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Ekkehart Schlicht:

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Department of Economics
University of Munich

Volkswirtschaftliche Fakultät
Ludwig-Maximilians-Universität München

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A Variant of Uzawa's Theorem

Ekkehart Schlicht

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Department of Economics, Ludwig-Maximilians-Universität,
Schackstr. 4, 80539 Munich, Germany, schlicht@lmu.de

Abstract

UZAWA (1961) has shown that balanced growth requires technological progress to be strictly HARROD neutral (purely labor-augmenting). This paper offers a slightly more general variant of the theorem that does not require assumptions about savings behavior or factor pricing and is much easier to prove.

UZAWA's (1961) theorem states, broadly speaking, that balanced growth requires technological progress to be HARROD neutral (purely labor-augmenting) along the equilibrium growth path. This is an extremely restrictive, and consequently extremely decisive, requirement, establishing that steady-state growth is a highly singular and therefore highly improbable case.[1] Yet textbooks mention

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[1] As AGHION and HOWITT (1998, 16 n.) remark, "there is no good reason that technological change takes that form." This singularity is *not* removed by theories about an induced bias in technological progress (KENNEDY 1964, SAMUELSON 1965, VON WEIZSÄCKER, 1966, DRANDAKIS and PHELPS 1966, ACEMOGLU 2003). Theses theories require a "innovation possibility frontier" remaining invariant over decades if not centuries. This seems even less probable than assuming HARROD-neutrality right away. On the other hand, disposing of the assumption would lead to a model that could be fitted to *any* devolopment, just by postulating a suitable bias in technological change. The "new" growth theory favors, perhaps for that reason, the direct assumption. I recollect that many theorists (including myself) abandoned "old" growth theory around 1970 because they were not prepared to build their theories on such shaky foundations.

the issue only in a cavalier manner, if at all.[2] This may be caused by the original proof being quite intricate. The purpose of this note is to provide a very short proof for a more general variant of the theorem. The theorem establishes that exponential growth implies HARROD neutrality. ("Exponential growth" refers to the case that all key variables grow exponentially; "balanced growth," requiring certain variables to grow in proportion, is covered as a special case.) In contrast to the classical statement by UZAWA (1961) and the more recent reformulation by JONES and SCRIMEGOUR (2004), the theorem does not involve assumptions about factor pricing (such as marginal productivity theory) or savings behavior.

Consider an economy with a neoclassical production function F. This function relates, at any point in time t, the quantity produced, denoted by Y_t, to labor input N_t and capital input K_t. The production function is assumed to exhibit, at any point in time, constant returns to scale. Due to technological progress, it shifts over time, and we write:

$$Y_t = F(N_t, K_t, t) \tag{1}$$

with

$$F(\lambda N, \lambda K, t) = \lambda F(N, K, t) \quad \text{for all} \ (N, K, t, \lambda) \in \mathbb{R}_+^4. \tag{2}$$

Labor input N grows exponentially at rate n:

$$N_t = e^{nt} N_0. \tag{3}$$

Consumption at time t is denoted by C_t. Investment equals savings ($Y_t - C_t$). The capital stock is augmented by savings and reduced by depreciation at the rate δ. Hence the capital stock changes over time according to

$$\dot{K}_t = Y_t - C_t - \delta K_t. \tag{4}$$

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[2]Books like ABEL and BERNANKE (2005, 362-5), AGÉNOR (2004, 440), AGHION and HOWITT (1998, 16, 65), BARRO (1997, 429), BLANCHARD (2006, 248), BLANCHARD and FISCHER (1989, 3-4), BRANSON (1989, 638 f.), BURMEISTER and DOBELL (1970, 78), BURDA and WYPLOSZ (1997, 112-24), FROYEN (2005, 78-85), GÄRTNER (2003, 238-41), HACCHE (1979, 101), MANKIW (2003, 208-9), ROMER (1996, 7), or WILLIAMSON (2005, 185-212) do not treat the problem in any intelligible way, while some older books like BARRO and SALA-I-MARTIN (1995, 54-5) and NEUMANN (1994, 40) try to convey an idea about the issue.

Theorem (Variant of UZAWA's theorem of 1961). If the system (1)-(4) possesses a solution where Y_t, C_t, and K_t are all nonnegative and grow with constant growth rates rates y, c, and k, respectively, we can write

$$F(N_t, K_t, t) = G\left(N_t e^{(y-n)t}, K_t\right). \tag{5}$$

According to this theorem, exponential growth requires technological progress to be HARROD neutral (purely labor augmenting) along the growth path, with a rate of progress of $y - n$.

Proof. By assumption we have

$$
\begin{aligned}
Y_t &= Y_0 e^{yt} \\
C_t &= C_0 e^{ct} \\
K_t &= K_0 e^{kt}.
\end{aligned}
\tag{6}
$$

From (4) and (6) we obtain

$$(k + \delta) K_t = Y_t - C_t \tag{7}$$

or

$$(k + \delta) K_0 = Y_0 e^{(y-k)t} - C_0 e^{(c-k)t} \tag{8}$$

for all t. Taking time derivatives yields

$$\left(y - k\right) Y_0 e^{(y-k)t} - (c - k) C_0 e^{(c-k)t} = 0$$

which implies

$$\left(y - k\right) Y_0 e^{(y-c)t} - (c - k) C_0 = 0$$

and therefore either $y = k$ and $c = k$, or $y = c$. If $y = c$, it follows that $(y - k)(Y_0 - C_0) = 0$. As $Y_0 = C_0$ would imply $K_0 = 0$ by (6) and (7) and this is ruled out by assumption, we must have $y = k$ in any case.

Define

$$G(N, K) := F(N, K, 0). \tag{9}$$

As $Y_0 = G(N_0, K_0)$, $Y_t = Y_0 e^{yt}$, $N_0 = N_t e^{-nt}$, $K_0 = K_t e^{-kt}$, and G is linear homo-

geneous, we can write

$$Y_t = G\left(N_t e^{(y-n)t}, K_t e^{(y-k)t}\right)$$

As $y = k$, this proves the theorem.

As noted in the proof, exponential growth requires production and consumption to grow at the common rate y. Hence the savings rate must be constant.

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