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A Visual Classification of Local Martingales

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Abstract. This paper considers the problem of when a local martingale is a martingale or a universally integrable martingale, for the case of time-homogeneous scalar diffusions. Necessary and sufficient conditions of a geometric nature are obtained for answering this question. These results are widely applicable to problems in stochastic finance. For example, in order to apply risk-neutral pricing, one must first check that the chosen density process for an equivalent change of probability measure is in fact a martingale. If not, risk-neutral pricing is infeasible. Furthermore, even if the density process is a martingale, the possibility remains that the discounted price of some security could be a strict local martingale under the equivalent risk-neutral probability measure. In this case, well-known identities for option prices, such as put-call parity, may fail. Using our results, we examine a number of basic asset price models, and identify those that suffer from the above-mentioned difficulties.

Mathematics Subject Classification (2000). Primary: 60J60, 60G44; Secondary: 60G40, 60J35, 60J50, 65L99.

Keywords. Diffusions, first-passage times, Laplace transforms, local martingales, ordinary differential equations.

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1. Introduction

A classical problem of long-standing importance is to formulate conditions that enable one to identify a given local martingale as one of the following: a uniformly integrable martingale, a non-uniformly integrable martingale, or a strict local martingale. Due to the importance of Girsanov's theorem, initial efforts to solve this problem focussed on exponential local martingales. In this regard, noteworthy sufficient conditions for guaranteeing that continuous exponential local martingales are in fact uniformly integrable martingales were obtained by Novikov [37] and Kazamaki [29, 30]. Extensions of these results to a general semimartingale framework were subsequently presented by Lépingle and Mémin [33] and Kallsen and Shiryaev [26].

The first conditions for classifying local martingales that are not necessarily of exponential type may be attributed to Azema et al. [2], where the objects of study were continuous local martingales that converge in L^1. Necessary and sufficient conditions were obtained there, under which such a local martingale is a uniformly integrable martingale. The conditions themselves were expressed in terms of the weak tails of the distributions of the supremum and quadratic variation processes. These results were later systematically refined and extended in a sequence of papers, including Galtchouk and Novikov [19], Novikov [38], Elworthy et al. [14, 15], Takaoka [47] and Kaji [25].

In this article our attention is restricted to local martingales within the realm of time-homogeneous scalar diffusions. So far only Delbaen and Shirakawa [11] and Kotani [32] appear explicitly to have considered the problem of identifying such processes as martingales or strict local martingales. The former article is concerned with local martingales expressed as Itô integrals, and solves the problem by an application of the first Ray-Knight theorem. Kotani [32], on the other hand, adopts an analytic approach, and shows that the problem can be solved by investigating the convergence of certain integrals (see Theorem 3.20).

In the interests of keeping the presentation as self-contained as possible, Section 2 begins with a review of the analytic theory of time-homogeneous scalar diffusions. We do, however, include a few results that are not readily found in the literature. We refer, in particular, to Lemmas 2.3 and 2.4, Corollary 2.5 and Proposition 2.6.

After a brief reacquaintance with some familiar concepts, Section 3 identifies the objects of our investigation as time-homogeneous scalar diffusions in natural scale, whose finite boundaries are natural or absorbing. Proposition 3.6 shows that these processes are local martingales, while Theorem 3.10 demonstrates that they are integrable. The discussion then builds up to Theorem 3.14, which obtains necessary and sufficient conditions under which the local martingales under consideration are martingales. These conditions are strikingly graphical by nature (hence the title of the paper), by which we mean that the problem of deciding whether a given local martingale is a martingale is simply a matter of inspecting the graphs of one or two functions. Next, Theorem 3.19 demonstrates that within the class of time-homogeneous scalar diffusions in natural scale, the difference between martingales and strict local martingales is purely a question of boundary behaviour. Theorem 3.20 then shows that the conditions derived by Kotani [32], Thm. 1 are equivalent to those presented in Theorem 3.14. Finally, Theorem 3.21 completes the section, by establishing that the question of whether a time-homogeneous scalar diffusion in natural scale is a uniformly integrable martingale is trivial.

Following the publication of Sin [46], which established the importance of strict local martingales in stochastic volatility models, ever more attention has been devoted to strict local martingales in stochastic finance. A popular idea is to interpret strict local martingales as models for asset price bubbles (see e.g. Cox and Hobson [6] and Heston et al. [22]). In Section 4 we study a number of popular diffusive local martingales that have appeared in the stochastic finance literature. In each case we apply the results from Theorem 3.14 to classify these processes as martingales or strict local martingales.

2. An Overview of Time-Homogeneous Scalar Diffusions

This section is devoted to a concise overview of the theory of time-homogeneous scalar diffusions, with the principal aims of establishing notation, and collecting a few key results. We lean heavily on the excellent expositions in Borodin and Salminen [5], Chap. II, and Rogers and Williams [44], § V.7, and direct the reader there for further details. The aspects of the theory we most wish to highlight are the boundary classification, attributed to Feller [17], and the elegant perspective offered by the Laplace transform, as elaborated by Itô and McKean [23]. Interspersed in the survey are a few results not typically available in textbooks. We refer, in particular, to Lemmas 2.3 and 2.4, Corollary 2.5 and Proposition 2.6, which all find application in Section 3.

Definition 2.1. Fix an interval $I \subseteq \mathbb{R}$, with left end-point $l \geq -\infty$ and right end-point $r \leq \infty$. Denote the canonical space of continuous I-valued paths by $\Omega := \mathsf{C}(\mathbb{R}_+, I)$, and let $X = (X_t)_{t\in\mathbb{R}_+}$ be the coordinate mapping process on this space, defined by $X_t(\omega) := \omega(t)$, for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$. Define the filtration $\mathfrak{F}^\circ = (\mathscr{F}_t^\circ)_{t\in\mathbb{R}_+}$, by setting $\mathscr{F}_t^\circ := \sigma(X_s \,|\, s \leq t)$, for all $t \in \mathbb{R}_+$, as well as the σ-algebra $\mathscr{F}_\infty^\circ := \sigma(X_t \,|\, t \in \mathbb{R}_+)$. Next, the shift operators $\vartheta = (\vartheta_t)_{t\in\mathbb{R}_+}$ are constructed, by setting $(\vartheta_t\omega)(s) := \omega(t+s)$, for all $\omega \in \Omega$ and $t, s \in \mathbb{R}_+$. Finally, let $\mathfrak{P} = \{\mathsf{P}_x \,|\, x \in I\}$ be a family of probability measures on $(\Omega, \mathscr{F}_\infty^\circ)$, satisfying

(i) $x \mapsto \mathsf{P}_x(A)$ is measurable, for all $A \in \mathscr{F}_\infty^\circ$;
(ii) $\mathsf{P}_x(X_0 = x) = 1$, for all $x \in I$; and
(iii) $\mathsf{E}_x(\eta \circ \vartheta_\sigma \,|\, \mathscr{F}_{\sigma+}^\circ) = \mathsf{E}_{X_\sigma}(\eta)$ P_x-a.s.,

for all bounded $\mathscr{F}_\infty^\circ$-measurable random variables η, and all $\mathfrak{F}^\circ$-stopping times σ. The tuple $(\Omega, \mathscr{F}_\infty^\circ, \mathfrak{F}^\circ, X, \vartheta, \mathfrak{P})$ is then called a canonical diffusion on I.

Unfortunately, the filtration $\mathfrak{F}^\circ$ introduced above has two deficiencies that lead to difficulties with respect to stopping times: Firstly, since it is not necessarily right-continuous, there is an unwelcome disparity between $\mathfrak{F}^\circ$-stopping times and $\mathfrak{F}^\circ$-optional times (see e.g. Karatzas and Shreve [28], Def. 1.2.1 and Prop. 1.2.3, p. 6). Secondly, since it is not necessarily complete, the first-entrance times of Borel sets are not guaranteed to be $\mathfrak{F}^\circ$-stopping times (see e.g. Rogers and Williams [43], § II.75, pp. 184–186). These shortcomings are easily remedied, by first introducing the right-continuous filtration $\mathfrak{F}^+ = (\mathscr{F}_t^+)_{t\in\mathbb{R}_+}$, defined by setting $\mathscr{F}_t^+ := \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}^\circ$, for all $t \in \mathbb{R}_+$. Next, we define the family of null-sets

$$\mathscr{N} := \{N \subseteq \Omega \,|\, N \subseteq A, \text{ for some } A \in \mathscr{F}_\infty^\circ \text{ satisfying } \mathsf{P}_x(A) = 0, \text{ for all } x \in I\}.$$

Finally, the filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is constructed, by setting $\mathscr{F}_t := \mathscr{F}_t^+ \vee \mathscr{N}$, for all $t \in \mathbb{R}_+$. Since none of the above affects the strong Markov property of X, as expressed by Definition 2.1 (iii), we shall henceforth regard $(\Omega, \mathscr{F}_\infty, \mathfrak{F}, X, \vartheta, \mathfrak{P})$ as the diffusion under consideration, where $\mathscr{F}_\infty := \mathscr{F}_\infty^\circ \vee \mathscr{N}$. Obviously, the probability measures P_x, for all $x \in I$, are easily extended to $\mathscr{F}_\infty$, by setting $\mathsf{P}_x(N) := 0$, for all $N \in \mathscr{N}$.

For any $x \in I$, P_x is the probability measure under which the initial value of X is x. We may also start X with a random initial value. To do so, let ν be a probability measure on I, referred to as the initial measure of X. Then there exists a unique probability measure P_ν on $(\Omega, \mathscr{F}_\infty)$, such that

$$\begin{aligned}\mathsf{P}_\nu(X_0 \in A_0, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)& \\ = \int_{A_0}\int_{A_1}\cdots\int_{A_n} \mathsf{P}_{y_{n-1}}(X_{t_n} \in dy_n)&\ldots \mathsf{P}_{y_0}(X_{t_1} \in dy_1)\,\nu(dy_0),\end{aligned}$$

for all $n \in \mathbb{N}$, $A_0, A_1, \ldots, A_n \in \mathscr{B}(I)$ and $t_1, \ldots, t_n \in \mathbb{R}_+$, with $0 < t_1 < \ldots < t_n$. As a special case of this construction, we obtain $\mathsf{P}_x = \mathsf{P}_{\varepsilon_x}$, for all $x \in I$, where the probability measure ε_x on I is defined by

$$\varepsilon_x(A) := \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $A \in \mathscr{B}(I)$.

For any $z \in I$, we denote the first-passage time of X to z by

$$\tau_z := \inf\{t > 0 \,|\, X_t = z\}. \tag{2.1}$$

We shall assume throughout that X is regular. Informally, this means that its state-space I cannot be decomposed into smaller subsets from which X cannot escape. Our definition of regularity does, however, allow for the possibility of absorbing (finite) boundaries, the importance of which will become clear later on:

Definition 2.2. X is said to be a regular diffusion if and only if

$$\mathsf{P}_x(\tau_z < \infty) > 0,$$

for all $x \in \operatorname{int}(I)$ and $z \in I$.

Note that since $\mathsf{P}_x(\Omega) = 1$, for all $x \in I$, it follows that X is a so-called honest diffusion. Therefore, its behaviour is completely determined by its speed measure $\mathfrak{m}$ and scale function $\mathfrak{s}$. In particular, X is not killed, and there is no need to specify a killing measure. In this sense, our setup is aligned with that of Rogers and Williams [44], § V.7, and differs somewhat from that of Borodin and Salminen [5], Chap. II.

The behaviour of X at the boundaries of its state-space is an important theme in this article. The following table employs the boundary classification scheme developed by Feller [17] to classify the end-points of I as exit or entrance boundaries for X, in terms of its speed measure and scale function, where $z \in \mathrm{int}(I)$ is arbitrary:

	Lower Boundary l	Upper Boundary r
Exit	$\int_{(l,z)} \mathfrak{m}(y,z)\,\mathfrak{s}(dy) < \infty$	$\int_{(z,r)} \mathfrak{m}(z,y)\,\mathfrak{s}(dy) < \infty$
Entrance	$\int_{(l,z)} \big(\mathfrak{s}(z) - \mathfrak{s}(y)\big)\,\mathfrak{m}(dy) < \infty$	$\int_{(z,r)} \big(\mathfrak{s}(y) - \mathfrak{s}(z)\big)\,\mathfrak{m}(dy) < \infty$

Boundaries that are both exit and entrance are called non-singular, while boundaries that are neither are called natural. Boundaries that are either exit or entry, but not both, are described as exit-not-entry or entry-not-exit, respectively. Natural boundaries and entry-not-exit boundaries are not part of the diffusion's state-space. If the process is started from the interior of its state-space, it reaches its exit boundaries with positive probability. It is also possible to start a diffusion at an entrance boundary.

Note that the behaviour of X at its non-singular boundaries is not uniquely determined by its speed measure and scale function, and must be specified separately, as part of the description of the process. Typical non-singular boundary conditions include reflection, killing and absorption (see e.g. Borodin and Salminen [5], § II.7, pp. 15–16). We shall specify absorption as the default condition for all finite non-singular boundaries. There are two reasons for this: Firstly, in Section 3 we shall see that absorption at finite exit boundaries is a necessary requirement to ensure that diffusions in natural scale are local martingales. Secondly, from an economic perspective, if X represents the value of an asset, then no-arbitrage considerations demand that once it has vanished, it should not reappear. In other words, if it is attainable, the origin is required to be an absorbing boundary for asset price processes.

We shall denote the transition density of X with respect to its speed measure by the function $q : \mathbb{R}_+ \times I \times I \to \mathbb{R}_+$, so that

$$\mathsf{P}_x(X_t \in A) = \int_A q(t,x,y)\,\mathfrak{m}(dy),$$

for all $t \in \mathbb{R}_+$, $x \in I$ and $A \in \mathscr{B}(I)$. For any fixed $\alpha > 0$, the associated Green's function $G_\alpha : I \times I \to \mathbb{R}_+$ is defined as the Laplace transform (with respect to time) of the transition density:

$$G_\alpha(x,y) := \mathscr{L}_\alpha\{q(t,x,y)\} = \int_0^\infty e^{-\alpha t} q(t,x,y)\,dt, \tag{2.2}$$

for all $x, y \in I$. It has the following representation:

$$G_\alpha(x,y) = \begin{cases} w_\alpha^{-1}\psi_\alpha(x)\phi_\alpha(y) & \text{if } x \le y; \\ w_\alpha^{-1}\phi_\alpha(x)\psi_\alpha(y) & \text{if } x \ge y. \end{cases} \tag{2.3}$$

The functions $\psi_\alpha, \phi_\alpha : I \to \mathbb{R}_+$ appearing above are strictly convex, continuous, strictly monotone, positive, and finite throughout $\operatorname{int}(I)$. Furthermore, they are the unique (up to a multiplicative constant) increasing and decreasing solutions, respectively, of the generalized ordinary differential equation (ODE)

$$(D_- f)(z) - (D_- f)(x) = 2\alpha \int_{[x,z)} f(y)\, m(dy), \tag{2.4}$$

for all $x, z \in \operatorname{int}(I)$, with $x < z$, that also satisfy appropriate conditions at the non-singular boundaries of I. The Wronskian w_α that appears in (2.3) is defined by

$$\begin{aligned} w_\alpha &:= \phi_\alpha(x)(D_-\psi_\alpha)(x) - (D_-\phi_\alpha)(x)\psi_\alpha(x) \\ &= \phi_\alpha(x)(D_+\psi_\alpha)(x) - (D_+\phi_\alpha)(x)\psi_\alpha(x) \end{aligned} \tag{2.5}$$

for all $x \in I$, and is independent of x. Finally, in (2.4) and (2.5), the expressions

$$(D_- f)(z) := \lim_{y \uparrow z} \frac{f(z) - f(y)}{\mathfrak{s}(z) - \mathfrak{s}(y)} \qquad \text{and} \qquad (D_+ f)(z) := \lim_{y \downarrow z} \frac{f(y) - f(z)}{\mathfrak{s}(y) - \mathfrak{s}(z)}$$

for all $z \in \operatorname{int}(I)$, denote the left- and right-hand derivatives relative to scale, respectively, of a convex function $f : I \to \mathbb{R}$.

For any $z \in I$, let $p_z : I \times \mathbb{R}_+ \to \mathbb{R}_+$ denote the density (with respect to Lebesgue measure) of the first-passage time τ_z, so that

$$\mathsf{P}_x(\tau_z < t) = \int_0^t p_z(x,s)\,ds,$$

for all $t \in \overline{\mathbb{R}}_+$ and $x \in I$. We shall also write $\tilde{q}_z : \mathbb{R}_+ \times I \times I \to \mathbb{R}_+$ for the transition density (with respect to speed measure) of X, with absorption at z. It then follows that

$$\mathsf{P}_x(X_t \in A, \tau_z \ge t) = \int_A \tilde{q}_z(t,x,y)\,\mathfrak{m}(dy),$$

for all $t \in \mathbb{R}_+$, $x \in I$ and $A \in \mathscr{B}(I)$. The following integral equation expresses the relationship between the transition densities of the diffusion with and without absorption at z:

Lemma 2.3. *Let $t \in \mathbb{R}_+$, and suppose that $x, y, z \in I$. Then*

$$q(t,x,y) = \tilde{q}_z(t,x,y) + \int_0^t p_z(x,s) q(t-s,z,y)\,ds. \tag{2.6}$$

Proof. From the Markov property of X, we get

$$\begin{aligned}\mathsf{P}_x(X_t \le y) &= \mathsf{P}_x(X_t \le y, \tau_z \ge t) + \mathsf{P}_x(X_t \le y, \tau_z < t)\\ &= \mathsf{P}_x(X_t \le y, \tau_z \ge t) + \int_0^t \mathsf{P}_x(\tau_z \in\, ds)\mathsf{P}_x(X_t \le y \,|\, \tau_z = s)\\ &= \mathsf{P}_x(X_t \le y, \tau_z \ge t) + \int_0^t \mathsf{P}_x(\tau_z \in\, ds)\mathsf{P}_z(X_{t-s} \le y).\end{aligned}$$

The result follows after differentiating with respect to y. □

Note that if $t \in \mathbb{R}_+$, and $x, y, z \in I$ satisfy $x \le z \le y$ or $x \ge z \ge y$, then $\tilde{q}_z(t, x, y) = 0$. Since the integral in (2.6) is a convolution, we then obtain the following Laplace transform identity:

$$\mathscr{L}_\alpha\{q(t, x, y)\} = \mathscr{L}_\alpha\{p_z(x, t)\}\mathscr{L}_\alpha\{q(t, z, y)\},$$

for all $\alpha > 0$, from which it follows that

$$\mathsf{E}_x\big(e^{-\alpha\tau_z}\big) = \mathscr{L}_\alpha\{p_z(x, t)\} = \frac{G_\alpha(x, y)}{G_\alpha(z, y)} = \begin{cases}\frac{\psi_\alpha(x)}{\psi_\alpha(z)} & \text{if } x \le z;\\ \frac{\phi_\alpha(x)}{\phi_\alpha(z)} & \text{if } x \ge z.\end{cases} \tag{2.7}$$

This well-known formula can be found in Itô and McKean [23], p. 128, where it is derived using a somewhat different argument. Note that our derivation of this identity is purely formal, illustrating the usefulness of (2.6). Our interest in (2.7) lies in the next result:

Lemma 2.4. *Let $x, z \in I$. Then*

$$\mathscr{L}_\alpha\{\mathsf{P}_x(\tau_z < t)\} = \begin{cases}\frac{1}{\alpha}\frac{\psi_\alpha(x)}{\psi_\alpha(z)} & \text{if } x \le z\\ \frac{1}{\alpha}\frac{\phi_\alpha(x)}{\phi_\alpha(z)} & \text{if } x \ge z,\end{cases} \tag{2.8}$$

for all $\alpha > 0$, and

$$\mathsf{P}_x(\tau_z < \infty) = \begin{cases}\lim_{\alpha\downarrow 0}\frac{\psi_\alpha(x)}{\psi_\alpha(z)} & \text{if } x \le z\\ \lim_{\alpha\downarrow 0}\frac{\phi_\alpha(x)}{\phi_\alpha(z)} & \text{if } x \ge z.\end{cases} \tag{2.9}$$

Proof. Equation (2.8) follows from

$$\mathscr{L}_\alpha\{\mathsf{P}_x(\tau_z < t)\} = \mathscr{L}_\alpha\bigg\{\int_0^t p_z(x, s)\, ds\bigg\} = \frac{1}{\alpha}\mathscr{L}_\alpha\{p_z(x, t)\},$$

for all $\alpha > 0$, together with (2.7). For (2.9), observe that

$$\mathsf{P}_x(\tau_z < \infty) = \int_0^\infty \lim_{\alpha\downarrow 0} e^{-\alpha t}p_z(x, t)\, dt = \lim_{\alpha\downarrow 0}\mathscr{L}_\alpha\{p_z(x, t)\},$$

and the result follows from (2.7). Note that since the Laplace transform is uniformly convergent with respect to the parameter α, we may freely interchange integral and limit in the above argument. □

As a first application of Lemma 2.4, we establish some asymptotic identities for the cumulative distributions of first-passage times. Note that these results can also be obtained in certain non-diffusion settings. For example, Doob's maximal inequalities (see e.g. Borodin and Salminen [5], Sec. I.19, p. 10) could be used to derive similar limits for general supermartingales and submartingales.

Corollary 2.5. *Suppose $l = -\infty$ and $r = \infty$. Then*

$$\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z < t) = \lim_{z\uparrow\infty} \mathsf{P}_x(\tau_{-z} < t) = \lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z \wedge \tau_{-z} < t) = 0, \tag{2.10}$$

for all $t \in \mathbb{R}_+$ and $x \in I$.

Proof. Fix $x \in I$ and $\alpha > 0$, and note that since ψ_α is strictly increasing, we may infer that $\psi_\alpha(\infty-) = \infty$. Consequently, by the dominated convergence theorem, in conjunction with identity (2.8), we obtain

$$\mathscr{L}_\alpha\left\{\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z < t)\right\} = \lim_{z\uparrow\infty} \mathscr{L}_\alpha\{\mathsf{P}_x(\tau_z < t)\} = \lim_{z\uparrow\infty} \frac{1}{\alpha}\frac{\psi_\alpha(x)}{\psi_\alpha(z)} = 0.$$

The uniqueness of Laplace transforms then implies that $\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z < t) = 0$, for all $t \in \mathbb{R}_+$. A similar argument, based on the fact that ϕ_α is strictly decreasing, gives $\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_{-z} < t) = 0$, for all $t \in \mathbb{R}_+$. Finally, note that

$$\mathsf{P}_x(\tau_z \wedge \tau_{-z} < t) \leq \mathsf{P}_x(\tau_z < t) + \mathsf{P}_x(\tau_{-z} < t),$$

for all $t \in \mathbb{R}_+$ and $z \in I$, to obtain the final identity. □

To complete this section, we briefly consider the situation when the state-space of the diffusion is truncated by the introduction of new absorbing boundaries. As the next result shows, the fundamental solutions of the generalized ODE (2.4) for the resulting absorbed diffusion can be read-off from the original solutions:

Proposition 2.6. *Suppose $a, b \in I$ satisfy $a < b$. Then the following statements are true:*

(i) *The fundamental increasing and decreasing solutions of* (2.4)*, corresponding to a lower absorbing boundary for X at a, are*

$$\tilde{\psi}_\alpha^a(x) := \psi_\alpha(x) - \frac{\psi_\alpha(a)}{\phi_\alpha(a)}\phi_\alpha(x) \quad \textit{and} \quad \tilde{\phi}_\alpha^a(x) := \phi_\alpha(x), \tag{2.11}$$

respectively, for all $x \in I \cap [a, \infty)$ and $\alpha > 0$.

(ii) *The fundamental increasing and decreasing solutions of* (2.4)*, corresponding to an upper absorbing boundary for X at b, are*

$$\tilde{\psi}_\alpha^b(x) := \psi_\alpha(x) \quad \textit{and} \quad \tilde{\phi}_\alpha^b(x) := \phi_\alpha(x) - \frac{\phi_\alpha(b)}{\psi_\alpha(b)}\psi_\alpha(x), \tag{2.12}$$

respectively, for all $x \in I \cap (-\infty, b]$ and $\alpha > 0$.

(iii) *If the process is absorbed upon reaching either boundary, then the relevant increasing and decreasing solutions of* (2.4) *are $\tilde{\psi}_\alpha^{a,b} := \tilde{\psi}_\alpha^a$ and $\tilde{\phi}_\alpha^{a,b} := \tilde{\phi}_\alpha^b$, respectively.*

In each case, the respective Green's function $\tilde{G}^a$, $\tilde{G}^b$ or $\tilde{G}^{a,b}$ for the absorbed diffusion is given by (2.3)*, with the appropriate functions replacing ψ_α and ϕ_α.*

Proof. Fix $\alpha > 0$, and suppose $x, y, z \in I$ satisfy $x, y \leq z$ or $x, y \geq z$. Combining (2.2) with (2.6), (2.3) and (2.7) yields

$$\begin{aligned}
\tilde{G}^z_\alpha(x,y) &:= \mathscr{L}_\alpha\{\tilde{q}_z(t,x,y)\} = \mathscr{L}_\alpha\{q(t,x,y)\} - \mathscr{L}_\alpha\{p_z(x,t)\}\mathscr{L}_\alpha\{q(t,z,y)\} \\
&= \begin{cases} G_\alpha(x,y) - \frac{\psi_\alpha(x)}{\psi_\alpha(z)} G_\alpha(z,y) & \text{if } x, y \leq z; \\ G_\alpha(x,y) - \frac{\phi_\alpha(x)}{\phi_\alpha(z)} G_\alpha(z,y) & \text{if } x, y \geq z \end{cases} \\
&= \begin{cases} w_\alpha^{-1}\psi_\alpha(x)\Big(\phi_\alpha(y) - \frac{\phi_\alpha(z)}{\psi_\alpha(z)}\psi_\alpha(y)\Big) & \text{if } x \leq y \leq z; \\ w_\alpha^{-1}\psi_\alpha(y)\Big(\phi_\alpha(x) - \frac{\phi_\alpha(z)}{\psi_\alpha(z)}\psi_\alpha(x)\Big) & \text{if } y \leq x \leq z; \\ w_\alpha^{-1}\Big(\psi_\alpha(x) - \frac{\psi_\alpha(z)}{\phi_\alpha(z)}\phi_\alpha(x)\Big)\phi_\alpha(y) & \text{if } y \geq x \geq z; \\ w_\alpha^{-1}\Big(\psi_\alpha(y) - \frac{\psi_\alpha(z)}{\phi_\alpha(z)}\phi_\alpha(y)\Big)\phi_\alpha(x) & \text{if } x \geq y \geq z. \end{cases}
\end{aligned} \tag{2.13}$$

Finally, set $z := b$ in the first two lines above, and $z := a$ in the second two, and then compare the resulting expressions for the Green's function with (2.3). □

Suppose $a, b, x \in I$ satisfy $a \leq x \leq b$. We shall write $\tilde{\mathsf{P}}^a_x$ (resp. $\tilde{\mathsf{P}}^b_x$) for the probability measure under which X is absorbed from below at a (resp. absorbed from above at b), after starting at x. Similarly, $\tilde{\mathsf{P}}^{a,b}_x$ will denote the probability measure under which X is absorbed at either a or b, after starting at x.

3. Classification of Diffusive Local Martingales

Throughout this section, let X be a time-homogeneous scalar diffusion with state-space I, as specified in Section 2. Our concern here is the identification of conditions under which X is a P_ν-local martingale, a P_ν-martingale, or a uniformly integrable P_ν-martingale, for any given initial measure ν. Unfortunately, there are some inconsistencies in the way these concepts are defined in the literature. The main difficulty arises from differing integrability assumptions concerning the initial values of local martingales. Although this is a technical issue, it is important in our setup, since it determines how we handle the initial measure of X. Therefore, to avoid ambiguity, and for easy reference later, we start with some familiar definitions:

Definition 3.1. X is a P_ν-martingale (resp. P_ν-supermartingale; P_ν-submartingale) if and only if

(i) $\mathsf{E}_\nu(|X_t|) < \infty$; and
(ii) $\mathsf{E}_\nu(X_t \,|\, \mathscr{F}_s) = X_s$ (resp. $\mathsf{E}_\nu(X_t \,|\, \mathscr{F}_s) \leq X_s$; $\mathsf{E}_\nu(X_t \,|\, \mathscr{F}_s) \geq X_s$) P_ν-a.s.,

for all $s, t \in \mathbb{R}_+$, with $s \leq t$, and any initial measure ν.

Definition 3.2. X is a uniformly integrable P_ν-martingale if and only if

(i) X is a P_ν-martingale; and

(ii) $\lim_{K\uparrow\infty}\sup_{t\in\mathbb{R}_+}\mathsf{E}_\nu\Big(\mathbf{1}_{\{|X_t|>K\}}|X_t|\Big)=0$,

for any initial measure ν.

Note that if $l>-\infty$ and $r<\infty$, then X clearly satisfies the second condition of the above definition, for any initial measure ν. If, in addition, X is a P_ν-local martingale (see Definition 3.3 below), then it is a uniformly integrable P_ν-martingale, by an application of the dominated convergence theorem. Uniformly integrable martingales may also be characterized in terms of convergence conditions: X is a uniformly integrable P_ν-martingale, for some initial measure ν, if and only if the P_ν-a.s. limit $X_\infty:=\lim_{t\uparrow\infty}X_t$ exists, and satisfies $\mathsf{E}_\nu(|X_\infty|)<\infty$ and $\mathsf{E}_\nu(X_\infty\,|\,\mathscr{F}_t)=X_t$, for all $t\in\mathbb{R}_+$.

Definition 3.3. X is a P_ν-local martingale if and only if there exists an increasing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ (called a localizing sequence), such that

(i) $\lim_{n\to\infty}\sigma_n=\infty$ P_ν-a.s.; and
(ii) $(X_{\sigma_n\wedge t}-X_0)_{t\in\mathbb{R}_+}$ is a uniformly integrable P_ν-martingale,

for each $n\in\mathbb{N}$ and any initial measure ν.

The feature of Definition 3.3 we wish to highlight is the fact that if X is a P_ν-local martingale, for some initial measure ν, it does not necessarily follow that $\mathsf{E}_\nu(|X_0|)<\infty$. This is a point on which some textbook definitions of local martingales differ. One may thus expect that the initial component of a local martingale must be handled with care, and the next result illustrates this point. It is a well-known consequence of Fatou's lemma, used repeatedly in the remainder of this article. We state a version that is consistent with the definitions above:

Lemma 3.4. *Suppose X is a P_ν-local martingale, for some initial measure ν. Then the following statements are true:*

(i) *X is a P_ν-supermartingale if $\mathsf{E}_\nu(|X_0|)<\infty$ and $l>-\infty$.*
(ii) *X is a P_ν-submartingale if $\mathsf{E}_\nu(|X_0|)<\infty$ and $r<\infty$.*

Proof. See Rogers and Williams [44], Lem. (14.3), p. 22. □

The following observation is used repeatedly in the sequel: Suppose X is a P_ν-local martingale, for some initial measure ν satisfying $\mathsf{E}_\nu(|X_0|)<\infty$, and assume that $l>-\infty$. Then, according to Lemma 3.4 (i), X is a P_ν-supermartingale that is bounded from below. The martingale convergence theorem (see e.g. Lipster and Shiryaev [34], Thm. 3.3, p. 61) then asserts that the P_ν-a.s. limit $X_\infty:=\lim_{t\to\infty}X_t$ exists, and satisfies $\mathsf{E}_\nu(|X_\infty|)<\infty$. Furthermore, we have

$$\mathsf{E}_\nu(X_\infty)=\mathsf{E}_\nu\left(\lim_{t\to\infty}X_t\right)\le\liminf_{t\to\infty}\mathsf{E}_\nu(X_t)=\liminf_{t\to\infty}\mathsf{E}_\nu(\mathsf{E}_\nu(X_t\,|\,\mathscr{F}_0))\le\mathsf{E}_\nu(X_0),$$

as a consequence of Fatou's lemma and Definition 3.1 (ii). A similar observation applies to the situation when X is a P_ν-local martingale, for some initial measure ν satisfying $\mathsf{E}_\nu(|X_0|)<\infty$, and $r<\infty$. In that case $-X$ is a P_ν-local martingale that is bounded from below, and the same argument as above asserts the existence of X_∞, such that $\mathsf{E}_\nu(|X_\infty|) < \infty$ and $\mathsf{E}_\nu(X_\infty) \geq \mathsf{E}_\nu(X_0)$.

Before continuing, we pause briefly to establish some conventions with respect to the use of notation and terminology. In this regard, for any given initial measure ν, the following notation and terminology will refer specifically to the associated probability measure P_ν: E_ν; P_ν-a.s.; P_ν-martingale; P_ν-supermartingale; P_ν-submartingale; uniformly integrable P_ν-martingale; and P_ν-local martingale. On the other hand, the statement that X is a martingale (resp. supermartingale; submartingale; uniformly integrable martingale; local martingale), without any reference to an initial measure, should be understood to mean that it is a P_x-martingale (resp. P_x-supermartingale; P_x-submartingale; uniformly integrable P_x-martingale; P_x-local martingale), for all $x \in I$. In a similar vein, unless explicitly indicated otherwise, all equalities and inequalities between $\mathscr{F}_\infty$-measurable random variables should be understood to hold P_x-a.s., for all $x \in I$.

The first objective of this section is to identify necessary and sufficient conditions under which X is a local martingale. In this regard, the concept of natural scale will play an important role:

Definition 3.5. X is said to be in natural scale if and only if $\mathfrak{s}(x) = x$, for all $x \in I$.

As observed by Borodin and Salminen [5], Sec. II.16, p. 23, being in natural scale imposes certain constraints on the boundary behaviour of X. In particular, the finite end-points of I are then either natural or exit, while the infinite end-points are either natural or entrance. The next proposition uses these observations to formulate a necessary and sufficient condition for X to be a local martingale, assuming that it is in natural scale. Remarkably, this boils down to a simple matter of the interaction between X and its finite boundaries:

Proposition 3.6. *Suppose X is in natural scale. Then it is a local martingale if and only if its finite boundaries are either natural or absorbing.*

Proof. ($\Rightarrow$) Suppose X is a local martingale. Since it is in natural scale, its finite boundaries are either natural or exit, as remarked above. So let $l > -\infty$ be an exit boundary, from which it follows that $l \in I$. By Lemma 3.4 (i), X is a supermartingale, and so the optional sampling theorem gives

$$\mathsf{E}_x\Big(\mathbf{1}_{\{\tau_l \leq t\}} X_t\Big) \leq \mathsf{E}_x\Big(\mathbf{1}_{\{\tau_l \leq t\}} X_{\tau_l \wedge t}\Big) = l\mathsf{P}_x(\tau_l \leq t),$$

for all $t \in \mathbb{R}_+$ and $x \in I$. Since $X_t \geq l$, for all $t \in \mathbb{R}_+$, it then follows that

$$\mathbf{1}_{\{\tau_l \leq t\}} X_t = \mathbf{1}_{\{\tau_l \leq t\}} l,$$

for all $t \in \mathbb{R}_+$. A similar argument applies when $r < \infty$.

($\Leftarrow$) Suppose the finite boundaries of X are either natural or absorbing. Since X is in natural scale, its infinite boundaries must be either natural or entrance, by the discussion preceding this result. This means that $\tau_l = \infty$ if $l = -\infty$ and $\tau_r = \infty$ if $r = \infty$. Consequently, we have $X = X^{\tau_l \wedge \tau_r}$, irrespective of whether the end-points of I are finite and/or infinite, and the result follows from Rogers and Williams [44], Cor. (46.15), p. 276. □

We now consider the question of how to recognize whether X is a P_ν-martingale, for any given initial measure ν. Since all martingales are also local martingales, it seems reasonable, in view of Proposition 3.6, that we should require the following two conditions to hold:

Assumption 3.7. X is in natural scale.

Assumption 3.8. The finite end-points of I are either natural or absorbing.

With Assumptions 3.7 and 3.8 in place, we are therefore assured that X is at minimum a local martingale. For it to be a martingale, it must meet the two conditions in Definition 3.1. The next theorem demonstrates that the first of these requirements is implicitly satisfied in our setup. Its proof requires that we briefly present the family of local time processes of X:

Definition 3.9. The family of local time processes $\{L_t^y \,|\, y \in I\}$ of X are given by

$$L_t^y := \lim_{\varepsilon \downarrow 0} \frac{1}{\mathfrak{m}(y-\varepsilon, y+\varepsilon)} \int_0^t \mathbf{1}_{\{X_s \in (y-\varepsilon, y+\varepsilon)\}} \, ds, \tag{3.1}$$

for all $t \in \mathbb{R}_+$ and $y \in I$.

It is clear from (3.1) that the process L^y is continuous and increasing, for all $y \in I$. Furthermore, given Assumptions 3.7 and 3.8, the family of local times is Hölder-continuous in the spatial variable, with exponent $\alpha \in \left(0, \frac{1}{2}\right)$, and this continuity is uniform in the temporal variable, over compact time intervals:

$$\sup_{s \in [0,t]} |L_s^y - L_s^z| \leq K(t) |y-z|^\alpha, \tag{3.2}$$

for all $t \in \mathbb{R}_+$ and $y, z \in I$, and for some function $K : \mathbb{R}_+ \to (0, \infty)$ (see e.g. Revuz and Yor [42], Cor. (1.8), p. 226).

Theorem 3.10. *Let Assumptions 3.7 and 3.8 be in force. Then the following statements are true:*

(i) *There exists a function $c : \mathbb{R}_+ \to (0, \infty)$, such that*

$$\mathsf{E}_x(|X_t|) \leq c(t) + |x|, \tag{3.3}$$

for all $t \in \mathbb{R}_+$ and $x \in I$.

(ii) *Let ν be an initial measure. Then $\mathsf{E}_\nu(|X_t|) < \infty$, for all $t \in \mathbb{R}_+$, if and only if*

$$\int_I |x| \, \nu(dx) < \infty. \tag{3.4}$$

Proof. (i) We start by fixing $z \in \mathrm{int}(I)$, and use Tanaka's formula (see e.g. Revuz and Yor [42], Thm. (1.2), p. 222) to obtain

$$|X_t - z| = |X_0 - z| + \underbrace{\int_0^t \mathrm{sgn}(X_s - z)\, dX_s}_{M_t} + L_t^z,$$

for all $t \in \mathbb{R}_+$. Observe that the process M, defined above, is a local martingale with initial value zero. Letting $(\sigma_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times for M, and applying Fatou's lemma, followed by the monotone convergence theorem, we then obtain

$$\begin{aligned} \mathsf{E}_x(|X_t - z|) &= \mathsf{E}_x\Big(\lim_{n\to\infty} |X_{\sigma_n\wedge t} - z|\Big) \\ &\leq |x - z| + \lim_{n\to\infty} \mathsf{E}_x\bigg(\int_0^{\sigma_n\wedge t} \mathrm{sgn}(X_s - z)\, dX_s\bigg) + \lim_{n\to\infty} \mathsf{E}_x\Big(L^z_{\sigma_n\wedge t}\Big) \\ &= |x - z| + \mathsf{E}_x\big(L_t^z\big), \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}_+$ and $x \in I$. Now fix $\alpha \in \big(0, \frac{1}{2}\big)$ and choose $\varepsilon > 0$, such that $(z - \varepsilon, z + \varepsilon) \subset \mathrm{int}(I)$. It then follows from (3.2) that

$$\sup_{s\in[0,t]} |L_s^y - L_s^z| \leq K(t)|y - z|^\alpha \leq K(t)\varepsilon^\alpha,$$

for all $t \in \mathbb{R}_+$ and $y \in (z - \varepsilon, z + \varepsilon)$. Using the occupation-measure formula (see e.g. Rogers and Williams [44], Thm. (49.1), p. 289), we thus obtain

$$\Big(L_t^z - K(t)\varepsilon^\alpha\Big)\mathfrak{m}(z-\varepsilon, z+\varepsilon) \leq \int_I \mathbf{1}_{\{y\in(z-\varepsilon,z+\varepsilon)\}} L_t^y\, \mathfrak{m}(dy) = \int_0^t \mathbf{1}_{\{X_s\in(z-\varepsilon,z+\varepsilon)\}}\, ds,$$

for all $t \in \mathbb{R}_+$. This yields the following upper bound for the expected local time of X at z:

$$\begin{aligned} \mathsf{E}_x\big(L_t^z\big) &\leq K(t)\varepsilon^\alpha + \frac{1}{\mathfrak{m}(z - \varepsilon, z + \varepsilon)} \int_0^t \mathsf{P}_x\Big(X_s \in (z - \varepsilon, z + \varepsilon)\Big)\, ds \\ &\leq K(t)\varepsilon^\alpha + \frac{t}{\mathfrak{m}(z - \varepsilon, z + \varepsilon)}, \end{aligned} \tag{3.6}$$

for all $t \in \mathbb{R}_+$ and $x \in I$, which is independent of x. Furthermore, this bound is finite, since $\mathfrak{m}(z - \varepsilon, z + \varepsilon) > 0$ (see e.g. Rogers and Williams [44], Def. (47.4), p. 277). Finally, combining (3.5) and (3.6) gives

$$\begin{aligned} \mathsf{E}_x(|X_t|) &\leq \mathsf{E}_x(|X_t - z|) + |z| \leq |x - z| + \mathsf{E}_x(L_t^z) + |z| \\ &\leq \underbrace{2|z| + K(t)\varepsilon^\alpha + \frac{t}{\mathfrak{m}(z - \varepsilon, z + \varepsilon)}}_{c(t)} + |x|, \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $x \in I$.

(ii) ($\Rightarrow$) Suppose ν satisfies $\mathsf{E}_\nu(|X_t|) < \infty$, for all $t \in \mathbb{R}_+$. It then follows that

$$\int_I |x|\,\nu(dx) = \int_I |x|\,\mathsf{P}_\nu(X_0 \in dx) = \mathsf{E}_\nu(|X_0|) < \infty,$$

in particular.

($\Leftarrow$) Suppose ν satisfies condition (3.4). It then follows from (3.3) and $\nu(I) = 1$ that

$$\mathsf{E}_\nu(|X_t|) = \int_I \mathsf{E}_x(|X_t|)\,\nu(dx) \le \int_I \Big(c(t) + |x|\Big)\,\nu(dx) = c(t) + \int_I |x|\,\nu(dx) < \infty,$$

for all $t \in \mathbb{R}_+$. □

A similar result to the above is obtained in Kotani [32], Lem. 1. However, the proof there is quite different, and relies heavily on the analytic theory of diffusions, whereas our argument applies general techniques from stochastic calculus (i.e. Tanaka's formula and the occupation-measure formula). In particular, our argument can be adapted to the case of arbitrary continuous local martingales, to establish an analogous and more general result.

We conclude from Theorem 3.10 that Defintion 3.1 (i) is satisfied if X conforms to Assumptions 3.7 and 3.8, and the initial measure ν satisfies condition (3.4). Our next step is to introduce a condition that, at first glance, appears to be strictly weaker than Definition 3.1 (ii). Nevertheless, as we shall see, it is precisely what we need to get a handle on martingales in our setup. Note that a similar condition was studied in detail by Elworthy et al. [15], but in the more general context of arbitrary continuous local martingales.

Definition 3.11. Define the function $\gamma : \mathbb{R}_+ \times I \to \mathbb{R}$, by setting

$$\gamma(t, x) := x - \mathsf{E}_x(X_t), \tag{3.7}$$

for all $t \in \mathbb{R}_+$ and $x \in I$. We say that X satisfies the constant mean condition if and only if $\gamma(t, x) = 0$, for all $t \in \mathbb{R}_+$ and $x \in I$.

Note that if X satisfies the constant mean condition, then for any initial measure ν, we have

$$\mathsf{E}_\nu(X_t) = \int_I \mathsf{E}_x(X_t)\,\nu(dx) = \int_I x\,\nu(dx) = \int_I x\,\mathsf{P}_\nu(X_0 \in dx) = \mathsf{E}_\nu(X_0),$$

for all $t \in \mathbb{R}_+$. In other words, the expected value of X remains constant, irrespective of its initial measure. Of course, having a constant expected value is necessary, but not in general sufficient, to ensure that a local martingale is a martingale (see e.g. Elworthy et al. [15], Prop. 3.8, p. 340). In the special case of diffusions, however, the situation is better:

Proposition 3.12. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4)*, if and only if it satisfies the constant mean condition.*

Proof. ($\Rightarrow$) Suppose X is a P_ν-martingale, for every initial measure ν satisfying condition (3.4). In particular, X is then a P_x-martingale, for all $x \in I$, and the constant mean condition is clearly satisfied.

($\Leftarrow$) Suppose X satisfies the constant mean condition, and let ν be an initial measure satisfying condition (3.4). By Theorem 3.10 (ii), we need only verify Definition 3.1 (ii). To do so, we use the Markov property of X (see Definition 2.1) and the constant mean condition, as follows:

$$\mathsf{E}_\nu\big(X_t \,\big|\, \mathscr{F}_s\big) = \mathsf{E}_\nu\big(X_{t-s} \circ \vartheta_s \,\big|\, \mathscr{F}_s\big) = \mathsf{E}_{X_s}(X_{t-s}) = X_s - \gamma(t-s, X_s) = X_s \quad \mathsf{P}_\nu\text{-a.s.},$$

for all $s, t \in \mathbb{R}_+$, such that $s \leq t$. □

As mentioned earlier, the behaviour of X at its boundaries is an important theme in this paper. As a useful short-hand, we therefore adopt the following simple scheme for describing the end-points of I:

	$l = -\infty$	$l > -\infty$
$r = \infty$	Type I	Type II
$r < \infty$	Type III	Type IV

Subject to Assumptions 3.7 and 3.8, the next two theorems derive testable conditions for determining whether X is a P_ν-martingale, given an initial measure ν satisfying condition (3.4). These results focus only on the cases when the boundaries of I are of Types I–III, since X is a bounded P_ν-local martingale when its boundaries are of Type IV, and hence a uniformly integrable P_ν-martingale, according to the discussion following Definition 3.2:

Theorem 3.13. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then the following statements are true:*

(i) *Let the end-points of I be of Type I. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\lim_{z \uparrow \infty} z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big) = 0, \tag{3.8}$$

for all $t \in \mathbb{R}_+$ and $x \in I$.

(ii) *Let the end-points of I be of Type II. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\lim_{z \uparrow \infty} z\mathsf{P}_x(\tau_z < t) = 0, \tag{3.9}$$

for all $t \in \mathbb{R}_+$ and $x \in I$.

(iii) *Let the end-points of I be of Type III. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\lim_{z \uparrow \infty} z\mathsf{P}_x(\tau_{-z} < t) = 0, \tag{3.10}$$

for all $t \in \mathbb{R}_+$ *and* $x \in I$.

Proof. (i) Fix $x \in I$, and note that $X^{\tau_z \wedge \tau_{-z}}$ is a bounded P_x-local martingale, and hence also a (uniformly integrable) P_x-martingale, for all $z \in (|x|, \infty)$. Thus, for any fixed $t \in \mathbb{R}_+$, we have

$$\begin{aligned} x &= \mathsf{E}_x\Big(X_t^{\tau_z \wedge \tau_{-z}}\Big) \\ &= \mathsf{E}_x\Big(\mathbf{1}_{\{\tau_z \wedge \tau_{-z} \geq t\}} X_t\Big) + z\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - z\mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t), \end{aligned}$$

for all $z \in (|x|, \infty)$. By Corollary 2.5, $\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z \wedge \tau_{-z} \geq t) = 1$, from which it follows that $\lim_{z\uparrow\infty} \mathbf{1}_{\{\tau_z \wedge \tau_{-z} \geq t\}} X_t = X_t$ P_x-a.s. Since $\mathsf{E}_x(|X_t|) < \infty$, by Theorem 3.10 (i), the dominated convergence theorem gives

$$\gamma(t, x) = \lim_{z\uparrow\infty} z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big),$$

and the result follows from Proposition 3.12.

(ii) Fix $x \in I$, and note that X^{τ_z} is a bounded P_x-local martingale, and hence also a (uniformly integrable) P_x-martingale, for all $z \in (x, \infty)$. Thus, for any fixed $t \in \mathbb{R}_+$, we have

$$x = \mathsf{E}_x(X_t^{\tau_z}) = \mathsf{E}_x\Big(\mathbf{1}_{\{\tau_z \geq t\}} X_t\Big) + z\mathsf{P}_x(\tau_z < t),$$

for all $z \in (x, \infty)$. By Corollary 2.5, $\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z \geq t) = 1$, from which it follows that $\lim_{z\uparrow\infty} \mathbf{1}_{\{\tau_z \geq t\}} X_t = X_t$ P_x-a.s. Since $\mathsf{E}_x(|X_t|) < \infty$, by Theorem 3.10 (i), the dominated convergence theorem gives

$$\gamma(t, x) = \lim_{z\uparrow\infty} z\mathsf{P}_x(\tau_z < t),$$

and the result follows from Proposition 3.12.

(iii) Since the state-space of $-X$ is of Type II, this result follows from part (ii). □

Note that, with the exception of the use of Proposition 3.12, the proof of the above theorem is not specific to diffusions. Rather, it relies almost entirely on general results for local martingales. In fact, Theorem 3.13 falls within a tradition of very similar results, which have appeared in e.g. Azema et al. [2], Novikov [38], Elworthy et al. [14, 15] and Takaoka [47]. The next theorem is, however, entirely new, and represents a view of the conditions in Theorem 3.13 from the perspective of Laplace transforms:

Theorem 3.14. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then the following statements are true:*

(i) *Let the end-points of* I *be of Type I. Then* X *is a* P_ν*-martingale, for every initial measure* ν *satisfying condition* (3.4)*, if and only if*

$$(D_-\psi_\alpha)(\infty-) = \infty \qquad \text{and} \qquad (D_-\phi_\alpha)(-\infty+) = -\infty, \tag{3.11}$$

for all $\alpha > 0$.

(ii) *Let the end-points of I be of Type II. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4)*, if and only if*

$$(D_-\psi_\alpha)(\infty-) = \infty, \tag{3.12}$$

for all $\alpha > 0$.

(iii) *Let the end-points of I be of Type III. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4)*, if and only if*

$$(D_-\phi_\alpha)(-\infty+) = -\infty, \tag{3.13}$$

for all $\alpha > 0$.

Proof. (i) ($\Rightarrow$) Suppose X is a P_ν-martingale, for every initial measure ν satisfying condition (3.4). In particular, X is then a P_x-martingale, for all $x \in I$. Now fix $x \in I$ and $\alpha > 0$, and consider imposing an absorbing lower boundary condition at some fixed $a \in (-\infty, x)$. According to Proposition 2.6, the fundamental increasing and decreasing solutions of (2.4), for the absorbed process, are given by $\tilde{\psi}^a_\alpha$ and $\tilde{\phi}^a_\alpha$, respectively, as defined by (2.11). We then see that

$$(D_-\psi_\alpha)(z) = \left(D_-\tilde{\psi}^a_\alpha\right)(z) + \frac{\psi_\alpha(a)}{\phi_\alpha(a)}(D_-\phi_\alpha)(z) \geq \left(D_-\tilde{\psi}^a_\alpha\right)(z) + \frac{\psi_\alpha(a)}{\phi_\alpha(a)}(D_-\phi_\alpha)(a),$$

for all $z \in [a, \infty)$, with the inequality following from the fact the ϕ_α is both strictly decreasing and strictly convex. However, since X is a $\tilde{P}^a_x$-martingale with Type II boundaries, part (ii) of this theorem gives $\left(D_-\tilde{\psi}^a_\alpha\right)(\infty-) = \infty$, from which the first identity in (3.11) follows. By a similar argument we may establish the limit $(D_-\phi_\alpha)(-\infty+) = -\infty$.

($\Leftarrow$) Fix $x \in I$ and $\alpha > 0$, and suppose that ψ_α and ϕ_α satisfy condition (3.11). From Lemma 2.4 and Proposition 2.6 we obtain

$$\begin{aligned}
&\mathscr{L}_\alpha\Big\{z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big)\Big\} \\
&= \mathscr{L}_\alpha\Big\{z\Big(\tilde{\mathsf{P}}^{-z}_x(\tau_z < t) - \tilde{\mathsf{P}}^z_x(\tau_{-z} < t)\Big)\Big\} = \frac{z}{\alpha}\left(\frac{\tilde{\psi}^{-z}_\alpha(x)}{\tilde{\psi}^{-z}_\alpha(z)} - \frac{\tilde{\phi}^z_\alpha(x)}{\tilde{\phi}^z_\alpha(-z)}\right) \\
&= \frac{z}{\alpha}\left(\frac{\psi_\alpha(x) - \frac{\psi_\alpha(-z)}{\phi_\alpha(-z)}\phi_\alpha(x)}{\psi_\alpha(z) - \frac{\psi_\alpha(-z)}{\phi_\alpha(-z)}\phi_\alpha(z)} - \frac{\phi_\alpha(x) - \frac{\phi_\alpha(z)}{\psi_\alpha(z)}\psi_\alpha(x)}{\phi_\alpha(-z) - \frac{\phi_\alpha(z)}{\psi_\alpha(z)}\psi_\alpha(-z)}\right) \\
&= \frac{z}{\alpha}\frac{\psi_\alpha(x)[\phi_\alpha(-z) + \phi_\alpha(z)] - \phi_\alpha(x)[\psi_\alpha(-z) + \psi_\alpha(z)]}{\phi_\alpha(-z)\psi_\alpha(z) - \phi_\alpha(z)\psi_\alpha(-z)} \\
&= \Bigg[\underbrace{\frac{z}{\alpha}\frac{\psi_\alpha(x)}{\psi_\alpha(z)}}_{A_1(z)}\Bigg(1 + \underbrace{\frac{\phi_\alpha(z)}{\phi_\alpha(-z)}}_{B_1(z)}\Bigg) - \underbrace{\frac{z}{\alpha}\frac{\phi_\alpha(x)}{\phi_\alpha(-z)}}_{A_2(z)}\Bigg(1 + \underbrace{\frac{\psi_\alpha(-z)}{\psi_\alpha(z)}}_{B_2(z)}\Bigg)\Bigg]\Bigg[1 - \underbrace{\frac{\phi_\alpha(z)\psi_\alpha(-z)}{\phi_\alpha(-z)\psi_\alpha(z)}}_{C(z)}\Bigg]^{-1},
\end{aligned}$$

for all $z \in (|x|, \infty)$. Recall that $\psi_\alpha(\infty-) = \phi_\alpha(-\infty+) = \infty$, since ψ_α and ϕ_α are strictly increasing and decreasing, respectively. It follows from this observation that $B_1(\infty-) = B_2(\infty-) = C(\infty-) = 0$. Furthermore, with the aid of L'Hôpital's rule, we obtain $A_1(\infty-) = A_2(\infty-) = 0$, from (3.11). So we have

$$\lim_{z\uparrow\infty} \mathscr{L}_\alpha\Big\{z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big)\Big\} = 0.$$

A similar argument gives

$$\begin{aligned}&\mathscr{L}_\alpha\Big\{z\Big|\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big|\Big\}\\ &\qquad\le \mathscr{L}_\alpha\Big\{z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) + \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big)\Big\}\\ &\qquad= \frac{A_1(z)(1-B)1(z)) + A_2(z)(1-B_2(z))}{1-C(z)},\end{aligned}$$

for all $z \in [|x|, \infty)$, from which it follows that

$$\lim_{z\uparrow\infty} \mathscr{L}_\alpha\Big\{z\Big|\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big|\Big\} = 0. \tag{3.14}$$

Therefore, the function

$$[|x|, \infty) \ni z \mapsto \mathscr{L}_\alpha\Big\{z\Big|\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big|\Big\}$$

is bounded. Consequently, we may apply the dominated convergence theorem applied to (3.14), giving

$$\mathscr{L}_\alpha\Big\{\lim_{z\uparrow\infty} z\Big(\mathsf{P}_x(\tau_z < \tau_{-z}, \tau_z < t) - \mathsf{P}_x(\tau_{-z} < \tau_z, \tau_{-z} < t)\Big)\Big\} = 0.$$

By the uniqueness of Laplace transforms, condition (3.8) thus holds, for all $t \in \mathbb{R}_+$, and so the result follows from Theorem 3.13 (i).

(ii) Fix $x \in I$ and $\alpha > 0$, and recall from the discussion following Lemma 3.4 that X is a P_x-supermartingale, and that X_∞ exists P_x-a.s. and satisfies $\mathsf{E}_x(|X_\infty|) < \infty$. Consequently, we obtain

$$0 \le z\mathsf{P}_x(\tau_z < t) \le z\mathsf{P}_x(\tau_z < \infty) \le |x| + \mathsf{E}_x(|X_\infty|),$$

for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}_+ \cap (x, \infty)$, from Doob's maximal inequalities (see e.g. Borodin and Salminen [5], Sec. I.19, p. 10), and we also see that

$$\mathscr{L}_\alpha\{|x| + \mathsf{E}_x(|X_\infty|)\} = \frac{|x| + \mathsf{E}_x(|X_\infty|)}{\alpha} < \infty.$$

Noting once again that $\psi_\alpha(\infty-) = \infty$, since ψ_α is strictly increasing, we now apply the dominated convergence theorem, followed by Lemma 2.4 and L'Hôpital's rule, to get

$$\begin{aligned}\mathscr{L}_\alpha\Big\{\lim_{z\uparrow\infty} z\mathsf{P}_x(\tau_z < t)\Big\} &= \lim_{z\uparrow\infty} \mathscr{L}_\alpha\{z\mathsf{P}_x(\tau_z < t)\} = \lim_{z\uparrow\infty} \frac{z}{\alpha}\frac{\psi_\alpha(x)}{\psi_\alpha(z)}\\ &= \frac{\psi_\alpha(x)}{\alpha} \lim_{z\uparrow\infty} \frac{1}{(D_-\psi_\alpha)(z)}.\end{aligned}$$

Finally, the result follows from Theorem 3.13 (ii), together with the uniqueness of Laplace transforms.

(iii) Note that $\hat{I} := \{-x \,|\, x \in I\}$ is the state-space of $-X$. Its end-points are clearly of Type II. Next, let $\hat{\psi}_\alpha, \hat{\phi}_\alpha : \hat{I} \to \mathbb{R}_+$, for all $\alpha > 0$, denote the fundamental increasing and decreasing solutions, respectively, of (2.11) for $-X$. It is easy to see that

$$\hat{\psi}_\alpha(x) = \phi_\alpha(-x) \qquad \text{and} \qquad \hat{\phi}_\alpha(x) = \psi_\alpha(-x),$$

for all $x \in \hat{I}$ and $\alpha > 0$. The result now follows from part (ii). □

Although conditions (3.11) and (3.12) appear rather forbidding at first glance, we shall see in Section 4 that they are generally quite easy to test in practice. Furthermore, they facilitate a remarkable extension of Proposition 3.6, which will be the content of Theorem 3.19. To get there, we require the following three propositions, which collectively elaborate on Rogers and Williams [44], Thm. (51.2) (iv), p. 295:

Proposition 3.15. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then the following statements are true:*

(i) *Let the end-points of I be of Types I or II. Then*

$$\lim_{x\uparrow\infty} \mathsf{E}_x(\tau_z) \le 2 \int_{[z,\infty)} (y - z)\, \mathfrak{m}(dy), \tag{3.15}$$

for all $z \in \mathrm{int}(I)$.

(ii) *Let the end-points of I be of Types I or III. Then*

$$\lim_{x\downarrow-\infty} \mathsf{E}_x(\tau_z) \le 2 \int_{(-\infty,z)} (z - y)\, \mathfrak{m}(dy), \tag{3.16}$$

for all $z \in \mathrm{int}(I)$.

Proof. (i) Fix $x, z \in \mathrm{int}(I)$, with $x > z$, and note that X^{τ_z} is a P_x-supermartingale, by Lemma 3.4 (i). Furthermore, we see from the discussion following Lemma 3.4 that the P_x-a.s. limit $X^{\tau_z}_\infty = X_{\tau_z}$ exists and satisfies $\mathsf{E}_x(|X_{\tau_z}|) < \infty$. Next, using Tanaka's formula (see e.g. Revuz and Yor [42], Thm. (1.2), p. 222), we obtain

$$(X_t - y)^- = (X_0 - y)^- + \underbrace{\int_0^t \mathbf{1}_{\{X_s \le y\}}\, dX_s}_{M_t} + \frac{1}{2} L^y_t,$$

for all $t \in \mathbb{R}_+$ and $y \in I$. Since the process M, defined above, is a P_x-local martingale with initial value zero, we may infer the existence of an associated localizing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$. Observe that

$$0 \le (X_{\sigma_n \wedge \tau_z} - y)^- \le (z - y)^- \quad \mathsf{P}_x\text{-a.s.},$$

for all $y \in I$ and $n \in \mathbb{N}$, and recall that L^y is a P_x-a.s. increasing process. Consequently, using the dominated convergence theorem, followed by the optional sampling theorem and the monotone convergence theorem, we get

$$\begin{aligned}\mathsf{E}_x\Big((X_{\tau_z}-y)^-\Big) &= \lim_{n\to\infty}\mathsf{E}_x\Big((X_{\sigma_n\wedge\tau_z}-y)^-\Big)\\ &= (x-y)^- + \lim_{n\to\infty}\mathsf{E}_x\bigg(\int_0^{\sigma_n\wedge\tau_z}\mathbf{1}_{\{X_s\le y\}}\,dX_s\bigg) + \frac{1}{2}\lim_{n\to\infty}\mathsf{E}_x\Big(L^y_{\sigma_n\wedge\tau_z}\Big)\\ &= (x-y)^- + \frac{1}{2}\mathsf{E}_x\Big(L^y_{\tau_z}\Big),\end{aligned}$$

for all $y\in I$. Rearranging this expression, we obtain

$$\mathsf{E}_x\Big(L^y_{\tau_z}\Big) \le 2\Big((z-y)^- - (x-y)^-\Big) = 2\Big((x-z)\wedge(y-z)^+\Big),$$

for all $y\in I$, since $X_{\tau_z}\ge z$ P_x-a.s. Finally, the occupation-measure formula (see e.g. Rogers and Williams [44], Thm. (49.1), p. 289) yields

$$\begin{aligned}\mathsf{E}_x(\tau_z) &= \mathsf{E}_x\bigg(\int_0^{\tau_z}\mathbf{1}_{\{X_s\ge z\}}\,ds\bigg) = \mathsf{E}_x\bigg(\int_I \mathbf{1}_{\{y\ge z\}}L^y_{\tau_z}\,\mathfrak{m}(dy)\bigg)\\ &= \int_{[z,\infty)}\mathsf{E}_x\Big(L^y_{\tau_z}\Big)\,\mathfrak{m}(dy) \le 2\int_{[z,\infty)}(x-z)\wedge(y-z)\,\mathfrak{m}(dy),\end{aligned}$$

and (3.15) follows as a consequence of the monotone convergence theorem.

(ii) The proof of (3.16) follows along the same lines as above. □

We remark that if the right-hand side of (3.15) is finite, for some $z\in\operatorname{int}(I)$, then $\mathsf{E}_x(\tau_z)<\infty$, for all $x\in(z,\infty)$. This in turn implies that $\mathsf{P}_x(\tau_z<\infty)=1$, for all $x\in(z,\infty)$, whence $X_{\tau_z}=z$ P_x-a.s. A quick inspection of the proof of Proposition 3.15 (i) reveals the inequality in (3.15) to be an equality in this case. A similar observation holds if the right-hand side of (3.16) is finite. In this sense, the inequalities in (3.15) and (3.16) are sharp.

Proposition 3.16. *Suppose Assumptions* 3.7 *and* 3.8 *are in force, and assume that the end-points of I are of Types I or II. Then the following conditions are equivalent:*

(i) $\phi_\alpha(\infty-)>0$*;*
(ii) $(D_-\psi_\alpha)(\infty-)<\infty$*; and*
(iii) $\int_{[z,\infty)}(y-z)\,\mathfrak{m}(dy)<\infty$*,*

for all $z\in\operatorname{int}(I)$ and $\alpha>0$.

Proof. (i)⇒(ii): Fix $\alpha>0$, and suppose that $\phi_\alpha(\infty-)>0$. Since ϕ_α is strictly decreasing and ψ_α is non-negative, we obtain the following inequality from (2.5):

$$\phi_\alpha(x)(D_-\psi_\alpha)(x) = w_\alpha + (D_-\phi_\alpha)(x)\psi_\alpha(x) < w_\alpha,$$

for all $x\in I$. Taking limits, it therefore follows that $(D_-\psi_\alpha)(\infty-)<\infty$.

(ii)⇒(iii): Fix $z\in\operatorname{int}(I)$ and $\alpha>0$, and suppose that $(D_-\psi_\alpha)(\infty-)<\infty$. We now get

$$\int_{[z,\infty)} (y-z)\,\mathfrak{m}(dy) \le \int_{[z,\infty)} \frac{\psi_\alpha(y)-\psi_\alpha(z)}{(D_-\psi_\alpha)(z)}\,\mathfrak{m}(dy)$$
$$\le \frac{1}{(D_-\psi_\alpha)(z)} \int_{[z,\infty)} \psi_\alpha(y)\,\mathfrak{m}(dy) = \frac{(D_-\psi_\alpha)(\infty-)-(D_-\psi_\alpha)(z)}{2\alpha(D_-\psi_\alpha)(z)} < \infty,$$

from (2.4), together with facts that ψ_α is non-negative, strictly increasing and strictly convex.

(iii)⇒(i): Fix $z \in \mathrm{int}(I)$ and $\alpha > 0$, and suppose that $\int_{[z,\infty)}(y-z)\,\mathfrak{m}(dy) < \infty$. We then obtain

$$\phi_\alpha(\infty-) = \phi_\alpha(z) \lim_{x\uparrow\infty} \mathsf{E}_x(e^{-\alpha\tau_z}) \ge \phi_\alpha(z) \lim_{x\uparrow\infty} e^{-\alpha \mathsf{E}_x(\tau_z)}$$
$$\ge \phi_\alpha(z) e^{-2\alpha\int_{[z,\infty)}(y-z)\,\mathfrak{m}(dy)} > 0,$$

from (2.7), followed by Jensen's inequality and (3.15). □

Proposition 3.17. *Suppose Assumptions* 3.7 *and* 3.8 *are in force, and assume that the end-points of I are of Types I or III. Then the following conditions are equivalent:*

(i) $\psi_\alpha(-\infty+) > 0$*;*
(ii) $(D_-\phi_\alpha)(-\infty+) > -\infty$*; and*
(iii) $\int_{(-\infty,z)}(z-y)\,\mathfrak{m}(dy) < \infty$*,*

for all $z \in \mathrm{int}(I)$ and $\alpha > 0$.

Proof. Similar to the proof of Proposition 3.16. □

Suppose Assumption 3.7 holds, and assume that the end-points of I are of Type I. By inspecting the boundary classification table in Section 2, we observe, with the aid of Propositions 3.16 and 3.17, that X has entrance boundaries if and only if condition (3.11) is violated. In order to add more insight to this observation, we first recall the following definition of entrance boundaries (see e.g. Revuz and Yor [42], Def. (3.9), p. 305):

Definition 3.18. The lower end-point l (resp. upper end-point r) is an entrance boundary if and only if $\lim_{x\downarrow l} \mathsf{P}_x(\tau_z < t) > 0$ (resp. $\lim_{x\uparrow r} \mathsf{P}_x(\tau_z < t) > 0$), for some $t \in \mathbb{R}_+$ and $z \in \mathrm{int}(I)$.

We now prove an extension of Proposition 3.6, which characterizes martingales in our setup purely in terms of their boundary behaviour. The proof is based on first principles, in the sense that we use Definition 3.18 directly, rather than the boundary classification table in Section 2.

Theorem 3.19. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4)*, if and only if its infinite boundaries are natural.*

Proof. Suppose the end-points of I are of Types I or II, and fix $\alpha > 0$. By an application of the dominated convergence theorem and Lemma 2.4, we obtain

$$\mathscr{L}_\alpha\Big\{\lim_{x\uparrow\infty} \mathsf{P}_x(\tau_z < t)\Big\} = \frac{1}{\alpha}\lim_{x\uparrow\infty}\frac{\phi_\alpha(x)}{\phi_\alpha(z)},$$

for all $z \in \mathrm{int}(I)$. It then follows from the uniqueness of Laplace transforms that $\phi_\alpha(\infty-) > 0$ if and only if there exists an $s \in \mathbb{R}_+$, such that $\lim_{x\uparrow\infty} \mathsf{P}_x(\tau_z < t) > 0$, for all $t \in [s,\infty)$ and $z \in \mathrm{int}(I)$. In particular, by Proposition 3.16, we have $(D_-\psi_\alpha)(\infty-) < \infty$ if and only if $r = \infty$ is an entrance boundary. Similarly, if the end-points of I are of Types I or III, then $(D_-\phi_\alpha)(-\infty+) > -\infty$ if and only if $l = -\infty$ is an entrance boundary. Finally, the result follows from Proposition 3.6 and Theorem 3.14. □

Elegant necessary and sufficient conditions for X to be a martingale, subject to Assumptions 3.7 and 3.8, have recently been obtained Kotani [32], Thm. 1. As a second application of Propositions 3.16 and 3.17, we now establish the correspondence between Kotani's conditions and the classical probability-theoretic conditions of Theorem 3.13:

Theorem 3.20 (Kotani [32]). *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then the following statements are true:*

(i) *Let the end-points of I be of Type I. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\int_{[z,\infty)} (y - z)\,\mathfrak{m}(dy) = \infty \qquad \text{and} \qquad \int_{(-\infty,z)} (z - y)\,\mathfrak{m}(dy) = \infty,$$

for all $z \in \mathrm{int}(I)$.

(ii) *Let the end-points of I be of Type II. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\int_{[z,\infty)} (y - z)\,\mathfrak{m}(dy) = \infty,$$

for all $z \in \mathrm{int}(I)$.

(ii) *Let the end-points of I be of Type III. Then X is a P_ν-martingale, for every initial measure ν satisfying condition* (3.4), *if and only if*

$$\int_{(-\infty,z)} (z - y)\,\mathfrak{m}(dy) = \infty,$$

for all $z \in \mathrm{int}(I)$.

Proof. This follows by using Propositions 3.16 and 3.17 to translate the conditions in Theorem 3.14. □

We conclude this section by searching for conditions under which X is a uniformly integrable P_ν-martingale, for some initial measure ν satisfying condition (3.4), subject to Assumptions 3.7 and 3.8. According to the discussion following Definition 3.2, a necessary requirement is that the P_ν-a.s. limit X_∞ should exist, and satisfy $\mathsf{E}_\nu(|X_\infty|) < \infty$ and $\mathsf{E}_\nu(X_\infty) = \mathsf{E}_\nu(X_0)$. As seen in the discussion following Lemma 3.4, this limit exists and belongs to L^1 if the end-points of I are of Types II–IV. Of course, the case when the end-points of I are of Type IV is trivial, since X is then a bounded P_ν-local martingale, and consequently also a uniformly integrable P_ν-martingale, by the discussion following Definition 3.2 again. However, the situation when the end-points of I are of Types II or III is perhaps a little surprising:

Theorem 3.21. *Suppose Assumptions* 3.7 *and* 3.8 *are in force. Then the following statements are true:*

(i) *Let the end-points of I be of Type II. Then $X_\infty = l$ P_ν-a.s., for every initial measure ν satisfying condition* (3.4).

(ii) *Let the end-points of I be of Type III. Then $X_\infty = r$ P_ν-a.s., for every initial measure ν satisfying condition* (3.4).

Proof. (i) Fix $x \in I$, and note that l is either natural or absorbing, since $l > -\infty$. We start, therefore, by assuming that l is absorbing. This implies that

$$\{\tau_z < \infty\} = \{\tau_z < \tau_l\},$$

for all $z \in \operatorname{int}(I)$. Since $l \in I$, the fact that X is in natural scale then yields

$$\mathsf{P}_x(\tau_z < \infty) = \mathsf{P}_x(\tau_z < \tau_l) = \frac{x-l}{z-l}, \tag{3.17}$$

for all $z \in (x, \infty)$, according to Rogers and Williams [44], Def. (46.10), p. 275. Next, suppose that l is a natural boundary. In particular, $l \notin I$, which implies that $\lim_{k\to\infty} \tau_{l+\frac{1}{k}} = \infty$, from which it follows that

$$\{\tau_z < \infty\} = \bigcup_{k=1}^{\infty} \Big\{\tau_z < \tau_{l+\frac{1}{k}}\Big\}.$$

Since probability measures are continuous from below and X is in natural scale, Rogers and Williams [44], Def. (46.10), p. 275 now gives

$$\begin{aligned} \mathsf{P}_x(\tau_z < \infty) &= \mathsf{P}_x\left(\bigcup_{k=1}^{\infty} \Big\{\tau_z < \tau_{l+\frac{1}{k}}\Big\}\right) = \lim_{n\to\infty} \mathsf{P}_x\left(\bigcup_{k=1}^{n} \Big\{\tau_z < \tau_{l+\frac{1}{k}}\Big\}\right) \\ &= \lim_{n\to\infty} \mathsf{P}_x\Big(\tau_z < \tau_{l+\frac{1}{n}}\Big) = \lim_{n\to\infty} \frac{x - \big(l + \frac{1}{n}\big)}{z - \big(l+\frac{1}{n}\big)} = \frac{x-l}{z-l}, \end{aligned} \tag{3.18}$$

for all $z \in (x, \infty)$. Next, observe that X^{τ_z} is in fact a uniformly integrable P_x-martingale, according to the discussion following Definition 3.2, since it is bounded. Consequently,

$$x = \mathsf{E}_x(X^{\tau_z}_\infty) = \mathsf{E}_x(\mathbf{1}_{\{\tau_z=\infty\}} X_\infty) + z\mathsf{P}_x(\tau_z < \infty). \tag{3.19}$$

Furthermore, it follows from (3.17) and (3.18) that

$$\lim_{z\uparrow\infty} \mathsf{P}_x(\tau_z = \infty) = 1 - \lim_{z\uparrow\infty} \frac{x-l}{z-l} = 1,$$

irrespective of whether l is natural or absorbing, whence $\lim_{z\uparrow\infty} \mathbf{1}_{\{\tau_z=\infty\}} X_\infty = X_\infty$ P_x-a.s. Finally, since $\mathsf{E}_x(|X_\infty|) < \infty$, an application of the dominated convergence theorem to (3.19) yields

$$x - \mathsf{E}_x(X_\infty) = \lim_{z\uparrow\infty} z\mathsf{P}_x(\tau_z < \infty) = \lim_{z\uparrow\infty} z\frac{x-l}{z-l} = x - l, \tag{3.20}$$

with the help of (3.17) and (3.18), once again. This implies that $\mathsf{E}_x(X_\infty) = l$, from which we obtain

$$\mathsf{E}_\nu(X_\infty) = \int_I \mathsf{E}_x(X_\infty)\, \nu(dx) = l\nu(I) = l,$$

for any initial measure ν satisfying condition (3.4), and the result follows from the fact that $X_\infty \geq l$ P_ν-a.s.

(ii) Similar to the proof of part (i). □

The rather startling implication of Theorem 3.21 is that there are no uniformly integrable martingales within the class of time-homogeneous diffusions in natural scale, with one finite boundary. Rather, such processes are all extreme supermartingales or submartingales, in the sense that all their paths converge to the finite boundary. A driftless geometric Brownian motion (see Example 4.3) provides an important example of this phenomenon, since Theorem 3.21 (i) indicates that all paths must converge to the origin. Of course, in the case of driftless geometric Brownian motions, this behaviour is also a well-known consequence of the law of large numbers (see e.g. Karatzas and Shreve [28], Rem., p. 193). We also note the discussion in Karatzas and Shreve [28], p. 192, which highlights the impact of Theorem 3.21 (i) on potential applications of Girsanov's theorem.

4. Some Examples Relevant to Finance

In this section we consider the situation where the P_x-dynamics of X are given by a stochastic differential equation (SDE) of the form

$$X_t = x + \int_0^{\tau_l \wedge \tau_r \wedge t} a(X_s)\, d\beta_s, \tag{4.1}$$

for all $t \in \mathbb{R}_+$ and $x \in I$, where $a : I \to \mathbb{R}_+$ and $\beta = (\beta_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. In this case the characteristics of X are absolutely continuous with respect to Lebesgue measure, and we have

$$\mathfrak{m}(dx) = 2a^{-2}(x)\, dx \qquad \text{and} \qquad \mathfrak{s}(x) = x, \tag{4.2}$$

for all $x \in I$ (see e.g. Borodin and Salminen [5], p. 17). It is also clear from (4.1) that X is absorbed at its finite boundaries, by construction. According to Proposition 3.6, we may thus conclude that X is a local martingale.

The next step is to determine whether X is in fact a martingale. We start by observing that the generalized ODE (2.4) now simplifies to

$$\frac{1}{2}a^2(x)f''(x) = \alpha f(x), \tag{4.3}$$

for all $x \in I$ and $\alpha > 0$ (see e.g. Borodin and Salminen [5], pp. 17–18). In general, this equation is easily solved for the increasing and decreasing fundamental solutions ψ_α and ϕ_α, respectively. Once these functions have been identified, Theorem 3.14 (i) and (ii) may be used to answer the above question.

In financial applications, it is usually the case that $I = \mathbb{R}_+$ or $I = (0, \infty)$. It then follows from Lemma 3.4 (i) that X is at minimum a supermartingale. Generally speaking, there are two situations where it is important to know whether X is also a martingale: Firstly, X may be the density process for a change of measure, associated with a corresponding change of numéraire (see e.g. Geman et al. [20]). In this case, if X is not a martingale, it is then a strict supermartingale, and the resulting measure will be deficient (i.e. its total mass will be less than one). In such circumstances the change of numéraire technique for pricing contingent claims is not viable, and one has to adopt an alternative methodology, such as the benchmark approach advocated by Platen and Heath [40].

The second situation where it is important to know whether X is a martingale occurs when (4.1) describes the risk-neutral behaviour of a discounted asset. A standard problem then is to determine the price of a European contingent claim, with payoff function $h : I \to \mathbb{R}_+$ and maturity $T > 0$. A well-established technique for obtaining the discounted pricing function $V : I \times [0, T] \to \mathbb{R}_+$ for the claim is to solve the associated Black-Scholes partial differential equation (PDE)

$$\frac{\partial V}{\partial t}(t, x) + \frac{1}{2}a^2(x)\frac{\partial^2 V}{\partial x^2}(t, x) = 0, \tag{4.4}$$

for all $(t, x) \in I \times [0, T]$, subject to the terminal condition

$$V(T, x) = h(x), \tag{4.5}$$

for all $x \in I$. Now, since X is a supermartingale, it follows that the function γ, given by (3.7), satisfies the inequality $\gamma(t, x) \geq 0$, for all $(t, x) \in \mathbb{R}_+ \times I$. Furthermore, we see from e.g. Friedman [18], Thm. 6.1, p. 124 that the function $\bar{\gamma} : [0, T] \times I \to I$, defined by $\bar{\gamma}(t, x) := \gamma(T - t, x)$, for all $(t, x) \in [0, T] \times I$, also satisfies the Black-Scholes PDE (4.4), with terminal condition $\bar{\gamma}(T, x) = 0$, for all $x \in I$. However, if X is not a martingale, then γ is not identically zero, according to Proposition 3.12. Consequently, there exists a maturity $T > 0$, such that for any solution V of (4.4)–(4.5), and any $a \in \mathbb{R}$, the function

$$\bar{V}_a := V + a\bar{\gamma}$$

provides another solution. This association between strict local martingales and the non-uniqueness of solutions of the Black-Scholes PDE is discussed at length by Heston et al. [22].

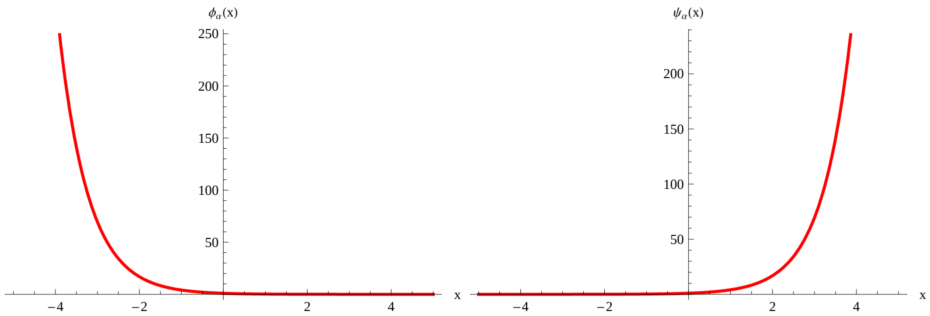

FIGURE 4.1. The functions ϕ_α and ψ_α for Brownian motion.

The question of whether X is a martingale or a strict local martingale is therefore of fundamental importance to stochastic finance. We now examine specific examples of (4.1), solving the ODE (4.3) in each case, and then applying Theorem 3.14 (i) and (ii) to determine whether X is a martingale, in each case. Our first example is simply Brownian motion itself:

Example 4.1 (Brownian motion). In this case $I = \mathbb{R}$ and $a(x) := 1$, for all $x \in I$. Solving (4.3) then yields

$$\phi_\alpha(x) = e^{-\sqrt{2\alpha}\,x} \qquad \text{and} \qquad \psi_\alpha(x) = e^{\sqrt{2\alpha}\,x},$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.1). It is easily seen that $\phi'_\alpha(-\infty+) = -\infty$ and $\psi'_\alpha(\infty-) = \infty$, for all $\alpha > 0$, from which we may deduce that X is a martingale, by Theorem 3.14 (i).

Our next example is based on the square-root process of Feller [16], which continues to play a significant role in stochastic finance. For example, it features in the Cox et al. [9] model for the term structure of interest rates, as well in the Heston [21] stochastic volatility model. However, this process first appeared in the finance literature in Cox and Ross [8], where it was considered as a model for stock price dynamics—later becoming a special case of the constant elasticity of variance (CEV) model (see e.g. Cox [7]). In this context, the question of whether X is a martingale is crucial for determining whether options on the stock are sensibly priced by solving the discounted Black-Scholes PDE (4.4)–(4.5).

Example 4.2 (Squared Bessel process of dimension zero). In this case $I = \mathbb{R}_+$ and $a(x) := 2\sqrt{x}$, for all $x \in I$. Solving (4.3) then yields

$$\phi_\alpha(x) = \sqrt{x}\,K_1\Big(\sqrt{2\alpha x}\Big) \qquad \text{and} \qquad \psi_\alpha(x) = \sqrt{x}\,I_1\Big(\sqrt{2\alpha x}\Big),$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.2). It is easily seen that $\psi'_\alpha(\infty-) = \infty$, for all $\alpha > 0$, from which we may deduce that X is a martingale, by Theorem 3.14 (ii).

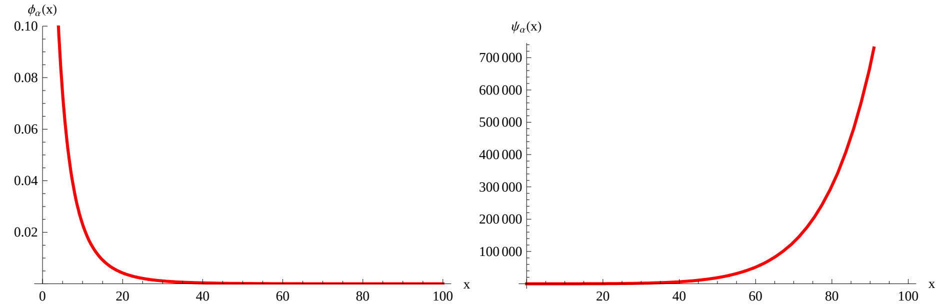

FIGURE 4.2. The functions ϕ_α and ψ_α for the squared Bessel process of dimension zero.

Geometric Brownian motion was originally proposed as a model for risky asset price dynamics by Samuelson [45]. Thereafter, Black and Scholes [4] and Merton [36] famously solved the pricing problem for European puts and calls on assets whose prices follow geometric Brownian motions. This gave geometric Brownian motion the impetus for becoming the most widely-used general asset price model today—both in practice and in the academic literature. Driftless geometric Brownian motion—which is a local martingale—appears to two contexts in financial modeling: Firstly, in a market containing a savings account yielding a constant risk-free rate of return, and a risky security whose price follows a geometric Brownian motion, the density process defining the equivalent risk-neutral probability measure is a driftless geometric Brownian motion. Secondly, under this probability measure, the discounted price process of the risky asset is itself a driftless geometric Brownian motion. It is therefore vitally important that driftless geometric Brownian motions are in fact proper martingales:

Example 4.3 (Driftless geometric Brownian motion). In this case $I = (0, \infty)$ and $a(x) := x$, for all $x \in I$. Solving (4.3) then yields

$$\phi_\alpha(x) = x^{-\frac{1}{2}\left(\sqrt{8\alpha+1}-1\right)} \qquad \text{and} \qquad \psi_\alpha(x) = x^{\frac{1}{2}\left(\sqrt{8\alpha+1}+1\right)},$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.3). It is easily seen that $\psi'_\alpha(\infty-) = \infty$, for all $\alpha > 0$, from which we may deduce that X is a martingale, by Theorem 3.14 (ii).

A new framework for asset price modeling and contingent claim valuation, called the benchmark approach, has recently been advocated by Platen and Heath [40]. The central idea of this approach is to focus on modeling the so-called growth-optimal portfolio (GOP), which originated in a study by Kelly [31]. The properties of this portfolio, and questions about its existence, have been elucidated in a stream of subsequent publications, including e.g. Long [35], Bajeux-Besnainou and

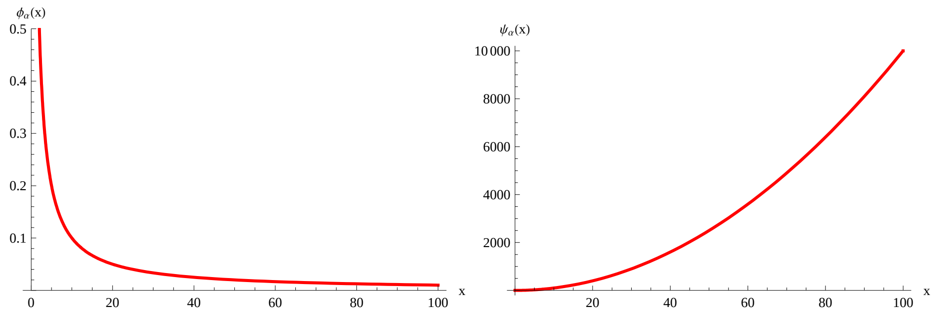

FIGURE 4.3. The functions ϕ_α and ψ_α for driftless geometric Brownian motion.

Portait [3], Platen [39] and Karatzas and Kardaras [27]. The most important of these properties, from the perspective of stochastic finance, is that the GOP acts as a numéraire for the real-world probability measure. This presents the enticing possibility of contingent claim pricing under the real-world probability measure (see Platen and Heath [40], § 9.1), without the sophisticated machinery of equivalent changes of probability measure. The only requirement is a realistic model of the GOP that is both realistic and tractable. This realization motivated the development of the minimal market model (MMM), presented in Platen and Heath [40], Chap. 13. The salient feature of this model is that the density process for a putative equivalent risk-neutral probability measure is an inverted time-changed squared Bessel process of dimension four. As the following example demonstrates, this density process is a strict local martingale, which implies that risk-neutral pricing is not possible with the MMM:

Example 4.4 (Inverted squared Bessel process of dimension four). In this case $I = (0, \infty)$ and $a(x) := 2x^{\frac{3}{2}}$, for all $x \in I$. Solving (4.3) then yields

$$\phi_\alpha(x) = \sqrt{x} I_1\left(\sqrt{\frac{2\alpha}{x}}\right) \qquad \text{and} \qquad \psi_\alpha(x) = \sqrt{x} K_1\left(\sqrt{\frac{2\alpha}{x}}\right),$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.4). It is easily seen that $\psi'_\alpha(\infty-) = \sqrt{2\alpha}$, for all $\alpha > 0$, from which we may deduce that X is a strict local martingale, by Theorem 3.14 (ii).

Next, we present a famous example of a strict local martingale, originally due to Johnson and Helms [24]. This process was later studied by Delbaen and Schachermayer [10], where it was used to illustrate some of the pathologies arising from strict local martingales in stochastic finance:

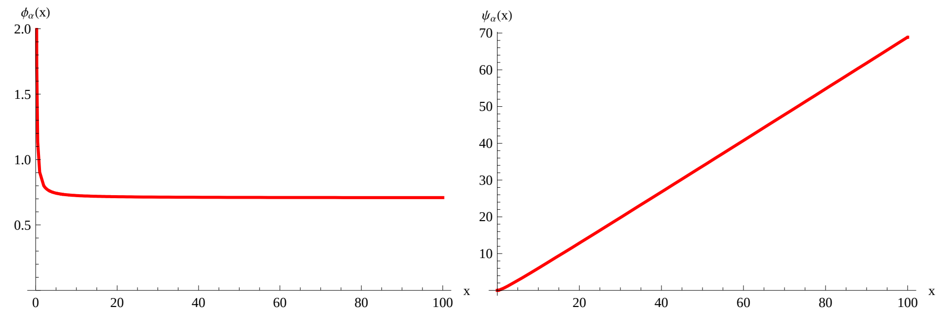

FIGURE 4.4. The functions ϕ_α and ψ_α for the inverted squared Bessel process of dimension four.

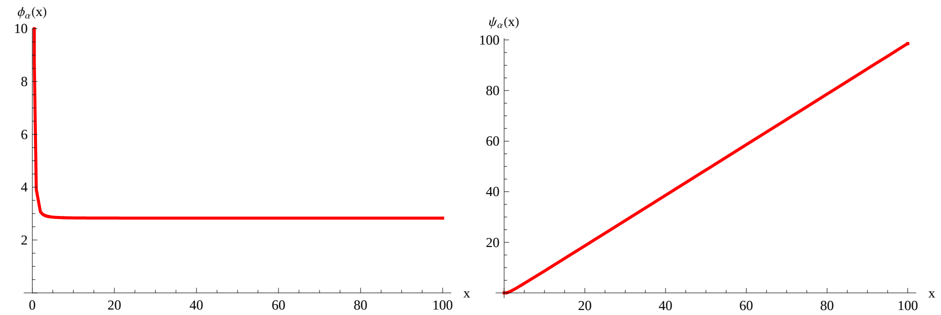

FIGURE 4.5. The functions ϕ_α and ψ_α for the inverted Bessel process of dimension three.

Example 4.5 (Inverted Bessel process of dimension three). In this case $I = (0, \infty)$ and $a(x) := x^2$, for all $x \in I$. Solving (4.3) then yields

$$\phi_\alpha(x) = x\left(e^{\frac{\sqrt{2\alpha}}{x}} - e^{-\frac{\sqrt{2\alpha}}{x}}\right) \qquad \text{and} \qquad \psi_\alpha(x) = xe^{-\frac{\sqrt{2\alpha}}{x}},$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.5). It is easily seen that $\psi'_\alpha(\infty-) = 1$, for all $\alpha > 0$, from which we may deduce that X is a strict local martingale, by Theorem 3.14 (ii).

It is interesting to note that even though the processes in Examples 4.4 and 4.5 are strict local martingales, they nevertheless satisfy strong integrability conditions. For example, the inverted Bessel process of dimension three from Example 4.5 is bounded in L^2 (see e.g. Revuz and Yor [42], Ex. (2.13), p. 194). On the other hand, the inverted squared Bessel process of dimension four from Example 4.4 is not square-integrable (see e.g. Platen and Heath [40], eqn. (8.7.14),

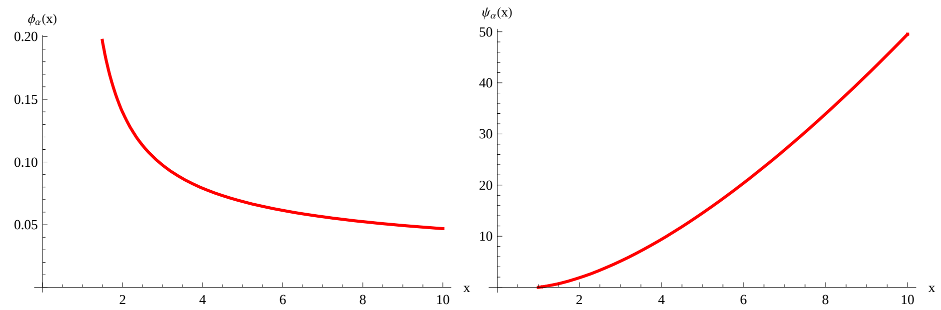

FIGURE 4.6. The functions ϕ_α and ψ_α for Kummer's local martingale.

p. 307), but it is bounded in L^1. Both of these processes are uniformly integrable, by an application of Doob [12], Thm. 11.3, p. 359.

Based on the above examples, it seems natural to speculate that the process defined by (4.1) is a martingale if and only if its diffusion coefficient is asymptotically sub-linear, in the sense that $\lim_{x\uparrow\infty} \frac{a(x)}{x} < \infty$. The following example from Ekström and Tysk [13]—who analyze it by considering super-solutions of (4.4) and (4.5), with h chosen as the identity function—is therefore slightly surprising:

Example 4.6 (Kummer's local martingale). In this example we set $a(x) := x\sqrt{\ln x}$, for all $x \in I$. This specification implies that $l = 1$, so that either $I = [1,\infty)$ or $I = (1,\infty)$. To resolve this issue, we examine the lower boundary behaviour of X, by using (4.2) to compute

$$\int_1^z \mathfrak{m}(y)\,dy = \int_1^z \frac{2}{y^2 \ln y}\,dy = 2\left(\operatorname{li}\frac{1}{z} - \operatorname{li}\frac{1}{1+\varepsilon}\right) = \infty,$$

for all $z \in (1,\infty)$, where the function li is the logarithmic integral (see e.g. Abramowitz and Stegun [1], Chap. 5). It therefore follows from the table in Section 2 that the lower boundary is not exit, and so $I = (1,\infty)$. To investigate the question of whether X is a martingale or not, we employ the transformation of variables $\ln x \mapsto \xi$, and set $v_\alpha(\xi) := u_\alpha(x)$, for all $x \in I$ and $\alpha > 0$, whereupon the ODE (4.3) becomes

$$\xi v_\alpha''(\xi) - \xi v_\alpha'(\xi) - 2\alpha v_\alpha(\xi) = 0,$$

for all $\xi \in (0,\infty)$ and $\alpha > 0$. This is recognizable as an instance of Kummer's equation—also known as a degenerate hypergeometric equation (see e.g. Polyanin and Zaitsev [41], pp. 137–139)—whose solutions may be expressed in terms of the confluent hypergeometric functions M and U (see e.g. Abramowitz and Stegun [1], Chap. 13). For our original equation (4.3), we then obtain

$$\phi_\alpha(x) = \ln x\, U(1+2\alpha, 2, \ln x) \qquad \text{and} \qquad \psi_\alpha(x) = \ln x\, M(1+2\alpha, 2, \ln x),$$

for all $x \in I$ and $\alpha > 0$ (see Figure 4.6). In this case we get $\psi_\alpha'(\infty-) = \infty$, for all $\alpha > 0$, from which it follows that X is a martingale, by Theorem 3.14 (ii).

As we have seen, the processes in Examples 4.2, 4.3 and 4.6 are all examples of martingales that are bounded from below. Each of these processes therefore possesses an almost sure limit X_∞, according to the discussion following Lemma 3.4, with $\mathsf{E}_x(|X_\infty|) < \infty$, for all $x \in I$. Although we already know that $X_\infty = l$, by Theorem 3.21 (i), we can verify this fact explicitly for the examples cited above. To do so, we use the first equality in (3.20), as well as (2.9), to get

$$\mathsf{E}_x(X_\infty) = x - \lim_{z\uparrow\infty} z\mathsf{P}_x(\tau_z < \infty) = x - \lim_{z\uparrow\infty} z \lim_{\alpha\downarrow 0} \frac{\psi_\alpha(x)}{\psi_\alpha(z)},$$

for all $x \in I$. In each of the three examples cited above, we obtain

$$\lim_{\alpha\downarrow 0} \frac{\psi_\alpha(x)}{\psi_\alpha(z)} = \frac{x-l}{z-l},$$

for all $x, z \in I$, with $x \leq z$, which implies that $\mathsf{E}_x(X_\infty) = l$.

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