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Abstract

Most real-life bargaining is resolved gradually; two parties reach intermediate agreements without knowing the whole range of possibilities. These intermediate agreements serve as disagreement points in subsequent rounds. Cooperative bargaining solutions ignore these dynamics and can therefore yield accurate predictions only if they are robust to its specification. We

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identify robustness criteria which are satisfied by four of the best-known bargaining solutions, the Nash, Kalai-Smorodinsky, Proportional and Discrete Raiffa solutions. We show that the "robustness of intermediate agreements" plus additional well-known and plausible axioms, provide the first characterization of the Discrete Raiffa solution and novel axiomatizations of the other three solutions. Hence, we provide a unified framework for comparing these solutions' bargaining theories.

JEL classification: C78; D74

Keywords: Nash's bargaining problem, robustness, intermediate agreements, the Discrete Raiffa solution, the Nash solution, the Kalai-Smorodinsky solution, Proportional solutions.

1 Introduction

Nash's bargaining problem is a pair (S, d), where $S \subset \mathbb{R}^2$ is a convex and compact utility possibility set and d is the disagreement point, the utility allocation that results if no agreement is reached by the parties. A bargaining solution f associates each problem (S, d) with a unique point in S. Since Nash's (1950) seminal solution and axioms, various other solutions and axioms have been proposed.

One prominent axiom is the Step-by-Step Negotiation (SSN) axiom (Kalai, 1977). SSN requires that the bargaining outcome be invariant under decomposition of the bargaining process into stages: if parties know that they will face two nested sets in sequence, first a subset S of the set of feasible alternatives T and then T, then the solution outcome of the initial problem can function as an intermediate agreement for the subsequent problem. Kalai (1977) emphasized the advantage of SSN as follows:

> This principle is observed in actual negotiations (e.g., Kissinger's

step-by-step), and it is attractive since it makes the implementation of a solution easier. It is also attractive because we can view every bargaining situation that we encounter in life as a first step in a sequence of predictable or unpredictable bargaining situations that may still arise. Thus, the outcome of the current bargaining situation will be the threat point for the future ones.

Indeed, most real-life bargaining is resolved gradually; parties reach intermediate agreements without knowing the whole range of future possibilities, and these intermediate agreements serve as new disagreement points and pave the way for subsequent negotiations.[1] Cooperative bargaining solutions ignore these dynamics and can therefore yield accurate predictions only if they are robust to its specification. SSN does provide a substantial robustness test. When two parties face an uncertain bargaining prospect in which the problem could be either (S, d) or (T, e) with $d = e$ and $S \subset T$, SSN then suggests that they can reach an intermediate agreement at $f(S, d)$ before uncertainty is resolved, as moving the disagreement point from d to $f(S, d)$ has no effect to the bargaining outcome in either (S, d) or (T, e). We argue, however, that this test is too strong. When two parties consent to reach a point d' as an intermediate agreement, it is legitimate to say that

[1] Uncertainty is also resolved gradually in a single-person decision-making situation. A shopper in a typical supermarket faces more than 200 varieties of cookies, soups and cereals (Schwarz, 2004). The marketing and economics literatures provide well-established analyses and evidence that consumers do not consider all brands in a given market at once when making a purchase decision, and that the set of brands they consider changes over time as they learn more about the product, since the set later includes brands that they were initially unaware of (Chiang, Chib and Narasimhan, 1999; Goeree, 2008). See Huberman and Regev (2001) regarding financial decisions involving plans in a set of hundreds of available funds, and Dawes and Brown (2004) regarding university choices. Also see Rubinstein and Salant (2006) for a theoretical model in which individuals encounter the alternatives sequentially.

they expect to maintain the same relative bargaining position when moving from d to d'. Accordingly, SSN implies that, regardless of the form of the expansion T of S, $f(S,d)$ and d give them the same relative bargaining position! One would not expect parties to reach intermediate agreements within S — and especially the solution outcome of S as an intermediate agreement — even if they knew that the ultimate set T would contain S, unless some additional things about the relationship between S and T were also known.

Here, a class of weaker robustness tests is proposed. We consider a situation where two parties face an uncertain bargaining prospect in which the problem could be either (S,d) or (T,e) with $d=e$. Each test requires the parties to be able to reach an intermediate agreement when S and T conform to a specific relationship. We require such an intermediate agreement to be *robust*, that is, if we replace the initial disagreement point by it, it has no effect to the bargaining outcome. Below is a list of the circumstances of the robustness tests that we will consider here for this purpose.

1. S is included in T.

2. S and T have the same *ideal payoffs* — i.e., each party's highest possible individually rational payoff is the same in both S and T — and S and T need not include one another.

3. The parties expect to receive the same relative payoff gains in S and T, and S and T need not include one another.

4. The parties expect to make the same relative concessions in S and T, and S and T need not include one another.

Thus, in the first weakening of SSN's robustness test, certain points can serve as intermediate agreements whenever the bargaining sets S and T are nested, but any such intermediate agreement is not necessarily the solution outcome of S. In our second robustness test, certain points can serve as intermediate agreements whenever S and T share the same ideal point. Likewise, in the third and fourth robustness tests, certain points can serve as intermediate agreements whenever the parties expect to have either the same relative gains (in Criterion 3) or the same relative concessions (in Criterion 4) in both S and T.

The above four robustness tests will be termed "Robustness of Intermediate Agreements" (RIA) axioms. Each of these RIA axioms — when combined with some other well-known and plausible axioms — will lead to the axiomatization of the Discrete Raiffa solution (the first such that we know of for this solution[2]), as well as of the Nash, Kalai/Smorodinsky and Proportional solutions. Our results can be briefly summarized as follows:

(1) The Discrete Raiffa solution is characterized by the Midpoint Domination (MD) axiom, an RIA axiom, and the Independence of Non-Midpoint Dominating Alternatives (INMD) axiom.

(2) The Nash solution is characterized by MD, an RIA axiom, and the Disagreement Point Continuity (DCONT) axiom.

(3) The Kalai/Smorodinsky solution is characterized by MD, an RIA axiom, DCONT and the Strong Disagreement Point Monotonicity (SDM) axiom.

(4) The Proportional solutions are characterized by an RIA axiom, DCONT, the Weak Pareto Optimality axiom (WPO), the Pareto Continuity (PCONT) axiom and the Strong Individual Rationality (SIR) axiom.

[2] The most desirable feature of Raiffa's contribution, namely its description of the bargaining process, has nevertheless led researchers to seek — and provide characterizations of — a "continuous" version of the Raiffa solution (Livne, 1989, and Peters and van Damme, 1991). In addition, we have very recently been informed by Walter Trockel that in a few months he will send us a manuscript of his which contains an axiomatic characterization of the Discrete Raiffa solution.

Hence, we provide a unified framework for comparing these solutions' bargaining theories.

We will now consider an example in which two agents face an uncertain negotiation prospect, such that writing a contingent binding agreement regarding either of the two potential utility possibility sets is not possible. Consider a country with two major political parties that could win the up-coming elections. Depending on which party wins, the new government will provide firms in major sectors with either (1) tax breaks or (2) trade protection. Suppose that there are two major automobile companies in the country with sufficient production synergies between them. Given a prevailing recession in the country, it is taken for granted that these firms will tacitly be allowed to collude in the product market, regardless of the policy that will be pursued after the elections.

Observe that each policy will generate a different utility possibility set between the two automobile companies. The "tax break" policy with tacit collusion will generate a utility possibility set S, and the "trade protection" policy with tacit collusion will generate the set T. The disagreement point d involves their current profits. If these companies choose to wait, doing nothing until the new government gets elected and announces its policy, they will obtain d (note that they cannot write an overt contingent binding agreement since overt collusion is prohibited by law). However, they can instead reach an intermediate agreement $d' > d$ in the meantime. The intermediate agreement d' may involve forming a research joint venture (RJV); such an RJV can pave the way for their tacit collusion, regardless of the actual policy that the new government will announce.[3]

[3] Another example considers two high-tech research engineers who work in the same field. In the future, the government will adopt one of the two possibilities: (1) to subsidize some expensive strategic equipment or (2) to provide generous progressive tax deductions to individuals' earnings that will accrue in research teams. Here too, each policy will generate a different utility possibility

Note that since an intermediate agreement d' may possibly affect the bargaining solution outcome of any potential utility possibility set differently from the way in which the initial disagreement point d would, the ideal situation would be for the agents to agree on an intermediate agreement $d' > d$ which is robust: if both agents adhere to the same particular sharing rule (in the form of a bargaining solution), regardless of which utility possibility set is realized, it would not matter later whether they move to the bargaining solution outcome of the realized utility possibility set from (i) the intermediate agreement d' or (ii) their initial disagreement point d. Otherwise — i.e., if the intermediate agreement affects their future bargaining solution outcome differently from the way in which the initial disagreement point would have — at least one of the agents would not be willing to agree to the intermediate agreement.[4]

The remainder of the paper is organized as follows. In the next section, we provide a brief review of the relevant literature. In Section 3, we define some basic solutions and axioms, and in Section 4, we propose and discuss the motives for

set between the two engineers who are the players. Suppose that the "subsidy" policy regarding the major strategic equipment will generate a particular utility possibility set S and the "tax deduction" policy will generate the set T. Their initial disagreement point d involves their current earnings. If these individuals choose to wait and do nothing until the actual policy is announced, they will be obtaining d in the meantime. However, they could instead decide to reach an interim outcome $d' > d$ in the meantime. The interim outcome d' might involve renting office space together; they can use the joint office space regardless of whether they then use the expensive strategic equipment together without forming a research team, or start working as a research team without buying the expensive strategic equipment, which would not pay off without the subsidy.

[4]When the uncertainty cannot be resolved quickly (or will re-surface frequently), at least one of the parties may be better off using a series of intermediate agreements in time instead of committing to binding long-term agreements. This may also be the case if the feasible intermediate agreements are not neutral.

our RIA axioms. Section 5 provides characterizations of the Discrete Raiffa, Nash, Kalai/Smorodinsky and Proportional solutions. The final section concludes.

2 Relevant Literature

The significance and fundamental role of bargaining[5] were recognized as early as 1881 by Edgeworth, but for a very long time it was deemed to lack a clear solution.[6] In 1950, Nash proposed a framework which allowed a unique feasible outcome to be selected as the solution outcome of each given problem. Nash (1950) also provided the first axiomatic derivation of a bargaining solution, characterized by four axioms — namely WPO, Symmetry (SYM), Scale Invariance (SI), and Independence of Irrelevant Alternatives (IIA). Raiffa (1953) later criticized the Nash solution (and especially the IIA axiom), and proposed another solution which essentially described a discrete bargaining process, but which has never been characterized axiomatically. More then twenty years later, Kalai and Smorodinsky (1975) raised similar criticisms and characterized a new solution which, like the Discrete Raiffa solution, emphasized the parties' ideal payoffs.[7]

Initially, all characterizations employed an *independence* or *monotonicity* axiom pertaining to changes in the feasible set (pioneered by Nash, 1950, and Kalai

[5]Binmore (1994, p. 21): "much negotiation in real life ... create[s] a surplus that would otherwise be unavailable ... If you have a fancy house to sell that is worth $2m to you and $3m to me, then ... a surplus of $1m is available for us to split."

[6]See Roth (1979b, p. 5).

[7]There have been other solution concepts characterized axiomatically since then: the Egalitarian solution (Kalai, 1977; Roth, 1979a), the Equal Sacrifice solution (Aumann and Maschler, 1985; Chun, 1988), the Perles/Maschler solution (Perles and Maschler, 1981), the Equal Area solution (Anbarci, 1993; Anbarci and Bigelow, 1994), the Average Payoff solution (Anbarci, 1995), and the Dictatorial solutions (Bigelow and Anbarci, 1993).

and Smorodinsky, 1975, respectively).[8] The second generation characterizations then shifted the focus to *changes in the disagreement payoffs*, as well as to considerations of *uncertain disagreement points* (pioneered by Thomson, 1987, and Chun and Thomson, 1990, respectively). These axioms, however, typically did not refer to any bargaining process.[9] By adding such a process to the bargaining framework, Nash's (1953) Demand Game established a new research agenda, which has commonly been referred to as the Nash program (see Binmore, 1998). It uses the strategic (non-cooperative) approach to provide non-cooperative foundations for axiomatic (cooperative) bargaining solutions.[10]

Later, MD (Sobel, 1981) and the Step-by-Step Negotiation (SSN) axiom (Kalai, 1977)[11] embedded a bargaining process by reaching intermediate agreements.[12]

[8]When the solution outcome does not respond to changes in the bargaining set, that axiom is known as the independence axiom; when at least one of solution payoffs may be altered following a change in the bargaining set, it is dubbed the monotonicity axiom. Indeed, the Nash, Kalai/Smorodinsky, Perles/Maschler, Equal Area, and Average Payoff solutions have all initially been characterized by SYM, WPO, SI and either an independence or a monotonicity axiom.

[9]Both generations of characterizations were essential, since a bargaining problem consists of a bargaining set and a disagreement point, i.e., (S, d).

[10]This strand of research produced very interesting work accompanying the cooperative bargaining solution concepts, starting with Nash (1953), followed by Moulin (1984), Binmore et al (1986), Howard (1992), Anbarci (1993), and Anbarci and Boyd III (2009), among others. Outside of the Nash bargaining framework, in the cooperative game theory area (with Transferable Utility in Characteristic Form Games), Gul (1989) provided non-cooperative foundations for the Shapley Value as well.

[11]Moulin (1983) used MD to characterize the Nash solution, and Kalai (1977) used SSN to characterize Proportional solutions (the Egalitarian and Dictatorial solutions are special cases of the latter).

[12]Intermediate agreements help eliminate the most lop-sided and/or inefficient portions of a utility possibility set S, which at least one of the parties would find undesirable (in effect,

However, MD and SSN have not subsequently been generalized to give rise to a class of axioms which would help in further understanding other prominent solutions. This paper aims to highlight the implications of the main idea and the crucial role of robustness in intermediate agreements in a unified way by proposing and describing such a class of axioms.

3 Basic Definitions

A two-person (*bargaining*) problem is a pair (S, d), where $S \subset \mathbb{R}^2$ is the *set of utility possibilities* that the players can achieve through cooperation and $d \in S$ is the *disagreement point*, which is the utility allocation that results if no agreement is reached. It is assumed that (1) S is compact, convex and comprehensive (if $x, z \in S$ implies that $y \in S$ for all $x \leq y \leq z$), and (2) $x > d$ for some $x \in S$.[13] Let Σ be the class of all two-person problems. Unless otherwise stated, we consider problems in Σ.[14]

For all $S \in \Sigma$, let $IR(S, d) \equiv \{x \in S | x \geq d\}$, $WPO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2$ and $x' > x \Rightarrow x' \notin S\}$, and $PO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2$ and $x' \geq x \Rightarrow x' \notin S\}$. Denote the *ideal point* of (S, d) as $b(S, d) \equiv (b_1(S, d), b_2(S, d))$, where $b_i(S, d) = \max\{x_i | x \in IR(S, d)\}$,[15] and the *midpoint* of (S, d) is $m(S, d) \equiv \frac{1}{2}(b(S, d) + d)$. A

the meta-bargaining models of van Damme, 1986, and Anbarci and Yi, 1992, also pertain to the elimination of portions of S which are deemed undesirable by at least one of the parties). Anbarci, Skaperdas and Syropoulos (2002) consider a setup in which each party can invest in his disagreement payoff to avoid such portions of S and to improve his solution outcome payoff.

[13] Given $x, y \in \mathbb{R}^2$, $x > y$ if $x_i > y_i$ for each i, and $x \geq y$ if $x_i \geq y_i$ for each i.

[14] A more restrictive class of bargaining problems is as follows: a bargaining problem (S, d) is *smooth* if S admits a unique supporting hyperplane at each utility vector on its boundary. $\Sigma^s \subset \Sigma$ denotes the class of all smooth problems.

[15] The ideal point is well defined, since S is assumed to be compact.

solution is a function $f : \Sigma \to \mathbb{R}^2$ such that for all $(S, d) \in \Sigma$, $f \in S$.

The *disagreement point set* of (S, d) with respect to f, $D(S, d, f) \equiv \{d' \in IR(S, d) | f(S, d')$
$= f(S, d)\}$, is the set of all points d' in S (weakly) dominating d, such that if we replace d with d' and keep the utility feasibility set S unchanged, we still reach the same solution outcome. As will be clear later, $D(S, d, f)$ is be a key element in our analysis.

Next, we list some basic axioms in the literature.

Weak Pareto Optimality (WPO): For all $(S, d) \in \Sigma$, $f(S, d) \in WPO(S)$.

Symmetry (SYM): For all $(S, d) \in \Sigma$, if $[d_1 = d_2$, and $(x, y) \in S \Rightarrow (y, x) \in S]$, then $f_1(S, d) = f_2(S, d)$.

Scale Invariance (SI): For all $(S, d) \in \Sigma$, $T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a positive affine transformation if $T(x_1, x_2) = (a_1x_1 + b_1, a_2x_2 + b_2)$ for some positive constants a_i and b_i. We require that for such a transformation T, $f(T(S), T(d)) = T(f(S, d))$.

Individual Rationality (IR): For all $(S, d) \in \Sigma$, $f(S, d) \geq d$.

Strong Individual Rationality (SIR): For all $(S, d) \in \Sigma$, $f_i(S, d) \geq d_i$, with strict inequality whenever $x_i > d_i$ for some $x \in S$ with $i = 1, 2$.

Independence of Non-individually Rational Alternatives (INIR): For all $(S, d) \in \Sigma$, $f(S, d) = f(IR(S, d), d)$.

Disagreement Point Monotonicity (DM): For all $(S, d) \in \Sigma$, if d and e are in S with $e_i = d_i$ and $e_j > d_j$, then $f_j(S, e) \geq f_j(S, d)$, for $i, j = 1, 2$ with $i \neq j$.

Strong Disagreement Point Monotonicity (SDM): The same as DM, but with "$>$" in the conclusion instead of "$\geq$", if such a point $f(S, e)$ exists.

Disagreement Point Continuity (DCONT): For all $(S, d) \in \Sigma$ and every sequence $d^1, d^2, ...$ in S, if $\lim_{n\to\infty} d^n = d \in S$ (in the Hausdorff topology), then $\lim_{n\to\infty} f(S, d^n) = f(S, d)$.

Pareto Continuity (PCONT): For all sequences $\{(S^n, d^n)\} \in \Sigma$ and $(S, d) \in$

Σ,if $WPO(S^n)$ converges to $WPO(S)$ in the Hausdorff topology and $d^n = d$ for all n, then $\lim_{n\to\infty} f(S^n, d^n) = f(S, d)$.

Midpoint Domination (MD): For any $(S, d) \in \Sigma$, $f(S, d) \geq m(S, d)$.

MD requires that the agreement Pareto dominates the outcome of the random dictatorship. Note that the relationship between MD and the next axiom is like the relationship between the IR and INIR axioms. IR and INIR are based on d, while MD and INMD are based on m.

Independence of Non-Midpoint-Dominating Alternatives (INMD): For all $(S, d), (T, d) \in \Sigma$, if $IR(S, m(S, d)) = IR(T, m(T, d))$, then $f(S, d) = f(T, d)$.

If the hypothesis $IR(S, m(S, d)) = IR(T, m(T, d))$ holds, then $m(S, d) = m(T, d)$ and $b(S, d) = b(T, d)$. INMD states that parties should focus on the alternatives dominating the midpoint.

We introduce four prominent solution concepts, the Nash, Kalai/Smorodinsky, Discrete Raiffa, and Proportional solutions.

The Nash solution N: For each $(S, d) \in \Sigma$, $N(S, d) = \arg\max\{(x_1 - d_1)(x_2 - d_2) | x \in IR(S, d)\}$.

The Kalai/Smorodinsky solution KS: For each $(S, d) \in \Sigma$, $KS(S, d) = \max\{u \in S |$ there exists $\alpha \in [0, 1]$ such that $u = \alpha b(S, d) + (1 - \alpha)d\}$.

The Discrete Raiffa solution DR: For each $(S, d) \in \Sigma$, consider a non-decreasing sequence $\{m_t\} \in S$ with $m_0 = m(S, d)$ and $m_t = m(S, m_{t-1})$; then $DR(S, d) = \lim_{t\to\infty} m_t$.

The Proportional solution P: For each $(S, d) \in \Sigma$, there are strictly positive constants p^1 and p^2 such that $f(S, d) = d + \lambda(S, d)p$, where $p = (p^1, p^2)$ and $\lambda(S, d) = \max\{t | tp \in S - d\}$.

4 The Robustness of Intermediate Agreements (RIA) Axioms

Consider the following axiom of Kalai (1977):

Step-by-step Negotiations (SSN): For all (S, d), $(T, d) \in \Sigma$, such that $T \subset S$ and $(S - f(T, d), 0) \in \Sigma$, $f(S, d) = f(T, d) + f(S - f(T, d), 0)$.

SSN is a very robust requirement and Kalai (1977) demonstrated that, when combined with WPO and SIR, it is sufficient to uniquely characterize the Proportional solutions.

Recall from the last section that $D(S, d, f)$ represents the set of all common intermediate agreements d' in S dominating d, such that, if we replace the initial disagreement point d with d', we still reach the same bargaining outcome. To link our axioms to SSN conceptually, we restate SSN as follows:

Given two problems $(S, d), (T, d) \in \Sigma$, whenever $S \subset T$, $f(S, d) \geq d$ and $(T, f(S, d)) \in \Sigma$, then $D(T, d, f) \supset \{f(S, d)\}$.

For a given problem (T, d), SSN requires that $D(T, d, f)$ is not only non-empty, but also contains $f(S, d)$ for *all* problems $(S, d) \in \Sigma$, with $S \subset T$ and $(T, f(S, d)) \in \Sigma$. In other words, $f(S, d)$ can serve as an intermediate agreement in reaching $f(T, d)$. However, as we argued in the Introduction, one would not expect parties to reach intermediate agreements — and especially the solution outcome of S as an intermediate agreement — even if they knew that the ultimate outcome T would contain S, unless some additional things about the relationship between S and T were also known.

Next, we propose four fairly intuitive axioms that are closely related to SSN but provide weaker robustness tests. All axioms consider the situation where two parties face an uncertain bargaining prospect with (S, d) and (T, d) as possible problems.

1. The first axiom, RIA-Inclusion, is a weaker version of SSN. As in SSN, the bargaining sets S and T are nested. However, RIA-Inclusion only requires the disagreement point set $D(T, d, f)$ of the larger utility possibility set T to include the disagreement point set $D(S, d, f)$ of the smaller set S.

Robustness of Intermediate Agreements with Inclusion (RIA-Inclusion): For all $(S, d), (T, d) \in \Sigma$ such that $S \subset T, D(S, d, f) \subset D(T, d, f)$.[16]

In other words, RIA-Inclusion does not require the solution outcome, $f(S, d)$, of the smaller set S to necessarily be an intermediate agreement. It only states that in this case the parties will be willing to reach any point $d' \in D(S, d, f)$ as an intermediate agreement instead of sticking to the status quo d. This intermediate agreement is robust, in that the bargaining outcome remains unchanged in either problem when the agents move from d to d'. Thus, certain points can serve as robust intermediate agreements when S and T are nested.

2. Now, consider two parties facing a bargaining situation where — unlike in SSN and RIA-Inclusion — the bargaining sets need not be nested. Apart from the disagreement outcome d, assume that the parties also know the maximal utility each of them can receive (i.e., the ideal point), but are uncertain about the Pareto optimal frontier. Therefore, they can not reach a final outcome yet. In this case also they will be willing to reach an intermediate agreement d' instead of sticking to the status quo. As in the RIA-Inclusion axiom, the intermediate agreement d' should be robust. Accordingly, we must have $\cap_{(S,d)\in\Sigma^{b,d}} D(S, d, f)\backslash\{d\} \neq \emptyset$, where $\Sigma^{b,d}$ is the collection of all problems in Σ with ideal point b and disagreement point d. The axiom stated below is a weaker version of this requirement.[17]

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[16] To see that RIA-Inclusion is weaker than SSN, pick any $\widetilde{d} \in D(S, d, f)$ (if it exists). SSN implies that $f(T, \widetilde{d}) = f(T, f(S, \widetilde{d})) = f(T, f(S, d)) = f(T, d)$.

[17] The terms $\{f(S, d)\}$ and $\{f(T, d)\}$ can be dropped if we work with an enlarged domain by

Robustness of Intermediate Agreements in the (d, b)-Box (RIA-Box): For all (S, d), $(T, d) \in \Sigma$ such that $b(S, d) = b(T, d)$, $(D(S, d, f) \cup \{f(S, d)\}) \cap (D(T, d, f) \cup \{f(T, d)\}) \backslash \{d\} \neq \emptyset$.

RIA-Box only requires that each pair of problems with the same disagreement point and ideal point have a non-empty intersection of their disagreement point sets union the final outcomes.[18] The agents know d and b, but they do not know whether they will face S or T. Again, we want them to be able to reach an intermediate agreement $d' > d$ in that whether S or T is realized tomorrow, the agents should be able to move from d' or d to $f(S, d)$ or $f(T, d)$. Thus, this robustness criterion states that certain points can serve as robust intermediate agreements when (S, d) and (T, d) share the same ideal point. Note that RIA-Box is satisfied by the Proportional solutions (including both the Egalitarian and Dictatorial solutions), as well as by the Discrete Raiffa solution.

3. Our third and fourth robustness criteria concern the concept of bargaining power. Both criteria state that a robust intermediate agreement exists when parties have the same relative bargaining power in S and T. Intuitively, when two parties face an uncertain bargaining circumstance, they agree to reach an intermediate agreement d' only if d and d' give them the same relative bargaining position (power) for later negotiations. When (S, d) and (T, d) are very divergent, it may be difficult to find an intermediate agreement that gives them the same relative bargaining power as d does. On the contrary, if (S, d) and (T, d) are *homogenous* in that both parties have the same relative bargaining power in both (S, d) and (T, d), then we should expect them to reach an intermediate agreement easily. The only

allowing d to be on the boundary of the bargaining set.

[18] The axiom can be modified to hold for all $(S, d) \in \Sigma$ instead of for only two problems $(S, d), (T, d) \in \Sigma$. Such a modification would surely strengthen intuition, but would also make it mathematically stronger.

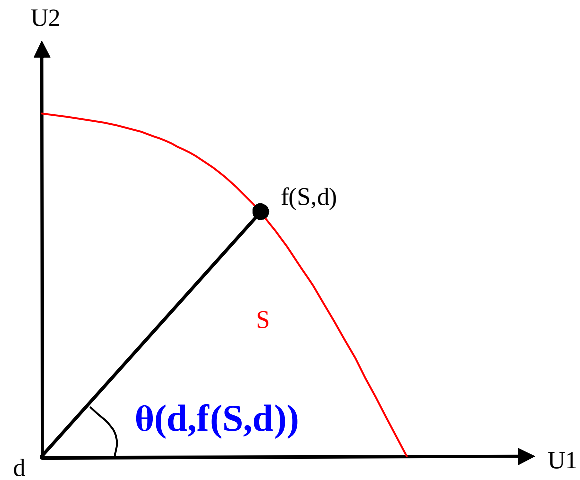

FIGURE 1

thing remaining is to determine how the parties perceive their relative bargaining power. In the following, we suggest two measures of bargaining power.

It may not be clear *ex-ante* what kinds of economic and non-economic factors will determine a party's bargaining power relative to that of the other. Nevertheless, it should be clear *ex-post* that one party's gain in a negotiation relative to the other's must increase monotonically with respect to that party's bargaining power relative to that of the other. This simple idea inspires our first definition of bargaining power. It is as follows: For any $x, y \in \mathbb{R}^2$ and $x \neq y$, let $l[x, y]$ be the line segment connecting x and y, and $\theta(x, y)$ be the gradient (slope) of $l[x, y]$. Suppose that the solution outcome is $f(S, d) \geq d$ for a given problem (S, d). Then the gradient $\theta(d, f(S, d))$, which measures the relative gains in bargaining, could be an appropriate index of bargaining power (see Figure 1).

Observe that $\theta(d, f(S, d)) = 0$ implies that Agent 1 has absolute bargaining power, $\theta(d, f(S, d)) = \infty$ implies that Agent 2 has absolute bargaining power, and Agent 1's bargaining power decreases monotonically with θ. If $\theta(d, f(S, d)) = \theta(d, f(T, d))$, then the parties receive the same relative gains over (S, d) and (T, d)

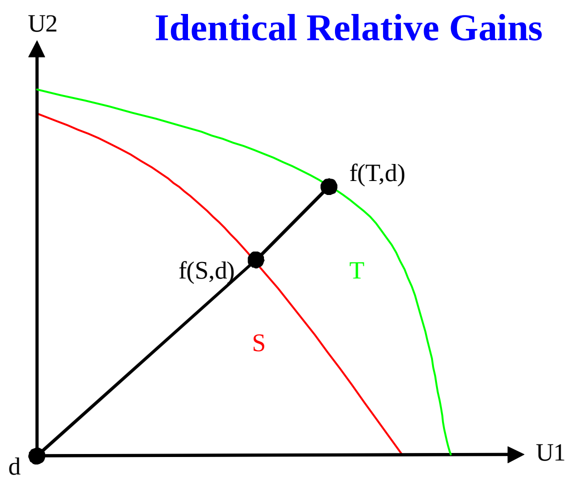

FIGURE 2

(see Figure 2).

Robustness of Intermediate Agreements with Identical Relative Gains (RIA-Gains): For all $(S,d),(T,d) \in \Sigma$, if (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(d, f(S,d)) = \theta(d, f(T,d))$, then $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$.

Beginning with the same disagreement point d, if two parties perceive (correctly) that they will receive the same relative gains in two problems (S,d) and (T,d), then there exists at least one allocation d' in $S \cap T$ that is agreeable to both parties as an intermediate agreement. As in the previous two RIA axioms, this intermediate agreement $d' > d$ should be robust, in that whether S or T is realized tomorrow, the agents should be able to move from either d' or d to $f(S,d)$ or $f(T,d)$. Any such d' therefore needs to be a common intermediate agreement in $D(S,d,f) \cap D(T,d,f)$. It is easy to see that RIA-Gains is satisfied by the Proportional solutions, as well as the Nash solution.

4. Our last RIA axiom, RIA-Concessions, is built on a measure of the relative concessions parties make. Suppose that two parties have reached an agreement.

This outcome can be viewed as a compromise that balances their concessions. A particular party's concessions are feasible outcomes — measured in a particular way that both parties agree on — that a party prefers to the negotiated outcome. Thus, concessions can be considered as bargaining chips here; possessing more bargaining chips in a particular negotiation would then yield more bargaining power to a party.

Suppose that the parties' environment now changes in such a way that new potential outcomes, mostly beneficial to Agent 1, have been added to the feasible set. Then, from both parties' points of view, maintaining the original compromise would amount to Agent 1 making relatively greater concessions than before. This in turn would give Agent 1 more bargaining power in the new environment. If the initial compromise was reached by "balancing" the concessions made by one party against those made by the other, then maintaining the original payoff ratio will result in an "imbalance" of concessions, and therefore at least one party — Agent 1 — will think that the original payoff ratio should no longer remain intact.

However, if the new environment is such that the new potential outcomes are equally beneficial to both parties (given the way in which parties agree to measure concessions), then maintaining the original payoff ratio still results in an outcome balancing relative concessions and thus relative bargaining powers. In that regard, $\theta(b(T, f(S,d)), f(S,d))$ can be used to measure the parties' relative concessions and relative bargaining powers in $(T, f(S,d))$ with respect to $f(S,d)$. As in RIA-Box and RIA-Gains, S and T need not include one another.

Robustness of Intermediate Agreements with Identical Relative Concessions (RIA-Concessions): For all $(S,d),(T,d) \in \Sigma$, if (i) $f(S,d) \in IR(S,d)\backslash\{d\}$ and $f(T,d) \in IR(T,d)\backslash\{d\}$ and (ii) $\theta(b(T, f(S,d)), f(T,d)) = \theta(b(T, f(S,d)), f(S,d))$, then $D(S,d,f) \cap D(T,d,f)\backslash\{d\} \neq \emptyset$; moreover, $f_i(T,d) = b_i(T,d)$ for some i only

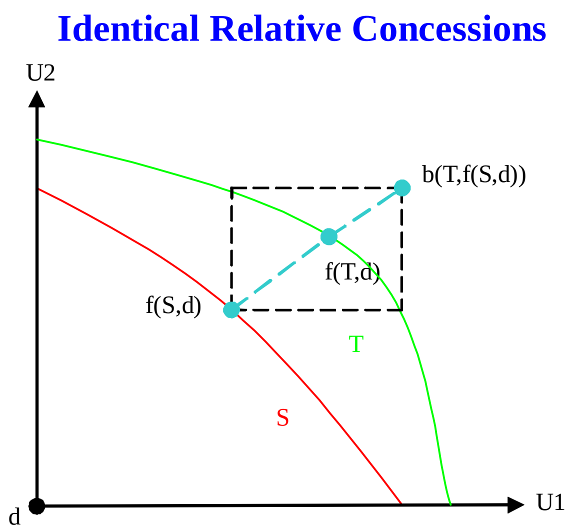

FIGURE 3

if $b(T,d) \in T$.[19]

In two problems (S,d) and (T,d), parties should *not* expect that their bargaining power will change if the added relative concessions are the same as before, i.e., when $\theta(b(T,f(S,d)),\ f(T,d)) = \theta(b(T,f(S,d)), f(S,d))$ (see Figure 3). RIA-Concessions requires that there exists at least one allocation d' in $S \cap T$ that is agreeable to both parties as an intermediate agreement when they perceive to make the same relative concessions in two problems.

[19]The last requirement "$f_i(T,d) = b_i(T,d)$ for some i only if $b(T,d) \in T$" is there to guarantee that $b(T,x) \neq f(T,d)$ will hold for all $x \in IR(T,d)\backslash\{f(T,d)\}$; otherwise $\theta(b(T,f(S,d)), f(T,d))$ may not be well-defined. This condition can be dropped if we restrict the domain of bargaining problems to be non-level or replace DCONT with PCONT in characterizing KS.

5 Characterizations of the Discrete Raiffa, Nash, Kalai/Smorodinsky, and Proportional Solutions

5.1 The Discrete Raiffa Solution

As was mentioned earlier, RIA-Box is satisfied by all Proportional solutions, as well as by the Discrete Raiffa solution. It is also known that MD is satisfied by the Nash, Kalai/Smorodinsky, Discrete Raiffa, Equal Area, and Average Payoff solutions. As has also been mentioned, INMD is satisfied by the Nash, Kalai/Smorodinsky, Discrete Raiffa and Dictatorial solutions. The following is the first axiomatic characterization of the Discrete Raiffa solution. All proofs from this section are relegated to the Appendix, Part A.

Proposition 1 *DR is the unique solution satisfying INMD, MD and RIA-Box.*

Thus, given INMD and MD, if two parties, whenever they face an uncertain bargaining situation with two possible underlying feasible sets with the same disagreement point d and ideal point b, are willing to reach intermediate agreements, then the bargaining outcome must be DR.

5.2 The Nash Solution

Recall that the RIA-Gains axiom is satisfied by all Proportional solutions and the Nash solution. MD is satisfied by a significant number of solutions, as was mentioned above, while DCONT, which is even more innocuous, is satisfied by all known solutions.

RIA-Gains is closely related to the axiom of *Disagreement Point Convexity* introduced by Peters and Van Damme (1991):

Disagreement Point Convexity (DPC): $f(S, \alpha d+(1-\alpha)f(S,d)) = f(S,d)$ for all $\alpha \in (0,1)$.

DPC requires that $D(S,d,f) \supset l(d, f(S,d))$. If the premises of RIA-Gains hold, then DPC implies that $D(S,d,f) \cap D(T,d,f) \supset l(d, \min\{f(S,d), f(T,d)\})$.[20] DPC therefore implies RIA-Gains. However, the following example shows that RIA-Gains does not imply DPC. Consider the ϵ-egalitarian solution, E^ϵ, defined as follows: (1) if $E_1^\epsilon(S,d)-d_1 = E_2^\epsilon(S,d)-d_2 \geq \epsilon$, it assigns $(E_1^\epsilon(S,d)-\epsilon, E_2^\epsilon(S,d)-\epsilon)$, where $\epsilon > 0$; (2) if $E_1^\epsilon(S,d) - d_1 = E_2^\epsilon(S,d) - d_2 < \epsilon$, it assigns d.[21] E^ϵ satisfies DCONT and RIA-Gains, but violates DPC.

Proposition 2 *N is the unique solution satisfying DCONT, MD and RIA-Gains.*[22]

Thus, given DCONT and MD, if two parties, whenever they face an uncertain bargaining environment consisting of two possible feasible sets with the same disagreement point d, are willing to reach intermediate agreements provided that they receive the same relative utility gains over these two possible problems, then the bargaining outcome must be N, the compromise that maximizes the product of their utility gains.

Remark 1 *Peters and Van Damme (1991) demonstrate that N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and DPC. The following proposition improves on their result, because DPC implies RIA-Gains but not vice versa.*

Proposition 3 *N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and RIA-Gains.*

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[20]Note that $\min\{f(S,d), f(T,d)\}$ is well-defined when $\theta(d, f(S,d)) = \theta(d, f(T,d))$.

[21]$E(S,d)$ stands for the Egalitarian solution.

[22]DCONT is merely a technical condition and can be dropped if we modify the axiom of RIA-Gains slightly. Please see the Appendix, Part B.

Proof. It is straightforward to show that SIR, DCONT, SYM, SI and RIA-Gains together imply DPC. ■

5.3 The Kalai/Smorodinsky Solution

SDM is satisfied by the Kalai/Smorodinsky solution, as well as by the Equal Area and Average Payoff solutions. We have already elaborated on the number of solutions satisfying MD and DCONT. Imposing these axioms, together with RIA-Concessions, yields a characterization of the Kalai/Smorodinsky solution:

Proposition 4 *KS is the unique solution satisfying SDM, DCONT, MD and RIA-Concessions.*[23,24]

Given SDM, DCONT, and MD, if two parties, whenever they face an uncertain bargaining situation with two possible underlying feasible sets with the same disagreement point d, are willing to reach intermediate agreements provided that they have the same relative concessions over these two possible problems, then the bargaining outcome must be KS.

Remark 2 *It can readily be seen that the axiom of MD can be replaced by PO in Proposition 4.*

Proposition 5 *KS is the unique solution satisfying SDM, DCONT, PO and RIA-Concessions.*

[23]As mentioned before, DCONT is merely a technical condition. It can be dropped if we modify the axiom of RIA-Concessions slightly. Please see the Appendix, Part B.

[24]Note that SDM, rather than the weaker version DM, is required in the characterization of KS. However, even though N does not satisfy SDM in Σ, it nevertheless does satisfy it in Σ^s. Hence, one clearly cannot distinguish KS, N, and DR from each other — at least in Σ^s — on the basis of SDM solely.

5.4 Proportional Solutions

DCONT, PCONT, SIR and WPO are satisfied by all known solution concepts. As was mentioned earlier, RIA-Inclusion is weaker than SSN. By imposing DCONT, PCONT, SIR, WPO and RIA-Inclusion together, we obtain the following result:

Proposition 6 *P is the only class of solutions satisfying DCONT, PCONT, SIR, WPO and RIA-Inclusion.*

Given DCONT, PCONT, SIR and WPO, if two parties, whenever they face an uncertain bargaining circumstance with two possible underlying problems with the same disagreement point d, are willing to reach intermediate agreements as long as the two problems S and T are nested, then the bargaining outcome must be P.[25]

We can now fully summarize our results:

DR	RIA-Box	MD		INMD
N	RIA-Gains	MD	DCONT	
KS	RIA-Concessions	MD	DCONT	SDM
P	RIA-Inclusion		DCONT	PCONT+WPO+SIR

That is, beside MD and RIA-Box, the Discrete Raiffa solution's axiomatic characterization uses only one more axiom, INMD. In addition MD and RIA-Gains, the Nash solution's characterization likewise involves only one more axiom, DCONT. Beside MD and RIA-Concessions, the Kalai/Smorodinsky solution's characteriza-

[25] Rachmilevitch (2009) recently wrote an interesting note on our paper, which at that time had not yet incorporated the characterization of the Proportional solutions. Rachmilevitch used IR, SYM, a slightly weaker version of DCONT, 'Translation Invariance' and a new axiom, 'Interim Improvement', to characterize the Egalitarian solution. His Interim Improvement axiom and our RIA-Inclusion axioms do not imply one another.

tion uses DCONT and SDM. Beside RIA-Inclusion, the Proportional solutions' characterizations use WPO, DCONT, PCONT and SIR.

6 Conclusion

Although there have been earlier non-unified attempts to bring the bargaining process into Nash's bargaining problem (via the SSN axiom of Kalai, 1977, and the MD axiom of Sobel, 1981), previous characterizations of solutions have typically relied on crucial axioms entailing changes in the utility possibility set and in the disagreement point, and have not described any bargaining process. In this paper, we relax the strong robustness criterion of SSN, and thereby highlight, in a unified way, the crucial role played by the robustness of the intermediate agreements (and the resulting bargaining process) which we propose here. By describing plausible circumstances under which such robust intermediate agreements can be obtained, our Robustness of Intermediate Agreements (RIA) axioms portray a bargaining process.

A major accomplishment of our formulation is the novel axiomatic characterization of the Discrete Raiffa solution, which had previously eluded researchers. All of our characterizations, except for that of the Proportional solutions, involve both MD and an RIA axiom, which each pertain to different aspects of the bargaining process.

This unified approach aims to bridge the gap between the axiomatic and strategic approaches to bargaining. The use of robustness of intermediate agreements is certainly one fruitful way of bringing these two approaches together.

7 Appendix

7.1 Part A: Proofs

Proof of Proposition 1. It is obvious that DR satisfies these three axioms. Suppose that f satisfies INMD, MD and RIA-Box. We show that $f = DR$. Let $(S,d) \in \Sigma$. By the convexity of S, $m(S,d) \in S$. If $m(S,d) \in PO(S)$, then by MD $f(S,d) = m(S,d) = DR(S,d)$. Now suppose that $m(S,d) \notin PO(S)$. By the convexity of S again, $m(S,d) \notin WPO(S)$, and hence, $(S, m(S,d)) \in \Sigma$. To show that $f(S,d) = DR(S,d)$ in this case, it is sufficient to show that $f(S,d) = f(S, m(S,d))$. Consider the problem (T,d), where $T = conv\{d, (d_1, b_2(S,d)), (b_1(S,d), d_2)\}$.[26] By MD, (i) $f(T,d) = m(S,d)$, and (ii) $D(T,d,f) = l[d, m(S,d))$. By RIA-Box, there exists a common intermediate agreement $a \in l[d, m(S,d)) \cup \{m(S,d)\}$ such that $f(S,d) = f(S,a)$. INMD excludes all points below $m(S,d)$ from being a common intermediate agreement. Hence, $a = m(S,d)$. ■

Proof of Proposition 2. It is obvious that N satisfies these three axioms. We will show that if f satisfies these three axioms, then $f = N$. The proof is based on the following nice characterization of the Nash solution by de Clippel (2007).

Lemma 1 *(Theorem 1 of de Clippel, 2007) N is the unique solution satisfying MD and DPC.*

With this Lemma in hand, it is sufficient to show that DCONT, MD and RIA-Gains imply DPC. Let $(S,d) \in \Sigma$. By MD, $f(S,d) > d$. Consider the problem (T^ε, d), where $T^\varepsilon = conv\{d, (2f_1(S,d) - d_1 - \frac{\varepsilon}{f_2(S,d)-d_2}, d_2), (d_1, 2f_2(S,d) - d_2 - \frac{\varepsilon}{f_1(S,d)-d_1})\}$. By MD, (i) $f(T^\varepsilon, d) = (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})$, which in turn implies that $\theta(d, f(S,d)) = \theta(d, f(T^\varepsilon, d))$, and (ii) $D(T^\varepsilon, d, f) =$

[26]Note that "conv" denotes "the convex hull of."

$l[d,(f_1(S,d)-\frac{\varepsilon}{2(f_2(S,d)-d_2)},f_2(S,d)-\frac{\varepsilon}{2(f_1(S,d)-d_1)}))$. By RIA-Gains, at least one point $a^1\in l(d,(f_1(S,d)-\frac{\varepsilon}{2(f_2(S,d)-d_2)},f_2(S,d)-\frac{\varepsilon}{2(f_1(S,d)-d_1)}))$ is in the disagreement point set of (S,d) with respect to f. Starting with a^1 as a new disagreement point and repeating the argument above gives us a strictly increasing sequence $\{a^n\}$ such that $a^n\in D(S,d,f)\ \forall n$. By RIA-Gains and DCONT, $\lim_{n\to\infty}a^n=(f_1(S,d)-\frac{\varepsilon}{2(f_2(S,d)-d_2)},f_2(S,d)-\frac{\varepsilon}{2(f_1(S,d)-d_1)})$. Invoking DCONT and RIA-Gains again, $(f_1(S,d)-\frac{\varepsilon}{2(f_2(S,d)-d_2)},f_2(S,d)-\frac{\varepsilon}{2(f_1(S,d)-d_1)})\in D(S,d,f)$. Letting ε vary from 0 to $2(f_1(S,d)-d_1)(f_2(S,d)-d_2)$ gives us DPC. ■

Proof of Proposition 4. It is straightforward to show that KS satisfies these four axioms. Suppose that f satisfies SDM, DCONT, MD and RIA-Concessions. We will show that $f=KS$ holds. The proof consists of two steps:

(I) If $T=conv\{d,(d_1+a,d_2),(d_1,d_2+c),(d_1+a,d_2+c)\}$ for some $a,c>0$, then $f(T,d)=b(T,d)=(d_1+a,d_2+c)$. Suppose on the other hand that $f(T,d)\neq(d_1+a,d_2+c)$. By MD, $f(T,d)\geq m(T,d)>d$. Denote by $L(f(T,d),(d_1+a,d_2+c))$ the straight line going through $f(T,d)$ and (d_1+a,d_2+c), and define $\eta\equiv\inf\{x\geq m(T,d)|\ x\in L(f(T,d),(d_1+b,d_2+c))\}$. As the partial order $\geq$ in $\mathbb{R}^2$ induces a linear order in $L(f(T,d),(d_1+b,d_2+c))$, η is well-defined. There are two cases:

(i) $\eta=m(T,d)$. Consider the problem (W,d) with $W=conv\{d,(d_1+a,d_2),(d_1,d_2+c)\}$. Notice that $b(T,d)=b(W,d)=(d_1+a,d_2+c)$. By MD, (a) $f(W,d)=(d_1+\frac{a}{2},d_2+\frac{c}{2})=m(T,d)$, and (b) $D(W,d,f)=l[d,m(T,d))$. Accordingly, $\theta(b(T,f(W,d)),f(T,d))=\theta(b(T,f(W,d)),f(W,d))$, and by RIA-Concessions there exists $y^1\in l(d,m(T,d))$ such that $f(T,d)=f(T,y^1)$ and $f(W,d)=f(W,y^1)$. Applying RIA-Concessions repeatedly, we get a strictly increasing sequence $\{y^i\}$, with $y^i\in l(d,m(T,d))$ such that $f(T,d)=f(T,y^i)$ and $f(W,d)=f(W,y^i)$ for all i. By DCONT and RIA-Concessions, $\lim y^i=m(T,d)$. Consequently, by DCONT, $f(T,d)=f(T,m(T,d))$. Taking $m(T,d)$ as a new disagreement point and iteratively applying the equation

$f(T,d) = f(T,m(T,d))$ shows that $f(T,d) = (d_1 + a, d_2 + c)$, contradicting our premise that $f(T,d) \neq (d_1 + a, d_2 + c)$.

(ii) If $\eta \neq m(T,d)$, then either $\eta = (\alpha, m_2(T,d))$ for some $\alpha \in (m_1(T,d), d_1 + a]$ or $\eta = (m_1(T,d), \beta)$ for some $\beta \in (m_2(T,d), d_2 + c]$. Without loss of generality, assume that $\eta = (\alpha, m_2(T,d))$ for some $\alpha \in (m_1(T,d), d_1 + a]$. There are two sub-cases:

Subcase 1. $\alpha \in (m_1(T,d), d_1 + a)$. Consider the problem (Φ, d), where $\Phi \equiv conv\{d, (2\alpha - d_1, d_2), (d_1, d_2 + c)\}$. Following the same steps as in (i), we have $\eta \in D(T,d,f)$. We then take η as a new disagreement point and iteratively apply the equation $f(T,\eta) = f(T, m(T,\eta))$, concluding that $f(T,d) = (d_1 + a, d_2 + c)$.

Subcase 2. $\alpha = d_1 + a$. Consider the problem (Ψ, d), where $\Psi = conv\{d, (d_1 + 2a, d_2), (d_1, d_2 + c)\}$. $m(\Psi, d) = \eta$, but note that $(T, \eta) \notin \Sigma$. Nevertheless, using the same argument as in (i), we are still able to get a strictly increasing sequence $\{x^i\}$ with $\lim_{i\to\infty} x^i = \eta$ such that $x^i \in D(T,d,f)$ for all i. By MD, $f_1(T,d) = d_1 + a$ and $f_2(T,d) \geq \frac{1}{2} m_2(T, m(T,d))$. Taking x^i sufficiently close to η as a new disagreement point and invoking the standard limiting argument (recursively) shows that $f_2(T,d) = d_2 + c$. Therefore, $f(T,d) = (d_1 + a, d_2 + c)$.

(II) Let $(S,d) \in \Sigma$. If $b(S,d) \in S$, then $IR(S,d) = conv\{d, (d_1, b_2(S,d)), (b_1(S,d), d_2), b(S,d)\}$. Therefore, $f(S,d) = b(S,d) = KS(S,d)$ from (I). Assume now that $b(S,d) \notin S$. By MD, $f(S,d) \geq m(S,d)$. We now show that $f(S,d) \in PO(S)$. Suppose on the other hand that $f(S,d) \notin PO(S)$. Consider the problem (W,d), where $W = conv\{d, (2f_1(S,d) - d_1, d_2), (d_1, 2f_2(S,d) - d_2)\}$. By MD, $f(W,d) = f(S,d)$. Hence $\theta(b(S, f(W,d)), f(S,d)) = \theta(b(S, f(W,d)), f(W,d))$. Repeatedly invoking RIA-Concessions and DCONT gives us a strictly increasing sequence $\{x^i\}$ with $\lim_{i\to\infty} x^i = f(S,d)$ such that $x^i \in D(S,d,f)$ for all i. Consequently, by MD, $f(S,d) \in PO(S)$.

Define $\Gamma \equiv \{x \in IR(S,d) |\ \theta(b(S,x), f(S,d)) = \theta(b(S,x), x)$ and $x \leq f(S,d)\}$.

Since $f_i(S,d) \neq b_i(S,d)$, $\Gamma \backslash \{f(S,d)\}$ is non-empty. It can be shown that either $\Gamma \cap \{x \in IR(S,d) | x_1 = d_1\} \neq \emptyset$ or $\Gamma \cap \{x \in IR(S,d) | x_2 = d_2\} \neq \emptyset$. There are two cases to be considered:

(i) If $d \in \Gamma$, then $f(S,d) = l[d, b(S,d)] \cap PO(S) = KS$.

(ii) If $d \notin \Gamma$, then either $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$ or (d_1, β) for some $\beta \in (d_2, f_2(S,d))$. Without loss of generality, assume $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$. It is straightforward to show that by MD, DCONT and RIA-Concessions, $\Gamma \backslash (f(S,d) \cup (\alpha, d_2)) \subset D(S,d,f)$. By DCONT, $(\alpha, d_2) \in D(S,d,f)$, which violates SDM as $\alpha > d_1$. Therefore, $d \in \Gamma$ and $f = KS$. ■

Proof of Proposition 6. It can easily be seen that P satisfies these five axioms. Suppose that f satisfies DCONT, PCONT, SIR, WPO and RIA-Inclusion; we then show that $f = P$. It is sufficient to show that DCONT, PCONT, SIR, WPO and RIA-Inclusion imply SSN. Denote by $\Sigma^{nl} \subset \Sigma$ the class of all problems that are non-level (see de Clippel, 2007). Every $(S,d) \in \Sigma$ can be approximated by a sequence of problems in Σ^{nl}. Then, by PCONT, it is sufficient to show that the claim is true in Σ^{nl}. Let (S,d) and (T,d) in Σ^{nl} such that $S \subset T$ and $(T, f(S,d)) \in \Sigma^{nl}$. First we show that there is a sequence $\{d^n\}$ with $d^n \in D(S,d,f)$ such that $\lim_{n\to\infty} d^n = f(S,d)$. Denote by $B_\epsilon(y) = \{x \in \mathbb{R}^2 | \, \|x - y\| < \epsilon\}$ the open ball of radius $\epsilon > 0$ centered at y. For any given $\epsilon > 0$, we show that $B_\epsilon(f(S,d)) \cap D(S,d,f) \neq \emptyset$. Pick any $z \in B_\epsilon(f(S,d)) \cap l(d, f(S,d))$. By SIR, $z < f(S,d)$. If $f(S,z) = f(S,d)$, the claim is established. Suppose now that $f(S,z) \neq f(S,d)$. Assume without loss of generality that $f_1(S,z) > f_1(S,d)$; then by WPO, $f_2(S,z) \leq f_2(S,d)$. Define $\zeta \equiv (z_1, f_2(S,d)) \in B_\epsilon(f(S,d))$. Since S is non-level, $(S,\zeta) \in \Sigma$, by SIR $f_2(S,\zeta) > f_2(S,d)$, and hence, by WPO, $f_1(S,\zeta) \leq f_1(S,d)$. $f(S, l[z,\zeta]) \subset WPO(S)$. Since $l[z,\zeta]$ is connected (in the Hausdorff topology), and, by DCONT, $f(S,\cdot)$ is continuous, $f(S, l[z,\zeta])$ is connected. Consequently, there exists $\gamma \in l[z, \zeta]$ such that $f(S, \gamma) = f(S, d)$. As ϵ is arbitrary, we can find a sequence $\{d^n\}$ such that $d^n \in D(S, d, f)$ for all n and $\lim_{n\to\infty} d^n = f(S, d)$. Then, by RIA-Inclusion, $d^n \in D(T, d, f)$ for all n; DCONT completes the proof. ■

7.2 Part B: $\alpha-$RIA Results

α-Robustness of Intermediate Agreements with Identical Relative Gains (α-**RIA-Gains**): Suppose that $(S, d), (T, d) \in \Sigma$, and pick any $\alpha \in [0, 1)$. A solution f satisfies α-**RIA-Gains** if (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$, and (ii) $\theta(d, f(S, d)) = \theta(d, f(T, d))$ implies that there exists $x \in D(S, d, f) \cap D(T, d, f)$ with $x \geq \alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$.

α-RIA-Gains strengthens RIA-Gains by requiring that at least one common intermediate agreement which dominates $\alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$ exists. This axiom can be seen as a condition on the speed of convergence. This common intermediate agreement can be arbitrarily close to d if we pick α sufficiently close to 1. Note that DPC implies α-RIA-Gains as well. Therefore, the following straightforward extension of Proposition 2 improves Theorem 1 of de Clippel (2007).

Proposition 7 *N is the unique solution satisfying MD and α-RIA-Gains for all $\alpha \in (0, 1)$.*

α-Robustness of Intermediate Agreements with Identical Relative Concessions (α-**RIA-Concessions**): Suppose that $(S, d), (T, d) \in \Sigma$ and $\alpha \in [0, 1)$. If (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$, and (ii) $\theta(b(T, f(S, d)), f(T, d)) = \theta(b(T, f(S, d)), f(S, d))$, then there exists $x \in D(S, d, f) \cap D(T, d, f)$ with $x \geq \alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$; moreover, $f_i(T, d) = b_i(T, d)$ for some i only if $b(T, d) \in T$.

It is straightforward to show the following:

Proposition 8 *KS is the unique solution satisfying SDM, MD and α-RIA-Concessions.*

Proposition 9 *KS is the unique solution satisfying SDM, PO and α-RIA-Concessions.*

References

[1] Anbarci, N. (1993), "Noncooperative Foundations of the Area Monotonic Solution," *Quarterly Journal of Economics*, 108, 245-258.

[2] Anbarci, N. (1995), "Reference Functions and Balanced Concessions in Bargaining," *Canadian Journal of Economics*, 28, 675-682.

[3] Anbarci, N., S. Skaperdas and C. Syropoulos (2002), "Comparing Bargaining Solutions in the Shadow of Conflict: How Norms against Threats Can Have Real Effects," *Journal of Economic Theory*, 106, 1-16.

[4] Anbarci, N. and J. Bigelow (1994), "The Area Monotonic Solution to the Cooperative Bargaining Problem," *Mathematical Social Sciences*, 28, 133-142.

[5] Anbarci N. and G. Yi (1992), "A Meta-Allocation Mechanism in Cooperative Bargaining," *Economics Letters*, 38, 175-179.

[6] Anbarci, N. and J. Boyd III (2009), "Nash Demand Game and the Kalai/Smorodinsky Solution," mimeo, Deakin University and Florida International University.

[7] Aumann, R. J. and M. B. Maschler (1985), "Game-theoretic Analysis of a Bankruptcy Problem from the Talmud," *Journal of Economic Theory*, 36, 195-213.

[8] Bigelow, J. and N. Anbarci (1993), "Non-Dictatorial, Pareto-Monotonic, Cooperative Bargaining: An Impossibility Theorem," *European Journal of Political Economy*, 9, 551-558.

[9] Binmore, K. G. (1998), Game Theory and the Social Contract II: Just Playing, MIT Press, Cambridge, MA.

[10] Binmore, K., A. Rubinstein and A. Wolinsky (1986), "The Nash Bargaining Solution in Economic Modeling," *Rand Journal of Economics*, 17, 176-188.

[11] Chiang, J., S. Chib, and C. Narasimhan (1999), "Markov Chain Monte Carlo and Models of Consideration Set and Parameter Heterogeneity," *Journal of Econometrics*, 89, 223-248.

[12] Chun, Y. (1988), "The Equal-loss Principle for Bargaining Problems," *Economics Letters*, 26, 103-106.

[13] Chun, Y. and W. Thomson (1990), "Bargaining with Uncertain Disagreement Points," *Econometrica*, 58, 951-959.

[14] Dawes, P. L. and Brown, J. (2004), "The Composition of Consideration and Choice Sets in Undergraduate University Choice: An Exploratory Study," *Journal of Marketing for Higher Education*, 14, 37-59.

[15] de Clippel, G. (2007), "An Axiomatization of the Nash Bargaining Solution," *Social Choice and Welfare*, 29, 201-210.

[16] Edgeworth, F. Y. (1881), Mathematical Psychics. Kegan Paul Publishers, London. Reprinted in 2003, P. Newman (ed.) F. Y. Edgeworth's Mathematical Psychics and Further Papers on Political Economy. Oxford University Press.

[17] Goeree, M. (2008), "Limited Information and Advertising in the US Personal Computer Industry," *Econometrica*, 76, 1017-1074.

[18] Gul, F. (1989), "Bargaining Foundations of Shapley Value," *Econometrica*, 57, 81-95.

[19] Howard, J. V. (1992), "A Social Choice Rule and Its Implementation in Perfect Equilibrium," *Journal of Economic Theory*, 56, 142-159.

[20] Huberman, G. and T. Regev (2001), "Contagious Speculation and a Cure for Cancer: A Nonevent That Made Stock Prices Soar," *Journal of Finance*, 56, 387-396.

[21] Kalai, E. (1977), "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," *Econometrica*, 45, 1623-1630.

[22] Kalai, E. and M. Smorodinsky (1975), "Other Solutions to Nash's Bargaining Problem," *Econometrica*, 43, 513-518.

[23] Livne, Z. (1989), "On the Status Quo Sets Induced by the Raiffa Solution to the Two-Person Bargaining Problem," *Mathematics of Operations Research*, 14, 688-692.

[24] Moulin, H. (1983), "Le Choix Social Utilitariste," Ecole Polytechnique Discussion Paper.

[25] Moulin, H. (1984), "Implementing the Kalai-Smorodinsky Bargaining Solution," *Journal of Economic Theory*, 33, 32-45.

[26] Nash, J. F. (1950), "The Bargaining Problem," *Econometrica*, 18, 155-162.

[27] Nash, J. F. (1953), "Two-person Cooperative Games," *Econometrica*, 21, 128-140.

[28] Perles, M. A. and M. Maschler (1981), "The Super-additive Solution for the Nash Bargaining Game," *International Journal of Game Theory*, 10, 163-193.

[29] Peters, H. and E. Van Damme (1991), "Characterizing the Nash and Raiffa Bargaining Solutions by Disagreement Point Axioms," *Mathematics of Operations Research*, 16, 447-461.

[30] Raiffa, H. (1953), "Arbitration Schemes for Generalized Two-person Games," In H. W. Kuhn and A. W. Tucker (eds.), Contributions to the Theory of Games II, Annals of Mathematics Studies No. 28 (361-387), Princeton University Press, Princeton, NJ.

[31] Rachmilevitch, S. (2009), "Interim Outcomes in Bargaining and the Egalitarian Solution," mimeo, Northwestern University, Department of Economics.

[32] Roth, A. E. (1979a), "Proportional Solutions to the Bargaining Problem," *Econometrica*, 47, 775-777.

[33] Roth, A. E. (1979b), Axiomatic models of bargaining, Springer, New York.

[34] Rubinstein, A. and Y. Salant (2006), "A Model of Choice From List," *Theoretical Economics*, 1, 3-17.

[35] Schwarz, N. (2004), "Metacognitive Experiences in Consumer Judgment and Decision Making," *Journal of Consumer Psychology*, 14, 332-348.

[36] Sobel, J. (1981), "Distortion of Utilities and the Bargaining Problem," *Econometrica*, 49, 597-620.

[37] Thomson, W. (1987), "Monotonicity of Bargaining Solutions with Respect to the Disagreement Point," *Journal of Economic Theory*, 42, 50-58.

[38] Thomson, W. (1994), "Cooperative Models of Bargaining," In R. J. Aumann and S. Hart (eds.), Handbook of Game Theory with Economic Applications, Vol. 2. Elsevier, New York.

[39] van Damme, E. C. C. (1986), "The Nash Bargaining Solution is Optimal," *Journal of Economic Theory*, 38, 78-100.