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Output Maximization of an Agency

Pahlaj Moolio *, Jamal Nazrul Islam **, and Haradhan Kumar Mohajan***

ABSTRACT

Considering Cobb-Douglas function in three variables as an explicit form of production function, in this paper an attempt has been made to maximize an output subject to a budget constraint, using Lagrange multipliers technique, as well as necessary and sufficient conditions for optimal value have been applied. We gave interpretation of Lagrange multiplier in this specific illustration, showing its positive value, and examined the behavior of the agency.

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Key words: Lagrange Multipliers; Economic Problems; Maximizing Output Function; Budget Constraints; Explicit Examples.

1. INTRODUCTION

The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus and has been used to facilitate the determination of necessary conditions; normally, this method was considered as device for transferring a constrained problem to a higher dimensional unconstrained problem (Islam 1997, Pahlaj and Islam 2008). Using this technique, Baxley and Moorhouse (1984) analyzed an example of utility maximization, and provided a formulation for nontrivial constrained optimization problem with special reference to application to economics. They considered implicit functions with assumed characteristic qualitative features and provided illustration of an example, generating meaningful economic behaviour. This approach and formulation may enable one to view optimization problems in economics from a somewhat wider

* Ph.D. Fellow, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh; and Professor and Academic Advisor, Pannasastra University of Cambodia, Phnom Penh, Cambodia. Corresponding Author: Pahlaj Moolio. E-mail: pahlajmoolio@gmail.com

** Emeritus Professor, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh. Phone: +880-31-616780.

*** Lecturer, Premier University, Chittagong, Bangladesh , E mail: haradhan_km@yahoo.com

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perspective. Using this technique, Pahlaj and Islam (2008) considered a problem of cost minimization in three variables subject to output function as a constraint, studying the behaviour of the firm, and hence extended the work of Pahlaj (2002). Detailed discussion of the Lagrange multipliers method and its use in economics is given in Islam (2008).

In this paper, we consider theoretically a variation of the problem considered by Pahlaj and Islam (2008), assuming that a government agency is allocated an annual budget B and required to maximize and make available some sort of services to the community. If the agency uses factors $K,\ L,$ and R in the same sense as used by Pahlaj and Islam (2008) to produce and provide services to the community, then its objective is to maximize the output function subject to a budget constraint. This problem is thus the dual to the cost minimization of a competitive firm, considered by Pahlaj and Islam (2008).

In section 2, we deal with formulating mathematical model for the problem, considering Cobb-Douglas production function in three variables (factors: capital, labour, and other inputs). Considering an explicit form of production function, we apply necessary conditions to this output maximization problem, and find stationary point as well as optimal value of the production function in section 3. In section 4, we give a reasonable interpretation of the Lagrange multiplier in the context of this particular illustration. Sufficient conditions are applied in section 5. In section 6, we analyze the comparative static results (Chiang 1984) and examine the behaviour of the agency; that is, how a change in the input costs will affect the situation, or if the budget for the services undergoes some changes. In final section 7, we provide conclusion and recommendations.

2. THE MATHEMATICAL MODEL

We consider that, for the fixed annual budget, a government agency is charged to produce and provide to the community with the quantity Q units of the services during a specified time, say for instance, in a year, with the use of K quantity of capital, L quantity of labour, and R quantity of other inputs into its service oriented production process. These other inputs (e.g., land and other raw materials) are combined to produce the production (Humphery 1997; Pahlaj and Islam 2008). If the agency uses factors $K,\ L,$ and R to produce and provide quantity Q units of the services (Baxley and Moorhouse 1984; Pahlaj and Islam 2008) to the community, then its objective is to maximize the output function:

$$Q = g\left(K, L, R\right), \tag{1}$$

subject to the budget constraint:

$$B = rK + wL + \rho R\,, \tag{2}$$

where r is the rate of interest or services per unit of capital K, w is the wage rate per unit of labour L, and ρ is the cost per unit of other inputs R, while g is a suitable production function. The government agency takes these and all other factor prices as given. We assume that second order partial derivatives of the function g with respect to the independent variables (factors) $K,\ L,$ and R exist.

Ignoring the actual form of the function Q, we now formulate the maximization problem for the output function given by (1) in terms of single Lagrange multiplier λ, by defining the Lagrangian function Z as follows:

$$Z(K, L, R, \lambda) = Q(K, L, R) + \lambda(B - rK - wL - \rho R). \tag{3}$$

This is a four dimensional unconstrained problem obtained from (1) and (2) by the use of Lagrange multiplier λ, as a device. Assuming that the government agency maximizes its output, the optimal quantities $K^*, L^*, R^*, \text{and } \lambda^*$ of $K, L, R, \text{and } \lambda$ that necessarily satisfy the first order conditions; which can be obtained by partially differentiating the Lagrangian function (3) with respect to four variables $\lambda, K, L, \text{and } R$ and setting them equal to zero:

$$Z_\lambda = B - rK - wL - \rho R = 0, \tag{4a}$$

$$Z_K = Q_K - \lambda r = 0, \tag{4b}$$

$$Z_L = Q_L - \lambda w = 0, \tag{4c}$$

$$Z_R = Q_R - \lambda \rho = 0, \tag{4d}$$

where

$$Z_K = \frac{\partial Z}{\partial K},\ Z_L = \frac{\partial Z}{\partial L},\ Z_R = \frac{\partial Z}{\partial R},\ Z_\lambda = \frac{\partial Z}{\partial \lambda},\ \text{and } Q_K = \frac{\partial g}{\partial K},\ Q_L = \frac{\partial g}{\partial L},\ Q_R = \frac{\partial g}{\partial R}.$$

It may be noted that the partial derivative with respect to λ is just the same as the constraint - this is always the case, so we get again $B = rK + wL + \rho R$, while from (4b-d), the Lagrange multiplier is obtained as follows:

$$\lambda = \frac{Q_K}{r} = \frac{Q_L}{w} = \frac{Q_R}{\rho}. \tag{5}$$

Considering the infinitesimal changes dK, dL, dR in K, L, R, respectively, and the corresponding changes dQ and dB, we get:

$$dQ = Q_K dK + Q_L dL + Q_R dR, \tag{6}$$

$$dB = r dK + w dL + \rho dR. \tag{7}$$

With the use of (4b-d) or (5), we obtain the following equation:

$$\frac{dQ}{dB} = \frac{Q_K dK + Q_L dL + Q_R dR}{r dK + w dL + \rho dR} = \lambda. \tag{8}$$

Thus, the Lagrange multiplier gives the change in total output consequent to change in the inputs. If, for example, one of the inputs, say K, is held constant, means $dK = 0$, then (8) represents the partial derivative: $\left(\frac{\partial Q}{\partial B}\right)_K$ (with $dK = 0$), and so.

3. AN EXPLICIT EXAMPLE

We now consider an explicit form of the output function g in (1), and provide a detailed discussion and intrinsic understanding of the problem at hand.
Let the function g given by

$$Q = g\left(K, L, R\right) = AK^{\alpha} L^{\beta} R^{\gamma}, \tag{9}$$

where A is assumed to be unchanged technology; and the exponents $\alpha, \beta,$ and γ are constants that constitute the output elasticities with respect to capital, labour, and other inputs (Humphery 1997, and Pahlaj and Islam 2008), respectively. Using (2) and (9), (3) takes the following form:

$$Z\left(K, L, R, \lambda\right) = AK^{\alpha} L^{\beta} R^{\gamma} + \lambda\left(B - rK - wL - \rho R\right). \tag{3a}$$

Therefore, (4a-d) become:

$$Z_{\lambda} = B - rK - wL - \rho R = 0, \tag{10a}$$

$$Z_{K} = \alpha AK^{\alpha-1} L^{\beta} R^{\gamma} - r\lambda = 0, \tag{10b}$$

$$Z_{L} = \beta AK^{\alpha} L^{\beta-1} R^{\gamma} - w\lambda = 0, \tag{10c}$$

$$Z_{R} = \gamma AK^{\alpha} L^{\beta} R^{\gamma-1} - \rho\lambda = 0. \tag{10d}$$

Using the method of successful elimination and substitution, we solve above set of equations and obtain the optimum values of $K,\ L,\ R,$ and λ:

$$K = K^{*} = \frac{\alpha B}{r\left(\alpha + \beta + \gamma\right)}. \tag{11a}$$

$$L = L^{*} = \frac{\beta B}{w\left(\alpha + \beta + \gamma\right)}. \tag{11b}$$

$$R = R^{*} = \frac{\gamma B}{\rho\left(\alpha + \beta + \gamma\right)} \tag{11c}$$

$$\lambda = \lambda^{*} = A\left(\frac{\alpha^{\alpha} \beta^{\beta} \gamma^{\gamma}}{r^{\alpha} w^{\beta} \rho^{\gamma}}\right)\left(\frac{B^{(\alpha+\beta+\gamma-1)}}{\left(\alpha + \beta + \gamma\right)^{(\alpha+\beta+\gamma-1)}}\right). \tag{11d}$$

Thus, the stationary point is as below:

$$\left(K^{*}, L^{*}, R^{*}\right) = \left(\frac{\alpha B}{r\left(\alpha + \beta + \gamma\right)}, \frac{\beta B}{w\left(\alpha + \beta + \gamma\right)}, \frac{\gamma B}{\rho\left(\alpha + \beta + \gamma\right)}\right). \tag{12}$$

Moreover, substituting the values of K^{*}, L^{*}, R^{*} from (11a-c) into (9), we get the optimal value of the production function in terms of $r, w, \rho, A, B,$ and α, β, γ as follows:

$$Q^{*}=A\left(\frac{\alpha^{\alpha}\beta^{\beta}\gamma^{\gamma}B^{(\alpha+\beta+\gamma)}}{r^{\alpha}w^{\beta}\rho^{\gamma}(\alpha+\beta+\gamma)^{(\alpha+\beta+\gamma)}}\right). \tag{13}$$

4. INTERPRETATION OF LAGRANGE MULTIPLIER

Before we discuss sufficient conditions and analyze comparative static results, we provide an interpretation of Lagrange multiplier. To some extent, this might seem a bit silly to talk about the meaning of an artificial variable added for computational convenience, but bear with me there is a reasonable interpretation of this variable. With the aid of chain rule, from (13) we get:

$$\frac{\partial Q^{*}}{\partial B}=Q_{K}\frac{\partial K}{\partial B}+Q_{L}\frac{\partial L}{\partial B}+Q_{R}\frac{\partial R}{\partial B} \tag{14}$$

From (9), we get: $Q_K=\alpha AK^{\alpha-1}L^{\beta}R^{\gamma}$, $Q_L=\beta AK^{\alpha}L^{\beta-1}R^{\gamma}$, $Q_R=\gamma AK^{\alpha}L^{\beta}R^{\gamma-1}$.

And from (10b-d), we get: $r\lambda=\alpha AK^{\alpha-1}L^{\beta}R^{\gamma}$, $w\lambda=\beta AK^{\alpha}L^{\beta-1}R^{\gamma}$, $\rho\lambda=\gamma AK^{\alpha}L^{\beta}R^{\gamma-1}$.

Therefore, we write (14) as follows:

$$\frac{\partial Q^{*}}{\partial B}=\lambda^{*}\left[r\frac{\partial K}{\partial B}+w\frac{\partial L}{\partial B}+\rho\frac{\partial R}{\partial B}\right]. \tag{15}$$

From (10a), we have: $B=rK+wL+\rho R$.

Differentiating above equation, keeping $K, L,$ and R constants, we get:

$$1=r\frac{\partial K}{\partial B}+w\frac{\partial L}{\partial B}+\rho\frac{\partial R}{\partial B},$$

which allows us to rewrite (15) as:

$$\frac{\partial Q^{*}}{\partial B}=\lambda^{*}. \tag{16}$$

Therefore, (16) verifies (8). Thus, the Lagrange multiplier λ^{*} may be interpreted as the marginal output, that is, the change in total output incurred from an additional unit of budget B. In other words, in this particular illustration, if the agency wants to increase (decrease) 1 unit of its output, it would cause the total budget to increase (decrease) by approximately λ^{*} units. This is a reasonable interpretation.

5. SUFFICIENT CONDITIONS

Now, in order to be sure that the optimal solution obtained in (12) is maximum; we check it against the sufficient conditions, which imply that for a solution $K^{*}, L^{*}, R^{*},$ and λ^{*} of (10a-d) to be a relative maximum, all the bordered principal minors of the following bordered Hessian,

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix},$$

should take the alternate sign, namely, the sign of $\left|\overline{H}_{m+1}\right|$ being that of $(-1)^{m+1}$, where m is the number of constraints. In our case $m=1$, therefore, in this specific case, if

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & -B_K & -B_L \\ -B_K & Z_{KK} & Z_{KL} \\ -B_L & Z_{LK} & Z_{LL} \end{vmatrix} > 0, \tag{17a}$$

and $$\left|\overline{H}_3\right| = \left|\overline{H}\right| = \begin{vmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix} < 0, \tag{17b}$$

with all the derivatives evaluated at the critical values of $K^*, L^*, R^*, \text{and}\ \lambda^*$, then the stationary value of Q obtained in (13) will assuredly be the maximum. We check this condition, through expanding the determinant (17a) first, noticing that the second partial derivative of $Z_{KL} = Z_{LK}$:

$$\left|\overline{H}_2\right| = -B_K B_K Z_{LL} + 2B_K B_L Z_{KL} - B_L B_L Z_{KK}. \tag{18}$$

From (2) and (10b-d), we get:

$$B_K = r;\ B_L = w;\ B_R = \rho. \tag{19a}$$

$$Z_{KK} = \alpha(\alpha-1)AK^{\alpha-2}L^{\beta}R^{\gamma};\ Z_{LL} = \beta(\beta-1)AK^{\alpha}L^{\beta-2}R^{\gamma};$$

$$Z_{RR} = \gamma(\gamma-1)AK^{\alpha}L^{\beta}R^{\gamma-2}. \tag{19b}$$

$$Z_{KL} = Z_{LK} = \alpha\beta AK^{\alpha-1}L^{\beta-1}R^{\gamma};\ Z_{KR} = Z_{RK} = \alpha\gamma AK^{\alpha-1}L^{\beta}R^{\gamma-1};$$

$$Z_{LR} = Z_{RL} = \beta\gamma AK^{\alpha}L^{\beta-1}R^{\gamma-1}. \tag{19c}$$

Substitution of the values of $B_K, B_L, Z_{KK}, Z_{LL}, Z_{KL}$ from (19a-c) into (18) yields:

$$\left|\overline{H}_2\right| = AK^{\alpha}L^{\beta}R^{\gamma}\left(-r^2\beta^2L^{-2} + r^2\beta L^{-2} + 2rw\alpha\beta K^{-1}L^{-1} - w^2\alpha^2K^{-2} + w^2\alpha K^{-2}\right).$$

Substitution of the critical values of K^*, L^*, R^* from (11a-c) into above equation, and after straightforward but tedious calculation yields:

$$\left|\overline{H}_2\right| = A\left(\frac{\alpha+\beta}{\alpha\beta}\right)\left(\frac{\alpha^\alpha \beta^\beta \gamma^\gamma B^\psi}{r^\alpha w^\beta \rho^\gamma \psi^\psi}\right)\left(\frac{r^2 w^2 \psi^2}{B^2}\right), \tag{20a}$$

where $\psi = \alpha + \beta + \gamma$.

Similarly, from (17b), we expand the determinant, noticing that the second partial derivative of $Z_{KL} = Z_{LK}, Z_{KR} = Z_{RK}$, and $Z_{LR} = Z_{RL}$:

$$\begin{aligned}\left|\overline{H}\right| = &-B_K B_K Z_{LL} Z_{RR} + B_K B_K Z_{LR} Z_{LR} + 2B_K B_L Z_{KL} Z_{RR} - 2B_K B_R Z_{KL} Z_{LR} \\ &- 2B_K B_L Z_{KR} Z_{LR} + 2B_K B_R Z_{KR} Z_{LL} - B_L B_L Z_{KK} Z_{RR} + 2B_L B_R Z_{KK} Z_{LR} \\ &+ B_L B_L Z_{KR} Z_{KR} - 2B_L B_R Z_{KR} Z_{KL} - B_R B_R Z_{KK} Z_{LL} + B_R B_R Z_{KL} Z_{KL}.\end{aligned}$$

Substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{LL}, Z_{RR}, Z_{KL}, Z_{KR}, Z_{LR}$ from (19a-c) into above equation, and after straightforward but tedious calculation, we get:

$$\left|\overline{H}\right| = A^2 K^{2\alpha} L^{2\beta} R^{2\gamma} \begin{Bmatrix} r^2\beta^2\gamma L^{-2}R^{-2} + r^2\beta\gamma^2 L^{-2}R^{-2} - r^2\beta\gamma L^{-2}R^{-2} - 2rw\alpha\beta\gamma K^{-1}L^{-1}R^{-2} \\ -2r\rho\alpha\beta\gamma K^{-1}L^{-2}R^{-1} + w^2\alpha^2\gamma K^{-2}R^{-2} + w^2\alpha\gamma K^{-2}R^{-2} \\ -w^2\alpha\gamma K^{-2}R^{-2} - 2w\rho\alpha\beta\gamma K^{-2}L^{-1}R^{-1} + \rho^2\alpha^2\beta K^{-2}L^{-2} \\ +\rho^2\alpha\beta K^{-2}L^{-2} - \rho^2\alpha\beta K^{-2}L^{-2} \end{Bmatrix}.$$

Similarly, by substituting the critical values of K^*, L^*, R^* from (11a-c) into above equation, and after straightforward but tedious calculation, we get:

$$\left|\overline{H}\right| = -A^2\left(\frac{\alpha^{2\alpha}\beta^{2\beta}\gamma^{2\gamma}B^{2\psi}}{r^{2\alpha}w^{2\beta}\rho^{2\gamma}\psi^{2\psi}}\right)\left(\frac{r^2 w^2 \rho^2 \psi^5}{\alpha\beta\gamma B^4}\right), \tag{20b}$$

where $\psi = \alpha + \beta + \gamma$.

Since $A > 0, \alpha > 0, \beta > 0, \gamma > 0,$ and r, w, ρ are the costs of inputs and hence are positive, while B is budget that will never be negative, therefore, from (20a) $\left|\overline{H}_2\right| > 0$ and from (20b) $\left|\overline{H}\right| < 0$, as required by (17a) and (17b), respectively. Equations (20a) and (20b) are sufficient conditions satisfied to state that the stationary point obtained in (12) is a relative maximum point. Thus, the value of the output function obtained in (13) is indeed a relative maximum value.

6. COMPARATIVE STATIC ANALYSIS

Now, since sufficient conditions are satisfied, we drive further results of economic interest. Mathematically, we solve the four equations in (10a-d) for $K, L, R, \text{and}\ \lambda$ in terms of $r, w, \rho, \text{and}\ B$, and compute sixteen partial derivatives: $\frac{\partial K}{\partial r}, \cdots, \frac{\partial L}{\partial r}, \cdots, \frac{\partial R}{\partial r}, \cdots, \frac{\partial \lambda}{\partial r}, \cdots,$ etc.

These partial derivatives are referred to as the comparative static of the model. The model's usefulness is to determine how accurately it predicts the adjustments in the agency's input behaviour, that is, how the agency will react to the changes in the costs of capital, labour, and other inputs. Since we have assumed that the left side of each equation in (10) is continuously differentiable and that the solution exists, then by the Implicit Function Theorem $K, L, R, \text{and}\ \lambda$ will each be continuously differentiable function of $r, w, \rho, \text{and}\ B$, if the Jacobian matrix

$$J = \begin{bmatrix} 0 & -B_K & -B_L & -B_R \\ -B_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RL} & Z_{RR} \end{bmatrix}, \qquad (21)$$

is non-singular at the optimum point $\left(K^*, L^*, R^*, \lambda^*\right)$. As the sufficient conditions are met, so the determinant of (21) does not vanish at the optimum, that is, $\left|J\right| = \left|\overline{H}\right|$; consequently we apply the Implicit Function Theorem. Let F be the vector-valued function defined for the point $\left(\lambda^*, K^*, L^*, R^*, r, w, \rho, B\right) \in R^8$, and taking the values in R^4, whose components are given by the left side of the equations in (10a-d). By the Implicit Function Theorem, the equation

$$F\left(\lambda^*, K^*, L^*, R^*, r, w, \rho, B\right) = 0, \qquad (22)$$

may be solved in the form of

$$\begin{bmatrix} \lambda^* \\ K^* \\ L^* \\ R^* \end{bmatrix} = G\left(r, w, \rho, B\right). \qquad (23)$$

Moreover, the Jacobian matrix for G is given by

$$\begin{bmatrix} \frac{\partial\lambda^*}{\partial r} & \frac{\partial\lambda^*}{\partial w} & \frac{\partial\lambda^*}{\partial\rho} & \frac{\partial\lambda^*}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial\rho} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial\rho} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial\rho} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -J^{-1}\begin{bmatrix} -K^* & -L^* & -R^* & 1 \\ -\lambda^* & 0 & 0 & 0 \\ 0 & -\lambda^* & 0 & 0 \\ 0 & 0 & -\lambda^* & 0 \end{bmatrix}, \qquad (24)$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left side in (10) with respect to $r,$ then w, then ρ, and then B. Let C_{ij} be the cofactor of the element in the ith row and jth column of J, and then inverting J using the method of cofactor gives: $J^{-1} = \frac{1}{|J|}C^T$, where $C = \left(C_{ij}\right)$.

Thus, following the matrix multiplication rule, (24) can further be expressed in the following form:

$$\begin{bmatrix} \frac{\partial\lambda^*}{\partial r} & \frac{\partial\lambda^*}{\partial w} & \frac{\partial\lambda^*}{\partial\rho} & \frac{\partial\lambda^*}{\partial B} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial\rho} & \frac{\partial K^*}{\partial B} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial\rho} & \frac{\partial L^*}{\partial B} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial\rho} & \frac{\partial R^*}{\partial B} \end{bmatrix} = -\frac{1}{|J|}\begin{bmatrix} -K^*C_{11}-\lambda^*C_{21} & -L^*C_{11}-\lambda^*C_{31} & -R^*C_{11}-\lambda^*C_{41} & C_{11} \\ -K^*C_{12}-\lambda^*C_{22} & -L^*C_{12}-\lambda^*C_{32} & -R^*C_{12}-\lambda^*C_{42} & C_{12} \\ -K^*C_{13}-\lambda^*C_{23} & -L^*C_{13}-\lambda^*C_{33} & -R^*C_{13}-\lambda^*C_{43} & C_{13} \\ -K^*C_{14}-\lambda^*C_{24} & -L^*C_{14}-\lambda^*C_{34} & -R^*C_{14}-\lambda^*C_{44} & C_{14} \end{bmatrix}. \; 25)$$

Now, we study the effects of changes in $r, w, \rho,$ and B on $K, L,$ and R. Firstly, we find out the effect on capital K when it's interest rate increases. From (25), we get:

$$\frac{\partial K^*}{\partial r} = -\frac{1}{|J|}\left[-K^*C_{12} - \lambda^*C_{22}\right] = -\frac{K^*}{|J|}\begin{vmatrix} -B_K & Z_{KL} & Z_{KR} \\ -B_L & Z_{LL} & Z_{LR} \\ -B_R & Z_{RL} & Z_{RR} \end{vmatrix} + \frac{\lambda^*}{|J|}\begin{vmatrix} 0 & -B_L & -B_R \\ -B_L & Z_{LL} & Z_{LR} \\ -B_R & Z_{RL} & Z_{RR} \end{vmatrix}.$$

Expansion of above determinants yields:

$$\frac{\partial K^*}{\partial r} = -\frac{K^*}{|J|}\left\{-B_K Z_{LL} Z_{RR} + B_K Z_{LR} Z_{LR} + B_L Z_{KL} Z_{RR} - B_R Z_{KL} Z_{LR} - B_L Z_{KR} Z_{LR} + B_R Z_{KR} Z_{LL}\right\}$$
$$+ \frac{\lambda^*}{|J|}\left\{-B_L B_L Z_{RR} + 2B_L B_R Z_{LR} - B_R B_R Z_{LL}\right\}.$$

Substituting the values of $B_K, B_L, B_R, Z_{LL}, Z_{RR}, Z_{KL}, Z_{KR}$, and Z_{LR} from (19a-c) into above equation, and after straightforward calculation, we get:

$$\frac{\partial K^*}{\partial r} = -\frac{K^*}{|J|}\left(\beta\gamma A^2 K^{2\alpha} L^{2\beta} R^{2\gamma}\right)\left\{r\beta L^{-2}R^{-2} + r\gamma L^{-2}R^{-2} - rL^{-2}R^{-2} - w\alpha K^{-1}L^{-1}R^{-2} - \rho\alpha K^{-1}L^{-2}R^{-1}\right\}$$
$$+ \frac{\lambda^*}{|J|}\left(AK^{\alpha}L^{\beta}R^{\gamma}\right)\left\{-w^2\gamma^2 R^{-2} + w^2\gamma R^{-2} + 2w\rho\beta\gamma L^{-1}R^{-1} - \rho^2\beta^2 L^{-2} + \rho^2\beta L^{-2}\right\}.$$

Since $|J| = |\overline{H}|$, therefore, by substituting the value of $|\overline{H}|$ from (20b), as well as the optimal values K^*, L^*, R^*, λ^* from (11a-d) into the above equation, and after straightforward but tedious calculation, we get:

$$\frac{\partial K^*}{\partial r} = -\frac{\alpha B}{r^2(\alpha + \beta + \gamma)}.$$

Since $\alpha > 0,\ \beta > 0,\ \gamma > 0,$ and $r > 0,$ and B is the budget of the agency that can never be negative, therefore,

$$\frac{\partial K^*}{\partial r} < 0, \tag{26}$$

which indicates that if the interest rate or services of capital K increases, the agency may consider decreasing the level of input K.

Secondly, we examine the effects on labour L when the interest rate of capital K increases. Again from (25), we get:

$$\frac{\partial L^*}{\partial r} = -\frac{1}{|J|}\left[-K^*C_{13} - \lambda^* C_{23}\right] = \frac{K^*}{|J|}\begin{vmatrix} -B_K & Z_{KK} & Z_{KR} \\ -B_L & Z_{LK} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RR} \end{vmatrix} - \frac{\lambda^*}{|J|}\begin{vmatrix} 0 & -B_K & -B_R \\ -B_L & Z_{LK} & Z_{LR} \\ -B_R & Z_{RK} & Z_{RR} \end{vmatrix}.$$

$$\frac{\partial L^*}{\partial r} = \frac{K^*}{|J|}\{-B_K Z_{LK} Z_{RR} + B_K Z_{RK} Z_{LR} + B_L Z_{KK} Z_{RR} - B_R Z_{KK} Z_{LR} - B_L Z_{KR} Z_{RK} + B_R Z_{KR} Z_{LK}\}$$
$$-\frac{\lambda^*}{|J|}\{-B_K B_L Z_{RR} + B_K B_R Z_{LR} + B_L B_R Z_{RK} - B_R B_R Z_{LK}\}.$$

By substituting the values of $B_K, B_L, B_R, Z_{KK}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (19a-c) into above equation, and after simplification, we get:

$$\frac{\partial L^*}{\partial r} = \frac{K^*}{|J|}\left(\alpha\gamma A^2 K^{2\alpha} L^{2\beta} R^{2\gamma}\right)\left\{r\beta K^{-1}L^{-1}R^{-2} - w\alpha K^{-2}R^{-2} - w\gamma K^{-2}R^{-2} + wK^{-2}R^{-2} + \rho\beta K^{-2}L^{-1}R^{-1}\right\}$$
$$-\frac{\lambda^*}{|J|}\left(AK^{\alpha}L^{\beta}R^{\gamma}\right)\left\{-rw\gamma^2 R^{-2} + rw\gamma R^{-2} + r\rho\beta\gamma L^{-1}R^{-1} + w\rho\alpha\gamma K^{-1}R^{-1} - \rho^2\alpha\beta K^{-1}L^{-1}\right\}.$$

By substituting the optimal values of K^*, L^*, R^*, λ^* from (11a-d) into above equation, and after straightforward but tedious calculation, we get:

$$\frac{\partial L^*}{\partial r} = \frac{A^2}{|J|}\left\{\left(\frac{1}{\gamma}\right)\left(\frac{\alpha^{2\alpha}\beta^{2\beta}\gamma^{2\gamma}}{r^{2\alpha}w^{2\beta}\rho^{2\gamma}}\right)\left(\frac{B^{2\psi}}{\psi^{2\psi}}\right)\left(\frac{rw\rho^2\psi^3}{B^3}\right) - \left(\frac{1}{\gamma}\right)\left(\frac{\alpha^{2\alpha}\beta^{2\beta}\gamma^{2\gamma}}{r^{2\alpha}w^{2\beta}\rho^{2\gamma}}\right)\left(\frac{B^{2\psi}}{\psi^{2\psi}}\right)\left(\frac{rw\rho^2\psi^3}{B^3}\right)\right\}$$

$$\frac{\partial L^*}{\partial r} = 0. \tag{27}$$

This indicates that there will be no effect on the level of labour L, if the interest rate of capital K increases. This also indicates that in this case labour and capital are complement to each other.

The above analysis relates to the effects of a change in interest rate of capital K; our results are readily adaptable to the case of a change in wage rate of labour L, as well as to a change in cost of other inputs R.

Next, we analyze the effect of a change in budget B. Suppose that the service-providing agency gets additional budget in order to increase it's services; then naturally, we can expect that there will be an increase in its inputs of K, L, and R. We examine and verify this mathematically as follows. From (25), we get:

$$\frac{\partial K^*}{\partial B} = -\frac{1}{|J|}[C_{12}] = \frac{1}{|J|}\begin{vmatrix} -B_K & Z_{KL} & Z_{KR} \\ -B_L & Z_{LL} & Z_{LR} \\ -B_R & Z_{RL} & Z_{RR} \end{vmatrix}.$$

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left\{-B_K Z_{LL} Z_{RR} + B_K Z_{LR} Z_{RL} + B_L Z_{KL} Z_{RR} - B_R Z_{KL} Z_{LR} - B_L Z_{KR} Z_{RL} + B_R Z_{KR} Z_{LL}\right\}$$

.

By substituting the values of $B_K, B_L, B_R, Z_{LL}, Z_{RR}, Z_{LK}, Z_{KR}, Z_{LR}$ from (19a-c) into above equation, and after simplification, we get:

$$\frac{\partial K^*}{\partial B} = \frac{1}{|J|}\left(\beta\gamma A^2 K^{2\alpha} L^{2\beta} R^{2\gamma}\right)\left\{r\beta L^{-2} R^{-2} + r\gamma L^{-2} R^{-2} - rL^{-2} R^{-2} - w\alpha K^{-1} L^{-1} R^{-2} - \rho\alpha K^{-1} L^{-2} R^{-1}\right\}.$$

Again, since $|J| = |\overline{H}|$, therefore, by putting the value of $|\overline{H}|$ from (20b), as well as the optimal values of K^*, L^*, R^* from (11a-c), and after straightforward calculation, we get:

$$\frac{\partial K^*}{\partial B} = \frac{\alpha}{r(\alpha + \beta + \gamma)}.$$

Again, since $\alpha > 0,\ \beta > 0,\ \gamma > 0,$ and $r > 0$, therefore,

$$\frac{\partial K^*}{\partial B} > 0. \tag{28}$$

which verifies our assumption and common sense that when the budget size increases, the agency may consider increasing its level of inputs: capital, labour, and other inputs, in order to increase the output services. Our results and discussion are true for $\frac{\partial L^*}{\partial B} > 0$, $\frac{\partial R^*}{\partial B} > 0$ as well.

7. CONCLUSION AND RECOMMENDATIONS

In this article, we have applied Lagrange multiplier method to an agency's output maximization problem subject to budget constraint, using necessary and sufficient conditions for optimal values – in this particular case, maximization of the output of an agency. It is demonstrated that value of the Lagrange multiplier is positive, providing its reasonable interpretation; that is, if the agency is asked to increase (decrease) 1 unit of its output, it would cause total budget to increase (decrease) by approximately λ^* units. With the help of comparative static analysis and application of Implicit Function Theorem, we mathematically showed the behaviour of the agency, and suggest that if the cost of a particular input increases, the agency needs to consider decreasing the level of that particular input; at the same time, and there is no effect on level of other inputs. As well as, we demonstrated mathematically that when the budget increases the agency may consider increasing its level of inputs: capital, labour, and other inputs, in order to increase the output.

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