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Extending the case for a beneficial brain drain

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Abstract. The recent literature about the so-called beneficial brain drain assumes that destination countries are characterized not only by higher wages than the source country, but also by a higher or at least not lower relative return to education. However, it is a well known stylized fact that the returns to education are higher in rich than in poor countries. Against this background, we assess whether the main prediction of this literature, namely the possibility of a beneficial brain gain, still holds under the reverse assumption. We show that there is a still a strong case for a beneficial brain drain, even if the returns to education in the source country exceed those in the destination country. Immigration policies that are biased against unskilled workers are not necessary for a beneficial brain drain to occur once one considers that agents face heterogeneous migration costs.

Keywords: migration; brain drain; skill premium; heterogeneous agents; selective immigration policies

JEL classification: F22; J24; O15

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1 Introduction

The recent literature on the brain drain has evidenced that the opportunity to migrate can increase the human capital endowment of the sending countries because, so the reasoning goes, "higher prospective returns to skills in a foreign country impinge on skill acquisition decisions at home" (Stark *et al.*, 1997). A higher skill premium – both in relative and absolute terms - in the destination country represents a necessary condition for a beneficial brain drain to occur in Stark *et al.* (1997), while most of this literature assumes that the skill premium is higher at destination in absolute terms, and constant across countries in relative terms (e.g. Mountford, 1997; Vidal, 1998; Beine *et al.*, 2001; Stark and Wang, 2002). This contradicts one of the key assumptions of the literature on the self-selection of immigrants, where the relative skill premium is expected to be lower in destination countries (e.g. Borjas, 1987; Chiquiar and Hanson, 2005).

Recently, Egger and Felbermayr (2007) claimed that there is no room for a beneficial brain drain if one conversely assumes that the relative skill premium is lower in the destination country. Such a conclusion could be potentially disruptive for the empirical relevance of the main prediction of this strand of literature, since it is a stylized fact – as Figure 1 shows – that the relative skill premium is inversely related with the income level of a country.¹

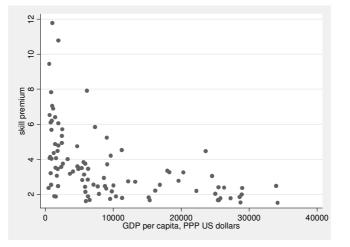


Figure 1. Skill premium and GDP per capita across countries

Source: Authors' elaboration on Freeman and Oostendorp (2000).

The objective of this brief paper is to show that the case for a beneficial brain drain actually does not hinge on any assumption about the skill premium. We demonstrate that a beneficial brain drain can occur even if the skill premium is lower in the destination than

¹ We follow Belot and Hatton (2008) for the identification of the sets of skilled and unskilled occupations.

it is in the sending country, provided that one realistically considers that migration costs are heterogeneous across agents. This result does not require immigration policies to be selective with respect to education.

2 The model

Consider a small isolated economy populated by one-period lived agents, who maximize a utility function that is linear in income. Agents have to decide whether to invest in education, and they face different educational costs due to heterogeneous innate learning abilities. As in Docquier and Rapoport (2007), an agent has to forego a fraction $(1-\eta)$ of the wage of a skilled worker in order to get an education, where η is uniformly distributed over the support [0,1].

Skilled and unskilled workers are given exogenous probabilities p and q to migrate to a high-wage country, with q=(1-s)p and $s \in [0,1]$. The parameter s measures the selectivity of the immigration policies: the higher s, the larger the bias against unskilled workers. The selectivity can influence the incentives to invest in education, as agents are induced to invest in education "in order to be eligible for emigration" (Mountford, 1997).

We assume that agents face heterogeneous migration costs: migration costs amount to a fraction δ of the foreign wage, where δ uniformly distributed over the support [0,1] independently from η .² We assume that his own realizations of δ and η are known to each worker before he takes any educational or migration decision.

In line with the literature, we assume that the labor supply of both skilled and unskilled workers is inelastic, while - differently from all papers but Egger and Felbermayr (2007) - we assume that skilled and unskilled workers are complementary production factors. The literature assumes either that labor is supplied by homogeneous workers (Mountford, 1997; Vidal, 1998), or that labor is supplied by heterogeneous workers whose skills are perfectly substitutable (Beine *et al.*, 2001; Stark and Wang, 2002; Docquier and Rapoport, 2007). While these assumptions in the existing literature entail that the high-wage destination country offers a relative skill premium that is not lower than the one prevailing in the source country, our modeling choice allows us to consider the case where the relative skill premium is higher in the source country.

² The main result of the model would not change if we assumed that migration costs are fixed rather than proportional, a specification that appears to be better able to explain the observed self-selection of international migrants Grogger and Hanson (2008).

Domestic production takes place through an aggregate production function that is homogenous of degree one in skilled and unskilled workers respectively, denoted by H and L respectively. Factor markets are perfectly competitive, and each type of labor is paid its marginal product.

The small economy assumption implies that foreign wages for skilled and unskilled workers, w_{i}^{*} and w_{ℓ}^{*} , are insensitive to immigration from the domestic country. We employ $w_{i}(h)$ and $w_{\ell}(h)$ to denote domestic wages which are endogenously determined by h, the ratio of the Lebesgue measure of the set H of domestically employed skilled workers over the Lebesgue measure of the set L of domestically employed unskilled workers. We omit time subscripts for notational ease, we normalize $w_{\ell}(h)$ to 1 and assume that foreign wages are higher than domestic ones for both skill groups.

Each worker regards domestic wages as unaffected by his own educational and migration choices; thus, for any given expected value of h, an unskilled agent is willing to migrate if his own realization of the parameter δ satisfies:

$$[1] \qquad \qquad \delta < \delta_{\ell} \left(h \right) = \frac{\mathsf{w}_{\ell}^* - 1}{\mathsf{w}_{\ell}^*},$$

while a skilled agent is willing to migrate only if:

$$[2] \qquad \qquad \delta < \delta_{i}(h) = \frac{w_{i}^* - w_{i}(h)}{w_{i}^*}$$

An agent characterized by a realization of δ higher than $\delta_{\ell}(h)$ and $\delta_{\ell}(h)$ does not opt for migration, hence only domestic wages influence his educational decision; he decides to invest in education only if:

$$[3] \qquad \qquad \eta > \eta \left(\delta, h \right) = \frac{1}{w_{i} \left(h \right)}, \ \delta \in \left(\max \left\{ \delta_{i} \left(h \right), \delta_{i} \left(h \right) \right\}, 1 \right]$$

If δ is intermediate between $\delta_{\ell}(h)$ and $\delta_{\hat{n}}(h)$, then only one of the two possible educational decisions will induce an agent to migrate; this implies that he will invest in education only if:

$$\begin{bmatrix} 4 \end{bmatrix} \qquad \eta > \eta \left(\delta, h \right) = \begin{cases} \delta_{\ell} \left(h \right) \leq \delta_{\delta_{\ell}} \left(h \right) \colon \frac{1}{p \left(1 - \delta \right) w_{\delta_{\ell}}^{*} + w_{\delta_{\ell}} \left(h \right)}, \ \delta \in \left(\delta_{\ell} \left(h \right), \delta_{\delta_{\ell}} \left(h \right) \right) \\ \delta_{\ell} \left(h \right) > \delta_{\delta_{\delta}} \left(h \right) \colon \frac{\left(1 - s \right) p \left(1 - \delta \right) w_{\ell}^{*} + \left[1 - \left(1 - s \right) p \right]}{w_{\delta_{\ell}} \left(h \right)}, \ \delta \in \left(\delta_{\delta_{\delta}} \left(h \right), \delta_{\ell} \left(h \right) \right) \\ . \end{cases}$$

Finally, an agent whose parameter δ is lower than $\delta_{\ell}(h)$ and $\delta_{\delta}(h)$ would always opt for a foreign employment, hence he decides to become skilled only if:

$$[5] \qquad \eta > \eta \left(\delta, h\right) = \frac{\left(1-s\right)p\left(1-\delta\right)w_{\ell}^{*}+1-\left(1-s\right)p}{p\left(1-\delta\right)w_{\ell}^{*}+\left(1-p\right)w_{\ell}\left(h\right)}, \ \delta \in \left[0, \min\left\{\delta_{\ell}\left(h\right), \delta_{\ell}\left(h\right)\right\}\right].$$

Let $\psi(h)$ represent the actual value of h that arises from a given expectation. Accounting for the share of skilled and unskilled workers who migrate gives then:

$$[6] \qquad \qquad \psi(h) = \frac{\left(1 - \int_{0}^{1} \eta(\delta, h) \partial \delta\right) - p \int_{0}^{\delta_{\ell}(h)} \left[1 - \eta(\delta, h)\right] \partial \delta}{\int_{0}^{1} \eta(\delta, h) \partial \delta - (1 - s) p \int_{0}^{\delta_{\ell}(h)} \eta(\delta, h) \partial \delta}.$$

A certain value of h can represent an equilibrium only if expectations are fulfilled:

$$[7] h = \psi(h).$$

It can be easily verified that an equilibrium value of h exists, and that it is unique. For existence, it suffices to observe that $\psi(h)$ diverges to infinity when h approaches 0, while it goes to 0 when h diverges to infinity, while uniqueness follows from the strict monotonicity of $\psi(h)$.

When the economy is closed to migration Eq. [6] reduces to:

[8]
$$\psi(h)|_{p=0} = w_{i}(h)|_{p=0} - 1$$
.

We denote with h^c the value of h that satisfies Eq. [8].

3 Immigration policies and the domestic equilibrium

We depart from the relevant literature assuming that the relative skill premium is higher in the domestic closed economy equilibrium than abroad, i.e. that

A fully anticipated change in either p or s determines a shift of the function $\psi(h)$; given Eq. [7], an upward (downward) shift determines an increase (decrease) in the equilibrium level of h. *A priori*, it is not possible to determine the sign of the partial derivative of $\psi(h)$ with respect to either p or s, has both variables exert two distinct and contrasting effects upon $\psi(h)$.

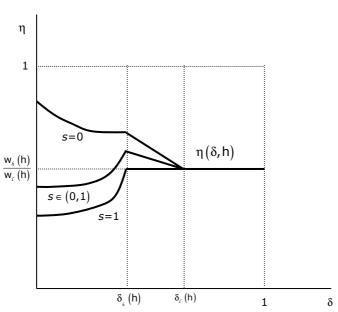


Figure 2 Educational and migration decisions with p>0

Figure 1 shows that a ban on unskilled migration, s=1, improves the incentives to invest in education, as a portion of $\eta(\delta,h)$ moves downwards as p increases, and the set of skilled workers who invest in education with any p>0 is larger than the level that characterizes the closed-economy equilibrium. Adopting the terminology introduced by Beine *et al.* (2001), we can say that an increase in p determines a beneficial *brain effect*, which is counteracted – and possibly offset – by a negative *drain effect*, as the share of skilled workers who migrate is higher than the corresponding share of unskilled workers, which is actually zero. A converse reasoning applies to a general immigration policy, s=0, while for any intermediate degree of selectivity, 0 < s < 1, it is not possible to determine the sign of either of the two effects.

4 The case for a beneficial brain drain

The previous discussion suggests that migration probabilities interact in a complex way in influencing the domestic skill composition. This is why the following Lemma is crucial for simplifying the analysis:

Lemma 1. The partial derivative of $\psi(h)$ with respect to p is an affine function of s when p converges to 0. Specifically:

$$\lim_{p \to 0^+} \frac{\partial \psi(h)}{\partial p} = (\alpha + \beta) - \beta s$$

where

$$\alpha = \frac{\delta_{\scriptscriptstyle \acute{n}}\left(h^{c}\right) \left[\left[w_{\scriptscriptstyle \acute{n}}^{*} - w_{\scriptscriptstyle \acute{n}}\left(h^{c}\right)\right] - 2\left[w_{\scriptscriptstyle \acute{n}}\left(h^{c}\right) - 1\right] \right]}{2} \text{ and } \beta = w_{\scriptscriptstyle \acute{n}}\left(h^{c}\right) w_{\scriptscriptstyle \ell}^{*} \left[\frac{1}{2} \left[\delta_{\scriptscriptstyle \acute{n}}\left(h^{c}\right)\right]^{2} + \frac{w_{\scriptscriptstyle \ell}^{*} - 1}{w_{\scriptscriptstyle \acute{n}}\left(h^{c}\right)} \left(w_{\scriptscriptstyle \acute{n}}\left(h^{c}\right)\left(2 - w_{\scriptscriptstyle \ell}^{*}\right) - 1\right) \right]$$

Proof. [See Annex]

Lemma 1 states that the impact of a marginal change in the probability to migrate p upon the equilibrium skill composition of the domestic workforce is a linear function of the degree of selectivity of immigration policies, when p approaches zero. If the partial derivative of $\psi(h)$ with respect to p is positive in the neighborhood of zero, then the equilibrium level of h is – at least for low migration probabilities – higher than the one emerging in the closed-economy case.

Lemma 1 also implies that a beneficial brain drain occurs – at least for low levels of p - under a general immigration policy if $\alpha + \beta > 0$, and we can demonstrate the following proposition:

Proposition 1. For any closed-economy equilibrium, there is a non-empty set in the (w_{ℓ}^*, w_{ℓ}^*) space such that a beneficial brain drain occurs under a general immigration policy.

Proof. From the definitions of the two parameters we have that:

$$\alpha + \beta = \frac{\delta_{\hbar} \left(h^{c}\right) \left[\left[w_{\hbar}^{*} - w_{\hbar} \left(h^{c}\right)\right] - 2 \left[w_{\hbar} \left(h^{c}\right) - 1\right] \right]}{2} + w_{\hbar} \left(h^{c}\right) w_{\ell}^{*} \left[\frac{1}{2} \delta_{\hbar} \left(h^{c}\right)^{2} + \frac{w_{\ell}^{*} - 1}{w_{\hbar} \left(h^{c}\right)} \left(w_{\hbar} \left(h^{c}\right) \left(2 - w_{\ell}^{*}\right) - 1\right) \right]$$

Observe that $\alpha + \beta$ is continuous in the foreign wage for high skilled labor. Assume now that w_{i}^{*} is such that the foreign skill premium is the same as in the domestic closed-economy equilibrium, i.e. $w_{i}^{*} = w_{i} (h^{c}) w_{i}^{*}$. Under this assumption, the expression for $\alpha + \beta$ becomes:

$$\alpha + \beta = \frac{\left(\frac{w_{\ell}^{*} - 1}{w_{\ell}^{*}}\right)\left[\left[w_{\hbar}\left(h^{c}\right)w_{\ell}^{*} - w_{\hbar}\left(h^{c}\right)\right] - 2\left[w_{\hbar}\left(h^{c}\right) - 1\right]\right]}{2} + w_{\hbar}\left(h^{c}\right)w_{\ell}^{*}\left[\frac{1}{2}\left(\frac{w_{\ell}^{*} - 1}{w_{\ell}^{*}}\right)^{2} + \frac{w_{\ell}^{*} - 1}{w_{\hbar}\left(h^{c}\right)}\left(w_{\hbar}\left(h^{c}\right)\left(2 - w_{\ell}^{*}\right) - 1\right)\right].$$

With some algebraic manipulations, the above expression simplifies to:

$$\alpha + \beta = \left(\frac{w_{\ell}^{*} - 1}{w_{\ell}^{*}}\right) \left(w_{\ell}^{*2} - 1\right) \left[\left(w_{\hbar} \left(h^{c}\right) \left(2 - w_{\ell}^{*}\right) - 1\right) \right]$$

As $w_{l}^{*} > 1$, it is obvious that $\alpha + \beta > 0$ if and only if:

$$w_{\ell}^{*} \leq 2 - \frac{1}{w_{\hbar}(h^{c})}$$

As $\alpha + \beta$ is continuous in the wage w_{i}^{*} , it is possible to find values of w_{i}^{*} lower than $w_{i}(h^{c})w_{i}^{*}$, such that $\alpha + \beta$ is still positive. This proves the Proposition.

5 Conclusion

This paper proves that a case for a beneficial brain drain can be made even if the returns to education are higher in the source than in the destination country. This is empirically highly relevant, since it can be considered as a stylized fact that the skill premium is higher in poor compared to rich countries. Heterogeneity in individual migration costs drives this result. If we would have considered homogeneous migration costs – as in Egger and Felbermayr (2007) – we would have concluded that the option to migrate reduces the endowment of source countries with human capital if immigration policies are not skill-selective.

Our result holds for a general immigration policy, i.e. an immigration policy which is not skill-selective. Skill-selective immigration policies have ambiguous effects on human capital endowments in source countries, they can both increase and reduce human capital endowments there.

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Annex

Proof of Lemma 1. We can compute the partial derivative of $\psi(h)$ with respect to p from Eq. [6] in the main text:

The limit of Eq. [A1] for p that converges to zero is given by:

$$\begin{bmatrix} A2 \end{bmatrix} \qquad \lim_{p \to 0^+} \frac{\partial \psi(h)}{\partial p} = (1-s) \,\delta_\ell \left(h^c\right) \left[w_{\ell} \left(h^c\right) - 1\right] - \delta_\ell \left(h^c\right) \left[w_{\ell} \left(h^c\right) - 1\right] - \left[w_{\ell} \left(h^c\right)\right]^2 \left(\lim_{p \to 0^+} \frac{\partial \int_0^1 \eta(\delta, h) \, \partial \delta}{\partial p}\right) \right]^2 \left(\lim_{p \to 0^+} \frac{\partial \int_0^1 \eta(\delta, h) \, \partial \delta}{\partial p}\right) = (1-s) \,\delta_\ell \left(h^c\right) \left[w_{\ell} \left(h^c\right) - 1\right] - \left[w_{\ell} \left(h^c$$

From Eqs. [3]-[5] and Eq. [9], the integral that appears on the right hand side of Eq. [A2] can be expressed as:

.

$$\begin{bmatrix} A3 \end{bmatrix} \qquad \int_{0}^{1} \eta(\delta,h) \partial \delta = (1-s) \frac{w_{\ell}^{*}}{w_{\hbar}^{*}} \delta_{\hbar}(h) + \frac{(1-s)p(w_{\hbar}^{*} - w_{\ell}^{*}w_{\hbar}(h)) + (1-s)w_{\ell}^{*}w_{\hbar}(h) - w_{\hbar}^{*}}{p(w_{\hbar}^{*})^{2}} ln \left[\frac{w_{\hbar}(h)}{w_{\hbar}(h) + p(w_{\hbar}^{*} - w_{\hbar}(h))} \right] + (\delta_{\ell}(h) - \delta_{\hbar}(h)) \frac{1 + (1-s)p(w_{\ell}^{*} - 1)}{w_{\hbar}(h)} - (\delta_{\ell}(h)^{2} - \delta_{\hbar}(h)^{2}) \frac{w_{\ell}^{*}}{2w_{\hbar}(h)} + (1-\delta_{\ell}(h)) \frac{1}{w_{\hbar}(h)}$$

where the integral of $\eta(\delta, h)$ for $\delta \in [0, \delta_i(h)]$ has been obtained through integration by substitution. The partial derivative of [A3] with respect to p is given by:

Taking the limit of [A4] for p that goes to zero yields:

$$\begin{bmatrix} A5 \end{bmatrix} \quad \lim_{p \to 0^{+}} \frac{\partial \int_{0}^{1} \eta\left(\delta,h\right) \partial \delta}{\partial p} = (1-s) \left[\frac{w_{\ell}^{*} - 1}{w_{\hat{k}}\left(h^{c}\right) - \delta_{\hat{k}}\left(h^{c}\right)} - \frac{1}{w_{\hat{k}}\left(h^{c}\right)} \delta_{\hat{k}}\left(h^{c}\right) + \frac{w_{\ell}^{*}\left(w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right)}{2w_{\hat{k}}^{*}w_{\hat{k}}\left(h^{c}\right)} \delta_{\hat{k}}\left(h^{c}\right) \right] - \frac{w_{\hat{k}}^{*} - w_{\hat{k}}\left(h^{c}\right)}{2\left[w_{\hat{k}}\left(h^{c}\right)\right]^{2}} \delta_{\hat{k}}\left(h^{c}\right) + \frac{w_{\ell}^{*}\left(w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right)}{2w_{\hat{k}}^{*}w_{\hat{k}}\left(h^{c}\right)} \delta_{\hat{k}}\left(h^{c}\right) \right] - \frac{w_{\hat{k}}^{*} - w_{\hat{k}}\left(h^{c}\right)}{2\left[w_{\hat{k}}\left(h^{c}\right)\right]^{2}} \delta_{\hat{k}}\left(h^{c}\right) + \frac{w_{\ell}^{*}\left(w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right)}{2w_{\hat{k}}^{*}w_{\hat{k}}\left(h^{c}\right)} \delta_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right]} \delta_{\hat{k}}\left(h^{c}\right) + \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right]} \delta_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right]} \delta_{\hat{k}}\left(h^{c}\right) + \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) + w_{\hat{k}}^{*}\right]} \delta_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c}\right) - \frac{1}{2\left[w_{\hat{k}}\left(h^{c$$

where we have applied De L'Hôpital theorem twice as the limit presented an undetermined form. Finally, plugging Eq. [A5] into Eq. [A2], and with some algebraic manipulations we get:

$$\begin{bmatrix} A6 \end{bmatrix} \qquad \lim_{p \to 0^+} \frac{\partial \psi(h)}{\partial p} = (\alpha + \beta) - \beta s$$

$$\text{where } \alpha = \frac{\delta_{\text{\tiny κ}}\left(h^{c}\right)\left[\left[w_{\text{\tiny κ}}^{*} - w_{\text{\tiny κ}}\left(h^{c}\right)\right] - 2\left[w_{\text{\tiny κ}}\left(h^{c}\right) - 1\right]\right]}{2} \text{ and } \beta = w_{\text{\tiny κ}}\left(h^{c}\right)w_{\text{\tiny ℓ}}^{*}\left[\frac{1}{2}\left[\delta_{\text{\tiny κ}}\left(h^{c}\right)\right]^{2} + \frac{w_{\text{\tiny ℓ}}^{*} - 1}{w_{\text{\tiny κ}}\left(h^{c}\right)\left(2 - w_{\text{\tiny ℓ}}^{*}\right) - 1\right)\right] \text{ .}$$

This proves the Lemma.