QUANTITATIVE FINANCE RESEARCH CENTRE

UNIVERSITY OF TECHNOLOGY SYDNEY

QUANTITATIVE FINANCE RESEARCH CENTRE

UTS

THINK.CHANGE.DO

QUANTITATIVE FINANCE RESEARCH CENTRE

Research Paper 247 April 2009

Asset Markets and Monetary Policy

Eckhard Platen and Willi Semmler

ISSN 1441-8010 www.qfrc.uts.edu.au

Asset Markets and Monetary Policy

Eckhard Platen[1] and **Willi Semmler**[2]

April 27, 2009

Abstract. Monetary policy has pursued the concept of inflation targeting. This has been implemented in many countries. Here interest rates are supposed to respond to an inflation gap and output gap. Despite long term continuing growth of the world financial assets, recently, monetary policy, in particular in the U.S. after the subprime credit crisis, was challenged by severe disruptions and a meltdown of the financial market. Subsequently, academics have been in search of a type of monetary policy that does allow to influence in an appropriate manner the investor's behavior and, thus, the dynamics of the economy and its financial market. The paper suggests a dynamic portfolio approach. It allows one to study the interaction between investors' strategic behavior and monetary policy. The article derives rules that explain how monetary authorities should set the short term interest rate in interaction with inflation rate, economic growth, asset prices, risk aversion, asset price volatility, and consumption rates. Interesting is that the inflation rate needs to have a certain minimal level to allow the interest rate to be a viable control instrument. A particular target interest rate has been identified for the desirable optimal regime. If the proposed monetary policy rule is applied properly, then the consumption rate will remain stable and the inflation rate can be kept close to a minimal possible level. Empirical evidence is provided to support this view. Additionally, in the case of an economic crisis the proposed relationships indicate in which direction to act to bring the economy back on track.

JEL Classification: G10, G13
1991 *Mathematics Subject Classification:* primary 65C20; secondary 60H10.
Key words and phrases: risk aversion, interest rate, dynamic portfolio, consumption rate, inflation, monetary policy, benchmark approach.

---

[1]University of Technology, Sydney, School of Finance & Economics and Department of Mathematical Sciences, PO Box 123, Broadway, NSW, 2007, Australia
[2]Department of Economics, New School, New York, 79 Fifth Ave, New York, NY 10003, USA, and Center for Empirical Macroeconomics, Bielefeld University, Germany, e-mail: semmlerw@newschool.edu

"... under the mandate it has been given by the Congress, the Federal Reserve has a responsibility to take monetary policy actions to minimize the damage that financial instability can do to the economy. .. Pursuing such policies does help financial markets recover from episodes of financial instability, and so it can help lift asset prices." Frederic Mishkin (2007)

1 Introduction

It seems appropriate in current critical times to acknowledge the fact that the financial market and economic activity strongly interact, representing some challenge for monetary policy. The question is what the quantitative relationships are and what appropriate monetary policy rules should be. In times of normal economic activity it has turned out that monetary policy, which sets the interest rate appropriately, is sufficient to keep the economy on a sound path.

One of the most important recently established monetary policy rules was the Taylor rule. Taylor (1993) has put forward the concept of inflation targeting, suggesting the short term interest rate to respond to the inflation and output gap. It has been made popular by economic research, for example by the works of Svensson (1997, 1999) and Woodford (2003). The latter, among others, extended the concept of inflation targeting to an optimizing framework, introducing some optimal Taylor rule.

Yet, from early on, the question came up of whether the monetary authorities should also give some weight to asset price movements. In the context of the Bernanke, Gertler & Gilchrist (1999) framework of an imperfect capital market, net worth moves procyclically and the financial accelerator can magnify disturbances. On the other hand, it has been proposed that asset price movements should only be taken into account as much as they affect the inflation rate. It was assumed that a central bank that responds strongly to inflation pressures eliminates much of the distortionary effects of asset price movements on economic activity. It should be mentioned that some monetary authorities refer to the Tinbergen rule which says that one needs as many instruments in monetary policy as there are targets.

Cecchetti et al. (2000) argued that monetary authorities would obtain beneficial effects when responding to asset prices. This, in particular, should hold if there are exogenous bubbles or non-fundamental asset price movements. In this context Dupor (2005) emphasizes that capital valuations are impacted by asset price bubbles which causes distortions in aggregate demand. Thus, the monetary authority, by reducing distortions owing to variations in the return on capital, may generate beneficial effects on real activity by also giving weight to asset prices.

Others have argued that these asset price bubbles could be seen as resulting from asset price booms that occur in conjunction with changes in the underlying economic fundamentals, see Beaudry & Portier (2004). Along those lines, some literature has attempted to construct models where bubbles arise due to misperceived technology shocks, see Christiano, Motto & Rostagno (2005), suggesting that asset prices should not be targeted by central banks.

Asset prices have become also important in the studies on the Japanese deflation of the 1990s. In the last ten years, the literature has focused on the Japanese deflationary experience and argued that when asset prices and product prices decline and inflation rates approach zero, or become negative, there will be a zero bound of the nominal interest rate that may create a liquidity trap for monetary authorities. Central banks may not be able any longer to effectively steer the economy with monetary policy to some targets, see Coenen & Wieland (2004) and Eggertsson & Woodford (2003). When interest rates hit the zero level bound, Bernanke, Reinhard & Sack (2004) have brought the monetary policy of "quantitative easing" into the discussion, which is to provide the private economy – suffering from a burst of an asset price bubble – with liquidity by providing capital for credit, purchasing "bad assets" of the banking sector[3]. Now fiscal policy has been added as policy tool reviving the economy through government spending.

The experience of the extensive subprime crisis in the U.S. since the middle of 2007, and the subsequent global financial market meltdown and banking crisis and their serious effect on real economic activity has shifted the emphasis to the study of monetary policy in the context of asset markets. Asset price movements since the beginning of 2008 entailed huge portfolio losses and led to extensive rebalancing of portfolios, namely movements out of complex derivative securities, real estate, stocks and commodities into treasury bonds and cash. The monetary authority in the U.S., but also other central banks and governments worldwide, reacted with extreme interest rate reductions, support for banks and substantial government spending plans. In the U.S. interest rates were reduced from 5.25% in 2007 to 0.25% in December 2008, yielding to virtually zero interest rates. Moreover, the asset price deflation, the large losses in portfolios, and the danger of a catastrophic meltdown, has led monetary policy to react strongly, not only in terms of interest rate reduction by the monetary authority, but also in terms of "quantitative easing".[4]

Under these extreme conditions the study of the effects of monetary policy in the context of asset accumulation and dynamic portfolio models have become a challenging issue. The approach of this paper uses optimal asset allocation in a dynamic setting. The resulting optimal solution implies certain relationships between key quantities. This dynamic portfolio approach makes visible, for instance, not only the transmission mechanism through which the financial

[3]The concept was already put forward by Bernanke, Reinhard & Sack (2004). It has been pursued in the U.S. since the Fall of 2007, when the financial market meltdown became visible.

[4]Quantitative easing improves banks' and firms' balance sheets and is supposed to unlock the credit channel.

meltdown is translated into a real economic meltdown, but will allow us to study properly the effectiveness and limitations of monetary policy. Such policy should be conducted recognisant of the optimal relationships we are revealing.

The application of a portfolio model to macroeconomics has already been favored by Tobin (1969, 1980) and Frankel (1995), to name a few. But this was mainly a portfolio theory of static type. What is needed is a dynamic portfolio theory that permits the economic agents to strategically respond to expected future returns, inflation, interest rate, and other factors. Campbell & Viceira (2002) have contributed in a seminal way to enhance our understanding of the potential of dynamic portfolio theory and strategic asset allocation. They give an important role to nominal and real bonds in financial investments and show how different types of investors, that is investors with different risk aversion and different time horizon, would react to a change of an investment environment created by monetary policy. They also show that inflation rates, which the central banks try to target, affect the composition of asset holdings strongly. They, for example, demonstrate how the anti-inflation interest rate policy of the Volcker-Greenspan area had substantial effects on the composition of asset holdings and, thus, on the economy.

Whereas Campbell & Viceira (2002) when exploiting martingale properties of asset prices use assumptions such as constant variances, constant expected risk premia,[5] a constant consumption-wealth ratio and the existence of an equivalent risk neutral probability measure, in order to undertake a log-linearization of their models, these assumptions do not seem to be too realistic if one studies the interaction of monetary policy and the asset markets in more general terms over long periods of time. Moreover, the impact of strategic portfolio shifts on the real economy, for example, caused by strong asset price movements and monetary policy changes, is not considered further in the literature. But this has become an important trigger for the current deep and extensive recession.

Our approach suggested here is closely related to the work of Campbell & Viceira (2002). We also introduce the distinction between nominal and real assets, and use a martingale approach, however, the latter very generally under the real world probability in the sense of Platen & Heath (2006). The key economic factors like inflation rate, consumption rate, expected return, and volatility can be general stochastic processes. Moreover we pursue more of a nonlinear approach when we spell out a new interest rate rule and new inflation rate rule as a consequence of the dynamic portfolio approach. As we show, there can arise important knife edge problems in the interrelation of economic growth, inflation rate, interest rate, risk aversion, return on the assets and consumption rate. These relations are nonlinear ones and will be studied as such.

Overall, what we aim for is a better understanding about the impact of monetary policy under extraordinary and, of course, also normal circumstances. It will turn

[5]For example, in Chapter 3 of the book by Campbell & Viceira (2002).

out that if monetary policy is not cautiously implemented with a long term view for a sustainable economic and financial development, then situations may arise where the economy cannot be controlled via monetary policy and can suffer for long periods of time even with strong government intervention.

The paper will describe an interest rate rule that gives an understanding where the interest rate should be set under given circumstances to ensure a sound long term evolution of the economy and its financial market. In particular, the interest rate i_t has to be set in the proposed *interest rate rule* at about

$$i_t = \big(a_t - \gamma\,\sigma_t^2\,(1 - |1 - \alpha_t|)\big)^+ ,$$

where a_t can be interpreted as the expected return of the equity index, γ as the risk aversion, α_t as the fraction of wealth invested in the equity market and σ_t as the volatility of the equity index, where $x^+ = \max(0, x)$. We will also indicate how to reach a situation where monetary policy can safely operate and how to maintain such a regime. In this context the importance of a critical minimal level of the inflation rate will be emphasized by the *inflation rate rule* suggesting a typical level of inflation at about

$$\pi_t = a_t - c_t + \frac{\gamma\,\sigma_t^2}{2}\,\alpha_t\,(\alpha_t - 1),$$

and not much smaller, where c_t is the consumption rate. A sufficient stimulus for the economy can be shown to be essential to bring the economy back on track in case zero interest rates emerge. Also the increase of the actual inflation which lowers the real interest rate, may become necessary to steer the economy back into an optimal dynamics. Removing "bad assets" and "quantitative easing" can calm the volatility down, which acts in a similar direction. Finally, more foreign debt in an economy which has already borrowed also helps to bring the interest rate back into positive territory.

The paper is organized as follows. Section 2 considers the asset market dynamics in the context of a dynamic portfolio decision model with the usual budget equations. Section 3 studies the optimal market dynamics and derives our proposed interest rate and inflation rate rules. Section 4 provides comments on monetary policy, pursued during the current and previous financial market meltdowns. Section 5 then discusses some empirical facts illustrating our proposed interest rate rule. Section 6 concludes the paper.

2 Asset Market Dynamics and Budget Equation

A common hypothesis about the behavior of asset prices is that they follow Itô processes, see Merton (1992) and Cochrane (2001). We consider a continuous-time market with continuous asset prices, where we examine a portfolio optimization problem for given capital consumption as fraction of total wealth. To simplify our analysis, we model the fixed income segment of the market by a locally riskless savings account and all risky investments via an aggregation into a risky asset index.

Income from capital is generated through dividends from the risky asset index and interest from the locally riskless savings. Additionally, capital gains on investments in the assets may benefit the investor. To model short term bonds we introduce the locally riskless asset β_t, which is also called the savings account, where

$$\beta_t = \exp\left\{ \int_0^t i_s \, ds \right\}, \tag{2.1}$$

with nominal interest rate process $i = \{i_t, t \geq 0\}$. The risky asset P_t shall be interpreted as market capitalization weighted total return asset index. It is assumed to satisfy the stochastic differential equation (SDE)

$$dP_t = P_t \left(a_t \, dt + \sigma_t \, dz_t \right) \tag{2.2}$$

for $t \geq 0$ with $P_0 > 0$. Here $z = \{z_t, t \geq 0\}$ denotes a standard Wiener process. We ensure that the volatility $\sigma_t > 0$ of the asset index is always strictly positive.

To bring the average cost of living, for instance housing, energy and food into the picture, we introduce a consumer price index I_t, satisfying the expression

$$I_t = \exp\left\{ \int_0^t \pi_s \, ds \right\}, \tag{2.3}$$

with inflation rate process $\pi = \{\pi_t, t \geq 0\}$.

The consumption rate process $c = \{c_t, t \geq 0\}$ models the rate at which capital is taken out of the financial market. This includes dividends and interest payments, but also share buy backs, new share issues and similar activities. We assume, for simplicity, that $c_t > 0$ satisfies the equation

$$c_t = c_0 \, \exp\left\{ \int_0^t e_s \, ds \right\} \tag{2.4}$$

for $t \geq 0$ with $c_0 > 0$. Here $e = \{e_t, t \geq 0\}$ denotes the growth rate process of the consumption rate. If the consumption rate does not change, then we simply have $e_t = 0$.

We assume that all processes a, σ, i, π, and e can be stochastic and there exists a unique market dynamics where the following manipulations and statements are meaningful. Therefore, these processes can have rather general dynamics which can be specified later to study, for instance, feedback effects.

The wealth process $W = \{W_t, t \geq 0\}$ shall be interpreted as the total wealth of all investors.[6] They invest at time t the fraction α_t in the risky asset index P_t

[6] Note that we can interpret total wealth here as the one generating income including equity,

and the remainder in the locally riskless savings account β_t. Note, for $\alpha_t > 1$ the investors borrow locally riskless to invest in the risky asset index. For convenience, we suppress in our notation the dependence of W on the choice of the fraction α, see also Campbell & Viceira (2002) (Ch. 5.2). Since financial capital is consumed with the given positive rate c_t at time t, the SDE for the total wealth process is of the form

$$dW_t = W_t \left((1-\alpha_t) i_t \, dt - c_t \, dt + \alpha_t (a_t \, dt + \sigma_t \, dz_t) \right) \tag{2.5}$$

for $t \geq 0$ with $W_0 > 0$.

It is a challenge to interpret and understand the interplay between inflation rate, interest rate, expected return, consumption rate, market volatility and the fraction invested in the risky asset index. The government and monetary authority have a responsibility to provide conditions so that the economy develops in the long run sustainably and efficiently. From the beginning it should not be ignored that in the long run the agents can only consume what is produced so that some intertemporal budget constraint holds.

In monetary policy the inflation rate π_t is typically aimed to be minimized, since high inflation makes it difficult to transfer wealth over time through the financial market. On the other hand, it is well-known that a too low inflation rate is not convenient for several reasons: Higher inflation makes the adjustment of real wages easier since there is a psychological hurdle against dropping wages. Low inflation could be easily mis-measured and there could be already a deflation even when the statistics does not show it. If necessary, higher inflation can make the real interest rate negative over certain periods of time. This raises the question, what is a sustainable level for the inflation rate under particular economic conditions? Obviously, the nominal interest rate can only be nonnegative, which is a major constraint for monetary policy. Are there other lower or higher bounds for the interest rate that when violated would make it a useless or counterproductive economic instrument? We will derive some relationships that may provide a better understanding of the role that the interest rate can play.

What counts for an investor is, how much real wealth can be consumed per unit of time sustainably. We will formulate this objective without introducing any particular time horizon. To achieve this, we will introduce a different criterion than typically employed in portfolio optimization, as for instance used in Merton (1973), without imposing any artificial terminal date or discount rate. More precisely, we assume that the objective of the investors in the market is the maximization of aggregate consumed real wealth

$$\frac{c_s W_s}{I_s}$$

real estate as well as human capital. This, in principle, would require heterogeneous assets, all generating different returns. In order to avoid such complicating modeling procedure we consider here only two representative assets: a risky asset index and a locally risk free asset. Extensions to heterogeneous assets could subsequently be undertaken.

per unit of time for a given consumption rate c_s at any given time $s \geq 0$. Since this quantity is random, we need to specify the aggregate preferences of the investors. This we can achieve by using a utility function $U(\cdot)$ for evaluating the utility from real wealth consumed per unit of time. Note that the consumption rate is given and not part of the optimization, which is different to typical portfolio-consumption optimizations. Our approach separates the optimal wealth evolution from the specification of the consumption rate, which can later be modeled and can depend on the evolution of all state variables and even external factors. We see no need to make it part of the total wealth optimization.

We aim to identify the optimal total wealth dynamics of the market such that it maximizes at each time the expected utility from consumed real wealth. This should give us some insight in how the key quantities relate to each other. More precisely, given some amount of wealth $W_t > 0$ at a given time $t \geq 0$, we will maximize at any such time the expected utility of consumed real wealth at any later time $s \in (t, \infty)$. Our objective is then expressed in the maximization

$$\max_{W} E_t \left(U \left(\frac{c_s W_s}{I_s} \right) \right) \tag{2.6}$$

for $0 \leq t < s < \infty$. Here the maximization is performed by identifying the self-financing total wealth process W that yields the above maximum. Note that E_t denotes the conditional expectation operator under the real world probability measure given the information at time t. Furthermore, for the utility function $U(\cdot)$ we assume monotonicity via $U'(\cdot) > 0$, concavity via $U''(\cdot) < 0$ and denote by $U'^{-1}(\cdot)$ the inverse function of $U'(\cdot)$. We allow rather general utility functions to express the preferences and, thus, the risk aversion of the investors. Of course, we request such mathematical properties that our following manipulations and statements will be meaningful. To be specific, we will later concentrate on the well-known power utility of the form

$$U(x) = \frac{x^{(1-\gamma)}}{1-\gamma}. \tag{2.7}$$

Here the risk aversion $\gamma > 0, \gamma \neq 1$, shall be constant. In the case $\gamma \to 1$ we have the logarithmic utility as limiting utility, which in the actual optimization and calculations does not create any problem for our outcome. Note that in our setting the notion of risk aversion may not be fully in line with what some readers may be used to.

Let us emphasize again that we do not follow the classical approach that maximizes for a representative agent expected utility from terminal wealth and consumption without the need to specify a time horizon. We focus here consciously on the aggregate behavior of investors who decide to invest either in fixed income or risky securities under a given consumption rate. Our proposed criterion (2.6) shall allow us to identify the optimal dynamics that the wealth process follows when investors maximize utility from the consumption of real wealth. As a result of keeping the consumption rate given, and out of the optimization, the resulting optimal wealth dynamics can be characterized very generally without specifying a time horizon and relying on any explicit evaluation of expectations of utilities. This will allow us to cover a wide range of utility functions and market dynamics. The results will be very robust and, in principle, model independent as a consequence of our new approach.

3 Market Dynamics

To solve the above portfolio optimization problem, we apply the martingale approach of Cox & Huang (1989) described, for instance, in Campbell & Viceira (2002) (Ch. 5). However, we combine it with the benchmark approach in Platen & Heath (2006), which relies only on the real world probability measure when forming expectations. This allows us to explore a much richer modeling world than prescribed under the classical no-arbitrage theory with the restrictive assumption on the existence of a risk neutral pricing measure when applied in the established no-arbitrage approach. Furthermore, trends in the real world dynamics of the market are taken into account, which are, in principle, ignored under the classical risk neutral methodology.

3.1 Optimal Wealth Dynamics

First, we introduce the so called benchmark, which is the numeraire portfolio S_t^* of the financial market in the sense of Platen & Heath (2006) and in generalization of Long (1990). It is the best performing portfolio in our investment universe in several ways. It equals the growth optimal portfolio, also known as the Kelly portfolio, see Kelly (1956) and Merton (1973), which maximizes expected logarithmic utility from terminal wealth. When pricing a contingent claim according to Platen & Heath (2006) its inverse will play the role of the stochastic discount factor

$$M_t = \frac{1}{S_t^*},$$

similar as in Cochrane (2001). In our setting the stochastic discount factor M_t satisfies the SDE

$$dM_t = -\theta_t \, M_t \, dz_t - i_t \, M_t \, dt \tag{3.1}$$

for $t \geq 0$, where we set for simplicity $M_0 = 1$, see Platen & Heath (2006). Here

$$\theta_t = \frac{a_t - i_t}{\sigma_t} \tag{3.2}$$

denotes the market price of risk. This is the central market invariant for the investors, which is also the volatility of the numeraire portfolio S_t^*. The accumulated total wealth at time t can be written as the product $W_t G_t$, where

$$G_t = \exp\left\{\int_0^t c_s\, ds\right\}. \tag{3.3}$$

By application of the Itô formula one obtains for the benchmarked accumulated total wealth

$$\frac{W_t\, G_t}{S_t^*} = M_t\, W_t\, G_t$$

the driftless SDE

$$d(M_t\, W_t\, G_t) = M_t\, W_t\, G_t\, (\alpha_t\, \sigma_t - \theta_t)\, dz_t. \tag{3.4}$$

According to Platen & Heath (2006) all benchmarked nonnegative self-financing wealth processes, when expressed in units of the numeraire portfolio S_t^*, are downward trending or have no trend. Those with no trend, which are so called martingales, are unique and deliver the minimal price of a corresponding replicable payoff, see Platen & Heath (2006). Consequently, for the replicable payoffs that we will consider it is sufficient to concentrate on total wealth processes where the following martingale property holds

$$E_t(W_s M_s G_s) = M_t\, W_t\, G_t \tag{3.5}$$

for $0 \leq t \leq s < \infty$. This is in line with the driftless SDE (3.4). The above type of wealth process provides the most cost efficient way to manage wealth under some given objective. Such objective could be expressed as a derivative payoff but it could also be the utility from consumed real wealth, as will be the case in our study.

Similar as under the classical martingale approach, see Campbell & Viceira (2002) (Ch. 5), the objective (2.6) translates with equation (3.5) into the constraint optimization problem

$$\begin{aligned} v_t &= \max_W E_t\left(U\left(\frac{c_s W_s}{I_s}\right)\right) - \ell_t E_t(W_s M_s G_s - W_t M_t G_t) \\ &= \max_W E_t\left(U\left(\frac{c_s W_s}{I_s}\right) - \ell_t (W_s M_s G_s - W_t M_t G_t)\right) \end{aligned} \tag{3.6}$$

for $0 \leq t < s < \infty$. Here ℓ_t is the Lagrange multiplier, which remains independent of s and will capture the budget constraint and facilitate that $M_t\, W_t\, G_t$ has to form a martingale. We emphasize that expectation is here taken under the real world probability measure and not under some putative risk neutral probability measure as in most of the literature.

To identify the candidate for the optimal total wealth process we interchange in (3.6) maximization and expectation and consider for each $t \geq 0$ and $s > t$ the static optimization problem

$$\max_{W_s} \left[U\left(\frac{c_s W_s}{I_s}\right) - \ell_t (W_s M_s G_s - M_t\, W_t\, G_t) \right],$$

which leads to the first order condition

$$U'\left(\frac{c_s W_s}{I_s}\right) \frac{c_s}{I_s} - \ell_t M_s G_s = 0.$$

By using the inverse function $U'^{-1}(\cdot)$ of $U'(\cdot)$ one obtains

$$\frac{c_s W_s}{I_s} = U'^{-1}\left(U'\left(\frac{c_s W_s}{I_s}\right)\right) = U'^{-1}\left(\ell_t M_s G_s \frac{I_s}{c_s}\right).$$

Consequently, the candidate for the optimal total wealth process is of the form

$$W_s = \frac{I_s}{c_s}\, U'^{-1}(\ell_t \phi_s) \tag{3.7}$$

with

$$\phi_s = \frac{M_s G_s I_s}{c_s}. \tag{3.8}$$

From the initial total wealth W_0 and initial consumption rate c_0 one obtains the corresponding constant Lagrange multiplier $\ell_t = \ell_0 = c_0 U'(c_0 W_0)$ for all $0 \leq t < s < \infty$. Note that the fixed initial wealth $W_0 > 0$ acts here as budget constraint. To avoid technicalities we assume now that from the initial wealth allocation onward the relation (3.7) for the candidate total wealth process is indeed feasible and describes the evolution of the resulting minimal self-financing portfolio that maximizes for any t and s the given expected utility

$$E_t\left(U\left(\frac{c_s W_s}{I_s}\right)\right).$$

For a wide range of models for which an equivalent risk neutral probability measure exists, this can be easily confirmed by classical methods, see Campbell & Viceira (2002), however it holds far more generally, see Platen & Heath (2006). This means that the candidate wealth process we consider maximizes the given expected utility.

It is important that we obtained the optimal total wealth W_s in formula (3.7) in an explicit manner as a function of known quantities. The optimal fraction α_t for the investment in the risky asset index will be described below. It will make the benchmarked accumulated total wealth

$$\frac{W_t\, G_t}{S_t^*} = M_t\, W_t\, G_t$$

a martingale. The current value of the benchmarked wealth process is then the best forecast of all its future values. Note, the SDE for ϕ_t given in (3.8) follows by the Itô formula as

$$d\phi_t = \phi_t([c_t + \pi_t - e_t - i_t]\, dt - \theta_t\, dz_t).$$

By the explicit relation (3.7) we can express the martingale

$$M_t\, W_t\, G_t = F(\phi_t) \tag{3.9}$$

via the function

$$F(\phi) = \phi\, U'^{-1}(\ell_0 \phi). \tag{3.10}$$

Therefore, it follows by the Itô formula the SDE

$$\begin{aligned} d(M_t\, W_t\, G_t) &= \frac{\partial}{\partial \phi} F(\phi_t)\phi_t([c_t + \pi_t - e_t - i_t]\, dt - \theta_t\, dz_t) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial \phi^2} F(\phi_t)\phi_t^2\, \theta_t^2\, dt. \end{aligned} \tag{3.11}$$

Comparison of the drift coefficients of the SDEs (3.4) and (3.11) yields the key relation

$$0 = c_t + \pi_t - e_t - i_t - \frac{\theta_t^2}{2\,\gamma_t}, \tag{3.12}$$

where

$$\gamma_t = -\frac{\frac{\partial}{\partial \phi} F(\phi_t)}{\phi_t\, \frac{\partial^2}{\partial \phi^2} F(\phi_t)}. \tag{3.13}$$

Note that equation (3.12) could be also interpreted as a relation involving the real interest rate $r_t = i_t - \pi_t$, which may be useful if some readers prefer to interpret our results more in real terms. For the case of power utility the quantity γ_t in (3.13) coincides with the risk aversion γ appearing in (2.7). We mention that for the case of power utility it follows by (2.7) that

$$U'^{-1}(y) = y^{-\frac{1}{\gamma}}$$

and thus

$$F(\phi_s) = \ell_0^{\frac{1}{\gamma}}\, \phi_s^{1-\frac{1}{\gamma}}.$$

The function $F(\phi_s)$ covers also the case of logarithmic utility which emerges for $\gamma = 1$.

The second key equation results from the comparison of the diffusion coefficients of the SDEs (3.4) and (3.11), which yields the equation

$$M_t\, W_t\, G_t(\alpha_t\, \sigma_t - \theta_t) = -\frac{\partial}{\partial \phi} F(\phi_t)\phi_t\, \theta_t.$$

Thus, the optimal fraction of total wealth at time t invested in the risky asset P_t is of the form

$$\alpha_t = \frac{\theta_t}{\bar{\gamma}_t \, \sigma_t} \tag{3.14}$$

with

$$\frac{1}{\bar{\gamma}_t} = 1 - \phi_t \, \frac{\frac{\partial}{\partial \phi} F(\phi_t)}{F(\phi_t)} \tag{3.15}$$

and using the function $F(\cdot)$ given in (3.10). Note that in the case of power utility one obtains $\bar{\gamma}_t = \gamma$ with γ denoting the risk aversion appearing in (2.7).

The fraction in (3.14) is consistent with the familiar optimal fraction that arises when maximizing expected power utility from terminal wealth, see for instance Merton (1973) and Campbell & Viceira (2002).

Above, we have derived the equations (3.12) and (3.14) which represent the two core relationships between relevant quantities under an optimal wealth dynamics. It is now possible to analyze how changes in one quantity may impact other quantities. We emphasize that we have not specified a_t, σ_t, c_t, π_t, i_t and e_t. Thus, the relations (3.12) and (3.14) hold generally when assuming an optimal wealth dynamics under the objective (2.6) that maximizes expected utility at all times. Of course, in reality there are perturbations distorting these relations. However, in the long run and for understanding in which direction the economy is likely to move under particular circumstances, the relationships could be rather useful.

3.2 Optimal Interest Rate

To make the following analysis specific we assume that the preferences of the investors can be expressed via power utility, which we substitute by logarithmic utility in the case when the risk aversion γ equals one. Attempting to obtain an optimal wealth dynamics we aim to reveal potential relationships between the interest rate and other key quantities. By (3.2) and (3.14) this leads us to an optimal interest rate $\tilde{i}_t$ in the form

$$\tilde{i}_t = a_t - \gamma \, \alpha_t \, \sigma_t^2. \tag{3.16}$$

Similar relations can be found, for instance, in Campbell & Viceira (2002) and Platen (1999). As we will see later, this is the interest rate that a monetary authority should approximately set when $\alpha_t < 1$, whereas in the case $\alpha_t > 1$ it will have to work with a higher interest rate to steer the economy. However, in the latter case the wealth dynamics will not be optimal since the interest rate will be not optimal.

The optimal interest rate increases when the expected return a_t increases. It increases when the fraction of wealth invested in the risky asset decreases or when the squared risky asset volatility σ_t^2 decreases, provided that the other quantities remain unchanged. The relation (3.16) gives some guidance, where the interest rate should be set for allowing the market to allocate its resources of capital investment for an optimal wealth evolution. The optimal interest rate level described in (3.16) is the one where the fluctuations of the wealth process coincide with those optimal under power utility with risk aversion γ.

In reality the nominal interest rate i_t cannot become negative. This imposes a constraint on monetary policy. Therefore, any theoretical interest rate $\hat{i}_t$, as suggested in (3.16), has to be modified towards an *adjusted interest rate* e.g.

$$i_t = (\hat{i}_t)^+. \tag{3.17}$$

For instance, in the case of a negative optimal interest rate $\tilde{i}_t < 0$ the wealth dynamics cannot be optimal since the interest rate can then only rest at zero. As a consequence, the evolution of wealth will not be optimal when this happens. The monetary authority should avoid market scenarios that lead to negative optimal interest rates. The expected utility from consumption of real wealth will be less than optimal as a consequence of such a market failure, which is rather typical in history during the time of recession after an extreme asset bubble.

In market situations when negative optimal interest rates arise, the expected return of the risky asset is by (3.16) too low when compared to its squared volatility for given risk aversion and fraction invested in the risky asset. Note that also an increase in debt, that is the fraction held in the risky asset index, can cause negative optimal interest rates. An example where in history zero interest rates have occurred is the Japanese stagnation in the 1990s until recently. This market situation emerged after the burst of the Japanese equity and housing bubble in the 1980s, see Svensson (1999). The period during the great depression in the 1930s after the 1929 market crash provides another typical example. Finally, the current financial crisis with virtually zero interest rates in the U.S. gives another example. This crisis was triggered by the burst of the U.S. housing bubble and led via a subsequent credit crunch to an equity market crash with a severe downturn in real economic activity.

Prior to all these crises, during the build-up of a preceding asset bubble, the expected returns were high, often enhanced by easily available credit. Such cheap credit increased substantially over time the fraction invested in the risky assets. Due to the leverage effect, where asset volatility is relatively small when asset prices are relatively high, the volatilities became rather low at the peak of the mentioned bubbles.

Due to the relatively high Sharp ratio during the build-up period of an asset bubble, the investors feel usually safe, consume much and expect similar high returns also in the future. Since risky assets become overvalued during a bubble, any small market correction can make the market aware of the potential magnitude of the looming necessary correction. Buyers may become rare, liquidity may shrink and lenders may not provide any further credit to highly leveraged investors. This may trigger a downward spiral for asset prices, resulting in a market crash. Consequently, the fraction invested in risky assets shrinks. In particular, when the real economy is slowing down as a result of a market correction the situation can become serious. In that case the expected return is low and the volatility becomes high, due to the already mentioned leverage effect. In summary, the likelihood that the optimal interest rate becomes negative has dramatically increased. This means that in the period after a market crash the market may be forced into a non-optimal wealth dynamics due to a not optimal zero interest rate policy. The potentially catastrophic consequences for the real economy, in particular employment, can be dramatic. Consumption may be reduced to relatively low volumes due to the loss in asset value even if the consumption rate may not decrease much.

Zero interest rate policy may arise together with a potential credit shortage because of looming defaults in highly leveraged industries. This potentially could delay a necessary increase in expected return for risky assets that could make the optimal interest rate positive again. Only over time, volatility may decrease and government stimulus packages together with "quantitative easing" may help to sufficiently increase expected returns so that the optimal interest rate finally becomes positive. The removal of "bad assets" may help to reduce volatility as part of the equation (3.16). We will see in the next section that positive optimal interest rates are necessary but not sufficient for a successful monetary policy.

3.3 Interest and Inflation Rate Rule

We have not discussed so far the role of inflation in monetary policy when targeting an optimal market dynamic. The inflation comes into consideration when we match the trend in the total wealth process with its optimal trend under power utility, resulting from equation (3.12). When exploiting (3.2) and (3.12) we obtain for the change in the consumption rate the differential equation

$$\begin{aligned} \frac{dc_t}{dt} &= c_t\, e_t \\ &= c_t \left(c_t + \pi_t - i_t - \frac{(a_t - i_t)^2}{2\gamma\, \sigma_t^2} \right) \\ &= c_t \left(\pi_t - \tilde{\pi}_t - \frac{(i_t - \phi_t)^2}{2\gamma\, \sigma_t^2} \right) \end{aligned} \tag{3.18}$$

with *critical inflation rate*

$$\tilde{\pi}_t = a_t - c_t - \frac{\gamma\, \sigma_t^2}{2} \tag{3.19}$$

and *intrinsic interest rate*

$$\phi_t = a_t - \gamma\, \sigma_t^2. \tag{3.20}$$

The sign of the combination of quantities on the right hand side of (3.18) determines the direction in which the consumption rate is moving in a given market situation that aims for an optimal wealth dynamic. We emphasize that c_t denotes the consumption rate, whereas the consumption per unit of time amounts to $c_t W_t$. This means that a dramatic decline in asset prices triggers even for a slowly moving consumption rate a sharp decline in real economic activity. This indicates that asset prices matter in monetary policy.

Note by (3.16) and (3.20) that for $\alpha_t = 1$ the intrinsic and optimal interest rates coincide. In this case all wealth is invested in the productive units of the economy. This is the important case when borrowing and lending offset each other, and we call this the case of *credit clearing* in the economy. If we consider an open economy, credit may not clear and it is possible that α_t may not equal one. For instance, the U.S. economy has built up substantial foreign debt after the 1980s, which yields an α_t greater than one. Other economies have been lending and are on the opposite side with $\alpha_t < 1$, for instance China and Japan.

When applying the optimal interest rate (3.16) in equation (3.18), it follows that

$$\frac{dc_t}{dt} = c_t \left(\pi_t - \tilde{\pi}_t - \frac{\gamma\, \sigma_t^2}{2} (1 - \alpha_t)^2 \right). \tag{3.21}$$

Consequently, if one assumes a stable consumption rate, which appears to be reasonable, together with the optimal interest rate, then the inflation rate should be about

$$\pi_t = \tilde{\pi}_t + \frac{\gamma\, \sigma_t^2}{2} (1 - \alpha_t)^2 \tag{3.22}$$

to support an optimal wealth evolution. We call this the *inflation rate rule.* We will show below that (3.22) suggests an inflation rate level for which a monetary policy, based on setting interest rates, may practically work. To see this, we discuss in the following several market situations under different inflation rate regimes:

1. **Subcritical inflation:**
   In this case, (3.22) is violated and the inflation rate is below the critical inflation rate, that is $\pi_t < \tilde{\pi}_t$. Since the consumption rate is always assumed to be above zero, that is $c_t > 0$, it follows from (3.18) under subcritical inflation that the consumption rate decreases, no matter where the interest rate is set. When the interest rate is close to the intrinsic interest rate, the decline is minimized. In this situation there is nothing in our equations that can stop this downward trend in the consumption rate. Due to the resulting decrease in consumption in the real economy the expected return a_t may decrease. A recession and eventually a depression could result if nothing else changes. This is in line with Bernanke, Gertler & Gilchrist (1999) who emphasize that investors or net worth moves procyclically and can magnify economic disturbances. One notes from this discussion that the scenario of

subcritical inflation can be rather dangerous. In such low inflation regime there is no possibility to counteract the decrease in the consumption rate by setting the interest rate at any level. We have seen that no change in the direction of a moving consumption rate can be achieved if the inflation rate is kept subcritical. This is an important insight which questions a rigid policy of low inflation rate targeting. Without enough reasoning for the choice of the targeted inflation rate level such monetary policy can have catastrophic effects.

2. **Supercritical inflation:**
   Let us now study the case when the inflation rate is supercritical, $\pi_t > \tilde{\pi}_t$. We discuss first the case without sufficient credit clearing, which means that α_t is significantly above or below one:

   (a) **Supercritical inflation without sufficient credit clearing:**
   In the following supercritical inflation rate regime the monetary authority implements the optimal interest rate (3.16) but does not have sufficient credit clearing such that it still remains

   $$\pi_t - \tilde{\pi}_t < \frac{\gamma\,\sigma_t^2}{2}\,(1-\alpha_t)^2, \tag{3.23}$$

   which means that the inflation rate is not optimal and lower than in (3.22). This yields by (3.21) and (3.23) a decreasing consumption rate. This downward trend in the consumption rate is not as strong as in the case under subcritical inflation, but it cannot be stopped, unless the inequality (3.23) is reversed. Such reversal can be achieved under the optimal interest rate (3.16) by making the inflation rate sufficiently high or balancing sufficiently the credit market. A complete balance between borrowing and lending for the economy, that is credit clearing with $\alpha_t = 1$, will always safely increase the consumption rate under supercritical inflation. However, increased borrowing for $\alpha_t > 1$ under lower than optimal inflation may further decrease the consumption rate and, thus, lead to a downturn in the real economy.

   (b) **Supercritical inflation and approximating the optimal interest rate:**
   Let us now relax the request on implementing exactly the optimal interest rate (3.16). Instead let the monetary authority deliberately deviate from it by setting the interest rate slightly above or below the optimal interest rate so that the consumption rate moves in one or the other desired direction. The aim is to identify a control mechanism that keeps the consumption rate almost constant.

   Under supercritical inflation we know from (3.18) that in the case

   $$(i_t - \phi_t)^2 \leq 2\gamma\,\sigma_t^2(\pi_t - \tilde{\pi}_t) \tag{3.24}$$

the consumption rate is increasing. It will not stop to increase if nothing else changes and the market may then overheat. However, by increasing the interest rate i_t above the level

$$\bar{i}_t = \phi_t + \sqrt{2\,\gamma\,\sigma_t^2(\pi_t - \tilde{\pi}_t)} \tag{3.25}$$

the monetary authority can stop the increase of the consumption rate. If it moves the interest rate back to a level below $\bar{i}_t$, then this increases again the consumption rate. This property of an approximately optimal wealth evolution provides a convenient mechanism for controlling the consumption rate via the setting of the interest rate. It is essential that the inflation rate is supercritical to make this control mechanism working. It represents the core feature that a successful monetary policy can sustainable exploit under normal circumstances.

When for $\alpha_t < 1$ the inflation rate π_t and risky fraction α_t are such that the inflation rate rule (3.22) holds, then the interest rate $\bar{i}_t$ becomes the optimal interest rate, see (3.16). In this case the *inflation gap* $\pi_t - \tilde{\pi}_t$ has by (3.22) the magnitude

$$\pi_t - \tilde{\pi}_t = \frac{\gamma}{2}\,\sigma_t^2\,(\alpha_t - 1)^2. \tag{3.26}$$

Note that for the case of credit clearing one obtains the minimal possible inflation rate, which is the critical inflation rate given in (3.19). In this case the total wealth dynamic is optimal.

For $\alpha_t > 1$ the interest rate $\bar{i}_t$ is not so close to the optimal one and the monetary authority should keep the inflation gap rather small because of the otherwise even less optimal wealth dynamics. The control mechanism still works similar as for the case $\alpha_t < 1$. We emphasize that in this case the total wealth evolution is not optimal.

Taking the nonnegativity of nominal interest rates into account leads by summarizing the above findings to the following *interest rate rule*, where the interest rate

$$i_t = (\bar{i}_t)^+ = \big(a_t - \gamma_t\,\sigma_t^2\,(1 - |1 - \alpha_t|)\big)^+ \tag{3.27}$$

should be targeted by the monetary.

In summary, to implement a successful monetary policy, the monetary authority has to create an economic environment where the optimal interest rate $\tilde{i}_t$ is safely located above zero. This means one needs to have

$$a_t > \gamma\,\alpha_t\,\sigma_t^2. \tag{3.28}$$

For this to happen, the expected return a_t of the risky asset index has to be large enough. Extremely high risk aversion but also excessive foreign debt can reverse this relationship. Governmental stimulus packages can increase expected returns, but increase also the fraction α_t in the risky asset due to additional debt. Fortunately, a substantial increase in debt for stimulating the economy in the case $\alpha_t > 1$ does not worsen the problem of negativity for the optimal interest rate. A calming down of the volatility of the risky asset index leads to a higher optimal interest rate. But this is not easily controllable by the monetary authority and happens only over longer periods of time. Removing "bad assets" from the market can, however, reduce the leverage and, thus, the risk that some institutions have and can bring the asset index volatility down. We emphasize again that it is the responsibility of the monetary authority to avoid the trap of negative optimal interest in the first place. This means, the monetary authority should act responsibly with a long term view by letting market corrections happen when they are due. Stimulating the development of asset bubbles by allowing cheap credit, which finally may lead to potentially extreme market corrections, is short sited.

Since politicians have rather short time horizons due to the election cycle, they should not be allowed to interfere in any way with monetary policy. Exceptions may arise in times of a crisis where the government should help to bring the economy back on track so that monetary policy can work again. It is irresponsible to overstimulate an economy for political or other reasons. It is particularly dangerous to let credit become too easily available. Cheap credit first pushes risky asset prices up and after the necessary market correction it makes it more likely that negative optimal interest rates occur and the above described economic trap emerges.

To make our findings even clearer we summarize below in other words how monetary policy should theoretically function:
When a positive optimal interest rate is secured for the economy, then the inflation rate has to be at least on its critical level $\tilde{\pi}_t$, shown in (3.19), but preferably on its optimal level, as described in the inflation rate rule (3.22). This means for $\alpha_t < 1$ by (3.22) and (3.19) that the monetary authority should target the inflation rate level

$$\pi_t = a_t - c_t + \frac{\gamma \, \sigma_t^2}{2} \, \alpha_t \, (\alpha_t - 1). \tag{3.29}$$

This could mean that after a crash there may be a deflationary period, in particular, for the case when α_t is located between zero and one, and relatively large volatility and risk aversion are present. In the case when α_t is greater than one a deflation is not so likely. The recent Japanese deflation is a typical example for a deflation when $\alpha_t < 1$. The US economy with its substantial debt, that is $\alpha_t > 1$, is not similarly likely to experience a deflation.

In the case when the inflation rate is on its target level (3.29) according to the inflation rate rule and the optimal interest rate is positive, then for $\alpha_t \leq 1$ the monetary authority can steer the economy in a sustainable optimal mode by controlling the consumption rate in the following way: If the consumption rate is considered to be too high for the economy, then the interest rate should be set slightly above the optimal interest rate

$$\tilde{i}_t = a_t - \gamma\, \alpha_t\, \sigma_t^2.$$

To have a strong impact on the market, the interest rate could be set to a level as high as the monetary authority wishes. This is also intuitive and applied in monetary policy.

In the case when the consumption rate is considered to be too low for the economy, the monetary authority should reduce the interest rate to a level below the optimal interest rate. For $\alpha_t < 1$, the interest rate level providing the strongest impact on an upward trend for the consumption rate is the intrinsic interest rate level

$$\phi_t = a_t - \gamma\, \sigma_t^2,$$

see (3.20).

A more subtle case arises in the situation when $\alpha_t > 1$, that is, when the fraction invested in the risky asset index is above one. To decrease here the consumption rate, the interest rate has to be set above $\bar{i}_t$, see (3.25). For increasing the consumption rate, i_t should be set below this rate, for instance, at the optimal interest rate $\tilde{i}_t$, see (3.16). The resulting wealth dynamic under such monetary policy is not as close to the optimal one as was the case for $\alpha_t < 1$. To minimize for $\alpha_t > 1$ the effect of non-optimality the monetary authority could target an inflation rate slightly above the critical inflation rate $\tilde{\pi}_t$, see (3.19). However, this is a dangerous ride, which could easily lead into an economy with subcritical inflation creating either an uncontrolled boom or bust, as discussed previously.

Under the above monetary policy rules, it follows that the interest rate has to be somehow close to the optimal interest rate $\tilde{i}_t$. This needs to be complemented by an inflation policy that targets an inflation rate slightly above the critical inflation rate $\tilde{\pi}_t$. If the economy is in disarray and such monetary policy is currently out of reach, then the economy has to be brought into a regime where the above described monetary policies can be applied again. Unfortunately, the above relationships do not suggest that there exists an automatically stabilizing mechanism that brings the economy by itself back on track. Most important seems to be the insight that it is necessary to allow inflation to increase to a level where monetary policy works. This questions the low inflation targeting policies that have been supported by many politicians and economists.

4 Evaluation of Current Monetary Policy

The currently popular monetary policy rule, the Taylor rule, is supposed to respond to the output and inflation gap, which are in our notation given by $a_t - \bar{a}$ and $\pi_t - \bar{\pi}$, respectively. Here $\bar{a}$ is the long run average expected return and $\bar{\pi}$

the long run average inflation rate. This may work reasonably well in normal circumstances, since it reflects roughly what our findings suggest in the case $\alpha_t > 1$ and $\tilde{i}_t > 0$. However, this may not be adequate for extreme situations as in a financial crisis or deflationary phase. When compared with the Taylor rule the interest rate rule (3.16) and the inflation rate rule (3.22) are clearly different in detail. Current monetary policy focuses primarily on holding the inflation rate low. For example, it has been set at a fixed target level of about 2% in several countries. Holding the inflation rate low is still reasonable as long as the inflation rate is above the critical inflation rate $\tilde{\pi}_t$ given in (3.19). Holding it dogmatically at a fixed 2% level may eventually bring it below the critical inflation rate and can then be extremely damaging to the economy. As discussed above, the consumption rate will then move uncontrolled in one direction and drives the economy either into a recession or into an asset bubble, depending on the course it already has.

The interest rate rule (3.27) links the productivity of the economy, reflected by the expected return a_t, with the risk aversion γ, the fraction α_t of wealth invested in the risky asset index, the market risk, which is expressed via the squared volatility σ_t^2, and the fixed income market governed by the interest rate itself. Under this rule and normal economic circumstances, higher expected returns trigger higher interest rates, and higher squared market volatility results in lower interest rates. These statements are in line with common perception where monetary policy is expected to move interest rates when one of these quantities changes. The interest rate rule (3.27) puts this perception into a clear quantitative relationship. Due to the quadratic dependence of the consumption rate on the interest rate, the formula is nonlinear and quite subtle.

The fraction α_t of wealth invested in the risky asset index plays in the interest rate rule (3.27) an important role which has been probably not much perceived among central bankers and economists. An economy that has foreign debt, that is $\alpha_t > 1$, is more susceptible with its interest rate policy to the volatility of its equity market than is one which runs a surplus. We saw in our previous discussion in Section 3 that for an economy with foreign debt it is more difficult to create an approximately optimal wealth evolution by setting the interest rate than it is for an economy with a surplus. On the other hand, an economy with $\alpha_t < 1$ faces for certain situations the danger of a deflation.

From the interest rate rule it follows that a combination of low expected returns a_t with high volatility σ_t, high risk aversion γ and high debt α_t can bring the interest rate to zero. This then makes the economy uncontrollable via interest rate policy. The nominal interest rate can in this case only rest at zero and the economy drifts on its own course. Investors react to signals other than the level of the interest rate. Consequently, the monetary authority has lost control, which could result in a massive meltdown of the real economy due to an almost unstoppable decrease in consumption. If the monetary authority and government do not gain control again, only after such catastrophic scenario has run its course, the market and the economy can pragmatically start from scratch. This means, if asset values are destroyed in relative economic terms and vulnerable leveraged companies defaulted, then expected returns will eventually come back to positive territory and volatilities may calm down. The inflation rate could pick up sufficiently so that the interest rate finally matters again and monetary policy starts to work.

The above analysis explains that the monetary authority always has to have the long term prosperity of an economy in mind and needs to avoid an economic trap of zero interest rate in the first place. Our results give the monetary authority some guidance how to avoid the indicated undesirable scenarios and how eventually to come out of such an economic trap.

One notes from the inflation rate rule (3.29) that the inflation rate is minimal in the case when the credit market clears, that is $\alpha_t = 1$. This represents an economy with balanced lending and borrowing. Consequently, a minimal inflation rate policy and a functioning monetary control of the economy via the interest rate favor the situation where all financial wealth is invested in the risky assets, and the fixed income market clears. In this way the economy uses all of its own wealth to support maximally its long term sustainable growth. A reliance on credit, that is $\alpha_t > 1$, for building economic growth increases in the long run the inflation rate needed to obtain a sustainable optimal wealth development. On the other hand, for $\alpha_t < 1$ inflation is lower, but deflation may emerge in certain situations. We may add that the interest rate rule (3.27) shows that the suggested interest rate level is highest for $\alpha_t = 1$. This means that an economy that clears its credit has the best chance to avoid the trap of negative optimal interest when following the described policies.

Finally, we remark that for the case of power utility with constant consumption rate $c_t = c$ under minimal inflation targeting and credit clearing one can show that the resulting utility of real capital consumption turns out to be trendless. That is, one has a martingale and, thus,

$$U\left(\frac{c\,W_t}{I_t}\right) = E_t\left(U\left(\frac{c\,W_s}{I_s}\right)\right)$$

for all $0 \leq t < s < \infty$. This is a reasonable relationship, because a fair monetary policy should provide similar expected utility from real consumption per unit of time also for future generations without any upward or downward trend. Under the prescribed objective there is no postponing of real consumption into the future or any consumption at the cost of some future generation if the monetary authority acts according to the derived interest rate rule and avoids extreme scenarios like the build up of significant risky asset bubbles with subsequent deep recessions.

It follows by relation (3.4) that the accumulated wealth $W_t G_t$, when expressed in units of the benchmark S_t^*, is trendless and thus a martingale with

$$\frac{W_t\,G_t}{S_t^*} = E_t\left(\frac{W_s\,G_s}{S_s^*}\right).$$

Since S_t^* is the best performing portfolio this guarantees that the utility we aimed to maximize is achieved in the least expensive manner. This is an important fact since more expensive strategies are possible under the benchmark approach, see Platen & Heath (2006). We emphasize that a wide range of market models can be employed under the provided framework. Furthermore, we have focussed here only on general relationships of the key quantities involved. When using particular models for the market dynamics additional insights can be expected.

Finally, we remark that if one is solving the optimization problem of Section 3 directly by calculating expected utility from terminal wealth for the case of power utility and assuming a Black-Scholes model for the risky asset index, then one obtains for this special case by classical dynamic programming, see Campbell & Viceira (2002) (Ch. 5.1.), the same formulae as derived in Section 3.1. We emphasize that the above interest rate and inflation rate rules hold for very general market dynamics and utility functions and are in this sense universal.

5 Some Empirical Facts

It makes sense to discuss the theoretically derived interest rate rule by looking at long term historical data. Since our results are independent of the particular market dynamics, we can leave substantial freedom for the modeling of the quantities involved.

Let us assume that the processes for the various rates and volatilities, this means the expected return, interest rate, inflation rate, asset volatility and consumption rate, are in a stationary mode, which shall reflect an equilibrium dynamic of the market. In this case one can estimate the relevant quantities. At least for the long term average and when taking an average over markets, the proposed equations have to be reasonably reflected. In Table 1 we show long run estimates for the nominal equity index return a, the nominal T-bill return i, the inflation rate π, the volatility σ, together with some dividend rate d taken for sixteen countries over the last century by Dimson, Marsh & Staunton (2002).

The long run estimate for the market price of risk $\theta \approx \frac{a-i}{\sigma}$, see (3.2), is calculated in the table. It reaches on average 23.6%, which is not far from a typical equity index volatility, which shows an average of 22.7%.

Of particular interest is the product $\gamma\alpha$ of the risk aversion parameter and the fraction held in the risky asset. It follows by (3.14) that

$$\gamma\,\alpha \approx \frac{a-i}{\sigma^2} = \frac{\theta}{\sigma},$$

which turns out to be on average at a level of about one for the sixteen economies. The risk aversion times the fraction invested in risky assets, $\gamma\alpha$, seems to be higher for Anglo-Saxon economies, which use a well developed equity market

Country	a	i	π	σ	d	θ	$\gamma\alpha$	c	π_{min}	$\pi - \pi_{min}$
Australia	0.119	0.045	0.041	0.177	0.048	0.418	2.362	0.091	0.034	0.007
Belgium	0.082	0.052	0.055	0.228	0.028	0.132	0.577	0.0056	0.039	0.016
Canada	0.097	0.049	0.031	0.168	0.041	0.286	1.701	0.059	0.032	-0.001
Denmark	0.089	0.07	0.041	0.201	0.044	0.095	0.47	0.0334	0.035	0.006
France	0.121	0.043	0.079	0.231	0.036	0.338	1.462	0.021	0.046	0.033
Germany	0.097	0.046	0.051	0.323	0.036	0.158	0.489	0.0075	0.035	0.016
Ireland	0.095	0.058	0.045	0.222	0.033	0.167	0.751	0.027	0.044	0.001
Italy	0.12	0.047	0.091	0.294	0.038	0.248	0.845	-0.013	0.045	0.046
Japan	0.125	0.054	0.076	0.303	0.057	0.234	0.773	0.0055	0.033	0.043
Netherlands	0.09	0.037	0.03	0.21	0.049	0.252	1.202	0.039	0.015	0.015
S. Africa	0.12	0.057	0.048	0.228	0.025	0.276	1.212	0.047	0.064	-0.016
Spain	0.1	0.065	0.061	0.22	0.047	0.159	0.723	0.030	0.036	0.025
Sweden	0.116	0.058	0.037	0.228	0.035	0.254	1.116	0.053	0.052	-0.015
Switzerland	0.076	0.033	0.022	0.204	0.029	0.211	1.033	0.033	0.026	-0.004
UK	0.101	0.051	0.041	0.2	0.038	0.25	1.25	0.041	0.038	0.003
US	0.101	0.041	0.032	0.202	0.041	0.297	1.47	0.053	0.030	0.002
Average	0.103	0.05	0.049	0.227	0.039	0.236	1.09	0.033	0.038	0.011

Table 1: Estimates from sixteen markets

for capitalizing their industries and have in most cases some foreign debt, that is $\alpha > 1$. Countries where during the last century the credit system provided primarily capital to industry or which have been lending to other economies, that is $\alpha < 1$, seem to give lower estimates for $\gamma\alpha$.

Since, for instance, the economies of the US, UK, Canada and also Australia have certain debt, setting α_t equal to one is probably not adequate in these cases. For these countries α_t should be higher, which would bring the corresponding estimates for γ closer to one. As long as these countries can serve their debt in the long run, a higher fraction $\alpha > 1$ is not a problem. However, these economies are with their wealth not growing optimally as could be when α and γ were both close to one. In the long run it seems to be possible that a risk aversion parameter not too far from one could be realistic, which opens an interesting line of thought. If one has then also credit clearing, that is, $\alpha \approx 1$, then the total wealth process of the economy has maximum expected growth. As we have discussed earlier, in the case $\alpha \approx 1$ one achieves the minimal possible inflation rate and is furthest away from a zero interest trap.

Let us estimate the long run consumption rate c for each country by assuming, for simplicity $\alpha = 1$. Based on relation (3.18) we have

$$c \approx i - \pi + \frac{(a-i)^2}{2\,\gamma\,\sigma^2}.$$

The resulting average of this value over all countries amounts to 3.3% which appears to be reasonable. Note that this is also close to the magnitude of the average dividend rate d of about 3.9%.

We show in Table 1 also the minimal possible or critical inflation rate $\tilde{\pi} \approx a -$

$c - \frac{\gamma\sigma^2}{2}$, see (3.19), where we again assume $\alpha \approx 1$. The resulting average critical inflation rate of 3.8% is roughly about 1% below the average inflation rate of 4.9% shown in Table 1. This makes sense, since the inflation rate rule and interest rate rule require an inflation rate slightly above the minimal possible rate so that monetary policy can work.

The theoretical relationships that we revealed appear to be rather plausible. Of course, empirical work needs to be undertaken to obtain more precise statements about their validity in a dynamical sense, which represents a challenging direction of research.

6 Conclusions

Monetary policy was challenged by the subprime credit crisis, severe financial disruptions and the meltdown of the financial market. There was panic and sudden shift of the investors‘ investment behavior. This led to a visible shift of monetary policy from inflation targeting to intervention into the financial market and economy. Academics have been in search of a type of monetary policy model that does allow to consider investor's behavior and the disruptive dynamics of the financial market. We suggested here a dynamic portfolio approach, which allows one to study the interaction between investors‘ strategic behavior and monetary policy. We derived policy rules that explain how monetary authorities should set the short term interest rate in interaction with inflation, economic growth, asset prices, risk aversion, market volatility, and consumption rate. We also discussed the dangers of deflation, too low inflation and zero interest rates. As we showed, the inflation rate needs to have a certain minimal level to allow the interest rate to be a viable control instrument. If our proposed monetary policy rules are applied properly, then the consumption rate will remain stable and the inflation rate can be kept at a minimal possible level. As the monetary policy focuses currently under the Taylor rule on the inflation and output gap, it should also have an eye on the necessary minimum level of inflation and the optimal interest rate. Our proposed relationships also indicate how to act in a crisis to bring the economy back on track.

We pursued a nonlinear approach when we spelled out the new inflation rule and interest rate rule in the context of portfolio optimization for investors. As we showed, there can arise important knife edge problems in the interrelation of inflation rate, interest rate, consumption rate, risk aversion, asset price volatility and expected return on the risky assets. Overall, we aimed to advance a better understanding about the impact of monetary policy under extreme financial and economic circumstances but also for an optimal long term wealth evolution.

References

Beaudry, P. & F. Portier (2004). An exploration into Pigou's theory of cycles. *J. Monetary Economy* **51**(6), 1183–1216.

Bernanke, B., M. Gertler, & S. Gilchrist (1999). The financial accelerator in a quantitative business cycle framework. In J. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*, pp. 1341–1393. Amsterdam, North-Holland.

Bernanke, B., V. R. Reinhard, & B. P. Sack (2004). Monetary policy alternatives at the zero bound: An empirical assessment. Federal Reserve Board, Washington, D.C. no 48.

Campbell, J. & L. Viceira (2002). *Strategic Asset Allocation, Portfolio Choice for Long Term Investors*. Oxford, Oxford University Press.

Cecchetti, S., H. Genberg, J. Lipsky, & S. Wadhwani (2000). Asset prices and central bank policy. Geneva Reports on the World Economy 2.

Christiano, L., R. Motto, & M. Rostagno (2005). Monetary policy and a stock market boom-bust cycle. Manuscript, Northwestern University and the European Central Bank.

Cochrane, J. H. (2001). *Asset Pricing*. Princeton University Press.

Coenen, G. & V. Wieland (2004). Exchange-rate policy and the zero bound on nominal interest rates. Working Paper Series 350, European Central Bank.

Cox, J. C. & C. F. Huang (1989). Optimum consumption and portfolio policies when asset prices follow a diffusion process. *J. Economic Theory* **49**, 33–83.

Dimson, E., P. Marsh, & M. Staunton (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*. Princeton University Press.

Dupor, B. I. (2005). Stabilizing non-fundamental asset price movements under discretion and limited information. *J. Monetary Economy* **52**, 727–747.

Eggertsson, G. B. & M. Woodford (2003). Optimal monetary policy in a liquidity trap. NBER Working Papers 9968, National Bureau of Economic Research.

Frankel, J. (1995). *Financial Market and Monetary Policy*. Cambridge: MIT Press.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica* **41**, 867–888.

Merton, R. C. (1992). *Continuous-Time Finance*. Blackwell, Oxford.

Platen, E. (1999). A short term interest rate model. *Finance Stoch.* **3**(2), 215–225.

Platen, E. & D. Heath (2006). *A Benchmark Approach to Quantitative Finance*. Springer Finance. Springer.

Svensson, L. E. O. (1997). Inflation forecast targeting: Implementation and monitoring inflation targets. *European Economic Review* **41**, 1111–1146.

Svensson, L. E. O. (1999). Inflation targeting: Some extensions. Stockholm School of Economics, manuscript.

Taylor, J. B. (1993). Discretion versus policyrules in practice. *Carnegie-Rochester Conference Series on Public Policy* **39**, 195–214.

Tobin, J. (1969). A general equilibirum approach to monetary theory. *J. of Money, Credit and Banking* **1**(1), 15–29.

Tobin, J. (1980). *Asset Accumulation and Economic Activity*. Oxford: Basic Blackwell.

Woodford, M. (2003). *Interest and Prices: Foundation of a Theory of Monetary Policy*. NJ, Princeton University Press.