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# BELIEF NON-EQUIVALENCE AND FINANCIAL TRADE: A COMMENT ON A RESULT BY ARAUJO AND SANDRONI*

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## Abstract

Aloisio Araujo and Alvaro Sandroni have shown in [1] that in a complete-markets economy in which there are no exogenous bounds to financial trade, existence of equilibrium requires agents with prior beliefs that agree on zero-probability events, and, therefore, with asymptotically homogeneous posteriors. This note illustrates the extent to which the result depends on market completeness: in general, equilibrium requires compatibility of beliefs only up to the revenue transfer opportunities allowed by the market; when the market is sufficiently incomplete, generically on the space of asset returns, even individuals who disagree on zero-probability events meet that "constrained-compatibility" requirement.

**Key words**: general equilibrium, heterogeneous beliefs, existence.

**JEL Classification**: D52, G1.

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# NO-EQUIVALENCIA DE EXPECTATIVAS E INTERCAMBIO FINANCIERO: UN COMENTARIO A UN RESULTADO DE ARAUJO Y SANDRONI


## Resumen

Aloisio Araujo y Alvaro Sandroni han mostrado en [1] que en una economía con mercados completos en la que no existen restricciones exógenas al intercambio en mercados financieros, la existencia del equilibrio requiere que las expectativas (priors) de los agentes sobre eventos de probabilidad cero coincidan y, por lo tanto, que sean asintóticamente homogéneas (las posteriores). Esta nota ilustra qué tanto el resultado depende de la completitud de los mercados: en general, la existencia del equilibrio requiere la compatibilidad de las expectativas (priors) condicionales a las transferencias permitidas por los mercados; cuando el mercado es lo suficientemente incompleto, genéricamente en el espacio de los retornos de los activos, aún individuos que discrepan en eventos de probabilidad cero, satisfacen esta restricción de “compatibilidad – restringida”.



**Palabras clave**: equilibrio general, expectativas heterogéneas, existencia.

**Clasificación JEL**: D52, G1.

In models that study the effects of belief heterogeneity in financial markets, it is oftentimes assumed that agents agree on what events have zero probability of occurring (e.g. [9], [7]), and/or that trades, in particular short sales, are exogenously bounded (e.g. [10], [5] and [4]). In [1], Aloisio Araujo and Alvaro Sandroni have shown that in an (infinite-horizon) economy with complete financial markets one of these two assumptions is indeed necessary: in their proof that competitive equilibrium exists only if all individuals have asymptotically homogeneous posterior beliefs, their argument is twofold: first, if prior beliefs do not coincide in the events to which zero probability is attached, then unbounded trading strategies, inconsistent with market clearing, are determined by individual optimization; and, second, when prior beliefs coincide in their null events, then, by the Blackwell-Dubbins theorem (see [3]), individual posterior beliefs will be asymptotically homogeneous.

Here, we study a similar problem without the assumption that markets are complete, and argue that, if financial markets are sufficiently incomplete, equilibrium trade generically exists, even when individual beliefs disagree on the null events. We consider a simple, two-period problem with a finite set of future states of the world. This setting suffices for our purposes, since, given the negative nature of our results, we will not invoke the Blackwell-Dubbins theorem. In contrast to the standard two-period problem, however, and since we want to allow unbounded trade, we assume that individuals *only* avoid bankruptcy *almost surely according to their own beliefs*.[1] The latter implies that the standard existence argument of [6] does not apply - indeed, neither does the general definition of no-arbitrage, nor its characterization via state-prices, since different individuals exhibit non-monotonic utility with respect to consumption in different states (see [2]).

We first introduce a condition on individual beliefs that makes them (pairwise) consistent up to the trading opportunities offered by the (possibly incomplete) asset market. We show

[1] But traders do not take into account the possibility that someone else may go bankrupt. In this sense, we only consider an ex-ante problem with somewhat naive consumers, unlike in [8].

that the condition is necessary for the existence of equilibrium prices for trade between two individuals. Using an argument similar to [6], and with some qualification, we also show that the condition is sufficient. When markets are complete, the condition is equivalent to requiring agreement on zero-probability events. However, we show that even when individual beliefs disagree in their null events, if markets are sufficiently incomplete then the condition is satisfied in a generic set of asset returns.

# 1 Constrained-compatible beliefs

$\mathcal{I} = \{1, ..., I\}$, with $I \in \mathbb{N}$, is a society.

$\mathcal{S} = \{1, ..., S\}$, with $S \in \mathbb{N}$, is the set of future states of the world. $s = 0$ is used to denote the present date.

$\mathcal{A} = \{1, ..., A\}$, with $A \in \mathbb{N}$, is the set of assets. The return of asset $a \in \mathcal{A}$ is the (column) vector $r^a \in \mathbb{R}^S$. For each $s \in \mathcal{S}$, denote $r_s = \left(r_s^1, ..., r_s^A\right) \in \mathbb{R}^A$, taken as a row vector. Assume that $r^1 \gg 0$. A portfolio is a (column) vector $z \in \mathbb{R}^A$.

For each $i \in \mathcal{I}$, $u^i : \mathbb{R} \longrightarrow \mathbb{R}$ is a monotone, continuous utility function, $\pi^i : \mathcal{P}(\mathcal{S}) \longrightarrow [0, 1]$ is a probability measure, with support denoted by $\mathcal{S}^i$,[2] and $\left(w_s^i\right)_{s=0}^S \in \mathbb{R}^{S+1}$ is state-contingent wealth. We assume that $w_0^i > 0$, and $w_s^i > 0$ for every $s \in \mathcal{S}^i$.

Asset prices are denoted by $q \in \mathbb{R}^A$, which is taken as a row vector.

Given prices $q$, define

$$B^i(q) = \left\{ z \in \mathbb{R}^A \middle| \, w_0^i - qz \geq 0 \text{ and } \forall s \in \mathcal{S}^i, w_s^i + r_s z \geq 0 \right\},$$

[2] This is, $\mathcal{S}^i = \{ s \in \mathcal{S} | \, \pi^i(\{s\}) > 0\}$

and $v^i_q : B(q) \longrightarrow \mathbb{R}$ by

$$v^i_q(z) = u^i\left(w^i_0 - qz\right) + \sum_{s\in\mathcal{S}^i} \pi^i(\{s\})\, u^i\left(w^i_s + r_s z\right).$$

Individual demand correspondences are, then, $Z^i : \mathbb{R}^A \rightrightarrows \mathbb{R}^A$, defined by

$$Z^i(q) = \arg\max_{z\in B^i(q)} v^i_q(z),$$

which implies that each individual considers as feasible trading strategies which are affordable and for which, according to her own beliefs, probability of bankruptcy is nil.[3]

**Lemma 1** *For each $i \in \mathcal{I}$, $Z^i(q) \neq \varnothing$ if, and only if,*

1. *There does not exist $z \in \mathbb{R}^A$ such that $qz < 0$ and $\pi^i(\{s|\, r_s z < 0\}) = 0$;*
2. *There does not exist $z \in \mathbb{R}^A$ such that $qz = 0$, $\pi^i(\{s|\, r_s z < 0\}) = 0$ and $\pi^i(\{s|\, r_s z > 0\}) > 0$.*

**Proof.** Proofs of lemmata are in the appendix. ■

We now impose a definition of belief compatibility that is mediated by the revenue transfers allowed by the financial markets.[4]

**Definition 1** *$j$'s beliefs are constrained-compatible with $i$'s beliefs if there does not exist $z \in \mathbb{R}^A$ such that $\pi^i(\{s|\, r_s z < 0\}) = 0$, $\pi^j(\{s|\, r_s z > 0\}) = 0$ and $\pi^i(\{s|\, r_s z > 0\}) > 0$.*

That is, $j$'s beliefs are constrained compatible with $i$'s beliefs if the financial markets do not allow $i$ to buy from $j$ a trade such that $i$ thinks she cannot lose revenue and may actually win some, and $j$ thinks he cannot lose either.

[3] That is, the budget constraint requires just that $\forall s \in S$, $\pi^i(\{s\})\left(w^i_s + r_s z\right) \geq 0$. Alternatively, we could lift all nonnegativivity constraints and impose Inada conditions on each $u^i$, which would yield the same results.

[4] Hence the name: here "constrained" signifies the same as in the definition of "constrained suboptimality" of [6].

A simple characterization of constrained compatibility of beliefs is provided by the theorem of the alternative: $j$'s beliefs are constrained compatible with $i$'s beliefs if, and only if, there exists $\left(\alpha^i, \beta^i, \gamma^i\right) \in \mathbb{R}_{++}^{\left|\mathcal{S}^i \setminus \mathcal{S}^j\right|} \times \mathbb{R}^{\left|\mathcal{S}^i \cap \mathcal{S}^j\right|} \times \mathbb{R}_{+}^{\left|\mathcal{S}^j \setminus \mathcal{S}^i\right|}$ such that

$$\sum_{s \in \mathcal{S}^i \setminus \mathcal{S}^j} \alpha_s^i r_s = \sum_{s \in \mathcal{S}^i \cap \mathcal{S}^j} \beta_s^i r_s + \sum_{s \in \mathcal{S}^j \setminus \mathcal{S}^i} \gamma_s^i r_s.$$

To see this, notice that, by definition, $j$'s beliefs are constrained compatible with $i$'s beliefs if, and only if, there exists no solution to the system

$$\begin{aligned} \forall s \in \mathcal{S}^i, r_s z &\geq 0 \\ \forall s \in \mathcal{S}^j, r_s z &\leq 0 \\ \exists s \in \mathcal{S}^i : r_s z &> 0 \end{aligned}$$

We can rewrite this system as

$$\begin{aligned} \left(-\sum_{s \in \mathcal{S}^i \setminus \mathcal{S}^j} r_s\right) z &< 0 \\ \forall s \in \mathcal{S}^i \cap \mathcal{S}^j, r_s z &= 0 \\ \forall s \in \mathcal{S}^i \setminus \mathcal{S}^j, \left(-r_s\right) z &\leq 0 \\ \forall s \in \mathcal{S}^j \setminus \mathcal{S}^i, r_s z &\leq 0 \end{aligned}$$

and it follows from [12, §22.2] that there exists no solution to the system if, and only if, for some $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}_{+}^{\left|\mathcal{S}^i \setminus \mathcal{S}^j\right|} \times \mathbb{R}^{\left|\mathcal{S}^i \cap \mathcal{S}^j\right|} \times \mathbb{R}_{+}^{\mathcal{S}^j \setminus \mathcal{S}^i} \times \mathbb{R}_{++}$ such that

$$\sum_{s \in \mathcal{S}^i \setminus \mathcal{S}^j} \alpha_s \left(-r_s\right) + \sum_{s \in \mathcal{S}^i \cap \mathcal{S}^j} \beta_s r_s + \sum_{s \in \mathcal{S}^j \setminus \mathcal{S}^i} \gamma_s r_s + \delta \left(-\sum_{s \in \mathcal{S}^i \setminus \mathcal{S}^j} r_s\right) = 0.$$

Letting $\alpha_s^i = \alpha_s + \delta > 0$, $\beta_s^i = \beta_s$ and $\gamma_s^i = \gamma_s \geq 0$ yields the result.

# 2 Trade

**Definition 2** *We say that agents $(i,j) \in \mathcal{I}^2$, $i \neq j$, trade, if there exists $q \in \mathbb{R}^A$ such that $Z^i(q) \cap -Z^j(q) \neq \varnothing$.*

We first show that constrained compatibility of beliefs is necessary for $(i,j)$ to trade.

**Proposition 1** *If $(i,j)$ trade, then $j$'s beliefs are constrained-compatible with $i$'s beliefs (and vice versa).*

**Proof.** Suppose not: let $q \in \mathbb{R}^A$ be such that $Z^i(q) \cap -Z^j(q) \neq \varnothing$ and let $z \in \mathbb{R}^A$ be such that $\pi^i(\{s | r_s z < 0\}) = 0$, $\pi^j(\{s | r_s z > 0\}) = 0$ and $\pi^i(\{s | r_s z > 0\}) > 0$.

If $qz < 0$, since $\pi^i(\{s | r_s z < 0\}) = 0$, then by lemma 1, part 1, $Z^i(q) = \varnothing$.

If $q(-z) < 0$, since $\pi^j(\{s | r_s(-z) < 0\}) = 0$, then, again by lemma 1, part 1, $Z^j(q) = \varnothing$.

Finally, if $q \cdot z = 0$, since $\pi^i(\{s | r_s z < 0\}) = 0$ and $\pi^i(\{s | r_s z > 0\}) > 0$, by lemma 1, part 2, $Z^i(q) = \varnothing$. ■

Since we are allowing bankruptcy in some states, sufficiency cannot be claimed from [6]. The proof is complicated by the fact that under beliefs that disagree on null events, the standard characterization of the set of no-arbitrage prices fails. For simplicity, we avoid these complications by using only positive, linearly independent assets. We follow [6], by imposing bounds on individual trades and apply the fixed-point argument to the bounded economy, where arbitrage poses no difficulty. We then relax the bounds asymptotically and use constrained compatibility to argue existence of trade even when only the nonnegativity constraints on states with positive probability, and not the artificial bounds, are imposed.

**Proposition 2** *Suppose that each $r^a > 0$ and that $\{r^a\}_{a \in \mathcal{A}}$ are linearly independent. Let $(i,j) \in \mathcal{I}^2$ be such that $\forall s \in \mathcal{S}$, $\pi^i(\{s\}) > 0$ or $\pi^j(\{s\}) > 0$. If j's beliefs are constrained-compatible with i's beliefs, and vice versa, then $(i,j)$ trade.*

**Proof.** Let $\mathbf{Q} = \left\{ \mathbf{q} = (q_0, q) \in \mathbb{R}_+ \times \mathbb{R}_+^A \middle| \sum_{a=0}^{A} q_a = 1 \right\}$.

Fix $n \in \mathbb{N}$.

For each $k \in \{i,j\}$, define the truncated budget correspondence $\mathbf{B}_n^k : \mathbf{Q} \rightrightarrows \mathbb{R}^{A+1}$, by

$$\mathbf{B}_n^k(\mathbf{q}) = \left\{ \mathbf{z} = (z_0, z) \in \mathbb{R} \times \mathbb{R}^A \middle| \begin{array}{c} q_0 \left(w_0^k - z_0\right) - qz \geq 0 \\ \forall s \in \mathcal{S}^k, w_s^k + r_s z \geq 0 \\ 0 \leq z_0 \leq (n+1)\, w_0^k \\ \forall a \in \mathcal{A}, -n \leq z_a \leq n \end{array} \right\},$$

which is a nonempty-, compact- and convex-valued correspondence, and is upper hemicontinuous. To show that $\mathbf{B}_n^k$ is lower hemicontinuous, let

$$\mathbf{T} = \left\{ \mathbf{z} = (z_0, z) \in \mathbb{R} \times \mathbb{R}^A \middle| \begin{array}{c} \forall s \in \mathcal{S}^k, w_s^k + r_s z \geq 0 \\ 0 \leq z_0 \leq (n+1)\, w_0^k \\ \forall a \in \mathcal{A}, -n \leq z_a \leq n \end{array} \right\},$$

which is convex, and define $\mathbf{f} : \mathbf{Q} \times \mathbf{T} \longrightarrow \mathbb{R}$ by $\mathbf{f}(\mathbf{q}, \mathbf{z}) = q_0 \left(w_0^k - z_0\right) - qz$. It is straightforward that $\mathbf{f}$ is continuous in $\mathbf{q}$ and concave in $\mathbf{z}$, and, since $w_0^k > 0$ and for every $s \in \mathcal{S}^k$, $w_s^k > 0$, for every $\mathbf{q} \in \mathbf{Q}$ there exists $z \in \mathbf{T}$ for which $\mathbf{f}(q, z) > 0$. This implies, by [11, §9.15], that $\mathbf{B}_n^k$ is lower hemicontinuous.

By the theorem of the maximum, it follows that the individual bounded demand correspondence $\mathbf{Z}_n^k : \mathbf{Q} \rightrightarrows \mathbb{R}^{A+1}$, defined as

$$\mathbf{Z}_n^k(\mathbf{q}) = \arg \max_{\mathbf{z} \in \mathbf{B}_n^k(\mathbf{q})} u_0^k(z_0) + \sum_{s \in \mathcal{S}^k} \pi^k(\{s\}) u_s^k\left(w_s^k + r_s z\right),$$

is upper hemicontinuous. This correspondence is also nonempty-, convex- and compact-valued.

Now, define $\mathbf{\Phi} : \mathbf{Q} \times \mathbf{N} \rightrightarrows \mathbf{Q} \times \mathbf{N}$, where

$$\mathbf{N} = \left[0, (n+1)\left(w_0^i + w_0^j\right)\right] \times [-n, n]^A,$$

as

$$\mathbf{\Phi}(\mathbf{q}, \mathbf{z}) = \left(\arg \max_{\mathbf{q} \in \mathbf{Q}} q_0\left(z_0 - w_0^i - w_0^j\right) + qz\right) \times \left(\mathbf{Z}_n^i(\mathbf{q}) + \mathbf{Z}_n^j(\mathbf{q})\right).$$

It follows by construction that $\mathbf{\Phi}$ is nonempty-, convex- and compact-valued, and upper hemicontinuous, so, by Kakutani's fixed-point theorem, there exists $(\mathbf{q}_n, \mathbf{z}_n) \in \mathbf{Q} \times \mathbf{N}$ such that $(\mathbf{q}_n, \mathbf{z}_n) \in \mathbf{\Phi}(\mathbf{q}_n, \mathbf{z}_n)$.

Since $\mathbf{z}_n \in \mathbf{Z}_n^i(\mathbf{q}_n) + \mathbf{Z}_n^j(\mathbf{q}_n)$, then $\mathbf{z}_n = \mathbf{z}_n^i + \mathbf{z}_n^j$, where $\mathbf{z}_n^i = (z_{n,0}^i, z_n^i) \in \mathbf{B}_n^i(\mathbf{q}_n)$, $\mathbf{z}_n^j = (z_{n,0}^j, z_n^j) \in \mathbf{B}_n^j(\mathbf{q}_n)$ and, by monotonicity of preferences, $q_{n,0}\left(z_{n,0} - w_0^i - w_0^j\right) + q_n z_n = 0$. Then, since

$$\mathbf{q}_n \in \arg \max_{\mathbf{q} \in \mathbf{Q}} q_0\left(z_{n,0} - w_0^i - w_0^j\right) + q z_n,$$

it follows that $z_{n,0} - w_0^i - w_0^j \leq 0$ and $z_n \leq 0$. Also, since $\mathbf{q}_n > 0$, if $z_{n,a} < 0$ for some $a \in \mathcal{A}$, then $q_{n,a} = 0$ (otherwise, $q_{n,0}\left(z_{n,0} - w_0^i - w_0^j\right) + q_n z_n < 0$) and, since $r^a > 0$ and for every $s \in \mathcal{S}$, by hypothesis, $\pi^i(\{s\}) > 0$ or $\pi^j(\{s\}) > 0$, it follows that for some $k \in \{i, j\}$, $z_a^k = n$ everywhere in $\mathbf{Z}_n^k(\mathbf{q}_n)$, so $z_{n,a} < 0$ would require $z_a^l < -n$ for $l \in \{i, j\} \setminus \{k\}$, for some $\mathbf{z}^l \in \mathbf{Z}_n^l(\mathbf{q}_n)$, which is impossible. It follows that $z_n = 0$.

Similarly, notice that if $q_{n,0} = 0$, then, by monotonicity, for all $k \in \{i, j\}$, everywhere in $\mathbf{Z}_n^k(\mathbf{q}_n)$, $z_0^k = (n+1)\,w_0^k$, so $z_{n,0} - w_0^i - w_0^j = n\left(w_0^i + w_0^j\right) > 0$ which is impossible; this implies that $q_{n,0} > 0$ and, therefore, that $z_{n,0} - w_0^i - w_0^j = 0$ (recall that $q_{n,0}\left(z_{n,0} - w_0^i - w_0^j\right) + q_n z_n = 0$ and $z_n = 0$).

It follows that $\frac{1}{q_{n,0}} q_n z_n^i = \frac{1}{q_{n,0}} q_n \left(-z_n^j\right)$ is bounded: since $\frac{1}{q_{n,0}} q_n z_n^k = w_0^k - z_{n,0}^k$, $z_{n,0}^k \geq 0$, then $-w_0^j \leq \frac{1}{q_{n,0}} q_n z_n^i \leq w_0^i$.

Then, $z_{0,n}^i = w_0^i - \frac{1}{q_{n,0}} q_n z_n^i$ is bounded and we now show that $z_n^i$ is bounded as well. First, notice that

1. If $\pi^i(\{s\}) > 0$, then $\left(r_s z_n^i\right)_{n=1}^{\infty}$ is bounded below;

2. If $\pi^j(\{s\}) > 0$, then $\left(r_s z_n^j\right)_{n=1}^{\infty}$ is bounded below, so $\left(r_s z_n^i\right)_{n=1}^{\infty}$ is bounded above.

Now, suppose that for some $\widehat{s} \in \mathcal{S}$, $\left(r_{\widehat{s}} z_n^i\right)_{n=1}^{\infty}$ is unbounded above. Then, for some $z \in \mathbb{R}^A$ with $r_{\widehat{s}} z > 0$, we have that, by 1,

$$\pi^i(\{s\}) > 0 \Longrightarrow r_s z \geq 0$$

and, by 2,

$$\pi^j(\{s\}) > 0 \Longrightarrow r_s z \leq 0$$

Again by 2, $\pi^j(\{\widehat{s}\}) = 0$, so it follows that $\pi^i(\{\widehat{s}\}) > 0$, which would contradict the fact that $j$'s beliefs are constrained compatible with $i$'s beliefs.

It follows that every $\left(r_s z_n^i\right)_{n=1}^{\infty}$ is bounded above, and since $i$'s beliefs are constrained compatible with $j$'s beliefs, by a similar argument, every $\left(r_s z_n^i\right)_{n=1}^{\infty}$ is bounded below. Since $\{r^a\}_{a\in\mathcal{A}}$ are linearly independent, it follows that $\left(z_n^i\right)_{n=1}^{\infty}$ is bounded, and since preferences are concave, it is immediate that, for $n$ large enough, $z_n^i \in Z^i\left(\frac{1}{q_{n,0}} q_n\right) \bigcap -Z^j\left(\frac{1}{q_{n,0}} q_n\right)$. ■

## 3 Constrained compatibility and equivalence of beliefs

Recall that $\pi^i$ is said to be absolutely continuous with respect to $\pi^j$ if $\pi^j(\{s\}) = 0$ implies that $\pi^i(\{s\}) = 0$. This is the property that [1] uses, via the Blackwell-Dubbins theorem, to imply convergence of conditional beliefs under complete markets. Notice first that absolute continuity is stronger than constrained compatibility.

**Proposition 3** *If $\pi^i$ is absolutely continuous with respect to $\pi^j$, then $j$'s beliefs are constrained-compatible with $i$'s beliefs.*

**Proof.** Let $z \in \mathbb{R}^A$ be such that $\pi^j(\{s | r_s z > 0\}) = 0$. By absolute continuity of $\pi^j$ with respect to $\pi^i$, it is immediate that $\pi^i(\{s | r_s z > 0\}) = 0$. ■

Now, the next proposition shows that, under complete markets, absolute continuity of $\pi^i$ with respect to $\pi^j$ is equivalent to constrained compatibility of $j$'s beliefs with $i$'s beliefs, which is to say that, when markets are complete, as in [1], trade requires absolutely continuous beliefs.

**Proposition 4** *Suppose that $S = A$, and $\{r^a\}_{a \in \mathcal{A}}$ are linearly independent. If $j$'s beliefs are constrained-compatible with $i$'s beliefs, then $\pi^i$ is absolutely continuous with respect to $\pi^j$.*

**Proof.** Suppose not: let $\widehat{s} \in \mathcal{S}$ such that $\pi^j(\{\widehat{s}\}) = 0$ and $\pi^i(\{\widehat{s}\}) > 0$. Let $z$ be such that $r_{\widehat{s}} z = 1$, while $r_s z = 0$ for every $s \in \mathcal{S} \setminus \{\widehat{s}\}$, which exists by linear independence. Then, $\pi^i(\{s | r_s z < 0\}) = 0$, $\pi^j(\{s | r_s z > 0\}) = 0$ and $\pi^i(\{s | r_s z > 0\}) > 0$. ■

The point of this note is that the necessity of absolute continuity only occurs by accident when markets are sufficiently incomplete.

Fix $(i, j)$ and assume that for every $s \in \mathcal{S}$, either $\pi^i(\{s\}) > 0$ or $\pi^j(\{s\}) > 0$.

Reorganizing states, let $\overline{s} \in \mathcal{S}$ be such that $\left(\pi^i(\{s\}) > 0 \Longleftrightarrow s \leq \overline{s}\right)$ and let $\underline{s} \in \mathcal{S}$ be such that $\left(\pi^j(\{s\}) > 0 \Longleftrightarrow s \geq \underline{s}\right)$. By assumption, $\underline{s} \leq \overline{s} + 1$.

Define the function $F^{i,j} : \mathbb{R}^A \times \mathbb{R}^{\underline{s}-1}_{++} \times \mathbb{R}^{S-\overline{s}}_{++} \times \mathbb{R}_{++} \times \left(\mathbb{R}^A\right)^S \longrightarrow \mathbb{R}^{S+1}$ as follows:

$$F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) = \begin{pmatrix} \left((r_s z - \alpha_s) r_s z\right)_{s=1}^{\underline{s}-1} \\ \\ (r_s z)_{s=\underline{s}}^{\overline{s}} \\ \\ \left((r_s z + \beta_s) r_s z\right)_{s=\overline{s}+1}^S \\ \\ \sum_{s=1}^{\underline{s}-1} (r_s z + \alpha_s) r_s z + \sum_{s=\underline{s}}^{\overline{s}} r_s z - \gamma \end{pmatrix}.$$

**Proposition 5** *$j$'s beliefs are not constrained-compatible with $i$'s beliefs if, and only if, there exists $\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) \in \mathbb{R}^A \times \mathbb{R}^{\underline{s}-1}_{++} \times \mathbb{R}^{S-\overline{s}}_{++} \times \mathbb{R}_{++} \times \left(\mathbb{R}^A\right)^S$ such that*

$$F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) = 0.$$

**Proof.** Suppose first that $j$'s beliefs are not constrained compatible with $i$'s beliefs, and fix $z \in \mathbb{R}^A$ such that $\pi^i(\{s | r_s z < 0\}) = 0$, $\pi^j(\{s | r_s z > 0\}) = 0$ and $\pi^i(\{s | r_s z > 0\}) > 0$. Define

$$\begin{aligned} (\forall s \in \{1, ..., \underline{s} - 1\}) \quad &: \quad \alpha_s = \begin{cases} r_s z \text{ if } r_s z > 0 \\ 1 \text{ otherwise} \end{cases} \\ (\forall s \in \{\overline{s} + 1, ..., S\}) \quad &: \quad \beta_s = \begin{cases} -r_s z \text{ if } r_s z < 0 \\ 1 \text{ otherwise} \end{cases} \\ \gamma \quad &= \quad \sum_{s: r_s z > 0} 2 (r_s z)^2 \end{aligned}$$

It is immediate that $\alpha \in \mathbb{R}_{++}^{\underline{s}-1}$ and $\beta \in \mathbb{R}_{++}^{S-\overline{s}}$, whereas, since $\pi^i\left(\{s|\, r_s z > 0\}\right) > 0$, $\gamma \in \mathbb{R}_{++}$. Notice that

$$\begin{aligned} r_s z &< 0 \Longrightarrow \pi^i\left(\{s\}\right) = 0 \Longrightarrow s > \overline{s} \\ r_s z &> 0 \Longrightarrow \pi^j\left(\{s\}\right) = 0 \Longrightarrow s < \underline{s} \end{aligned}$$

Since $\underline{s} < \overline{s} + 1$, it follows that

$$\begin{aligned} s \leq \underline{s} - 1 &\implies r_s z \geq 0 \\ \underline{s} \leq s \leq \overline{s} &\implies r_s z = 0 \\ s \geq \overline{s} + 1 &\implies r_s z \leq 0 \end{aligned}$$

Let $s \leq \underline{s} - 1$. If $r_s z > 0$, then $\alpha_s = r_s z$, so $(r_s z - \alpha_s)\, r_s z = 0$; alternatively, $r_s z = 0$ and the same conclusion is immediate. For $\underline{s} \leq s \leq \overline{s}$, it is immediate that $r_s z = 0$. Now, if $s \geq \overline{s} + 1$ and $r_s z < 0$, then $\beta_s = -r_s z$, so $(r_s z + \beta_s)\, r_s z = 0$; alternatively, $r_s z = 0$ and the same conclusion follows. It is also clear that $\sum_{s=1}^{\overline{s}} (r_s z + \alpha_s)\, r_s z - \gamma = 0$.

Now, suppose that $F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) = 0$ for $\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) \in \mathbb{R}^A \times \mathbb{R}_{++}^{\underline{s}-1} \times \mathbb{R}_{++}^{S-\overline{s}} \times \mathbb{R}_{++} \times \left(\mathbb{R}^A\right)^S$. Let $s \in \mathcal{S}$ be such that $r_s z < 0$; then, since $\alpha \gg 0$, it follows that $s \geq \overline{s} + 1$, and hence $\pi^i\left(\{s\}\right) = 0$. If, on the other hand, $s \in \mathcal{S}$ is such that $r_s z > 0$, then, because $\beta \gg 0$, it follows that $s \leq \underline{s} - 1$ and hence $\pi^j\left(\{s\}\right) = 0$. Finally, since $\sum_{s=1}^{\underline{s}-1} (r_s z + \alpha_s)\, r_s z + \sum_{s=\overline{s}}^{\overline{s}} r_s z = \sum_{s=1}^{\underline{s}-1} (r_s z + \alpha_s)\, r_s z = \gamma > 0$, it follows that for some $s \leq \underline{s} - 1$, $r_s z \neq 0$, which implies that $\pi^i\left(\{s\}\right) > 0$ and $r_s z > 0$. ■

**Lemma 2** *$F^{i,j}$ is transverse to* 0.

Let $\mathcal{R}^{i,j} \subseteq \left(\mathbb{R}^A\right)^S$ be the set of asset returns on which $j$'s beliefs are constrained compatible with $i$'s beliefs.

**Proposition 6** *If $A < \overline{s} - \underline{s} + 1$, then $\mathcal{R}^{i,j}$ contains an open subset of full Lebesgue measure.*

**Proof.** By lemma 2 and the Transversality theorem, it follows that the subset $\mathcal{R} \subseteq \left(\mathbb{R}^A\right)^S$, of $(r_s)_{s=1}^S$ such that $F^{i,j}\left(\cdot, (r_s)_{s=1}^S\right) \pitchfork 0$, is open and has full Lebesgue measure. Let $(r_s)_{s=1}^S \in \mathcal{R}$ and suppose that $F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) = 0$. By definition, this implies that $D_{z,\alpha,\beta,\gamma} F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right)$ has full row rank. Now, $D_{z,\alpha,\beta,\gamma} F^{i,j}$ has $S+1$ rows and $A + \underline{s} - 1 + S - \overline{s} + 1$ columns, and, since $A < \overline{s} - \underline{s} + 1$, it follows that it has strictly more rows than columns, which is impossible. It follows that $(r_s)_{s=1}^S \in \mathcal{R}$ only if it is not true that

$$\left(\exists\,(z, \alpha, \beta, \gamma) \in \mathbb{R}^A \times \mathbb{R}_{++}^{\underline{s}-1} \times \mathbb{R}_{++}^{S-\overline{s}} \times \mathbb{R}_{++}\right) : F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^S\right) = 0$$

which means, by proposition 5, that $(r_s)_{s=1}^S \in \mathcal{R}$ only if $(r_s)_{s=1}^S \in \mathcal{R}^{i,j}$. ■

The important implication is that constrained compatibility holds, generically, for any $\pi^i$ and $\pi^j$, regardless of absolute continuity. Since $\mathcal{I}$ is finite, pairwise constrained compatibility holds generically.

# 4 Concluding remarks:

[1] has argued that

| Existence of equilibrium and Market completeness | $\Longrightarrow$ | Consistency of prior beliefs | $\Longrightarrow$ | Convergence of posterior beliefs |
|---|---|---|---|---|

Modulo the differences in the models, we have shown that under market incompleteness traders may disagree in their beliefs. First,

$$\text{Existence of equilibrium trade} \Longrightarrow \text{Constrained consistency of beliefs},$$

while

$$\begin{array}{c}\text{Constrained consistency of beliefs}\\ \text{and}\\ \text{Positivity of Returns}\end{array} \Longrightarrow \text{Existence of equilibrium trade}\ .$$

So it is the (pairwise) consistency of individual beliefs, merely up to the trading opportunities offered by the asset market, that determines existence of equilibrium trade. As expected,

$$\text{Consistency of beliefs} \Longrightarrow \text{Constrained consistency of beliefs}$$

and,

$$\begin{array}{c}\text{Constrained consistency of beliefs}\\ \text{and}\\ \text{Market completeness}\end{array} \Longrightarrow \text{Consistency of beliefs}\ ,$$

so, as in [1],

$$\begin{array}{c}\text{Existence of equilibrium trade}\\ \text{and}\\ \text{Market completeness}\end{array} \Longrightarrow \text{Consistency of beliefs}\ .$$

However, enough incompleteness destroys last implication in a very robust way: *even with inconsistent beliefs, for almost all possible values of asset returns, beliefs are still constrained consistent.*

# Appendix

**Proof of lemma 1.** First, let $\overline{z} \in Z^i(q)$.

If 1 does not hold, let $\widetilde{z} = \overline{z} + z$. Then, $-q\widetilde{z} = -q\overline{z} - qz > -q\overline{z}$, and

$$r_s\overline{z} > r_s\widetilde{z} \Longleftrightarrow 0 > r_s z \Longrightarrow \pi^i(\{s\}) = 0.$$

It follows that $v^i(\widetilde{z}) > v^i(\overline{z})$, which is impossible, since $w_0^i - q\widetilde{z} > w_0^i - q\overline{z} \geq 0$, and, for every $s \in \mathcal{S}^i$, $w_s^i + r_s\widetilde{z} \geq w_s^i + r_s\overline{z} \geq 0$.

If 2 does not hold, let $\widetilde{z} = \overline{z} + z$. Then, $-q \cdot \widetilde{z} = -q \cdot \overline{z}$ and, as before, $\pi^i(\{s\}) > 0$ only if $r_s\overline{z} \leq r_s\widetilde{z}$. Moreover, for some $s \in \mathcal{S}$, $\pi^i(\{s\}) > 0$ and $r_s z > 0$, so $r_s\widetilde{z} > r_s\overline{z}$. Again, $v^i(\widetilde{z}) > v^i(\overline{z})$, which is a contradiction, because $w_0^i - q\widetilde{z} = w_0^i - q\overline{z} \geq 0$, and, for every $s \in \mathcal{S}^i$, $w_s^i + r_s\widetilde{z} \geq w_s^i + r_s\overline{z} \geq 0$.

Now, suppose that 1 and 2 hold. Renumber states if necessary, so that $\pi^i(\{s\}) > 0 \Longleftrightarrow s \leq \overline{s}$. Suppose first that the matrix $R^i = \begin{bmatrix} r_1^\top & \cdots & r_{\overline{s}}^\top \end{bmatrix}^\top$ has full column rank. Since $r^1 \gg 0$, it is straightforward that $R^i z > 0$ implies $qz > 0$, which is well known to imply that for some $(\mu_s)_{s=1}^{\overline{s}} \in \mathbb{R}_{++}^{\overline{s}}$, $q = \sum_{s=1}^{\overline{s}} \mu_s r_s$. Set

$$B = \left\{ x \in \mathbb{R}_+^{\overline{s}+1} \middle| \left(x_0 - w_0^i\right) + \sum_{s=1}^{\overline{s}} \mu_s \left(x_s - w_s^i\right) = 0 \text{ and } \left(\exists z \in \mathbb{R}^A\right) : \left(x_s - w_s^i\right)_{s=1}^{\overline{s}} = R^i z \right\}$$

is compact. Since $u^i$ is continuous, there exists

$$\overline{x} \in \arg\max_{x \in B} \left( u^i(x_0) + \sum_{s=1}^{\overline{s}} \pi^i(\{s\}) u^i\left(x_s^i\right) \right).$$

Let $\overline{z} \in \mathbb{R}^A$ be such that $\left(\overline{x}_s - w_s^i\right)_{s=1}^{\overline{s}} = V\overline{z}$. It is straightforward that $\overline{z} \in Z^i(q)$. If $\exists \widehat{z} \in \mathbb{R}^A$ such that $R^i\widehat{z} = 0$, then, by 1, $q\widehat{z} = 0$, so $Z^i(q) = \arg\max_{z \in B(q) \setminus [R^i]^\perp} v_q^i(z)$ and the result follows from the argument above. ■

**Proof of lemma 2.** With the following representative components of the function:

$$\left(\begin{array}{c} (r_1 z - \alpha_1)\, r_1 z \\ \vdots \\ \left(r_{\underline{s}-1} z - \alpha_{\underline{s}-1}\right) r_{\underline{s}-1} z \\ r_{\underline{s}} z \\ \vdots \\ r_{\overline{s}} z \\ \left(r_{\overline{s}+1} z + \beta_{\overline{s}+1}\right) r_{\overline{s}+1} z \\ \vdots \\ (r_S z + \beta_S)\, r_S z \\ \sum_{s=1}^{\overline{s}} (r_s z + \alpha_s)\, r_s z - \gamma \end{array}\right),$$

and with the representative arguments in the following order,

$$\left(\left[\begin{array}{cccccccccccccccc} z & \alpha_1 & \dots & \alpha_{\underline{s}-1} & \beta_{\overline{s}+1} & \dots & \beta_s & r_1 & \dots & r_{\underline{s}-1} & r_{\underline{s}} & \dots & r_{\overline{s}} & r_{\overline{s}+1} & \dots & r_S & \gamma \end{array}\right]\right),$$

we have that $DF^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^{S}\right)$ is the following matrix

**[Insert attached Matrix here]**

Fix $\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^{S}\right) \in \mathbb{R}^A \times \mathbb{R}_{++}^{\underline{s}-1} \times \mathbb{R}_{++}^{S-\overline{s}} \times \mathbb{R}_{++} \times \left(\mathbb{R}^A\right)^S$ such that $F^{i,j}\left(z, \alpha, \beta, \gamma, (r_s)_{s=1}^{S}\right) = 0$. Since $\sum_{s=1}^{\underline{s}-1} (r_s z + \alpha_s)\, r_s z + \sum_{s=\overline{s}}^{\overline{s}} r_s z = \gamma > 0$, it follows that $z \neq 0$.

Fix $s \leq \underline{s} - 1$. If $r_s z = 0$, it follows that $(2 r_s z - \alpha_s)\, z^\top = -\alpha_s z^\top$ and since $\alpha_s > 0$ and $z \neq 0$, we can perturb the $s$-th row, without perturbing any of the other first $S$ rows of the matrix. If $r_s z \neq 0$, we can obtain the same result by perturbing the column corresponding to $\alpha_s$.

Consider now $\underline{s} \leq s \leq \overline{s}$. Since $z \neq 0$, we can perturb the $s$-th row, without perturbing any of the other first $S$ rows of the matrix.

Now, for $s \geq \overline{s}+1$, suppose that $r_s z = 0$; it follows that $(2r_s z + \beta_s)\, z^\top = \beta_s z^\top$, and, since $\beta_s > 0$ and $z \neq 0$, that we can perturb the $s$-th row, without perturbing any of the other first $S$ rows of the matrix. If, alternatively, $r_s z \neq 0$, we can obtain the same result by perturbing the column corresponding to $\beta_s$.

This implies that the matrix, without its last column and its last row, has full row rank. It follows from the introduction of the last row and the last column, that $DF^{i,j}$ has full row rank. ■

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$$
\begin{bmatrix}
(2r_1z-\alpha_1)r_1 & -r_1z & \dots & 0 & 0 & \dots & 0 & (2r_1z-\alpha_1)z^\top & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(2r_{\underline{s}-1}z-\alpha_{\underline{s}-1})r_{\underline{s}-1} & 0 & \dots & -r_{\underline{s}-1}z & 0 & \dots & 0 & 0 & \dots & (2r_{\underline{s}-1}z-\alpha_{\underline{s}-1})z^\top & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
r_{\underline{s}} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & z^\top & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
r_{\overline{s}} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & z^\top & 0 & \dots & 0 & 0 \\
(2r_{\overline{s}+1}z+\beta_{\overline{s}+1})r_{\overline{s}} & 0 & \dots & 0 & r_{\overline{s}+1}z & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & (2r_{\overline{s}+1}z+\beta_{\overline{s}+1})z^\top & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(2r_Sz+\beta_S)r_S & 0 & \dots & 0 & 0 & \dots & r_Sz & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & (2r_Sz+\beta_S)z^\top & 0 \\
* & * & \dots & * & * & \dots & * & * & \dots & * & * & \dots & * & * & \dots & 0 & -1
\end{bmatrix}
$$