcal{J}$ and closure operator $\sigma(.)$. Then, any atom ordering that begins with some atoms $F_1 = \sigma(x_1), \ldots, F_r = \sigma(x_r)$ such that the set $\{x_1, \ldots, x_r\}$ is a basis of $\mathcal{J}$ is a recursive atom ordering.*

Proposition 8.15. *Let $\mathcal{J}$ be a pure* IS *of rank r having the* 2*-union property. Let $\mathcal{G}^*$ be the greatest element of $\mathcal{L}(\mathcal{J})$, so $\mathcal{G}^*$ is a bouquet of* 2 *matroids with roofs X_1, X_2. Let $\mathcal{G}$ be a bouquet of $\mathcal{L}(\mathcal{J})$ with closure operator $\sigma(.)$. Assume that the basis graph G_B is connected, then any atom ordering of $\mathcal{G}$ that begins with some atoms $F_1 = \sigma(x_1), \ldots, F_{r-1} = \sigma(x_{r-1})$ such that the set $\{x_1, \ldots, x_{r-1}\}$ is a basis of $X_1 \cap X_2$ is a recursive atom ordering.*

Proof. We prove the theorem by induction on the rank r of the IS $\mathcal{J}$ (or of any $\mathcal{G} \in \mathcal{L}(\mathcal{J})$). Let $r(.)$ denote the rank function of $\mathcal{J}$, then, the rank function of $\mathcal{G}$ or $\mathcal{G}^*$ coincides with $r(.)$ on finite rank sets. By assumption, G_B is connected, i.e. from Proposition 8.8, $r(X_1 \cap X_2) = r - 1$ and the roof graph G_R of $\mathcal{G}$ is connected. We can suppose that $r \geq 2$.

We first verify that one can find $r-1$ atoms of $\mathscr{G}$ as in Theorem 8.15. Since $r(X_1 \cap X_2) = r-1$, take a basis $I = \{x_1, \ldots, x_{r-1}\}$ of $X_1 \cap X_2$, then the flats $F_1 = \sigma(x_1), \ldots, F_{r-1} = \sigma(x_{r-1})$ are atoms of $\mathscr{G}$ as in Theorem 8.15. Observe that the flat $\sigma(I) = \sigma(F_1 \cup \cdots \cup F_{r-1})$ is well defined. Define an atom ordering Ω of $\mathscr{G}$ that begins with the atoms $F_1, \ldots, F_{r-1}$. We prove that Ω is a recursive atom ordering of $\mathscr{G}$.

We show that Ω verifies (8.13). For this, take an atom $F = \sigma(x)$ of $\mathscr{G}$ which is distinct from F_1. If $F = F_i$ with $2 \leq i \leq r-1$, then $\sigma(F_1 \cup F)$ is well defined. Else, F is after all F_i's in the order Ω; by applying axiom ($I4$) to the independent sets $\{x\}$ and I (in the matroid on X_j when $x \in X_j$), we deduce that $\{x, x_i\} \in \mathscr{I}$ for some $1 \leq i \leq r-1$ and thus $\sigma(F \cup F_i) = \sigma(\{x, x_i\})$ is well defined.

We now prove that (8.12) is satisfied. For this, let F be an atom of $\mathscr{G}$. Then, the intervals $[F, \multimap)$ in $\mathscr{G}$ and $\mathscr{G}^*$ are bouquets isomorphic, respectively, to $\mathscr{G} - F$ and $\mathscr{G}^* - F$, of rank $r-1$ and with IS $\mathscr{I} \cdot (X - F)$ (Theorem 6.5). Furthermore, when F is contained in $X_1 \cap X_2$, the interval $[F, \multimap)$ in $\mathscr{G}^*$ is a bouquet of 2 matroids with roofs X_1, X_2 and the IS $\mathscr{I} \cdot (X - F)$ has the 2-union property; note that its basis graph is still connected since $X_1 \cap X_2$ has rank $r-2$ in $[F, \multimap)$. When, for instance, $F \subseteq X_1$ and $F \nsubseteq X_2$, then the interval $[F, \multimap)$ in $\mathscr{G}^*$ is a matroid with roof X_1 and the IS $\mathscr{I} \cdot (X - F)$ is matroidal. Note that the atoms of the interval $[F, \multimap)$ in $\mathscr{G}$ are of the form $G = \sigma(\{x, y\})$ with $y \notin F$ and $\{x, y\} \in \mathscr{I}$. Define the set $B(F)$ of atoms of $[F, \multimap)$ in $\mathscr{G}$ that cover some atom F' of $\mathscr{G}$ which is before F in the order Ω. We show how to construct a recursive atom ordering of the interval $[F, \multimap)$ in $\mathscr{G}$ satisfying (8.12); for this, we distinguish three cases:

Case 1. *$F \subseteq X_1$ and $F \nsubseteq X_2$*. Then, $F = \sigma(x)$ with $x \in X_1 - X_2$ and, since the independent set $I = \{x_1, \ldots, x_{r-1}\}$ is contained in $X_1 \cap X_2$, from axiom ($I3$), $I + x \in \mathscr{I}$. The flats $G_1 = \sigma(\{x, x_1\}), \ldots, G_{r-1} = \sigma(\{x, x_{r-1}\})$ are atoms of $[F, \multimap)$ such that the set $\{x_1, \ldots, x_{r-1}\}$ is a basis of $\mathscr{I} \cdot (X - F)$. Note that $\{G_1, \ldots, G_{r-1}\} \subseteq B(F)$ holds. Consider an atom ordering of $[F, \multimap)$ that begins with the atoms $G_1, \ldots, G_{r-1}$ and then with the remaining atoms of $B(F)$; then, from Proposition 8.14, it is a recursive atom ordering and it satisfies (8.12).

Case 2. *$F \subseteq X_1 \cap X_2$ and $F = F_i$* for $i \in [1, r-1]$. Then the flats $G_k = \sigma(\{x_i, x_k\})$ for $k \in [1, r-1]$, $k \neq i$, are atoms of $[F, \multimap)$ such that the set $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r-1}\}$ is a basis of $X_1 \cap X_2$ in $\mathscr{I} \cdot (X - F)$. Note that $B(F) \subseteq \{G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_{r-1}\}$ holds. Consider an atom ordering of $[F, \multimap)$ that begins with atoms of $B(F)$ and then continues with the remaining atoms of $\{G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_{r-1}\}$; from the induction assumption, this is a recursive atom ordering and it satisfies (8.12).

Case 3. *$F \subseteq X_1 \cap X_2$ and $F \neq F_i$ for $i \in [1, r-1]$*. Let $F = \sigma(x)$, then, by axiom ($I4$) applied to the independent sets $\{x\}$ and $I = \{x_1, \ldots, x_{r-1}\}$, we deduce that

$I - x_i + x \in \mathcal{J}$ for some $i \in [1, r-1]$. The sets $G_k = \sigma(\{x, x_k\})$ for $k \in [1, r-1]$, $k \neq i$, are atoms of $[F, \multimap)$ such that the set $I - x_i$ is a basis of $X_1 \cap X_2$ in $\mathcal{J} \cdot (X - F)$ and $\{G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_{r-1}\} \subseteq B(F)$ holds. One obtains a recursive atom ordering of $[F, \multimap)$ satisfying (8.12) by putting first the atoms $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_{r-1}$, then the remaining atoms of $B(F)$ and finally all other atoms. □

Proposition 8.16. *Let $\mathcal{J}$ be a pure* IS *with the* 2*-union property and whose basis graph is connected. Then the* IS *$\mathcal{J}$ satisfies the Hirsch conjecture.*

Proof. For two distinct bases B, B' of $\mathcal{J}$, we denote by $d(B, B')$ the distance between B, B' in the basis graph; by assumption, it is finite. We first observe that, if B, B' are bases in a matroidal IS of rank r, then it follows from the basis exchange axiom (B) that:

$$d(B, B') = r - |B \cap B'|. \tag{8.17}$$

Consider now a pure IS $\mathcal{J}$ of rank r on X, $|X| = n$, with the 2-union property and connected basis graph. Hence, the greatest element $\mathcal{G}^*$ of $\mathcal{L}(\mathcal{J})$ is a bouquet of two matroids with roofs X_1, X_2 such that $r(X_1 \cap X_2) = r - 1$. Let B, B' be two distinct bases of $\mathcal{J}$. If $B, B' \subseteq X_i$ for $i = 1$ or 2, then, from (8.17), $d(B, B') = r - |B \cap B'| \leqslant n - r$. We now suppose that $B \subseteq X_1$, $B' \subseteq X_2$ and consider elements $x \in B - X_2$, $x' \in B' - X_1$. Let I be a maximal independent set such that: $B \cap B' \subseteq I \subseteq X_1 \cap X_2$; then, $|I| = r - 1$ and, from axiom $(I3)$, the sets $B_1 = I + x$ and $B_2 = I + x'$ are bases of $\mathcal{J}$ respectively contained in X_1, X_2. From (8.17), we have that: $d(B, B_1) = r - |B \cap B_1| = r - 1 - |I \cap B|$ and $d(B', B_2) = r - |B' \cap B_2| = r - 1 - |I \cap B'|$. Using the relations: $d(B, B') \leqslant d(B, B_1) + d(B_1, B_2) + d(B_2, B')$ and $d(B_1, B_2) = 1$, we deduce that:

$$d(B, B') \leqslant 2r - 1 - |I \cap B| - |I \cap B'|, \tag{8.18}$$

For completing the proof, we show that the right hand side of (8.18) is less or equal to $n - r$. For this, observe that: $n \geqslant |B \cup B' \cup I|$ and $|B \cup B' \cup I| = |B \cup B'| + |I| - |(B \cap I) \cup (B' \cap I)| = 3r - 1 - |B \cap I| - |B' \cap I|$, which therefore implies that: $n - r \geqslant 2r - 1 - |B \cap I| - |B' \cap I|$; this concludes the proof. □

The next theorem follows from Propositions 8.8, 8.9, 8.15 and 8.16.

Theorem 8.19. *Let $\mathcal{J}$ be a pure* IS *of rank r having the* 2*-union property, $\mathcal{G}^*$ be the greatest element of $\mathcal{L}(\mathcal{J})$ with roofs X_1, X_2 and $\mathcal{G}$ be an arbitrary bouquet of $\mathcal{L}(\mathcal{J})$. The following assertions are equivalent*:

(i) *the basis graph is connected*

(ii) $r(X_1 \cap X_2) = r - 1$

(iii) *the roof graph of $\mathcal{G}$ is connected*

(iv) FL($\mathscr{G}$) *is shellable*
(v) FL($\mathscr{J}$) *is shellable*
(vi) $\mathscr{J}$ *satisfies the Hirsch conjecture.*

The shellability of the flat complex FL($\mathscr{G}$) of a bouquet of matroids $\mathscr{G}$ seems therefore to be an intrinsic property of its IS $\mathscr{J}$, i.e. to depend only on properties of $\mathscr{J}$ and not on the flat configuration of the specific bouquet $\mathscr{G} \in \mathscr{L}(\mathscr{J})$. This is indeed the case for geometric semilattices and bouquets with the 2-union property for which a sufficient and necessary condition for shellability is the connectivity of the basis graph. We conjecure that this is still the case for general bouquets – at least when $\mathscr{L}(\mathscr{J})$ is a lattice – so, we conjecture that a bouquet is shellable if and only if the flat complex of its IS is shellable. We address the related open question of finding a necessary and sufficient condition for the shellability of FL($\mathscr{J}$), or $\mathscr{G} \in \mathscr{L}(\mathscr{J})$.

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