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Asset Price Dynamics with Time-Varying Second Moment

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A BEHAVIOURAL ASSET PRICING MODEL WITH A TIME-VARYING SECOND MOMENT

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ABSTRACT. We develop a simple behavioural asset pricing model with fundamentalists and chartists to study price behaviour in financial markets when chartists estimate both conditional mean and variance by using a weighted averaging process. Through a stability, bifurcation, and normal form analysis, the market impact of the weighting process is examined. It is found that the weighting process leads to different price dynamics when the fundamental price becomes unstable, depending on whether the chartists act as either trend followers or contrarians. It is also found that a time varying second moment of the chartists impacts differently on the stability of the bifurcated price dynamics, but has no impact on the stability of the fundamental price. Near the flip bifurcation boundary, the bifurcated period-two price dynamics are stable for all time varying second moments, but near the Hopf bifurcation, the bifurcated (quasi)periodic cycle is stable (unstable) when the time varying second moment value is high (low). Different routes to complicated price dynamics are also observed. The analysis provides an analytical foundation for the statistical analysis of the corresponding stochastic version of this type of behavioural model.

JEL classifications: D83; D84; E21; E32, C60

Keywords: Fundamentalists, chartists, stability, bifurcation, norm form analysis, investors' under- and over-reactions, stylized facts.

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1. Introduction

As a result of a growing dissatisfaction with (i) models of asset price dynamics based on the representative agent paradigm, as expressed for example by Kirman (1992), and (ii) the extreme informational assumptions of rational expectations, research into the dynamics of financial asset prices resulting from the interaction of heterogeneous agents has developed strongly over the last decade and a half.[1] In particular, Brock and Hommes (1997*b*, 1998) have introduced the concept of an *adaptively rational equilibrium*, where agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their fitness functions measured by realized profits. The resulting dynamical system is capable of generating the entire *zoo* of complex behaviour from local stability to high order cycles and chaos as various key parameters of the model change. The Brock and Hommes framework has been extended further in Gaunersdorfer (2000) and Chiarella and He (2001, 2002, 2003*b*) to incorporate time-varying (homogeneous) variance and heterogeneous risk and learning under both Walrasian auctioneer and market maker scenarios. It is found that the relative risk attitudes, different learning mechanisms and different market clearing scenarios affect asset price dynamics in a very complicated way. It has been shown (e.g. Hommes (2002)) that such simple nonlinear adaptive models are capable of explaining important stylized facts, including fat tails, volatility clustering and long memory, of real financial time series.

It is well recognised that heterogeneous expectations play a key role in the dynamical behaviour of asset price. Most of the literature cited above focuses on heterogeneous expectations of the first moment (of either price or return), rather than the second moment. Empirically, it is believed that[2] the second moments (i.e. conditional variances) are much easier to estimate than the first moments (i.e. conditional mean) and hence there should be more disagreement about the mean than about the variance among traders. Mathematically, the second moments are associated with higher order terms and they do not in general change the nature of local stability and bifurcations. However, when traders are heterogeneous, they may be heterogenous not only in the first moment but also in the second moment. This heterogeneity may come from their different information sets or different trading strategies they are using. The aim of this paper is to study the dynamical behaviour of prices when traders are heterogeneous with respect to both moments.

Given the variety of technical trading rules and differing risk aversion of various investors, this paper introduces a risk adjustment into the demand function for the chartists by assuming that they use a weighted average process of past prices to estimate and update both conditional mean and variance. Therefore their demand function is a nonlinear function of the conditional mean and variance, instead of a linear function of the conditional mean only. It is found that the mechanism of a variance adjusted demand function of the chartists is a natural way to prevent the price from

[1]See, for example Arthur *et al.* (1997), Brock and Hommes (1997*a*, 1997*b*), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Chiarella (1992), Chiarella *et al.* (2002), Chiarella and He (2001, 2002, 2003*b*), Dacorogna *et al.* (1995), Day and Huang (1990), Farmer and Joshi (2002), Gaunersdorfer (2000), Gaunersdorfer *et al.* (2003), Hommes (2001, 2002), LeBaron *et al.* (1999), Lux (1995, 1997, 1998) and Lux and Marchesi (1999).

[2]See, for example, Nelson (1992) and Bollerslev *et al.* (1994) for some justification on this.

getting stuck in a speculative bubble[3]. Similar to Brock and Hommes (1997*b*, 1998), an adaptive model based on the fitness function is obtained. We then examine how the price dynamics of the risky asset are affected by the reactions of investors, the switching intensity of the fitness function, and the weighting process and risk adjustment of the chartists.

This paper is closely related to Gaunersdorfer *et al.* (2003) who consider a simple asset pricing model of fundamentalists and chartists. In their model, the fundamentalists believe that tomorrow's price will move in the direction of the fundamental price, while chartists derive their beliefs from a simple technical trading rule using only the latest observed price (rather than a weighted average of the latest history of prices in our paper) and extrapolation of the latest observed price change. They assume a homogeneous time-varying conditional variance for both types of traders (as opposed to a constant variance for the fundamentalists and the adjusted time-varying variance which is estimated from the weighted average process for the chartists in our paper) and impose a penalty function in the fitness function of the chartists (to ensure that speculative bubbles cannot last forever). They go on to show that volatility clustering can be characterized by the coexistence of a stable steady state and a stable limit cycle, which arises as a consequence of a so-called Chenciner bifurcation.

To study the impact of the time-varying second moment, we use a normal form analysis associated with the standard stability and bifurcation analysis. Our analysis shows that, when the fundamental price becomes unstable, the time varying second moment and the weighting process lead to different price dynamics. This is due to the fact that the loss of local stability is accompanied by either flip or Hopf bifurcations, depending on whether the chartists act as either trend followers or contrarians. The time-varying second moment has no influence on the stability of the fundamental price. However, it does affect the stability of the bifurcated dynamics. Near the flip bifurcation boundary, the bifurcated period-two price dynamics are stable for all time varying second moments, but near the Hopf bifurcation, the bifurcated (quasi)periodic cycle is stable (unstable) when the time varying second moment value is high (low). Different routes to complicated price dynamics are also observed. The analysis provides an analytical foundation for the statistical analysis of the corresponding stochastic version of this behavioural model in Chiarella *et al.* (2005).

The plan of the paper is as follows. Section 2 develops a simple fundamentalist and chartist asset pricing model in a Walrasian market clearing scenario. The dynamics of the deterministic system, including stability and bifurcation analysis, when chartists are either trend followers or contrarians are examined in Sections 3 and 4, respectively. Section 5 concludes. All proofs are included in the Appendix.

2. THE MODEL

2.1. **Portfolio Optimization and Walrasian Equilibrium Price.** Following the framework of Brock and Hommes (1998), consider an asset pricing model with one risky

[3] In Gaunersdorfer *et al.* (2003), this is ensured by artificially adding a penalty function in the fitness function of the chartists, which is unnecessary in our model. A similar *stabilizing* force is added to the fitness function for the fundamentalists in Gaunersdorfer (2000) where a homogeneous time-varying second moment is updated through an exponential moving average process.

asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at the risk-free (annualised) rate r. Let p_t be the price (ex dividend) per share of the risky asset at time t and $\{y_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of investor h at $t+1$ is given by

$$W_{h,t+1} = RW_{h,t} + (p_{t+1} + y_{t+1} - Rp_t)z_{h,t} = RW_{h,t} + R_{t+1}z_{h,t}, \tag{2.1}$$

where

$$R_{t+1} = p_{t+1} + y_{t+1} - Rp_t \tag{2.2}$$

is the excess capital gain/loss, $R = 1 + r/K$, K is the trading frequency per annum[4], $W_{h,t}$ is the wealth at time t and $z_{h,t}$ is the number of shares of the risky asset purchased at t. Denote by $F_t = \{p_{t-1}, \cdots ; y_{t-1}, \cdots\}$ the information set formed at time t. Let $E_{h,t}, V_{h,t}$ be the *beliefs* of investor type h about the conditional expectation and variance, based on F_t. Then it follows from (2.1) and (2.2) that

$$E_{h,t}(W_{t+1}) = RW_t + E_{h,t}(R_{t+1})z_{h,t}, \qquad V_{h,t}(W_{t+1}) = z_{h,t}^2 V_{h,t}(R_{t+1}). \tag{2.3}$$

Assume each investor has a CARA (constant absolute risk aversion) utility function $u(W) = -e^{-a_h W}$ but with different risk aversion coefficient a_h, and maximises his/her expected utility of wealth, leading the optimal demand

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{2.4}$$

As in Brock and Hommes (1998), a Walrasian scenario is used to derive the demand equation, i.e. each trader is viewed as a price taker (see Brock and Hommes (1997*a*) and Grossman (1989) for detailed discussion). The market is viewed as finding the price p_t that equates the sum of these demand schedules to the supply. Let $z_{s,t}$ denote the supply of (risky) shares. Given bounded rationality, we classify all investors into H different types in terms of their conditional expectations on both mean and variance. Denote by $n_{h,t}$ the fraction of investors of type h at t (so that $\sum_h n_{h,t} = 1$). Then the equilibrium of demand and supply implies

$$\sum_h n_{h,t} \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})} = z_{st}. \tag{2.5}$$

Now assume zero supply of outside shares, i.e. $z_{st} = 0$, then (2.5) leads to

$$\sum_h n_{h,t} \frac{E_{ht}(R_{t+1})}{a_h V_{ht}(R_{t+1})} = 0. \tag{2.6}$$

Under the conditions that $E_t(y_{t+1}) = \bar{y}$ and $\lim_{t\to\infty} Ep_t/R^t = 0$, it can be shown that the *fundamental price* is constant and given by $p^* = \bar{y}/(R-1)$, which corresponds to the bench mark notion of the rational expectation; see Brock and Hommes (1998) for related discussion.

2.2. **Heterogeneous Beliefs.** In the following discussion, we adopt the popular fundamentalist/chartist model by assuming that all investors can be grouped as either fundamentalists (type 1) or chartists (type 2).

[4]Typically, annually, quarterly, monthly, weekly and daily trading periods correspond to $K = 1, 4, 12, 52$ and 250, respectively.

2.2.1. *Fundamentalists.* The fundamentalists are assumed to believe that the expected market price p_t is mean reverting to their perceived fundamental value p^* and the conditional variance of the market price is constant. That is,

$$\begin{cases} E_{1,t}(p_{t+1}) = p^* + v(p_{t-1} - p^*), \quad 0 \le v \le 1 \\ V_{1,t}(p_{t+1}) = \sigma_1^2, \end{cases} \tag{2.7}$$

where p^* is the fundamental price of the risky asset estimated by the fundamentalists at some cost, v is the speed of mean reversion estimated by the fundamentalists, and $\sigma_1 > 0$ is a constant. In particular, $E_{1,t}(p_{t+1}) = p^*$ for $v = 0$ and $E_{1,t}(p_{t+1}) = p_{t-1}$ for $v = 1$. The conditional expectation of the fundamentalists (2.7) can also be written as

$$E_{1,t}(p_{t+1}) = (1 - v)p^* + vp_{t-1}, \qquad 0 \le v \le 1,$$

which is a weighted average of the fundamental price and latest price. Hence small (large) values of v indicate that the fundamentalists give more (less) weight to the fundamental price and less (more) weight to the latest price, believing that price moves quickly (slowly) towards its fundamental value p^*. For convenience of discussion, we say the fundamentalists **over(under)-react** (to the market price) when more (less) weight v is given to the market price.

2.2.2. *Chartists.* Unlike the fundamentalists who are able to work out the fundamental value, chartists base their trading strategy on signals generated from the costless information contained in recent prices. The signal may be generated by comparing the latest price p_{t-1} with some reference price trends $\tilde{p}_{t-1}$, such as a moving average process. Specifically, the chartists consider the realizations of $\tilde{p}_{t-1}$ as random draws from some distribution. The distribution can be conditional on past realized values. For simplicity, we assume that $\tilde{p}_{t-1}$ is conditionally distributed on prices p_{t-2} and p_{t-3} with weighting probabilities w and $1-w$, respectively. Then the conditional mean and variance of the trend can be estimated by, respectively,

$$\begin{cases} \bar{p}_{t-2} \equiv wp_{t-2} + (1 - w)p_{t-3}, \quad 0 \le w \le 1, \\ \bar{\sigma}^2_{t-2} \equiv w[p_{t-2} - \bar{p}_{t-2}]^2 + (1 - w)[p_{t-3} - \bar{p}_{t-2}]^2. \end{cases} \tag{2.8}$$

Based on the trading signals $p_{t-1} - \tilde{p}_{t-1}$ and the conditional mean and variance estimates (2.8), we make the following assumptions for the chartists:

$$\begin{cases} E_{2,t}(p_{t+1}) = p_{t-1} + g(p_{t-1} - \bar{p}_{t-2}), \quad g \in \mathbb{R}, \\ V_{2,t}(p_{t+1}) = \sigma_1^2[1 + b\bar{\sigma}^2_{t-2}], \quad b \ge 0, \end{cases} \tag{2.9}$$

where $g \in \mathbb{R}$ is the estimated extrapolation rate of the chartists. That is, the chartists' beliefs are based on the latest price and their extrapolation of the trading signals generated from the trend. In particular, chartists are called *trend followers* when $g > 0$ and are *contrarians* when $g < 0$. For $w = 1$, $E_{2,t}(p_{t+1}) = p_{t-1} + g(p_{t-1} - p_{t-2})$ which is the case discussed in Gaunersdorfer *et al.* (2003) and, for $w = 0$, $E_{2,t}(p_{t+1}) = p_{t-1} + g(p_{t-1} - p_{t-3})$. Similarly, for convenience of discussion, we say the chartists **over(under)-react** when they extrapolate strongly (weakly), that is when $|g|$ is large (small). With regard to the chartists' estimate of the variance, they use the historical variance to scale up the fundamental variance through the parameter b. High b reflects a greater sensitivity to variance risk.

2.2.3. *Optimal Demand for the Fundamentalists and Chartists.* For the dividend process, we assume that for all agents

$$E(y_t) = \bar{y}, \qquad V(y_t) = \sigma_y^2. \tag{2.10}$$

Note that $\bar{y} = rp^*/K$. Based on the above assumptions, one obtains that for the fundamentalists

$$\begin{cases} E_{1,t}(R_{t+1}) = p^* + v(p_{t-1} - p^*) + \bar{y} - Rp_t, \quad 0 \le v \le 1 \\ V_{1,t}(R_{t+1}) = a_1(\sigma_1^2 + \sigma_y^2), \end{cases} \tag{2.11}$$

and for the chartists

$$\begin{cases} E_{2,t}(R_{t+1}) = p_{t-1} + g(p_{t-1} - \bar{p}_{t-2}) + \bar{y} - Rp_t, \quad g \in \mathbb{R} \\ V_{2,t}(R_{t+1}) = a_2(\sigma_y^2 + \sigma_1[1 + b\bar{\sigma}_{t-2}^2]), \quad b \ge 0. \end{cases} \tag{2.12}$$

Therefore, the optimal demands for the fundamentalists and chartists are given, respectively, by

$$\begin{cases} z_{1,t} = [p^* + v(p_{t-1} - p^*) + \bar{y} - Rp_t]/A_1, \\ z_{2,t} = [p_{t-1} + g(p_{t-1} - \bar{p}_{t-2}) + \bar{y} - Rp_t]/A_{2,t}, \end{cases} \tag{2.13}$$

where

$$A_1 = a_1(\sigma_1^2 + \sigma_y^2), \qquad A_{2,t} = a_2[\sigma_y^2 + \sigma_1^2(1 + b\bar{\sigma}_{t-2}^2)] \tag{2.14}$$

and $\bar{p}_t$ and $\bar{\sigma}_t^2$ are defined by (2.8).

2.3. **Performance Measure and Agent Adaptation.** Let $U_{1,t}$ and $U_{2,t}$ be the realized profit of the fundamentalists and chartists, respectively, defined by

$$U_{i,t} = R_t z_{i,t-1} - C_i, \qquad i = 1, 2$$

where $C_i \ge 0$ measures the total cost. Let the updated fractions be formed on the basis of discrete choice probability (see Manski and McFadden (1981), Anderson, de Palma and Thisse (1993), Brock and Hommes (1997*b*, 1998),

$$n_{i,t} = exp[\beta U_{i,t-1}]/Z_{t-1}, \qquad (i = 1, 2), \qquad Z_{t-1} = \sum_{i=1}^{2} exp[\beta U_{i,t-1}], \tag{2.15}$$

where $\beta (> 0)$ is the *intensity of choice* measuring how fast agents switch among different prediction strategies.[5] Let $m_t = n_{1,t} - n_{2,t}$. Then $n_{1,t} = (1 + m_t)/2, n_{2,t} = (1 - m_t)/2$ and

$$\begin{aligned} m_t &= \tanh\left[\frac{\beta}{2}(U_{1,t-1} - U_{2,t-1}) - \frac{\beta}{2}(C_1 - C_2)\right] \\ &= \tanh\left[\frac{\beta}{2} R_{t-1}(z_{1,t-2} - z_{2,t-2}) - \frac{\beta}{2} C\right], \end{aligned} \tag{2.16}$$

where $C = C_1 - C_2$. Because the cost of information to work out the fundamental values, the constant $C \ge 0$ in general.

[5]To prevent the price from getting stuck in a speculative bubble, Gaunersdorfer (2000) introduces a stabilizing force into the fitness function $U_{i,t}$, while Gaunersdorfer *et al.* (2003) adds a penalty function in the changing population fraction. In our model, this is achieved naturally by incorporating the conditional variance into the demand function of the chartists.

2.4. **The Complete Model.** To sum up, assuming zero net supply the Walrasian equilibrium price p_t satisfies

$$(1+m_t)z_{1,t} + (1-m_t)z_{2,t} = 0. \tag{2.17}$$

Substituting (2.13) and (2.16) into (2.17), one obtains that

$$p_t = \frac{A_{2,t}[p^* + v(p_{t-1} - p^*) + \bar{y}]e^{\beta U_{t-1}} + A_1[p_{t-1} + g(p_{t-1} - \bar{p}_{t-2}) + \bar{y}]}{R[A_{2,t}e^{\beta U_{t-1}} + A_1]}, \tag{2.18}$$

where

$$\begin{aligned} U_t &= R_t(z_{1,t-1} - z_{2,t-1}) - C \\ &= (p_t + y_t - Rp_{t-1})\bigg(\frac{1}{A_1}[p^* + v(p_{t-2} - p^*) + \bar{y} - Rp_{t-1}] \\ &\qquad - \frac{1}{A_{2,t-1}}[p_{t-2} + g(p_{t-2} - \bar{p}_{t-3}) + \bar{y} - Rp_{t-1}]\bigg) - C, \end{aligned}$$

and $\bar{p}_t, A_1$ and $A_{2,t}$ are defined by (2.8) and (2.14). Equation (2.18) determines the Walrasian market cleaning price p_t, which is summarized in the following Proposition 2.1.

Proposition 2.1. *Denote $x_{1,t} = x_t = p_t - p^*, x_{2,t} = x_{t-1}, x_{3,t} = x_{t-2}, x_{4,t} = x_{t-3}$ and $x_{5,t} = x_{t-4}$. Then the market cleanning price $p_t = x_t + p^*$ is governed by the following dynamical system in terms of $X = (x_1, x_2, x_3, x_4, x_5)^T$:*

$$X_t = G(X_{t-1}), \tag{2.19}$$

where

$$\begin{aligned} G(X) &= (F(X), x_1, x_2, x_3, x_4)^T, \\ F(X) &= f_1(X)/f_2(X), \\ f_1(X) &= vA_{21}x_1e^{\beta U(X)} + A_1(x_1 + g(x_1 - wx_2 - (1-w)x_3)), \\ f_2(X) &= R[A_{21}e^{\beta U(X)} + A_1], \\ U(X) &= (x_1 - Rx_2)\bigg(\frac{vx_3 - Rx_2}{A_1} \\ &\qquad - \frac{x_3 + g(x_3 - wx_4 - (1-w)x_5) - Rx_2}{A_{22}}\bigg) - C \end{aligned}$$

and

$$\begin{aligned} &A_1 = a_1(\sigma_y^2 + \sigma_1^2), \qquad A_{21} = a_2[\sigma_y^2 + \sigma_1^2(1 + b\bar{\sigma}_{21}^2)], \qquad A_{22} = a_2[\sigma_y^2 + \sigma_1^2(1 + \bar{\sigma}_{22}^2)], \\ &\bar{x}_{21} = wx_2 + (1-w)x_3, \qquad \bar{\sigma}_{21}^2 = w[x_2 - \bar{x}_{21}]^2 + (1-w)[x_3 - \bar{x}_{21}]^2, \\ &\bar{x}_{22} = wx_4 + (1-w)x_5, \qquad \bar{\sigma}_{22}^2 = w[x_4 - \bar{x}_{22}]^2 + (1-w)[x_5 - \bar{x}_{22}]^2. \end{aligned}$$

In addition, the population fraction difference $m_t = n_{1,t} - n_{2,t}$ evolves according to

$$m_t = M(X_{t-1}),$$

where

$$M(X) = \tanh[\beta U(X)/2].$$

In the following discussion, we examine the dynamics of the nonlinear deterministic system (2.19) by considering the chartists as either the trend followers or contrarians. We first undertake a theoretical study of the existence of the steady state, its stability, bifurcation, and normal form properties, followed by numerical simulations of the nonlinear system (2.19) to obtain some insight into its global properties.

3. DYNAMICS OF FUNDAMENTALISTS AND TREND FOLLOWERS

In this section, we consider the situation in which the chartists are trend followers, that is $g > 0$ in the model (2.19). We first have the following result on the existence of steady state that corresponds to the fundamental price p^*.

Proposition 3.1. *For system (2.19), the fundamental price $p_t = p^*$ is the unique steady state, that is $X^* = 0$.*

Proof. See Appendix A.1. □

Results on the local stability of the unique steady state and its bifurcation when the chartists are trend followers are given in the following Proposition.

Proposition 3.2. *Assume $g > 0$. Then the fundamental price p^* is locally asymptotically stable (LAS) if*

- *either*

$$v < v_3(g) \equiv c_1 \frac{1}{g} - c_2 - c_3 g \qquad \text{for} \qquad w \in [0, 1), \tag{3.1}$$

- *or $g < g_1 \equiv (a + x_o)R/a$ for $w = 1$,*

where

$$c_1 = \frac{(x_0 + a)^2 R^2}{a x_0 (1 - w)}, \qquad c_2 = \frac{x_0 w R + a w R + a(1 - w)}{x_0 (1 - w)},$$

$$c_3 = \frac{a(2 - w)}{x_0}, \qquad x_0 = e^{-\beta C}, \qquad a = \frac{a_1}{a_2}.$$

If $v = v_3(g)$ for $w \in [0, 1)$ or $g = g_1$ for $w = 1$, then there exists a pair of complex eigenvalues $\lambda = e^{\pm i\theta}$ for the linearized system at the steady state. If, in addition, $d \neq 0$, where

$$d \equiv \frac{1}{2} \frac{1}{e^{-3i\theta} - e^{-i\theta} + \gamma_3(1 - e^{-2i\theta})} \sum_{j,k,l=1}^{5} \frac{\partial^3 F(0)}{\partial x_j \partial x_k \partial x_l} e^{(j+k-l-5)i\theta} \tag{3.2}$$

and

$$\gamma_3 = -\frac{g a (1 - w)}{R(x_o + a)},$$

then, in the absence of strong resonances (i.e., $e^{ik\theta} \neq 1$ for $k = 1, 2, 3, 4$), a Hopf (Neimark-Sacker) bifurcation emanates out of the steady state $X^* = 0$. Furthermore the bifurcated periodic cycle or quasi-periodic cycle is supercritical when $d < 0$ and subcritical when $d > 0$.

Proof. See Appendix A.2. □

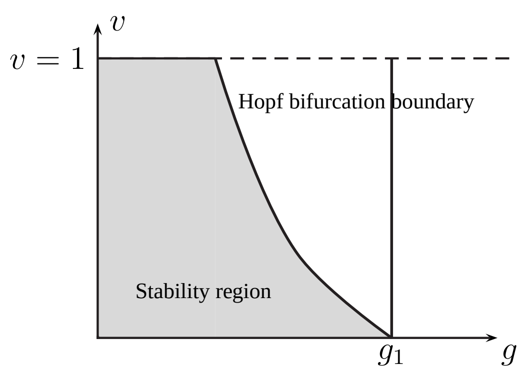

FIGURE 3.1. Local stability region of the steady state and the Hopf bifurcation boundary for $g > 0$.

The local stability region of the steady state and the corresponding Hopf bifurcation boundary are plotted in the (g, v) parameter plane in Fig. 3.1. It can be verified that, along the Hopf bifurcation curve, $g \to g_1$ as $w \to 1$[6]. In this limiting case, the Hopf bifurcation boundary is independent of the speed of price adjustment of the fundamentalists. In general, for $w \in [0, 1)$, Proposition 3.2 indicates that the condition for the Hopf bifurcation depends on the speed of price adjustment of the fundamentalists, the extrapolation rate of the trend followers, and the weighting process used by the trend followers to form the moving average[7]. More precisely, we have the following observations:

- Along the bifurcation boundary, $v_3(g)$ decreases as g increases and this is illustrated by the bifurcation value g^* in Table 3.1 for fixed $w = 0.5$ and different values of v. This implies that the fundamental steady state price is locally stable as long as reactions of both types of investors are balanced. Over-reactions from both fundamentalists and trend followers lead to instability of the fundamental price. This result is very intuitive. The same result obtained in Gaunersdorfer *et al.* (2003) when $w = 1$, which is a special case of our result.
- Based on the fact that $g = v_3^{-1}(v) \to g_1$ as $w \to 1$, we can see that the stability region of the fundamental steady state in (g, v) parameter space is enlarged as w increases, in particular, this becomes even more significant when v is close to 1, as verified by bifurcation values g^* in Table 3.2 for fixed $v = 0.5$ and different values of w. This indicates a stabilizing role when more weight is given to the most recent price.
- Along the Hopf bifurcation boundary, the type of bifurcation depends on values of θ in the pair of complex eigenvalues $\lambda_{2,3} = e^{\pm\theta i}$ of the linearized system at

[6]It follows from $v = v_3(g)$ that $g = v_3^{-1}(v)$. Then it can be verified that $g \to g_1$ as $w \to 1$

[7]See Appendix A.2 for detailed discussion.

the steady state, which in turn depends on values of $\rho = 2\cos(\theta)$ satisfying[8]

$$\rho = \frac{R(x_0 + a) - gaw}{ga(1-w)},$$

as illustrated in Tables 3.1 and 3.2. Similar to the findings in Chiarella and He (2003*a*), depending on values of ρ (and hence of θ), periodic (when $\theta/(2\pi)$ is a rational number) and quasi-periodic (when $\theta/(2\pi)$ is an irrational number) cycles can be generated through the Hopf bifurcation. The stability of the bifurcated cycles depends on the invariant expression d of the norm form analysis in Proposition 3.2. More details on normal form analysis in higher dimension can be found from Kuznetsov (2004).

Guided by the stability, bifurcation and normal form results in Proposition 3.2, we now turn to a numerical simulation study of the dynamics of the nonlinear system (2.19).

Parameter selection—In the following, we choose the fundamental price $p^* = \$100$, annual risk-free rate $r = 5\%$, annual volatility of the fundamental price $\sigma = 20\%$. For the trading frequency, we choose $K = 250$, which corresponds to a daily trading period. As a consequence[9], the total risk-free return per trading day $R = 1 + r/K = 1.0002$, daily price volatility $\sigma_1^2 = (p^*\sigma)^2/K = 8/5$ and daily dividend volatility $\sigma_y^2 = r^2\sigma_1^2 = 1/250$. We also choose the risk aversion coefficients for both types of investor as $a_1 = a_2 = 0.8$ and the cost difference $C = 0$. We examine how the dynamics of the nonlinear system is affected by the speed of price adjustment of the fundamentalists (the parameter v), the extrapolation rate of the trend followers (the parameter g), the coefficient of the variance adjusted demand (the parameter b), the weighting parameter used by the trend followers (the parameter w), and the switching intensity (the parameter β).

v	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
g^*	1.5169	1.4971	1.4775	1.4583	1.4395	1.4210	1.4029	1.3851	1.3676	1.3505	1.3337
ρ	1.63789	1.6725	1.7078	1.7434	1.7792	1.8154	1.8518	1.8884	1.9253	1.9625	2.0000

TABLE 3.1. Hopf bifurcation values of g^* and the corresponding values of ρ for various v and $w = 0.5, \beta = 2, C = 0$.

For fixed $w = 0.5$, Table 3.1 lists the bifurcation values g^* for various values of v. It clearly indicates that the fundamental price is always locally stable for any speed of price adjustment $v \in [0,1]$ when the trend followers under-react (i.e., extrapolate weakly with $g < g_1^*(= 1.3337)$). Otherwise (i.e., when $g \geq g_1^*$), the stability region for the extrapolation rate of the trend followers is enlarged as the fundamentalists under-react, as suggested by Proposition 3.1. Consequently, over-reactions from both the trend followers and fundamentalists can make the fundamental price become unstable through a Hopf bifurcation.

Similarly, for fixed $v = 0.5$, Table 3.2 lists some Hopf bifurcation values g^* for various values of w, indicating that an increase of the weighting parameter w enlarges the stability region for the extrapolation rate of the trend followers. This implies that, in

[8]See Appendix A.2 for the details.

[9]Motivated by the relation between annual mean dividend $\bar{Y} = rp^*$, we approximate the annual dividend volatility σ_Y by $r(p^*\sigma)$ and hence the daily dividend volatility σ_y by $r(p^*\sigma)/\sqrt{K}$.

w	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
g^*	1.0883	1.1422	1.2016	1.2671	1.3400	1.4208	1.5118	1.6137	1.7284	1.8572	2.0004
ρ	1.8380	1.8348	1.8310	1.8270	1.8219	1.8156	1.8079	1.7986	1.7867	1.7711	1.7499

TABLE 3.2. Hopf bifurcation values of g^* and the corresponding values of ρ for various w and $v = 0.5, \beta = 2, C = 0$.

terms of the stability of the fundamental steady state, a simple trend following strategy based on the difference of the latest two prices $[p_{t-1} - p_{t-2}]$ is better (for stability) than a more complicated strategy based on the difference $p_{t-1} - [wp_{t-2} + (1-w)p_{t-3}]$, illustrating that a simple *ad hoc* expectation scheme may work better than more sophisticated ones for the chartists to learn the steady state.

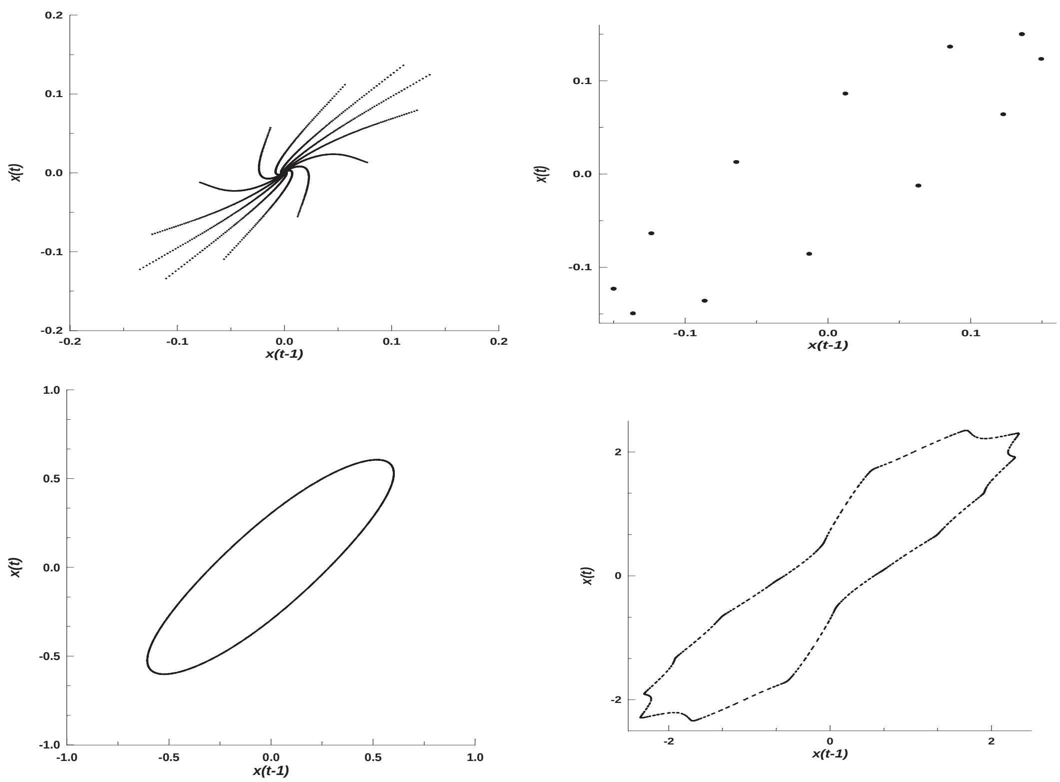

FIGURE 3.2. Phase plots of (x_t, x_{t-1}) for $w = 0.5, b = 2, \beta = 1, v = 0.2682$ and four values of $g = 1.46$ (upper left), g^* (upper right), 1.48 (lower left), and 1.52 (lower right).

Proposition 3.2 states that the only bifurcation occurring at an unstable steady state is the Hopf bifurcation. The nature of the (quasi)-periodicity of the Hopf bifurcation is determined by values of θ, which are in turn determined by values of ρ, illustrated in Table 3.1. For example, let $\theta/(2\pi) = 1/12$, then $\rho = 1.73205$ and $g^* = 1.464395$, which corresponds to $v = 0.2682$ with fixed $w = 0.5$. To see the price dynamics near the Hopf bifurcation boundary, we choose $\beta = 1, b = 2, w = 0.5, v = 0.2682$ and four different values of $g = 1.46, g^*, 1.48, 1.52$. Fig. 3.2 gives the limiting phase plots in the (x_{t-1}, x_t) plane for these four different values. For $g = 1.46 < g^*$, the upper left panel shows that the trajectories converge periodically to the steady state. For $g = g^*$, the upper right panel shows that the trajectories converge to a period-12 cycle, which verified the above calculation. For $g = 1.48 > g^*$, the lower left panel shows that the trajectories converge to a quasi-periodic orbit, which is characterized by a closed orbit in the phase plane. For $g = 1.52(> g^*)$, the lower right panel shows that the trajectories converge to a strange attractor. The corresponding bifurcation plot for the parameter g is given in Fig. 3.3. This illustrates one of many routes to complicated price dynamics through the Hopf bifurcation.

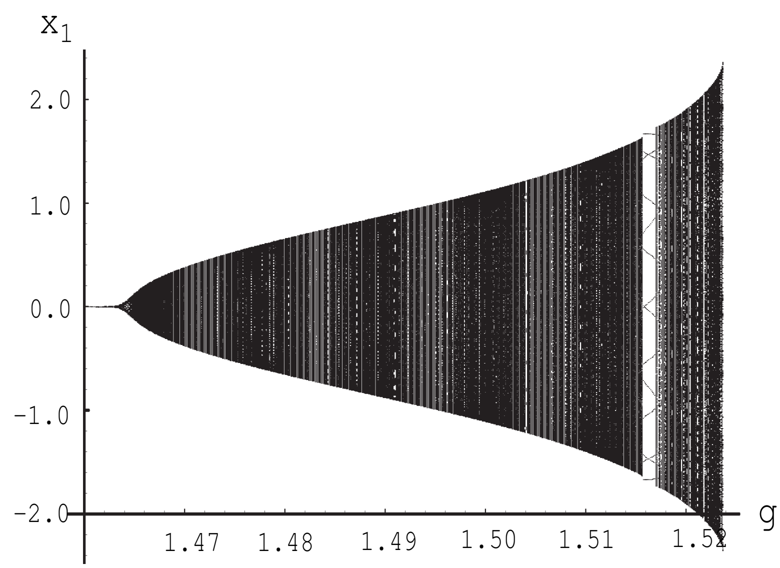

FIGURE 3.3. Bifurcation plot for the parameter g with $w = 0.5, b = 2, \beta = 1, v = 0.2682$.

The effect of the switching intensity β on the price dynamics is similar to that observed in Brock and Hommes (1998). That is, when the fundamental price becomes unstable, low switching intensities may lead solutions to converge to quasi-periodic cycles; however, as the intensity increases, solutions becomes more volatile, leading to explosive behaviour.

We now examine the effect of the variance adjustment coefficient b of the trend followers (see equation (2.9)). Note that the local stability conditions are independent of the variance coefficient b. However, the normal form analysis indicates that the existence and stability of the bifurcated cycle turn out to depend on b. This observation leads to a very interesting result—changes in b do affect the price dynamics when the steady state is unstable, but do not affect the stability of the steady state. To illustrate, we select $v = 0.3, w = 0.5$ and $\beta = 1$. It follows from Table 3.1 that the corresponding Hopf bifurcation value[10] is given by $g^* = 1.4583$. Applying Proposition 3.2 to this bifurcation value, we obtain $d = 0.1508 - 0.05169b$, implying $d = 0$ when $b = b^* =$

[10]Since $C = 0$, the bifurcation value g^* is independent of β.

2.917. Hence, the bifurcated (periodic or quasi-periodic) cycle is unstable for $b < b^*$ and stable for $b > b^*$. This can be verified by numerical simulations.

In summary, when the chartists are trend followers, the stability of the fundamental steady state of the nonlinear deterministic system can be characterized by stability, bifurcation, and normal form analysis. It is found that (i) under-reactions from both the fundamentalists and trend followers and low switching intensity can stabilize the fundamental steady state; (ii) the fundamental price becomes unstable when traders over-react; (iii) the stability of the bifurcated cycle (from the Hopf bifurcation), rather than that of the fundamental price, depends on the coefficient b of the time-varying second moment and a high (low) value of b implies a stable (unstable) bifurcated cycle; (iv) because of the trend following strategy, prices tend to the positively correlated (as indicated by the phase plots in Figure 3.2.

4. Dynamics of Fundamentalists and Contrarians

In this section, we consider the situation in which the chartists are contrarians, that is, $g < 0$ in model (2.19). Obviously the deterministic system has the same unique steady state, which corresponds to the fundamental price p^*. However, the dynamics are different from the previous case.

Proposition 4.1. *Assume $g < 0$ for the deterministic system (2.19). There exist w_1 and w_2 satisfying $0 < w_1 < w_2 < 1/2$ such that:*

(i) *for $w > w_2$, the fundamental price p^* of the deterministic system (2.19) is LAS if*

$$v > v_2(g) \equiv -\frac{R(x_0 + a) + a}{x_0} - \frac{2aw}{x_0} g.$$

In addition, if $v = v_2(g)$ and $\bar{d} \neq 0$ with

$$\bar{d} \equiv \frac{1}{6} \frac{1}{3 + 2\gamma_1 - \gamma_2} \sum_{j,k,l=1}^{5} \frac{\partial^3 F(0)}{\partial x_j \partial x_k \partial x_l} (-1)^{j+k+l+1}, \tag{4.1}$$

where

$$\gamma_1 = \frac{vx_0 + a(1 + g)}{R(x_0 + a)}, \qquad \gamma_2 = -\frac{gaw}{R(x_0 + a)},$$

then a period two cycle bifurcates out of the steady state $X^* = 0$. Furthermore, the period two cycle is stable when $\bar{d} < 0$ and unstable when $\bar{d} > 0$.

(ii) *for $w < w_1$, the fundamental price p^* of the deterministic system (2.19) is LAS if*

$$v > v_3(g) \equiv c_1 \frac{1}{g} - c_2 - c_3 g.$$

If $v = v_3(g)$, then there exists a pair of complex eigenvalue $\lambda = e^{\pm i\theta}$ for the linearized system at the steady state. If, in addition, $d \neq 0$, where d is defined by (3.2), then, in the absence of strong resonances (i.e., $e^{ik\theta} \neq 1$ for $k = 1, 2, 3, 4$), a Hopf (Neimark-Sacker) bifurcation emanates out of the steady state $X^* = 0$. Furthermore the bifurcated periodic cycle or quasi-periodic cycle is supercritical when $d < 0$ and subcritical when $d > 0$.

(iii) *for $w_1 \leq w \leq w_2$, there exists a $g^* < 0$ such that the fundamental price p^* of the deterministic system (2.19) is LAS if*

$$v > v_2(g) \quad \text{for} \quad g \in (g^*, 0), \qquad v > v_3(g) \quad \text{for} \quad g < g^*.$$

In addition, for $g \in (g^*, 0)$ and $v = v_2(g)$, flip bifurcations occur when $\bar{d} \neq 0$, while for $g < g^*$ and $v = v_3(g)$ Hopf (Neimark-Sacker) bifurcations occur when $d \neq 0$. In addition, the bifurcated orbit is stable (unstable) when $\bar{d} < 0(\bar{d} > 0)$ near the flip boundary and $d < 0(d > 0)$ near the Hopf bifurcation boundary.

Proof. See Appendix A.3. □

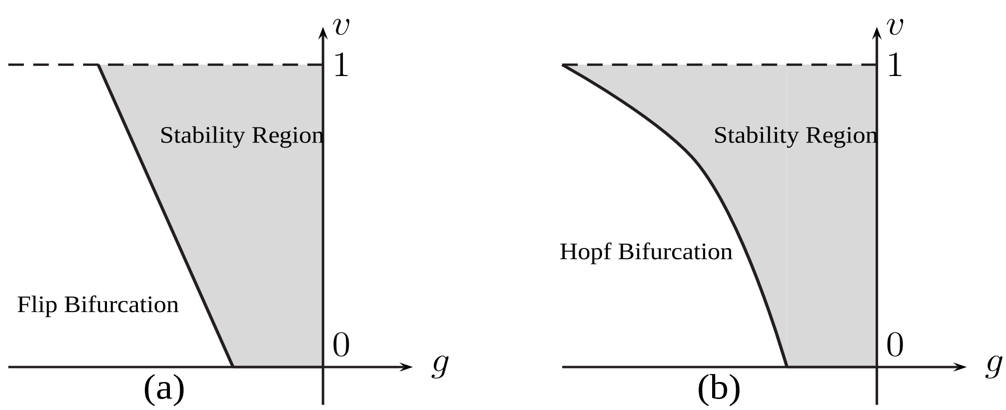

FIGURE 4.1. Local stability region of the steady state for $g < 0$ and (a) the flip bifurcation boundary for $w \in [w_2, 1]$, (b) the Hopf bifurcation boundary for $w \in [0, w_1]$.

Based on Proposition 4.1, the local stability region of the steady state and the corresponding bifurcation boundaries for the three different cases are plotted in (g, v) parameter space in Figs. 4.1 and 4.2, respectively. They illustrate that the fundamental price is locally stable when the contrarians under-react. Note that both $v_2(g)$ and $v_3(g)$ decrease as g increases. This implies that the stability region of the extrapolation rate of the contrarians (i.e. parameter g) is enlarged when the fundamentalists over-react (i.e. when v increases). However, it is very interesting to notice that, different from the case when chartists are trend followers, the fundamental steady state price becomes unstable through either flip (for $w > w_2$), or Hopf (for $w < w_1$), or both types of bifurcation (for $w_1 < w < w_2$), depending on the weight parameter w.

When the contrarians put less weight (i.e. $w < w_1$) on the more recent price, the steady state fundamental price becomes unstable through a Hopf bifurcation. For fixed $w = 0.3$, Table 4.1 lists the Hopf bifurcation values g_h^* and the corresponding values of ρ for various v. One can see that, as v increases, the stability region for the extrapolation rate g_h^* is enlarged. Also, numerical calculations show that, as w increases (up to $w = 0.46$), the Hopf bifurcation is shifting to the left in Fig. 4.1(b). In other words, as

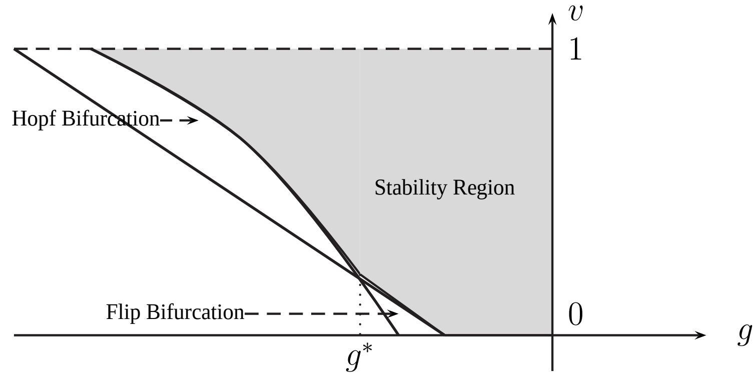

FIGURE 4.2. Local stability region of the steady state and the bifurcation boundaries for $g < 0$ and $w \in (w_1, w_2)$.

the weight w increases (up to a certain value), the stability region for the extrapolation rate is enlarged.

v	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
g_h^*	-2.4597	-2.4977	-2.5361	-2.5749	-2.6142	-2.6538	-2.6938	-2.7342	-2.7749	-2.8161	-2.8576
ρ	-1.5904	-1.5727	-1.5538	-1.5384	-1.5217	-1.5054	-1.4894	-1.4738	-1.4584	-1.4434	-1.4286

TABLE 4.1. Hopf bifurcation values g_h^* for various v and $w = 0.3, a_1 = a_2 = 0.8, \beta = 0, C = 0$.

When the contrarians put more weight (i.e. $w > w_2$) on the more recent price, the steady state loses local stability through a flip bifurcation. For fixed $w = 0.9$, Table 4.2 lists the flip bifurcation values g_f^* for various v. As v increases, the stability region for the extrapolation rate g is enlarged. Numerical calculations also show that the flip bifurcation boundary in Fig. 4.1(a) is shifting to the left as w increases, enlarging the stability region for the extrapolation rate.

v	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
g_f^*	-1.6669	-1.7224	-1.7780	-1.8336	-1.8891	-1.9447	-2.0002	-2.0558	-2.1113	-2.1669	-2.2224

TABLE 4.2. Flip bifurcation values g_f^* for various v and $w = 0.9, a_1 = a_2 = 0.8, \beta = 0, C = 0$.

When the weight parameter $w \in (w_1, w_2)$ with $0 < w_1 < w_2 < 1/2$, the fundamental steady state price loses its stability through both flip and Hopf bifurcations as indicated in Fig. 4.2. Table 4.3 lists the bifurcation values g^* for various w with fixed $v = 0.5$. In this case, $w^* \in (0.4, 0.5)$ and g^* correspond to Hopf bifurcation values for $w < w^*$ and flip bifurcation values for $w > w^*$. A similar result is also found in Chiarella and He (2003*a*) when agents learn from a weighted average process. In other words, Hopf bifurcations occur when less weight is given to the most recent price, while flip bifurcations occur when more weight is given to the most recent price.

w	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
g^*	-2.0487	-2.3127	-2.6538	-3.1110	-3.5004	-2.917	-2.5003	-2.1876	-1.9447

TABLE 4.3. Hopf bifurcation values g^* for $w < w^*$ and flip bifurcation values g^* for $w > w^*$ with $w^* \in (0.4, 0.5)$ and $v = 0.5, a_1 = a_2 = 0.8, \beta = 1, C = 0$.

Guided by the stability, bifurcation, and normal form analysis, we now examine the dynamics of the nonlinear system (2.19) by using numerical simulations.

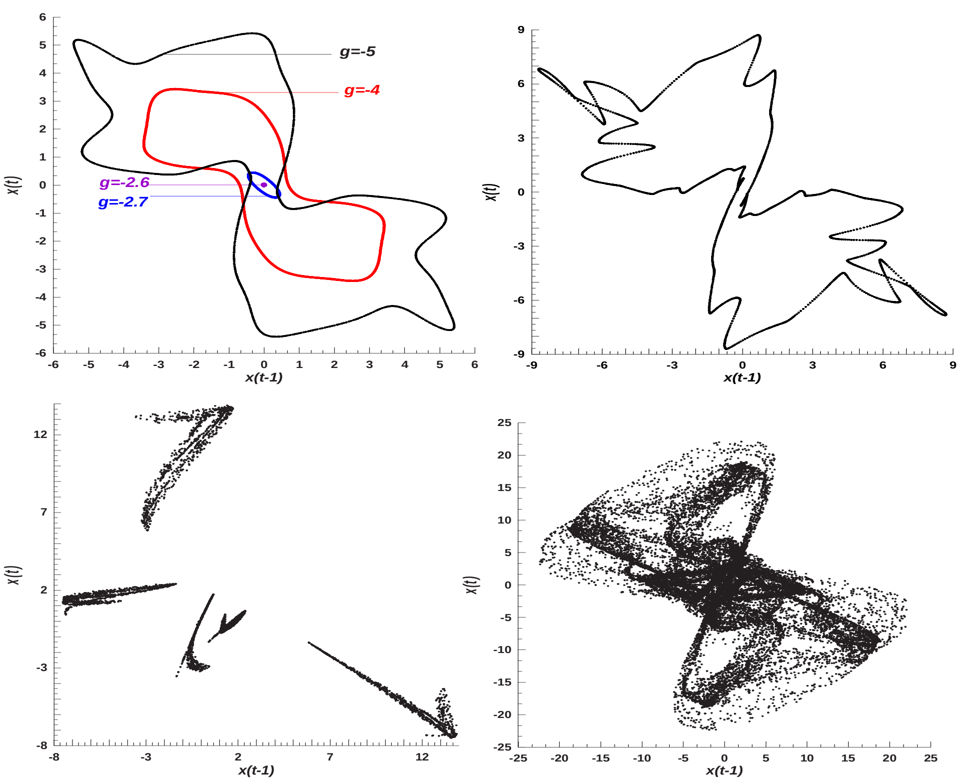

FIGURE 4.3. Phase plots of (x_t, x_{t-1}) for $g = -2.6, -2.7, -4, -5$ (top left), $g = -6$ (top right), $g = -7$ (bottom left) and $g = -8$ (bottom right) with fixed $w = 0.3, v = 0.5, \beta = 0, b = 1, a_1 = a_2 = 0.8, C = 0$.

We first examine the effect of extrapolation when the weighting parameter is low. To illustrate, we select $w = 0.3, v = 0.5, C = 0, \beta = 0, b = 1$. In this case, the fundamental steady state is locally stable for $g > g_h^* = -2.6538$ and loses its stability through a Hopf bifurcation at $g = g_h^*$. Similarly, the nature of the Hopf bifurcation is determined by the values of ρ. In addition, using (3.2), one can verify that there exists a $b^* \in (0, 1)$ such that $d < (>)0$ for $b > (<)b^*$, implying that the bifurcated cycles are stable for $b > b^*$ and unstable for $b < b^*$. Fig. 4.3 illustrates phase plots of the system for various values of g. It shows that, as the contrarians extrapolate weakly, the solutions converge to either the steady state (for $g > g^*$) or (quasi-)periodic cycles (for $g = -2.7, -4, -5 < g^*$). The corresponding bifurcated cycle is stable for g

near g_h^*. However, as the contrarians extrapolate strongly, solutions converge to some strange (for $g = -6, -7$) and even chaotic (for $g = -8$) attractors. The corresponding bifurcation diagram in the parameter g is plotted in Fig. 4.4.

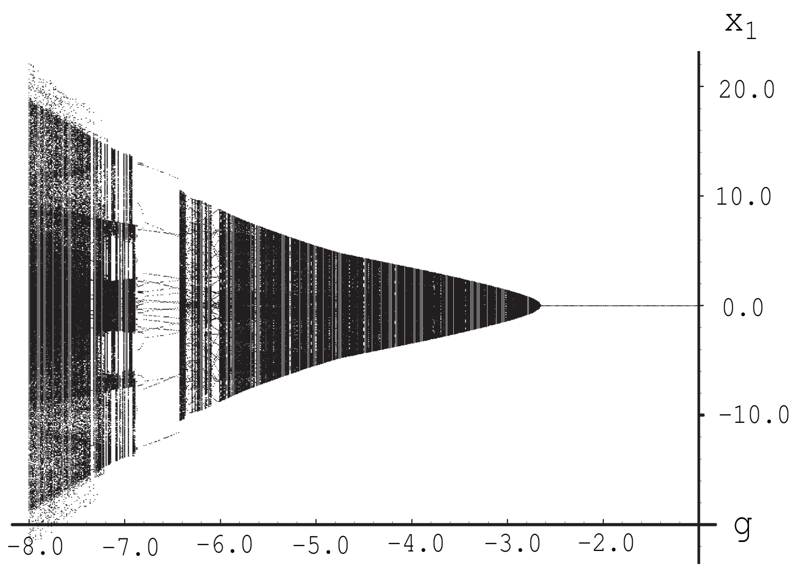

FIGURE 4.4. Bifurcation plot for the parameter g with fixed $w = 0.3, v = 0.5, \beta = 0, b = 1, a_1 = a_2 = 0.8, C = 0$.

The effect of extrapolation when the weighting parameter is high can be different from the previous case. To illustrate, we choose $w = 0.9$ and $v = 0.5, C = 0, \beta = 0, b = 1$. Then the steady state is locally stable for $g > g_f^* = -1.9447$ and loses its stability through a flip bifurcation at $g = g_f^*$. Using (4.1), one can verify that $\bar{d} < 0$ for all $b \geq 0$, implying that the bifurcated 2-period cycle is always stable for all $b \geq 0$ when $g = g_f^*$. Fig. 4.5 illustrates phase plots of the system for $g = -3, -4, -5, -5.5, -6$. The corresponding bifurcation diagram for the parameter g is given in Fig. 4.6. It shows that, as the contrarians increase their extrapolation activity, the dynamics lead to a period doubling bifurcation (for $g = -3$) first, and then two symmetric closed quasi-periodic cycles (for $g = -4$), and then to strange attractors (for $g = -5, -6$). Note that, for $g = -5.5$, solutions converge to two 14-periodic cycles. This shows a different route to complicated dynamics.

The effect of switching intensity β is also different. To illustrate, we choose $w = 0.48, v = 0.5, C = 0, b = 1$. In this case the steady state is locally stable for $g > g^* = -3.6057$ and a Hopf bifurcation occurs for $g = g^*$. For fixed $g = -6$, Fig. 4.7 illustrates phase plots of the system for $\beta = 0$ (top left), $\beta = 0.01$ (top right), $\beta = 0.03$ (bottom left), and $\beta = 0.1$ (bottom right). Different from the findings in Brock and Hommes (1998) and in the previous section, an increase in the switching intensity may stabilize the dynamics. This may due to the stabilizing nature of both fundamental and contrarian strategies.

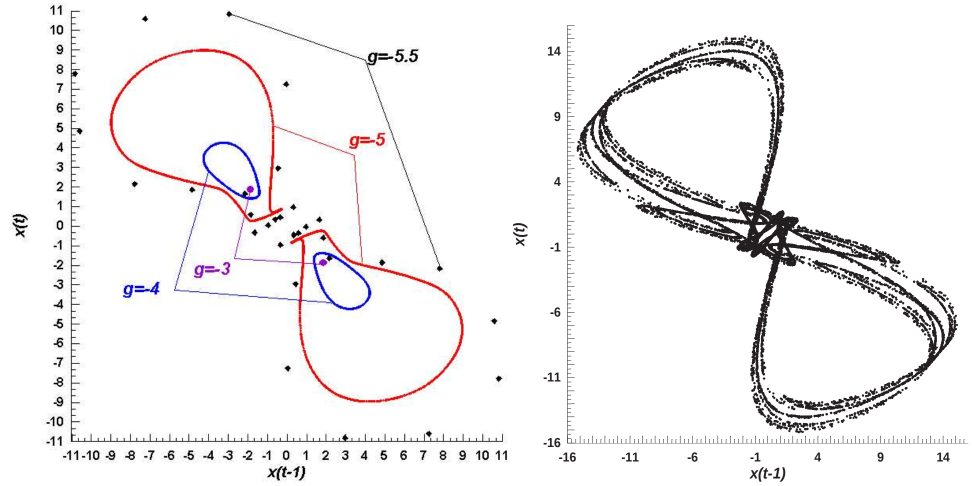

FIGURE 4.5. Phase plots of (x_t, x_{t-1}) for $g = -3, -4, -5, -5.5$ (left) and $g = -6$ (right) with fixed $w = 0.9, v = 0.5, \beta = 0, b = 1, a_1 = a_2 = 0.8, C = 0$.

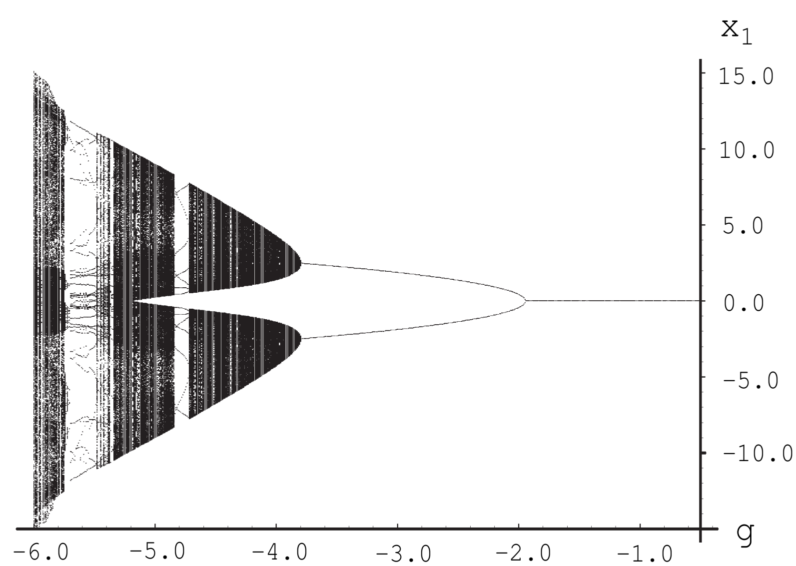

FIGURE 4.6. Bifurcation plot for the parameter g with fixed $w = 0.9, v = 0.5, \beta = 0, b = 1, a_1 = a_2 = 0.8, C = 0$.

In summary, for the model with contrarians, it is found that (i) within the local stability region, low (high) extrapolations of the contrarians are associated with under(over)-reactions of the fundamentalists; (ii) depending on the weight parameter, there are different routes (through either flip, or Hopf, or both bifurcations) to complicated dynamics when the fundamental price becomes unstable; (iii) the bifurcated period-two

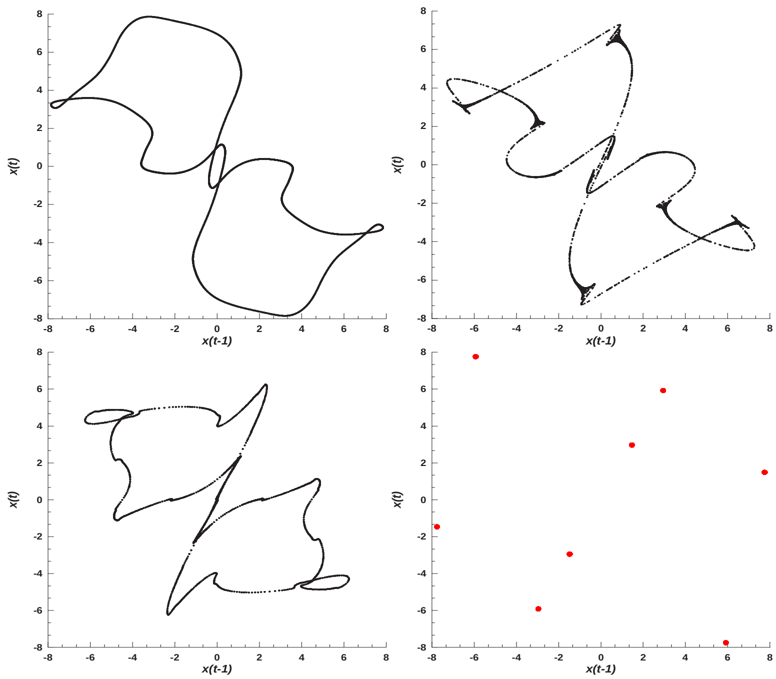

FIGURE 4.7. Phase plots of (x_t, x_{t-1}) for $\beta = 0$ (top left), $\beta = 0.01$ (top right), $\beta = 0.03$ (bottom left) and $\beta = 0.1$ (bottom right) with fixed $w = 0.48, v = 0.5, g = -6, b = 1$.

cycle through a flip bifurcation is always stable near the bifurcation value, while the bifurcated (quasi-)periodic cycle through a Hopf bifurcation is stable (unstable) for high (low) values of b near the bifurcation value; (iv) because of the stabilizing nature of both the fundamental and contrarian strategies, the price dynamics are less explosive, but more sensitive to the switching intensity between the two strategies; (v) because of the contrarian strategy, prices tend to negatively correlated (as indicated by the phase plots); (vi) an increase in the variance coefficient b has a stabilizing effect.

5. CONCLUSIONS

We present a simple asset pricing model with fundamentalists and chartists to study market price behaviour of the risky asset when chartists estimate both conditional mean and variance by using a weighted averaging process. Within our model, the under-reactions from both the fundamentalists and chartists can stabilize the fundamental price. However, when the fundamental price becomes unstable, the weighting process has a different effect on the market price behaviour, depending on whether the chartists behave like trend followers or contrarians. When the chartists are trend followers, the fundamental price becomes unstable through Hopf bifurcation only for all values of the weighting parameter. However, when the chartists are contrarians, the fundamental price becomes unstable through a Hopf bifurcation only when the weighting parameter is low, a flip bifurcation only when the weighting parameter is high, and either Hopf or flip bifurcations when the weighting parameter is near 0.5. Using a normal form analysis, it is found that the time-varying second moment coefficient is clearly related to the stability of bifurcated orbits, rather than to the stability of the fundamental price. Near the flip bifurcation boundary, the bifurcated period-two cycle is stable for all values of the coefficients, but near the Hopf bifurcation boundary, the bifurcated (quasi)periodic cycle is stable (unstable) when the value of the coefficient is high (low). To our knowledge, this is the first theoretical result on the role of a time-varying second moment on the stability of both the fundamental price and bifurcating orbits within the heterogeneous agent asset pricing framework. Different routes to complicated price dynamics from this weighting process are also observed.

The interplay between stochastic elements and deterministic dynamics is an interesting and important issue for model calibration. By assuming a random walk fundamental price process and introducing a noise trader to clear the market, a stochastic version of the deterministic model established in this paper is examined in Chiarella *et al.* (2005). The relationship between the statistical properties of the stochastic version and the stability and bifurcation of the underlying deterministic version, such as return autocorrelation patterns of the stochastic version and types of bifurcation of the underlying deterministic version, is examined. It is found that the model displays some of the stylised facts observed in high frequency financial data, such as fat tails, skewness and high kurtosis. The model also has the potential to generate volatility clustering and long memory features, which are the focus of much current research.

Appendix A. Proof of Propositions

A.1. **Proof of Proposition 3.1.** Let $X = X^*$ be the steady state of the corresponding deterministic system of (2.19). Then X^* satisfies $X^* = G(X^*)$, which is equivalent to $x^* = F(x^*, x^*, x^*, x^*, x^*)$. Let

$$m^* = \tanh\left[\frac{\beta}{2}(1-R)\left(\frac{v-R}{A_1} - \frac{1-R}{A_2}\right)x^{*2} - \frac{\beta}{2}C\right].$$

Then x^* satisfies

$$x^* = \frac{v(1+m^*)x^* + a(1-m^*)x^*}{R(1+m^*) + Ra(1-m^*)}. \tag{A.1}$$

It follows that either $x^* = 0$ or

$$(Ra - R + v - a)m^* = RA + R - v - a. \tag{A.2}$$

If $Ra - R + v - a = 0$, then (A.2) cannot hold. If $Ra - R + v - a \neq 0$, then

$$m^* = \frac{Ra + R - v - a}{Ra - R + v - a}.$$

It is easy to see that $|m^*| > 1$. Therefore $x^* = 0$ is unique fixed point.

A.2. **Proof of Proposition 3.2.** Let $m^0 = \tanh(-\frac{\beta}{2}C)$, and

$$x_0 = \frac{1+m^0}{1-m^0} = e^{-\beta C} > 0. \tag{A.3}$$

We first examine the local stability and bifurcation. At the steady state $X = 0$,

$$\gamma_1 = \frac{\partial F}{\partial x_1}\bigg|_{X=0} = \frac{vx_0 + a(1+g)}{R(x_0+a)},$$
$$\gamma_2 = \frac{\partial F}{\partial x_2}\bigg|_{X=0} = -\frac{gaw}{R(x_0+a)},$$
$$\gamma_3 = \frac{\partial F}{\partial x_3}\bigg|_{X=0} = -\frac{ga(1-w)}{R(x_0+a)},$$
$$\frac{\partial F}{\partial x_4}\bigg|_{X=0} = 0, \qquad \frac{\partial F}{\partial x_5}\bigg|_0 X = 0 = 0.$$

Correspondingly, the Jacobin matrix of the system at the steady state is given by

$$J \equiv \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence the characteristic equation satisfies

$$\Gamma(\lambda) \equiv \lambda^2(\lambda^3 - \gamma_1\lambda^2 - \gamma_2\lambda - \gamma_3) = 0.$$

Denote

$$\pi(\lambda) = \lambda^3 - \gamma_1\lambda^2 - \gamma_2\lambda - \gamma_3.$$

Then two of the eigenvalues of $\Gamma(\lambda)$ are $\lambda_{4,5} = 0$ (double), while the rest of three eigenvalues satisfy $\pi(\lambda) = 0$. Following from Jury's test, all the eigenvalues of $\pi(\lambda)$ satisfy $|\lambda| < 1$ iff

(i) $\pi_1 \equiv \pi(1) = 1 - \gamma_1 - \gamma_2 - \gamma_3 > 0$;
(ii) $\pi_2 \equiv (-1)^3\pi(-1) = 1 + \gamma_1 - \gamma_2 + \gamma_3 > 0$;
(iii) $\pi_3 \equiv 1 + \gamma_2 + \gamma_3(\gamma_1 - \gamma_3) > 0$;
(iii) $\pi_4 \equiv 1 - \gamma_2 - \gamma_3(\gamma_1 + \gamma_3) > 0$.

Note that

$$\pi_1 = \frac{(R-v)x_0 + (R-1)a}{R(x_0+a)} > 0, \tag{A.4}$$

$$\pi_2 = \frac{(R+v)x_0 + a(R+1+2gw)}{R(x_0+a)}, \tag{A.5}$$

$$\pi_3 = \frac{R^2(x_0+a)^2 - gawR(x_0+a) - (vx_0+a+ga)ga(1-w) - g^2a^2(1-w)^2}{R^2(x_0+a)^2}, \tag{A.6}$$

and

$$\pi_4 = \frac{R^2(x_0+a)^2 + gawR(x_0+a) + (vx_0+a+ga)ga(1-w) - g^2a^2(1-w)^2}{R^2(x_0+a)^2}. \tag{A.7}$$

Hence $\pi_1 > 0$ is always satisfied.

For $g > 0$, $\pi_2 > 0$ and $\pi_3 > 0$ implies $\pi_4 > 0$. Hence, all four conditions are reduced to $\pi_3 > 0$. Therefore, the steady state is LAS if $\pi_3 > 0$.

For $w = 1$, $\pi_3 > 0$ is reduced to the condition $g < g_1 \equiv R(x_o + a)/a$ and hence the stability condition becomes $g < g_1$ for $w = 1$. For $w \in [0, 1)$, solving v from the condition $\pi_3 > 0$ leads to

$$v < v_3(g) \equiv c_1 \frac{1}{g} - c_2 - c_3 g, \tag{A.8}$$

where c_1, c_2 and c_3 are defined as in Proposition 3.1. Hence the steady state is LAS if $v < v_3(g)$ for $w \in [0, 1)$.

For $w \in [0, 1)$, solving g from $v = v_3(g)$ leads to

$$g_+ = \frac{1}{2c_3}\Big[-(c_2 + v) + \sqrt{(c_2 + v)^2 + 4c_1c_3}\Big].$$

One can verify that $g_+ \to g_1$ as $w \to 1$.

It follows from $\pi_1 > 0$ and $\pi_2 > 0$ that both saddle-node and flip bifurcations can not occur. Hence the steady state becomes unstable only through the Hopf bifurcation boundary $v = v_3(g)$. Along the Hopf bifurcation boundary, the three eigenvalues must satisfy $\lambda_1 \in (-1, 1)$ and $\lambda_{2,3} = e^{\pm\theta i}$. This leads to

$$\rho + \lambda_1 = \gamma_1, \qquad 1 + \rho\lambda_1 = -\gamma_2, \qquad \lambda_1 = \gamma_3, \tag{A.9}$$

where $\rho = 2\cos(\theta)$. For $w \in [0, 1)$, using $\pi_3 = 0$, one obtains that

$$\rho = \gamma_1 - \gamma_3 = -\frac{1 + \gamma_1}{\gamma_3} = \frac{R(x_0 + a) - gaw}{ga(1 - w)}. \tag{A.10}$$

For $w = 1$, the Hopf bifurcation boundary is given by $g = g_1, v \in [0, 1]$. In this case, $\lambda_1 = \gamma_3 = 0$, $\gamma_2 = -1$ and $\rho = \gamma_1 = [vx_o + a(1 + g_1)]/[R(x_o + a)] = 1 + (vx_o + a)/[R(x_o + a)]$. The nature of the Hopf bifurcation depends on θ, and hence on the values of ρ. Note that, along the Hopf bifurcation boundary, g is a function of v. Therefore types of Hopf bifurcation depend on the speed of the price adjustment of the fundamentalists towards the fundamental price, the extrapolation rate, and the weighting parameter w of the trend followers.

We now conduct a normal form analysis to show the existence and stability of the Hopf bifurcation. Let the eigenvalues of J be $\lambda_1 \in (-1, 1)$, $\lambda_2 = e^{i\theta}, \lambda_3 = \bar{\lambda}_2$. Then

$$\begin{aligned} \gamma_3 &= \lambda_1, \\ \gamma_1 &= e^{i\theta} + e^{-i\theta} + \lambda_1, \\ \gamma_2 &= 1 + (e^{i\theta} + e^{-i\theta})\lambda_1. \end{aligned} \tag{A.11}$$

Let $q, p \in C^5$ be such that $Jq = e^{i\theta}q, J^Tp = e^{-i\theta}p$ and $\langle p, q\rangle = 1$, where $\langle ., .\rangle$ is the usual inner product. Then we can take $q = (e^{4i\theta}, e^{3i\theta}, e^{2i\theta}, e^{i\theta}, 1)^T$ and $p = (1, e^{-i\theta} - \gamma_1, e^{i\theta}\gamma_3, 0, 0)^T p_1$, where $p_1 = \frac{e^{4i\theta}}{2+e^{3i\theta}\gamma_3 - e^{i\theta}\gamma_1}$. A Neimark-Sarker bifurcation occurs if the following value is not equal to zero (see Kuznetsov (2004)):

$$\begin{aligned} d \equiv &\frac{1}{2} Re\{e^{-i\theta}[\langle p, C(q, q, \bar{q})\rangle + 2\langle p, B(q, (E - A)^{-1}B(q, \bar{q})\rangle \\ &+ \langle p, B(\bar{q}, (e^{2i\theta}E - A)^{-1}B(q, q)\rangle]\}, \end{aligned}$$

where $C(q, q, q), B(q, q) \in R^5$ and

$$C_i(q, q, q) = \sum_{j,k,l=1}^{5} \frac{\partial^3 G_i(0)}{\partial x_j \partial x_k \partial x_l} q_j q_k q_l,$$

$$B_i(q,q) = \sum_{j,k=1}^{5} \frac{\partial^2 G_i(0)}{\partial x_j \partial x_k} q_j q_k,$$

E is the identity matrix. Note that $\frac{\partial^2 G_i(0)}{\partial x_j \partial x_k} = 0, i, j, k = 1, ...5$. Hence

$$\begin{aligned} d = & \ \tfrac{1}{2} Re\{e^{-i\theta}[\langle p, C(q,q,\bar{q})\rangle]\} \\ = & \ \tfrac{1}{2} \frac{1}{e^{-3i\theta} - e^{-i\theta} + \gamma_3(1-e^{-2i\theta})} \sum_{j,k,l=1}^{5} \frac{\partial^3 F(0)}{\partial x_j \partial x_k \partial x_l} e^{(j+k-l-5)i\theta}. \end{aligned} \tag{A.12}$$

A.3. **Proof of Proposition 4.1.** Following from the first part of the proof of Proposition 3.1 in Appendix A.2, all the three eigenvalues of $\pi(\lambda) = 0$ satisfy $|\lambda| < 1$ iff $\pi_i > 0$ for $i = 1, 2, 3, 4$. $\pi_1 > 0$ is always satisfied. Hence the stability boundaries of the local stability region of the steady state depend on the relative positions of $\pi_2 = \pi_2 = \pi_4 = 0$ on the (g, v) parameter plane. Solving for v in terms of g from $\pi_2 = \pi_3 = \pi_4 = 0$ leads to the following equations, respectively,

$$v_2(g) = -\frac{R(x_0+a)+a}{x_0} - \frac{2aw}{x_0} g,$$

$$v_3(g) = c_1 \frac{1}{g} - c_2 - c_3 g,$$

$$v_4(g) = -c_1 \frac{1}{g} - c_2 - \frac{aw}{x_0} g.$$

For $w = 1$, $\pi_2 > 0, \pi_3 > 0$ and $\pi_4 > 0$ are equivalent to

$$g > -\frac{1}{2a}[R(x_o+a) + vx_0 + a], \quad g < \frac{R(x_o+a)}{a}, \quad g > -\frac{R(x_o+a)}{a},$$

respectively. Given $v \in [0, 1]$ and $R > 1$, these three conditions are reduced to $g > -\frac{1}{2a}[R(x_o + a) + vx_0 + a]$, which is equivalent to

$$v > -\frac{R(x_o+a) + a + 2ag}{x_o}.$$

In addition, along the boundary, one of the eigenvalues is -1, and hence a flip bifurcation occurs.

Now let $w \in [0, 1)$. Note that $v = v_2(g)$ is a straight line with slop $-\frac{2aw}{x_0}$, which is decreasing from 0 to $-\frac{2a}{x_0}$ as w varies from 0 to 1; $v = v_3(g)$ is concave up and decreasing from $+\infty$ to $-\infty$ for $g \in (-\infty, 0)$, and it moves to the left as w varies from 0 to 1; $v = v_4(g)$ is concave down and $\lim_{g\to -\infty} v_4(g) = \lim_{g\to 0^-} v_4(g) = +\infty$.

The relative position of the two curves $v = v_3(g)$ and $v = v_4(g)$ is determined by

$$v_{43} = v_4(g) - v_3(g) = -2c_1 \frac{1}{g} + (c_3 - \frac{aw}{x_0})g.$$

Hence $v_{43} = 0$ implies that $g = g_{43} = -\frac{(x_0+a)R}{a(1-w)}$. Note that

$$v_4(g_{43}) = v_3(g_{43}) = \frac{(x_0+a)R - a}{x_0} > 1.$$

Therefore two curves $\pi_3 = 0$ and $\pi_4 = 0$ intersect at one point, which is always above the line $v = 1$. It is easy to see that $\frac{\partial v_4}{\partial g}(g_{43}) = a(1-2w)/x_0 < 0, = 0, > 0$ if $w > \frac{1}{2}, = \frac{1}{2}, < \frac{1}{2}$ respectively. The relative positive positions of those curves are plotted in Fig. A.1. Obviously, for $w \le 1/2$, the stability conditions are reduced to

$\pi_2 > 0$ and $\pi_3 > 0$. It can be shown [11] that this is also true for $w > 1/2$. Therefore, the boundaries of the stability of the steady state are determined by $\pi_2 = \pi_3 = 0$.

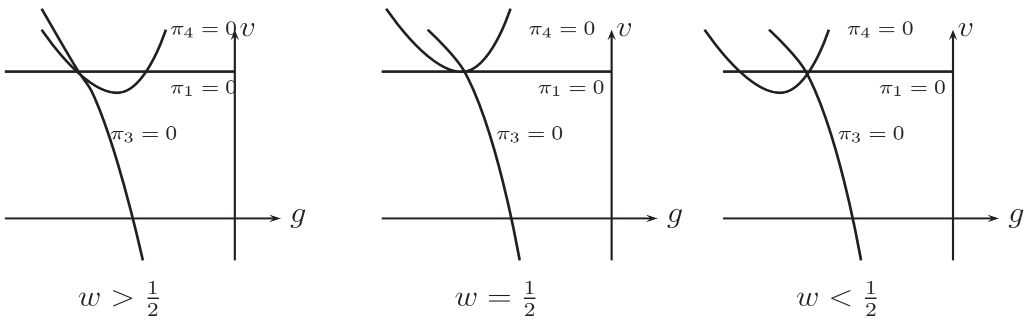

FIGURE A.1. Relative position of $\pi_i = 0$ for $i = 1, 3, 4$.

The relative position of the curves $v = v_2(g)$ and $v = v_3(g)$ is determined by

$$v_{32} = v_3(g) - v_2(g) = \frac{(x+a)^2 R^2}{a x_0 (1-w)} \frac{1}{g} + \frac{(Rx_0 + a)(1-2w)}{x_0(1-w)} + \frac{a(3w-2)}{x_0} g.$$

It is easy to see that, for $g < 0$, $v_{32} < 0$ for $w \geq \frac{2}{3}$, and $v = v_{32}(g)$ is decreasing from $+\infty$ to $-\infty$ for $g \in (-\infty, 0)$ if $w \in [0, \frac{2}{3})$. Therefore, when $w \geq \frac{2}{3}$, the line $v = v_2(g)$ is above the curve $v = v_3(g)$. Hence the stability conditions are reduced to $\pi_2 > 0$ and the stability region is bounded by flip bifurcation boundary only. When $w < \frac{2}{3}$, there is a unique intersection for these two curves, and the relative position is shown as Figure A.2. Consequently, there exist w_1 and w_2 satisfying $0 < w_1 < w_2 < 2/3$ such that the intersection of these two curves $v = v_2(g)$ and $v = v_3(g)$ is located in the band region $0 \leq v \leq 1$ if $w_1 \leq w \leq w_2$.

Note that the curves $v = v_1(g)$ and $v = v_2(g)$ intersect when $g = g_{12} = -(2Rx_0 + Ra + a)/(2aw)$. Obviously, g_{12} increases and g_{43} decreases for $w \in (0, 1)$. Also, $g_{12} < g_{43}$ for $w = 0$ and $g_{12} > g_{43}$ for $w = 1/2$. Hence $w_2 < 1/2$.

We now conduct the normal form analysis. J has an eigenvalue of -1 if and only if $\gamma_3 = -1 - \gamma_1 + \gamma_2$. Now we assume that this condition holds. Then a flip bifurcation may occur, depending on the value of coefficient of the third order term in the normal form. We now give the computation for the flip bifurcation.

Let $q = (1, -1, 1, -1, 1)^T$ and $p = (1, -1-\gamma_1, 1+\gamma_1-\gamma_2, 0, 0)^T/(3 + 2\gamma_1 - \gamma_2)$. Then $Aq = -q$, $A^T p = -p$ and $\langle p, q \rangle = 1$. The type of the flip bifurcation depends on the value

$$\bar{d} \equiv \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\langle p, B(q, (A-E)^{-1}B(q,q))\rangle,$$

[11] In fact, $\pi_2 = \pi_4 = 0$ intersect at $g = g_{24} = -R(x_0 + a)/(aw)$. Note that $v_4'(g_{24}) = aw(2w - 1)/(x_0(1-w))$, which is positive for $w > 1/2$ and negative for $w < 1/2$. Hence, for $w > 1/2$, if $g_{24} < g < 0$, the curve $\pi_4 = 0$ is above the curve $\pi_2 = 0$. Therefore, $\pi_4 > 0$ when $v \leq 1$ and $g > g_{24}$. The stability boundaries are then determined by $v = 0, v = 1$ and $\pi_2 = 0$.

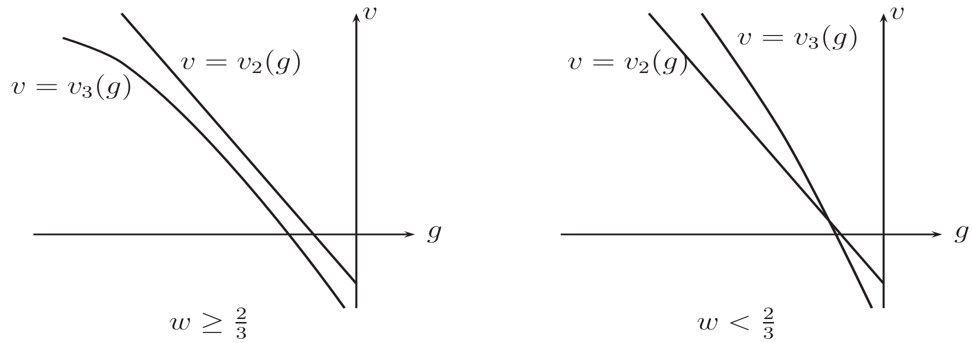

FIGURE A.2. The relative position of $v = v_2(g)$ and $v = v_3(g)$ for both $w \geq 2/3$ and $w < 2/3$.

where $C(q, q, q), B(q, q) \in R^5$ are defined in the previous proof. Note that $\frac{\partial^2 G_i(0)}{\partial x_j \partial x_k} = 0, i, j, k = 1, ...5$. So $\bar{d} = \frac{1}{6}\langle p, C(q, q, q)\rangle$. Hence

$$\bar{d} = \frac{1}{6}\frac{1}{3 + 2\gamma_1 - \gamma_2}\sum_{j,k,l=1}^{5}\frac{\partial^3 F(0)}{\partial x_j \partial x_k \partial x_l}(-1)^{j+k+l+1}.$$

The normal form calculation for the flip bifurcation is given in the proof of Proposition 3.2 in Appendix A.2.

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