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Endogenous Leverage: VaR and Beyond.

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Abstract

We study endogenous leverage in a general equilibrium model with incomplete markets. We prove that in any binary tree leverage emerges in equilibrium at the maximum level such that *VaR=0*, so there is no default in equilibrium, provided that agents get no utility from holding the collateral. When the collateral does affect utility (as with housing) or when agents have sufficiently heterogenous beliefs over three or more states, *VaR=0* fails to hold in equilibrium. We study commonly used examples: an economy in which investors have heterogenous beliefs and a CAPM economy consisting of investors with different risk aversion. We find two main departures from *VaR=0*. First, both examples show that with enough heterogeneity among the investors, equilibrium default is normal. Second, we find that more than one contract is actively traded in equilibrium on the same collateral, that is, the same asset is bought at different margin requirements by different agents. Finally, we study the relationship between leverage and asset prices. We provide an example that shows that as the regulatory authority gradually relaxes leverage restrictions from low levels and permits leverage to rise, asset prices start to rise, but eventually increased leverage paradoxically tends to reduce asset prices because the risky bonds become substitutes for the asset used as collateral.

Keywords: Endogenous Leverage, Collateral Equilibrium, VaR, Asset Prices.
JEL Codes: D52, D53, E44, G01, G11, G12

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1 Introduction

The recent economic turmoil has brought to the forefront the impact of leverage on financial system stability. The crisis might well be understood as the bottom of a *Leverage Cycle* in which leverage and asset prices crashed. It was preceded by years in which asset prices and the amount of leverage in the financial system increased dramatically. A new theoretical literature has begun to explain how leverage is determined in financial markets and how it affects asset prices, but the results so far have been quite special. The main goal of this paper is to extend the literature to more general and more realistic situations.

To attack the leverage endogeneity problem we follow the techniques developed by Geanakoplos (1997). Agents have access to a menu of contracts, each of them characterized by a non-contingent promise of size j and one unit of asset as collateral to back the promise. When an investor sells a contract j she is borrowing money and putting up collateral, when she is buying a contract j, she is lending money. In equilibium each contract j will have a price π_j, and the collateral will have a price p. The equilibrium loan-to-value of contract j thus emerges endogenously as π_j/p. The key is that even if all contracts are priced in equilibrium, because collateral is scarce, only a few will be actively traded. In this sense, leverage becomes endogenous.

Geanakoplos (2003, 2010) and Fostel-Geanakoplos (2010) studied a special class of economies with a continuum of risk neutral agents with common discount but heterogenous priors and only two states of nature. These papers found that out of the whole menu of contracts only one would ever be actively traded in equilibrium, namely the contract that made the maximum promise without ever defaulting, which we now call the $VaR = 0$ contract. Many papers in the more recent leverage literature arbitrarily assume that margins are set so that the chance of default is, say, 5%, or 0% (*VaR* equals to 5%, or 0%). But which should it be? Is *VaR* a useful variable in describing endogenous leverage?

In this paper we study general conditions under which $VaR{=}0$ arises as the only contract traded in equilibrium. We prove that in an economy with just two states of the world (or more generally, two successor nodes to every node in the date-event tree), and an arbitrary number of traders, collateral equilibrium always rules out default as long as the collateral does not directly affect the utility of any agent. Risk neutrality, common discounting, and the continuum of agents are not required for the conclusion and play no role in the theorem.

The theorem shows that there is a tremendous difference between physical collateral (like houses and cars) that generate contemporaneous utility and financial collateral that gives utility only through dividends or other cash flows that appear later. Our theorem might explain why there are some markets (like for mortgages) in which defaults are to be expected while in others (like Repos) margins are set so strictly that default is almost ruled out. We provide two examples, commonly used in the literature, to illustrate the proposition. The first involves an economy with heterogenous beliefs and the second is a CAPM economy with differences in risk aversion.

Next we explore equilibrium leverage in the general case with many investors and many states of nature. We extend the previous examples to an economy with three successor states. We find two main departures from the previous case. First, both examples show that with enough heterogeneity among the investors, equilibrium default is then normal, even if the collateral does not affect utility. Second, we find that more than one contract is actively traded in equilibrium on the same collateral, that is, the asset is bought at different loan to values by different agents. This is precisely what we observe in the real world: prime borrowers typically bought houses with high down-payments and low interest rates while subprime borrowers were putting almost no money down but paying a high interest rate on exactly the same kinds of houses. In light of this reversal when going from two states to many states, we must interpret the condition that there are just two branches at every node of the time-event tree. Two nodes suggest a world with very short maturity loans and no jumps in collateral values (Brownian motion can be approximated by binary trees with short intervals).

Finally, we study the relationship between leverage and asset prices. Geanakoplos (2003, 2010) argued that higher leverage creates higher asset prices. As the regulatory authority relaxes leverage limits, the most eager buyers can get access to more cash and so spend more money on the asset, driving up its price. Fostel-Geanakoplos (2008) gave a deeper reason why more leverage should drive up asset prices. They demonstrated that the price of any asset can be decomposed into two parts: its payoff value and its collateral value. The payoff value reflects the asset owner's valuation of the future stream of payments, i.e. it is the value attached to the asset due to its investment role. However, assets can also be used as collateral to borrow money. The collateral value reflects the asset owner's valuation of this second role. It will be positive when collateral is scarce. This decomposition into payoff value and collateral value is extremely important because it shows very clearly how financial models with leverage imply definite departures from fundamental asset valuation, in the form of deviations from the law of one price and from the efficient markets hypothesis. It explains why assets that can be used as collateral will sometimes trade for higher prices than identical assets that cannot. An asset that can be used to make a big promise will have greater collateral value than an asset that cannot. The formal analysis in Geanakoplos (2003, 2010) and Fostel-Geanakoplos (2008) was all in the context of binary trees for which $VaR = 0$ emerges endogenously. Subsequent papers linking leverage and asset prices, such as Brunnermeier and Pedersen (2009) and Garleanu and Pedersen (2009) assume $VaR = 0$.

In this paper we consider economies with more than two states of nature and we allow leverage to emerge endogenously in equilibrium. We gradually relax the leverage limits (that is an exogenously imposed ceiling on j that can be traded) and investigate how the asset prices change. We find that when the ceiling on leverage is high enough to allow a lot of default in equilibrium, further increases in leverage paradoxically tend to reduce asset prices when investors have different priors.

More precisely, the example shows that as leverage is allowed to increase from lower levels without default, asset prices increase. This is consistent with the previous literature with *VaR= 0*. But as leverage is allowed to increase to the point where default emerges in equilibrium, the span of the bond payoffs will change, giving investors more alternatives in which to invest their money. The effect of leverage on asset prices can be reversed, and asset prices can go down with more leverage. The intuition is that as leverage increases the bonds default more and grow riskier, coming to be better and better substitutes for the underlying asset. Some investors eventually switch out of the asset and buy the bonds instead, driving down the asset price.

The paper is related to a literature on collateral and credit constraints as in Bernanke, Gertler and Gilchrist (1999), Caballero and Krishnamurthy (2001), Fostel and Geanakoplos (2008a), Holmstrom and Tirole (1997), Kiyotaki and Moore (1997) and Shleifer and Vishny (1992). More closely, our paper is related to a literature on leverage as in Araujo, Kubler and Schommer (2009), Acharya and Viswanathan (2009), Adrian and Shin (2009), Brunnermeier and Pedersen (2009), Cao (2010), Fostel and Geanakoplos (2008b and 2010), Geanakoplos (1997, 2003 and 2010), Gromb and Vayanos (2002) and Simsek (2010).

It is also related to work that studies the asset price implications of leverage as Hindy (1994), Hindy and Huang (1995) and Garleanu and Pedersen (2009).

Some of these papers focus on *investor-based leverage* as in Acharya and Viswanathan (2009), Adrian and Shin (2009) and Gromb and Vayanos (2002), and others such as Brunnermeier and Pedersen (2009), Cao (2010), Fostel and Geanakoplos (2008b and 2010), Geanakoplos (1997, 2003 and 2009) and Simsek (2010) focus on *asset-based leverage*. Not all these models present a theory of endogenous leverage; most of them assume a *VAR=0* rule and study the cyclical properties of leverage as well as its asset pricing implications. In Acharya and Viswanathan (2009) and Adrian and Shin (2009) the endogeneity of leverage relies on asymmetric information and moral hazard problems between lenders and borrowers. Asymmetric information is important in many loan markets, especially those for which the borrower is also a manager who exercises control over the value of the collateral. The recent crisis, however, was centered not in the corporate bond world, where managerial control is central, but in the mortgage securities market, where the owner/borrrower generally has no control or specialized knowledge over the cash flows of the collateral.

In Araujo, Kubler and Schommer (2009), Cao (2010), Geanakoplos (1997, 2003, 2009), Fostel-Geanakoplos (2008, 2010) and Simsek (2010) endogeneity does not rely on asymmetric information, rather financial contracts are micro founded by a collateralized loan market. Geanakoplos (1997) showed how to make leverage endogenous by defining a contract as an ordered pair (promise, collateral) and requiring that every contract be priced in equilibrium, even if it is not actively traded. Geanakoplos (1997, 2003, 2010) and Fostel-Geanakoplos (2008), and Cao (2010) show that only the *VaR= 0* contract is traded in a concrete example with heterogenous priors. On the other hand, in the setting of Geanakoplos 1997, with two states of the world but where agents derived utility from the collateral (like living in the house), it turned out that equilibrium leverage was high enough that there was anticipated default. Geanakoplos (2003) gives an example with a continuum of risk neutral investors with different priors and three states of nature in which the only contract traded in equilibrium involved default. Simsek (2010) showed an example with two types of risk-averse investors and a continuum of states of nature with equilibrium default. Araujo et.al (2009) provided a two period example of an asset which is used as collateral in two different actively traded contracts when agents have utility over the asset.

The paper is organized as follows. Section 2 presents the general model of endogenous leverage. Section 3 presents the main theorem which shows conditions under which only the *VaR=0* contract is traded and two examples. Section 4 extends the previous examples and shows deviations from the theorem. Section 5 studies the relationship between asset prices and leverage.

2 A General Equilibrium Model of Endogenous Leverage

The model is a two-period general equilibrium model, with time $t = 0, 1$. Uncertainty is represented by the existence of different states of nature $s \in S$ including a root $s = 0$. Finally, we denote the time of s as $t(s)$, so $t(0) = 0$ and $t(s) = 1, \forall s \in S_T$, the set of terminal nodes of S. Suppose there is a single storable consumption good c and one asset Y which pays dividends d_s in each final state $s \in S_T$.

2.1 Investors

Each investor $h \in H$ is characterized by a utility, u^h, a discounting factor, δ^h, and subjective probabilities, q^h_s, $s \in S_T$. We assume that the Bernoulli utility function for consumption in each state $s \in S$, $u^h : R_+ \to R$, is differentiable, concave, and monotonic. The von-Neumann-Morgenstern expected utility to agent h is

$$U^h = u^h(c_0) + \delta^h \sum_{s \in S_T} q^h_s u^h(c_s) \quad (1)$$

Investor h's endowment of the consumption good is denoted by $e_s^h \in R_+$ in each state $s \in S$. His endowment of the only asset at time 0 is $a^h \in R_+$. We assume that the consumption good is present, $\sum_{h\in H} e_0^h > 0, \sum_{h\in H}(e_s^h + d_s a_s^h) > 0, \forall s \in S_T$.

Observe that we do not allow utility to depend on the asset. This assumption will play a crucial role in proving our main result, that with two states of nature, equilibrium can always be taken to be without default.

Every agent has direct access to an inter-period constant-returns-to-scale warehousing technology. This is a simple way to model consumption good durability in the economy. A unit of consumption warehoused in state 0 yields one unit of consumption in all final states. There is no depreciation. This warehousing greatly simplifies the computation of equilibrium in our examples by fixing the riskless interest rate at zero.

2.2 Financial Contracts and Collateral

We take the consumption good as numeraire and denote the price of Y at time 0 as p. At time 0 agents can trade financial contracts. A financial *contract* (A, C) consists of both a promise, A, and collateral backing it, C. Collateral consists of durable goods, which will be called assets. The lender has the right to seize as much of the collateral as will make him whole once the loan comes due, but no more. Since there is only one asset, we shall always take C to be one unit of Y.

A non-contingent contract j is of the form $(j \cdot \widetilde{1}, 1)$, where $\widetilde{1} \in R^{S_T}$ stands for the vector of ones with dimension equal the number of final states. Contract j promises j units of consumption good in each final state and the promise is backed by one unit of asset Y. Let J be the set of all contracts that use one unit of asset Y as collateral.

The price of contract j is π^j. An investor can borrow π^j today by selling contract j in exchange for a promise of j tomorrow. Since the maximum a borrower can lose is his collateral if he does not honor his promise, the actual delivery of contract j in state $s \in S_T$ is $min\{j, d_s\}$. If the collateral is so big that $j \leq d_s, \forall s \in S_T$, then the contract will not default. In this case its price defines a riskless rate of interest $(1 + r^j) = 1/\pi^j$.

The *Loan to Value* (LTV) associated to contract j is given by

$$LTV^j = \frac{\pi^j}{p} \tag{2}$$

The margin requirement m^j associated to contract j is $1 - LTV^j$, and the *leverage* associated to contract j is the inverse of the margin, $1/m^j$.

We define the *asset loan to value*, LTV for asset Y, as the trade-value weighted average of LTV^j across all contracts actively traded in equilibrium.

2.3 Budget Set

Given the asset and contract prices $(p, (\pi^j)_{j \in J})$, each agent $h \in H$ decides consumption in each state, c_s, and at time 0, warehousing, w, asset holding, y, contract sales (borrowing), $\varphi_j > 0$, or purchases (lending), $\varphi_j < 0$, in order to maximize utility (3) subject to the budget set defined by

$$B^h(p, \pi) = \{(c, w, y, \varphi) \in R_+^{S+1} \times R_+ \times R_+ \times R^J :$$

$$(c_0 + w - e_0^h) + p(y - a^h) \leq \textstyle\sum_{j \in J} \varphi_j \pi^j$$

$$(c_s - e_s^h - w) \leq y d_s - \textstyle\sum_{j \in J} \varphi_j min(j, d_s), \forall s \in S_T$$

$$\textstyle\sum_{j \in J} max(0, \varphi_j) \leq y\}$$

At time 0 expenditures on consumption and warehousing minus endowments, plus total net expenditures on the asset, can be at most equal to the money borrowed selling contracts using the asset as collateral. In the final period, at each state s expenditure on consumption net of initial endowments and warehousing can be at most equal to the dividend payment minus debt repayment. Finally, those agents who borrow must hold the required collateral at time 0.

First, notice that there is no sign constraint on φ_j; a positive (negative) φ_j indicates the agent is selling (buying) contracts or borrowing (lending) π^j. Second, notice that we are assuming that short selling of assets is not possible. This assumption is crucial for the results in the paper, since if we were to allow short selling in the examples we work in the paper, markets would be complete. Though a crucial assumption, we do not think it is an implausible one. It is impossible to short sell many assets in the real world, though the CDS market is beginning to change that. In Fostel-Geanakoplos (2011) we investigate the significance of CDS for asset pricing.

The set H of agents can be taken as finite (in which case we really have in mind a continuum of agents of each of the types), or we might think of $H = [0, 1]$ as a continuum of distinct agents, in which case we must think of all the agent characteristics as measurable functions of h. In the latter case we must think of the summation $\sum$ over agents in the next section as an integral over agents, and all the optimization conditions as holding with Lebesgue measure one.

2.4 Collateral Equilibrium

A *Collateral Equilibrium* in this economy is a vector of asset price and contract prices, individual consumption and warehousing, asset holding, and contract trades $((p,\pi),(c^h,w^h,y^h,\varphi^h)_{h\in H}) \in (R_+ \times R_+^J) \times (R_+^{S+1} \times R_+ \times R_+ \times R^J)^H$ such that

1. $\sum_{h\in H}(c_0^h + w^h - e_0^h) = 0$
2. $\sum_{h\in H}(c_s^h - w^h - e_s^h) = \sum_{h\in H} y^h d_s, \forall s \in S$
3. $\sum_{h\in H}(y^h - a^h) = 0$
4. $\sum_{h\in H} \varphi_j^h = 0, \forall j \in J$
5. $(c^h, w^h, y^h, \varphi_j^h) \in B^h(p,\pi), \forall h$
   $(c,w,y,\varphi) \in B^h(p,\pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h$

Markets for the consumption good in all states clear, assets and promises clear in equilibrium at time 0, and agents optimize their utility in their budget sets. As shown by Geanakoplos and Zame (1997), equilibrium in this model always exists under the assumption we have made so far.

3 When *VaR=0* Emerges Endogenously

3.1 Main Result

In this section we study conditions under which *VaR=0* arises endogenously in equilibrium: among all possible contracts, the only one actively traded in equilibrium is the one that promises the worst case scenario in the future. In this way, default never occurs in equilibrium, even though it could. We call this the maxmin contract because then the promise is equal to $j^* = \max\{j : j \leq \min_s d_s\}$.

Consider the situation in which $S = \{0, U, D\}$. Asset Y pays d_U units of the consumption good in state $s = U$ and $d_D < d_U$ in state $s = D$. Figure 1 depicts the asset payoff. One might imagine that some agents may value the asset much more than others, say because they attach very high probability q_U^h to the up state, or because they are more risk tolerant, or because they put a high value δ^h on the future. These agents might be expected to want to borrow a lot, promising $j > d_D$ so as to get their hands on more money to buy more assets at time 0. Indeed it is true that for $d_U > j > j^* = d_D$, any agent can raise more money $\pi^j > \pi^{j^*}$ by selling asset j rather than j^*. Nonetheless, the following result holds

Theorem:

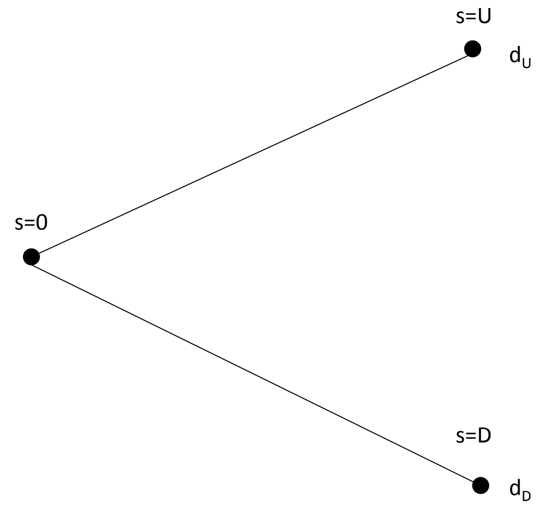

Figure 1: Asset payoff description.

Suppose that $S = \{0, U, D\}$ and that the max min contract $j^* = d_D < d_U$ is available to be traded, that is $j^* \in J$. Then given any equilibrium $((p, \pi), (c^h, w^h, y^h, \varphi^h)_{h \in H})$, we can find another equilibrium $((p, \pi), (c^h, w^h, \bar{y}^h, \bar{\varphi}^h)_{h \in H})$ with the same asset and contract prices and the same consumptions, in which j^* is the only contract traded, $\bar{\varphi}^h_j = 0$ if $j \neq j^*$.

Proof:

Note first that if $d_D = 0$ then the promises only deliver in U and they are perfect substitutes for the asset, so there is no point in trading them. Sellers of the contracts could simply hold less of the asset and reduce their borrowing to zero while buyers of the contracts could buy the asset instead, giving another equilibrium as claimed. So we might as well assume $0 < j^* = d_D < d_U$. The proof is organized into a series of claims and their proofs.

1. Let $a = \frac{p - \pi^{j^*}}{d_U - j^*}$ and $b = \pi^{j^*}/j^* - a$. Then $\pi^{j^*} = aj^* + bj^*$ and $p = ad_U + bd_D$.

$$aj^* + bj^* = aj^* + (\pi^{j^*}/j^* - a)j^* = \pi^{j^*}$$

Using the definitions of π^{j^*}, a and j^*

$$
\begin{aligned}
ad_U + bd_D - \pi^{j^*} &= a(d_U - j^*) + b(d_D - j^*) = (p - \pi^{j^*}) + 0 \\
ad_U + bd_D &= p
\end{aligned}
$$

2. Suppose j with $d_U > j > j^* = d_D$ is traded in equilibrium. Then $\pi^j = aj + bj^*$.

   Contract j pays fully in the up state, but defaults and pays only $j^* = d_D$ in the down state. The *seller* of the contract must have put up the collateral of one unit of the asset, and therefore on net is effectively holding an Arrow security in the U state, paying a price per dollar of

   $$\bar{a} = \frac{p - \pi^j}{d_U - j}$$

   The *seller* of contract j could instead have acquired U Arrow securities by buying the asset while borrowing π^{j^*}, that is making the riskless promise j^*. Hence

   $$\frac{1}{\bar{a}} = \frac{d_U - j}{p - \pi^j} \geq \frac{d_U - j^*}{p - \pi^{j^*}} = \frac{1}{a}$$

   The *buyer* of contract j could have instead bought j^* and bought $(j - j^*)$ U Arrow securities via the risky promise as above, hence it must be that

   $$\pi^j \leq \pi^{j^*} + (j - j^*)\frac{p - \pi^j}{d_U - j}$$

   and hence that

   $$\frac{(j - j^*)}{\pi^j - \pi^{j^*}} \geq \frac{d_U - j}{p - \pi^j}$$

   It follows that all the previous inequalities must be equalities, otherwise we would have[1]

   $$\frac{(j - j^*) + d_U - j}{\pi^j - \pi^{j^*} + p - \pi^j} > \frac{d_U - j^*}{p - \pi^{j^*}}$$

   a contradiction.

   Thus if contract $d_U > j > j^*$ is traded, then

   $$\pi^j = \pi^{j^*} + (j - j^*)a = aj + bj^*$$

3. If contract $j < j^*$ is traded, then $\pi^j/j = \pi^{j^*}/j^*$, hence $\pi^j = aj + bj$.

   If $\pi^j/j > \pi^{j^*}/j^*$, then the buyer of j made a mistake; instead he should have bought j/j^* units of j^*. On the other hand, if $\pi^j/j < \pi^{j^*}/j^*$,then the seller

[1]We make use of the arithmetic property that if $a, b, c, d > 0$, and $\frac{a}{b} > \frac{c}{d}$ then $\frac{a+c}{b+d} > \frac{c}{d}$.

of j made a mistake; instead he should have sold $j/j^* < 1$ units of j^*. Since $j < j^*$, this uses less collateral than is available and so is feasible, and brings more money, a contradiction.

4. The portfolio of assets and contracts that any agent h holds in equilibrium delivers (z_U^h, z_D^h), with $z_U^h \geq z_D^h \geq 0$ and costs $az_U^h + bz_D^h$.

   Any held portfolio payoff (z_U, z_D) is the sum of payoffs from individual transactions. The possible transactions include precisely contracts $j > j^*, j = j^*, j < j^*$, the asset, and the asset bought on margin by selling some security j. In every such transaction we have shown that the net payoff is a vector (r_U, r_D), with $r_U \geq r_D$ and costs $ar_U + br_D$. The conclusion follows from addition and the distributive law of arithmetic.

5. Define

$$\bar{y}^h = \frac{z_U^h - z_D^h}{d_U - d_D}$$
$$\bar{\varphi}_{j^*}^h = [\bar{y}^h d_D - z_D^h]/j^* = \bar{y}^h - z_D^h/j^*$$

   Then if y^h is replaced by $\bar{y}^h$ and φ_j^h is replaced by 0 for $j \neq j^*$ and by $\bar{\varphi}_{j^*}^h$ for $j = j^*$, and all prices and other individual choices are left the same, then we still have an equilibrium.

   Note first that $\bar{\varphi}_{j^*}^h \leq \bar{y}^h$, so this portfolio choice satisfies the collateral constraint. Observe from the second equation above that

$$\bar{y}^h d_D - \bar{\varphi}_{j^*}^h j^* = z_D^h$$

   and so

$$\bar{y}^h d_U - \bar{\varphi}_{j^*}^h j^* = \bar{y}^h d_U - \bar{y}^h d_D + z_D^h$$

   From the very first equation defining $\bar{y}^h$,

$$\bar{y}^h (d_U - d_D) + z_D^h = (z_U^h - z_D^h) + z_D^h = z_U^h$$

   Hence the portfolio choice $(\bar{y}^h, \bar{\varphi}_{j^*}^h)$ gives the same payoff (z_U^h, z_D^h). From the previous observations it must have the same cost as well. Hence every agent is optimizing. Summing over individuals we must get

$$\sum_h \bar{y}^h (d_U, d_D) - \sum_h \bar{\varphi}_{j^*}^h (j^*, j^*) =$$

$$\sum_h (z_U^h, z_D^h) = \sum_h y^h (d_U, d_D)$$

   where the last equality follows from the fact that $\sum_h \varphi_j^h = 0$ in the original equilibrium for each contract j. By the linear independence of the vectors

(d_U, d_D) and (j^*, j^*) we deduce that

$$\sum_h \bar{y}^h = \sum_h y^h$$
$$\sum_h \bar{\varphi}^h_{j^*} = 0$$

proving the theorem.■

As discussed before, *leverage* is endogenously determined in equilibrium. In particular, the proposition *derives* the conclusion that although all contracts will be priced in equilibrium, the only contract actively traded is the maxmin contract, which corresponds to the Value at Risk equal zero rule *assumed* by many other papers in the literature.

Geanakoplos (2003) stated a slightly stronger theorem (that equilibrium is also unique) in the special case where the set of agents was taken to be the continuum $H = [0, 1]$ and every agent was risk neutral and did not discount the futute, and q^h_U was taken to be continuous and strictly monotonic in h. Fostel-Geanakoplos (2010) formally proved that theorem and generalized it to all binary trees. The theorem in this paper is more general in that it does not depend on a continuum of agents and continuity of preferences across agents, or on identical discount rates, or on risk neutrality, or on any assumption about endowments (for example it does not assume that agent endowments in terminal periods are spanned by the asset). It includes the case where there is a finite number of agent types.

Note also that the theorem does not presume that markets are complete. Indeed, in the examples below we see that equilibrium is different from the Arrow-Debreu equilibrium. It does not say that equilibrium is unique, only that each equilibrium can be replaced by another with the same asset price and the same consumption by each agent, in which there is no default. The theorem could easily be extended to multiple periods and multiple assets provided that every asset took on at most two total values (capital value plus dividend value) immediately following each state; such a situation necessarily obtains if the tree is binary.

One key assumption in the theorem is that the tree is binary, so that the maxmin promise plus the U Arrow security (obtained by buying the asset while selling the minmax contract), positively spans the set of feasible portfolio payoffs. To take one example, if an agent wishes to leverage his asset purchases less than the maxmin, he can always leverage some of his holdings to the maxmin, and the others not at all.

The idea of the proof is that in equilibrium each agent must be indifferent to replacing his portfolio with another in which on each unit of collateral that he holds, he either leverages to the maximum amount without risk of default, or does not leverage at all. If every agent switches to this *VaR = 0* leverage, markets must still clear. From a simple spanning argument it is clear that the new portfolio each agent holds gives the same payoffs the next period. But it is important to realize that this new portfolio may involve each agent holding a different amount of the collateral asset. Agents are indifferent to switching to the new portfolio because of the assumption that the asset does not directly enter any agent's utility (but only indirectly through its payoffs). If the collateral were housing, for example, the theorem would not hold; it might well be that even with only two states agents would leverage in equilibrium to the point where they would default in one of the states (as shown in an example in Geanakoplos (1997, 2010)).

The theorem has a sort of Modigliani-Miller feel to it. But the theorem does *not* assert that the debt-equity ratio is irrelevant. Agents are indifferent to leveraging as in their equilibrium portfolio, or the point where *VaR = 0*. But they may not be indifferent to other levels of leverage. We shall shortly give an example with a unique equilibrium in which every borrower leverages to the *VaR = 0* point, but no agent would be indifferent to leveraging any less.

After illustrating the theorem with some examples, we go on to analyze the situation when there are more than two states. We find that in equilibrium there may be active default, and that different agents may make different promises on the same collateral.

3.2 Examples

Now we provide two examples, extensively used in the financial literature, to illustrate the theorem: i) heterogenous beliefs and ii) CAPM investors with differences in risk aversion.

3.2.1 Heterogenous Beliefs

Agents differ only in their subjective probabilities. There is a continuum of heterogenous agents indexed by $h \in H = [0, 1]$. The only source of heterogeneity is in subjective probabilities, $q_U^h = h$, so the higher the h the more optimistic the agent is with respect to the future.

Agents are risk neutral and do not discount the future. They start at $t = 0$ with an endowment of 1 unit of the consumption good and 1 unit of the asset. More formally, $U^h = \sum_{s \in S} q_s^h c_s$, $e_0^h = 1$ and $e_s^h = 0, s \neq 0$, and $a^h = 1, \forall h$.

Let us describe the system of equations that characterizes the equilibrium. Because of linear utilities and the continuity of utility in h and the connectedness of the set of agents $H = [0, 1]$, at state $s = 0$ there will be a *marginal buyer*, h_0, who will be indifferent between buying or selling Y. All agents $h > h_0$ will buy all they can afford of Y, i.e., they will sell all their endowment of the consumption good and borrow to the max using Y as collateral. On the other hand, agents $h < h_0$ will sell all their endowment of Y and lend to the more optimistic investors. The risk-less interest rate is zero.

At $s = 0$ aggregate revenue from sales of the asset is given by $p \times 1$.[2] On the other hand, aggregate expenditure on the asset is given by $(1 - h_0)(1 + p) + d_D$. The first term is total income (endowment plus revenues from asset sales) of buyers $h \in [h_0, 1]$. The second term is borrowing, which from the theorem is d_D (recall that the interest rate is zero). Equating we have

$$p = (1 - h_0)(1 + p) + d_D \tag{3}$$

The next equation states that the price at $s = 0$ is equal to the marginal buyer's valuation of the asset's future payoff.

$$p = q_U^{h_0} d_U + q_D^{h_0} d_D \tag{4}$$

Hence we have a system of two equations and two unknowns: the price of the asset, p, and the marginal buyer, h_0.

We solve the equilibrium for $d_U = 1$ and $d_D = .2$. Table 1 presents the equilibrium values for the asset price and marginal buyer. By the theorem, every agent who leverages chooses to sell the same contract using the asset as collateral, hence asset leverage and contract leverage are the same and described in the table. It is easy to check that this is a genuine equilibrium, this is, everybody is maximizing and markets clear.

Note that in this example equilibrium is unique (as shown in Fostel-Geanakoplos (2010)). Leverage is uniquely pinned down by equilibrium. There is no other equilibrium in which any agent leverages more or less than he does in the above equilibrium. If agents were forced to issue less debt and hold more equity, they would rise in anger: no borrower $h > h_0$ is indifferent to issuing less debt. Those agents are indifferent to leveraging more at the market prices, and then defaulting, but nobody would buy their promises. The point is that collateral is in short supply; there is no excess of it. If any agent were to issue fewer bonds but hold the same collateral, nobody else could step in to take his place and issue the missing bonds.

[2] All asset endowments add to 1 and without loss of generality are put up for sale even by those who buy it.

Table 1: Equilibrium: Heterogenous Beliefs.

Price, p	0.75
Marginal buyer, h_0	0.67
Leverage	
Margin, m	0.73
Leverage, 1/m	1.36
Loan to Value, 1-m	0.27

3.2.2 CAPM with Differences in Risk Aversion

There are two types of agents who agree on the probabilities; the heterogeneity is in their appetite for risk. There are risk tolerant and risk averse agents, $h = T, A$, and both are mean-variance investors. More formally, $U^h = \sum_{s \in S} q_s u^h(c_s)$, where $u^h(c_s) = c_s - \frac{1}{2}\alpha^h c_s^2$, and $\alpha^T < \alpha^A$. Suppose $q_U = .5$ and $\alpha^T = .02$ and $\alpha^A = .1$. As before agents do not discount the future. Suppose agents each own one unit of the asset, $a^h = 1, h = T, A$. Suppose $e_0^T = e_U^T = e_D^T = 1$ and $e_0^A = e_D^A = 1, e_U^A = 4$.

Let us describe how we solve for the equilibrium in this example. We guess a regime and calculate the equilibrium values according to this guess. At the end we check that our guess is correct. We guess that in equilibrium the tolerant agents buy all the asset in the economy and leverage to the max. On the other hand, the risk averse investors sell all their asset, buy all the consumption good and lend to the more tolerant investors. We solve for two variables in this case: the price of the asset, p, and the price of the only contract traded in equilibrium, π. For that we have two equations.

The first equation is the first order condition for lending corresponding to the risk averse investor. The price of the contract equals the payoff d_D of the contract weighted by the probabilities of the states and the marginal utilities tomorrow normalized by the marginal utility of consumption at $t = 0$.

$$\pi = \frac{q_U(1 - \alpha^A c_U^A) d_D + q_D(1 - \alpha^A c_D^A) d_D}{1 - \alpha^A c_0^A} \tag{5}$$

The second equation is the first order condition of the tolerant investor for purchasing the asset via the maxmin leverage.

$$p - \pi = \frac{q_U(1-\alpha^T c_U^T)(d_U - d_D) + q_D(1-\alpha^T c_D^T)(d_D - d_D)}{1-\alpha^T c_0^T} \tag{6}$$

Notice that this is not the usual first order condition for holding the asset. In fact, the relevant first order condition when the investor leverages all his asset holdings is on the *net* position. The marginal utility of the down-payment today has to equal the expected marginal utility of the net payoff (dividend minus delivery) in the future. Given proposition 1, the net payoff in state D is zero, hence buying an asset leveraging via the maxmin contract is equivalent to buying the state U Arrow security. As shown by in Fostel-Geanakoplos (2008) the asset pricing implications of this type of first order condition are very rich, explaining among other things different pricing kernels and deviations from classical laws of one price.

We solve the equilibrium for $d_U = 1$ and $d_D = .2$. Table 2 presents the equilibrium values for the asset and contract price. As before, by the theorem, asset leverage and contract leverage are the same and described in the table. It is easy to check that this is a genuine equilibrium, this is, that our guess was correct.[3]

Table 2: Equilibrium: CAPM.

Asset price, p	0.63
Contract price, π	0.24
Leverage	
Margin, m	0.68
Leverage, 1/m	1.47
Loan to Value, 1-m	0.32

[3] We need to check two things: First that the tolerant investor really wants to leverage to the max, for this to be the case, $\pi > \frac{q_U(1-\alpha^T c_U^T)d_D + q_D(1-\alpha^T c_D^T)d_D}{1-\alpha^T c_0^T}$. Second, we need to check that the averse investor is optimizing not holding the asset, i.e. $p > \frac{q_U(1-\alpha^A c_U^A)d_U + q_D(1-\alpha^A c_D^A)d_U}{1-\alpha^A c_0^A}$. These two conditions are satisfied in the equilibrium. The parameter choice of endowments was made so that this particular regime would be the equilibrium one. Of course, we could have chosen other parameters and the regime would have not be optimal. For example, agents would have liked to share the asset. These would have only complicated the system of equations without adding any conceptual insight. Fostel-Geanakoplos (2008) extensively studies the robustness of all these regimes.

4 Beyond *VaR=0*: Default and Multiple Contracts in Equilibrium.

We extend the previous examples to three states and we find that the theorem fails to hold. Now $S = \{0, U, M, D\}$. Asset Y pays d_s units of the consumption good in state s, where $d_U \geq d_M \geq d_D$. Figure 2 depicts the asset payoffs.

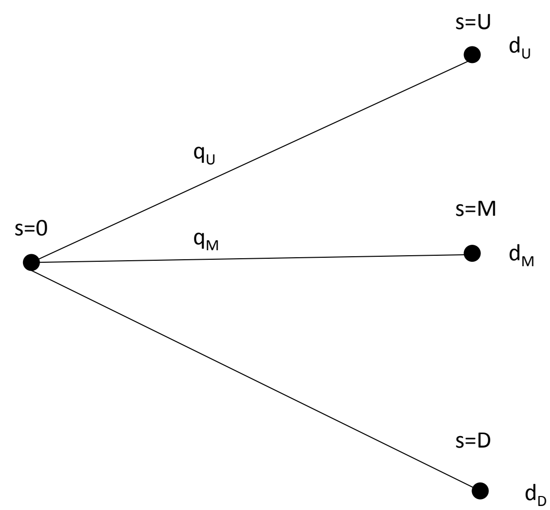

Figure 2: Asset payoff description.

We study a continuum economy with heterogenous beliefs and a two agent economy with different risk aversions and endowments, as in the previous examples. We find that we have both default and multiple contracts traded in equilibrium. This exercise shows that when other papers in the literature *assume VAR=0* contracts, they very likely are not considering genuine equilibria.

4.1 Heterogenous Beliefs: An Example of Multiple Contracts.

We consider first the analogue of the heterogeneous continuum in example 3.2.1, where each agent begins with one unit of the asset and one consumption good at time 0 and no other endowments, but we extend the example to three states of nature. As before, a (possibly different) price must emerge in equilibrium for *each* possible non-contingent promises backed by one unit of the asset as collateral. In example 3.2.1 with two states of nature we found that among all these contracts, just one contract need be traded, and that it involved no default. But with three states, we give robust examples where two contracts will be traded: a risk-less contract as before that promises d_D, the worst-case scenario in the future, and a risky contract that promises d_M in all states but defaults and delivers only d_D in $s = D$.

For concreteness we display the equilibrium for the following probabilities and asset payoffs: $q_U^h = h, q_M^h = h(1-h)$ and $q_D^h = (1-h)^2$, and $d_U = 3, d_M = 2$ and $d_D = 1$. Notice that the higher the h, the more optimisitc the agent. As before we assume no discounting, and we can guess the riskless interest rate will be zero.

Buying an asset on margin using a financial contract defines a down-payment at time 0 and a profile of net payoffs in the future. Two contracts will be traded for the asset, so that in addition to the asset we can think of five securities in total at time 0, three risky and two risk-less:

1. buying Y on margin using the risky bond (the one that promises d_M). In this case the down-payment is $p - \pi^{d_M}$ and the payoffs in the future are given by $d_U - d_M, 0, 0$ in states U, M and D respectively.
2. buying Y on margin using the risk-less bond (which promises d_D). In this case the down-payment is $p - \pi^{d_D} = p - d_D$ and the payoffs in the future are given by $d_U - d_D, d_M - d_D, 0$ in states U, M and D respectively.
3. the risky bond that promises d_M, and is collateralized by one unit of the asset. In this case the price is π^{d_M} and the payoffs in the future are given by d_M, d_M, d_D in states U, M and D respectively.
4. the risk-less bond that promises d_D, and is collateralized by one unit of the asset. In this case the payment is π^{d_D} and the payoffs in the future are given by d_D, d_D, d_D in states U, M and D respectively.
5. cash (that is holding the durable consumption good)

We guess that at $s = 0$, because of linear utilities, there will be a marginal buyer for each of the risky securities above. With this in mind, we need to find the value of 6 variables:

- Asset price: p.
- Risky bond price: π^{d_M}.
- Asset marginal buyers: h_M, h_D where h_M corresponds to the marginal buyer of Y leveraging with the risky bond, and h_D to the marginal buyer of Y leveraging with the risk-less bond.

- Risky bond marginal buyer: h_B.
- Total quantity of assets purchased by leveraging with the risky bond: y

Next, we will guess a regime in order to be able to define a system of equations, and once we get a solution we need to check that the regime is genuine, i.e. all agents are maximizing with those choices. In particular, agents $h > h_M$ buy Y and promise d_M, so they buy Y leveraging with the risky bond. Agents with $h_M > h > h_D$ buy Y and promise d_D, so they leverage using the risk-less bond. Agents $h_D > h > h_B$ sell Y and buy the risky bond (so lend in the risky market collateralized by Y). Finally, agents $h < h_B$ sell everything, hold risk-less securities (so lend in the risk-less markets).

Suppose the top $1 - h_M$ agents altogether buy y units of Y. The system of equations is given by

$$\frac{(d_U - d_M)q_U^{h_M}}{p - \pi^{d_M}} = \frac{(d_U - d_D)q_U^{h_M} + (d_M - d_D)q_M^{h_M}}{p - d_D} \tag{7}$$

$$\frac{(d_U - d_D)q_U^{h_D} + (d_M - d_D)q_M^{h_D}}{p - d_D} = \frac{d_M q_U^{h_D} + d_M q_M^{h_D} + d_D q_D^{h_D}}{\pi^{d_M}} \tag{8}$$

$$\frac{d_M q_U^{h_B} + d_M q_M^{h_B} + d_D q_D^{h_B}}{\pi^{d_M}} = 1 \tag{9}$$

$$1 - h_M + (1 - h_M + y)\pi^{d_M} = py \tag{10}$$

$$h_M - h_D + ((h_M - h_D) + (h_D - y))d_D) = p(h_D - y) \tag{11}$$

$$h_D - h_B + p(h_D - h_B) = (1 - h_M + y)\pi^{d_M} \tag{12}$$

Equation (7) states that the marginal buyer h_M is indifferent between buying the asset leveraging with the risky bond and leveraging the asset with the risk-less bond. Equation (8) states that the marginal buyer h_D is indifferent between buying the asset leveraging with the risk-less bond and buying the risky bond. Equation (9) says that the marginal buyer h_B is indifferent between buying the risky bond and holding risk-less assets like the risk-less bond or cash. Equation (10) says that the total amount of money spent by the top $1 - h_M$ agents on the asset in cash and by leveraging both their endowment of assets and their purchased assets via the risky bond should equal the total revenues from those asset purchases. Equation (11) is the analog but for the risk-less market, noting that the price of the riskless bond is h_D and that all sales of the asset come from the bottom h_D agents, so that purchases via leverage on the risk-less bond must be $h_D - y$. The last equation states that the total amount of money spent on the risky bond comes from the endowment (of goods and assets) by the agents in the interval (h_D, h_B) equals the total sales revenues.

Tables 3 shows the results. All agents above $h_M = .93$ leverage all their asset holdings, $1 - h_M + y = .35$,[4] using the risky bond. More pessimistic agents but above $h_D = .66$ leverage all their asset holdings, .65, but with the risk-less bond. Investors with less optimistic view than the previous ones but more optimistic than $h_B = .48$ do not hold the asset and lend in the risky market, i.e. they buy the risky bond, whereas the most pessimistic investors hold cash and lend in the default-free market.

Table 3: Equilibrium: Heterogenous Beliefs.

Marginal Buyers		**Asset Price**	
h_M	0.9307	p	2.4197
h_D	0.6589		
		Bond Price	
h_B	0.4839		
		π^{dM}	1.7336
	Asset purchases on risky market		
	y	0.276	

When the asset can take on at most two immediate successor values, equilibrium determines a unique actively traded promise and hence leverage. With three or more successor values, we cannot expect a simple promise. In this example there is default in equilibrium, and different agents buy the same asset with different leverage. But equilibrium still determines the economy-wide average leverage used to buy the asset. Equilibrium leverage is presented in table 4. There are four securities in total, three risky securities and one risk-less security (without considering warehousing). Columns 2 and 3 show the holdings and value of such holdings for each of the securities. Most importantly, column 4 shows the LTV of each of the two traded

[4] Notice that total asset holdings consist of initial endowments, $1 - .93$, plus new purchases, .27.

contracts. As was expected, LTV is higher for the risky contracts (they have a higher promise). Finally, column 5 shows the asset LTV. As defined in section 2, asset LTV is a weighted average, so it is obtained from the total amount borrowed using all contracts, $.5986 + .6547$ divided by the total value of collateral, 2.4197×1.

Table 4: Equilibrium Leverage: Heterogenous Beliefs.

Security	Holdings	Holdings Value	Contract LTV	Asset LTV
Y lev Medium	0.3453	0.8355	0.7165	0.5180
Y lev Min	0.6547	1.5842	0.4133	
Risky Bond	0.3453	0.5986		
Riskless bond	0.6547	0.6547		

4.2 CAPM: An Example with Default.

We now consider the same asset payoff tree as in the previous example but with mean variance investors as in section 3. In this case, we show an example in which only one contract is traded in equilibrium, but not the maxmin contract. Instead leverage takes place exclusively via the promise d_M which defaults in state D.

We calculate the equilibrium in the same way we did in section 3. We will guess a regime and calculate the equilibrium values according to this guess. Finally, of course, we will have to check that our guess is correct. We guess that in equilibrium the tolerant agents buy all the asset in the economy and it all using the contract that promises d_M. On the other hand, the averse investors sell all their asset, buy all the consumption good and lend to the more tolerant investors. We will have to solve for two variables in this case: the price of the asset, p, and the price of the only contract traded in equilibrium, π. For that we have two equations.

The first equation is the first order condition for lending corresponding to the averse investor. The marginal utility at $t = 0$ of the price of the contract has to equal the expected marginal utility in the future of the payoff of the contract.

$$\pi = \frac{q_U(1 - \alpha^A c_U^A)d_M + q_M(1 - \alpha^A c_M^A)d_M + q_D(1 - \alpha^A c_D^A)d_M}{1 - \alpha^A c_0^A} \tag{13}$$

The second equation is the first order condition for holding the asset corresponding to the tolerant investor.

$$p - \pi = \frac{q_U(1 - \alpha^T c_U^T)(d_U - d_M)}{1 - \alpha^T c_0^T} \tag{14}$$

We solve the equilibrium for the following parameter values: asset payoffs $d_U = 3, d_M = 2, d_D = 1$, asset endowments, $a^T = 0, a^A = 1$, good endowments, $e_s^T = 2, s \neq U$, and $e_U^T = 1$, and $e_s^A = 2, s \neq U$, and $e_U^A = 6$, risk aversion $\alpha^T = .06, \alpha^A = .1$, finally, probabilities, $q_s^h = 1/3, \forall h, \forall s$. Table 5 shows the equilibrium. As we did in section 3, we check that the regime is genuine and also that they do not want to borrow or lend using the risk-less bond, contrary to section 3.[5]

Table 5: Equilibrium: CAPM and default.

Asset price, p	2.4086
Contract price, π	2.0852
Leverage	
Margin, m	0.1343
Leverage, 1/m	7.4477
Loan to Value, 1-m	0.8657

[5] All results area available upon request.

5 Leverage and Asset Prices

In this section we study the effect of leverage on asset prices. As volatility of asset payoffs fall, increasing the maxmin payoff and thus increasing equilibrium leverage, or as regulators relax the maximum allowable leverage, what should happen to asset prices? Geanakoplos (2003, 2010) and Fostel-Geanakoplos (2008) argued, in the context of our two state model, that increased leverage would raise asset prices. But as we have seen, this is the context in which *VaR=0* is the endogenous leverage rule. Similar conclusions were reached later by Brunnermeier and Pedersen (2009), Garleanu and Pedersen (2009). But in these papers the *VaR=0* rule is imposed by assumption.

The following example allows for four states and endogenous leverage. *VaR=0* is not the rule; in equilibrium there is default and multiple promises on the same collateral. The example demonstrates that the relationship between leverage and asset prices is subtle. In the example, as the government relaxes a maximum leverage constraint, asset prices rises for awhile, but eventually go down. The intuition is that as leverage increases at the beginning, there is still no default. The bond payoff just increases proportionately, enabling optimists to borrow more without changing the span of available assets from which to choose. The price of the asset rises. But as the leverage constraint is further relaxed, agents endogenously choose to make promises that will involve a positive probability of default. The promise itself becomes riskier and more like the asset. Eventually some buyers switch from buying the asset to buying the bond. The higher the leverage, the more the bond becomes an asset substitute, competing away some of the asset demand.

Consider an extension of our heterogenous priors example to four states. Agent endowments and utilities are described exactly as before. State probabilities are now given by $q_s^h = \frac{3!}{(s-1)!(4-s)!} h^{s-1}(1-h)^{4-s}$. The asset payoff is now described in figure 3.

Asset payoffs are given by $d_4 = 4, d_3 = 3, d_2 = 2$ and $d_1 = 1$. We first study the equilibrium in an economy, we call economy 1, in which regulation limits bond promises collateralized by one unit of the asset to a risk-less contract that promises $d_1/2 = 1/2$. In a second step we study the equilibrium in an economy, economy 2, in which the maximum allowed promise is $d_1 = 1$. In economies 1 and 2, only a single risk-less contract is endogenously traded, namely the one with the maximum allowed promise.

In economy 3, we raise the maximum allowable promise to $d_2 = 2$. In equilibrium, two contracts are endogenously traded: the risk-less traded promise of 1 traded in economy 2 and a risky contract that promises $d_2 = 2$ and defaults in state $s = 1$. Economy 4 further relaxes the maximum promise to $d_3 = 3$. In equilibrium three

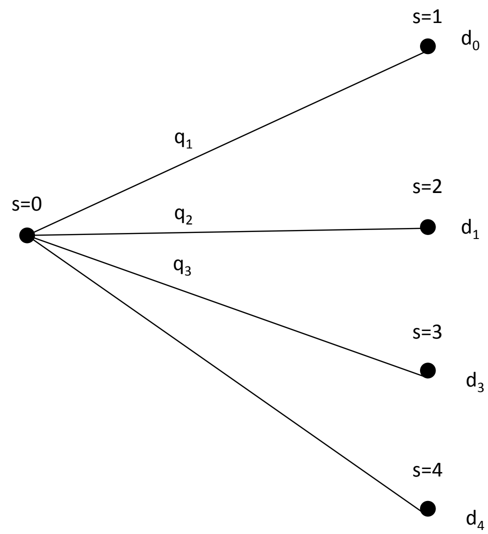

Figure 3: Asset Payoff Description.

contracts are endogenously traded, namely a contract promising 1, another promising 2, and a third promising 3. The second contract defaults in state 1, and the third contract defaults in states $s = 1, 2$. We calculate the equilibrium in these four different economies. The system of equations solved in each case and all the equilibrium values are presented in the appendix.

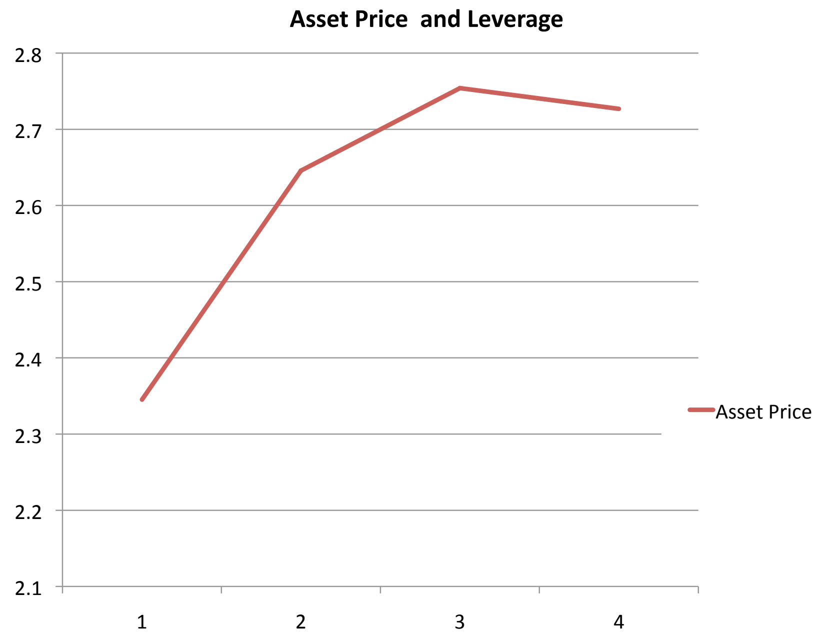

Figure 4: Leverage and Asset Price.

Figure 4 shows a graph corresponding to the asset price in each economy. We can clearly see that as we increase leverage from economy 1 to economy 2 and 3, the asset price increases. This increase is more dramatic at the beginning, when the asset span remains unchanged. However, as we increase leverage further in economy 4, the asset price decreases. In fact, the riskiest bond that promise 3 looks very much like the asset itself, and hence competes with it.

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7 Appendix

7.1 Economy 1

The only bond traded is the one that promises $d_3/2$. There are two variables to solve for, the asset price, p, and the marginal buyer, h_3. The system of equations is the following:

$$p - d_3/2 = q_0^{h_3}(d_0 - d_3/2) + q_1^{h_3}(d_1 - d_3/2) + q_2^{h_3}(d_2 - d_3/2) + q_3^{h_3}(d_3 - d_3/2) \quad (15)$$

$$p = (1 - h_3)(1 + p) + d_3/2 \quad (16)$$

The equilibrium is given by $p = 2.345$ and $h_3 = 0.448$.

7.2 Economy 2

The only bond traded is the one that promises d_3. There are two variables to solve for, the asset price, p, and the marginal buyer, h_3. The system of equations is the following:

$$p - d_3 = q_0^{h_3}(d_0 - d_3) + q_1^{h_3}(d_1 - d_3) + q_2^{h_3}(d_2 - d_3) + q_3^{h_3}(d_3 - d_3) \quad (17)$$

$$p = (1 - h_3)(1 + p) + d_3 \quad (18)$$

The equilibrium is given by $p = 2.646$ and $h_3 = 0.549$.

7.3 Economy 3

The are two bonds, a risk-less one that promises d_3 and a risky one that promises d_2. There are six variables: the asset price, p, the risky bond price, π, the marginal buyer that leverage with the risky bond, h_2, the marginal buyer that leverage with the risk-less bond, h_3, the marginal buyer of the risky bond, h_{B2}, and asset purchases leveraging with the risky bond, y. The regime is the following: $h > h_2$ buy Y and promise d_2, so they buy Y leveraging to the max with the risky bond. $h_2 > h > h_3$ buy Y and promise d_3, so they leverage using the risk-less bond. $h_3 > h > h_{B2}$ sell Y and buy the risky bond (so lend in the risky market collateralized by Y). Finally, $h < h_{B2}$ sell everything, hold risk-less securities (so lend in the risk-less markets). The system of equations is the following:

$$\frac{q_0^{h_2}(d_0 - d_2) + q_1^{h_2}(d_1 - d_2)}{p - \pi} = \frac{q_0^{h_2}(d_0 - d_3) + q_1^{h_2}(d_1 - d_3) + q_2^{h_2}(d_2 - d_3)}{p - d_3} \tag{19}$$

$$\frac{q_0^{h_3}(d_0 - d_3) + q_1^{h_3}(d_1 - d_3) + q_2^{h_3}(d_2 - d_3)}{p - d_3} = \frac{q_0^{h_3} d_2 + q_1^{h_3} d_2 + q_2^{h_3} d_2 + q_3^{h_3} d_3}{\pi} \tag{20}$$

$$\frac{q_0^{h_{B2}} d_2 + q_1^{h_{B2}} d_2 + q_2^{h_{B2}} d_2 + q_3^{h_{B2}} d_3}{\pi} = 1 \tag{21}$$

$$(1 - h_2) + (1 - h_2 + y)\pi = py \tag{22}$$

$$(h_2 - h_3) + (h_2 - h_3 + h_3 - y)d_3 = (h_3 - y)p \tag{23}$$

$$(h_3 - h_{B2}) + p(h_3 - h_{B2}) = \pi(1 - h_2 + y) \tag{24}$$

The equilibrium is given by table 6.

7.4 Economy 4

The are three bonds, a risk-less one that promises d_3 and two risky bond that promise d_2 and d_1. There are ten variables: the asset price, p, the risky bond price that promises d_2 , π_2, the risky bond price that promises d_1 , π_1, the marginal buyer that leverage with the risky bond d_1, h_1, the marginal buyer that leverage with the risky bond d_2, h_2,the marginal buyer that leverage with the risk-less bond, h_3, the marginal buyer of the risky bond that promises d_1, h_{B1},the marginal buyer of the risky bond that promises d_2, h_{B2}, asset purchases leveraging with the risky bond d_1 , y_1 and asset purchases leveraging with the risky bond d_2 , y_2. The regime is the following:

Table 6: Equilibrium

Marginal Buyers		Asset Price	
h_2	0.8064	p	2.7541
h_3	0.6675	**Bond Price**	
h_{B2}	0.3456	π	1.7198
Asset purchases on risky market			
y	0.5090		

$h > h_1$ buy Y and promise d_1, $h_1 > h > h_2$ buy Y and promise d_2. $h_2 > h > h_3$ buy Y and promise d_3, so they leverage using the risk-less bond. $h_3 > h > h_{B1}$ sell Y and buy the risky bond that promises d_1, $h_{B1} > h > h_{B2}$ sell Y and buy the risky bond that promises d_2. Finally, $h < h_{B2}$ sell everything, hold risk-less securities (so lend in the risk-less markets). The system of equations is the following:

$$\frac{q_0^{h_1}(d_0 - d_1)}{p - \pi_1} = \frac{q_0^{h_1}(d_0 - d_2) + q_1^{h_1}(d_1 - d_2)}{p - \pi_2} \tag{25}$$

$$\frac{q_0^{h_2}(d_0 - d_2) + q_1^{h_2}(d_1 - d_2)}{p - \pi_2} = \frac{q_0^{h_2}(d_0 - d_3) + q_1^{h_2}(d_1 - d_3) + q_2^{h_2}(d_2 - d_3)}{p - d_3} \tag{26}$$

$$\frac{q_0^{h_3}(d_0 - d_3) + q_1^{h_3}(d_1 - d_3) + q_2^{h_3}(d_2 - d_3}{p - d_3} = \frac{q_0^{h_3} d_1 + q_1^{h_3} d_1 + q_2^{h_3} d_2 + q_3^{h_3} d_3}{\pi_1} \tag{27}$$

$$\frac{q_0^{h_{B1}} d_1 + q_1^{h_{B1}} d_1 + q_2^{h_{B1}} d_2 + q_3^{h_{B1}} d_3}{\pi_1} = \frac{q_0^{h_{B1}} d_2 + q_1^{h_{B1}} d_2 + q_2^{h_{B1}} d_2 + q_3^{h_{B1}} d_3}{\pi_2} \tag{28}$$

$$\frac{q_0^{h_{B2}} d_2 + q_1^{h_{B2}} d_2 + q_2^{h_{B2}} d_2 + q_3^{h_{B2}} d_3}{\pi_2} = 1 \tag{29}$$

$$(1 - h_1) + (1 - h_1 + y_1)\pi_1 = py_1 \tag{30}$$

$$(h_1 - h_2) + (h_1 - h_2 + y_2)\pi_2 = py_2 \tag{31}$$

$$(h_2 - h_3) + (h_2 - h_3 + h_3 - (y_1 + y_2))d_3 = p(h_3 - (y_1 + y_2)) \tag{32}$$

$$(h_3 - h_{B1}) + p(h_3 - h_{B1}) = \pi_1(1 - h_1 + y_1) \tag{33}$$

$$(h_{B1} - h_{B2}) + p(h_{B1} - h_{B2}) = \pi_2(h_1 - h_2 + y_2) \tag{34}$$

The equilibrium is given by table 7.

Table 7: Equilibrium

Marginal Buyers		**Asset Price**	
h_1	0.9688	**p**	2.7269
h_2	0.8203	**Bond Prices**	
h_3	0.7159	**π_1**	2.2360
h_{B1}	0.5736	**π_2**	1.6976
h_{B2}	0.3288		

Asset purchases on risky markets	
y_1	0.2058
y_2	0.3891