THE CALIBRATION OF STOCK OPTION PRICING MODELS USING INVERSE PROBLEM METHODOLOGY

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ABSTRACT. We analyse the procedure for determining volatility presented by Lagnado and Osher, and explain in some detail where the scheme comes from. We present an alternative scheme which avoids some of the technical complications arising in Lagnado and Osher's approach. An algorithm for solving the resulting equations is given, along with a selection of numerical examples.

1. INTRODUCTION

Arbitrage free pricing of a derivative security begins with the specification of a stochastic process describing the dynamics of the underlying asset price. For many options, such as stock options, the most commonly used model is that of Black and Scholes. We assume that the asset price S, evolves according to a Stochastic Differential Equation (SDE) of the form

$$dS = \mu S dt + \sigma S dW_t \tag{1.1}$$

where W_t is a Wiener process, μ is the drift and σ is the volatility.

Once the model has been chosen it must be calibrated to the market. That is, the numerical values of the model's parameters must be determined. The price of an option written on the asset is given by the Black-Scholes equation

$$\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{1.2}$$

where the only parameter not freely observable in the market is the volatility σ. Here r is the constant rate of interest and D is the continuously compounded dividend on the underlying asset. A common approach to determining σ is to observe the market prices for a range of options and, using the price provided by the Black-Scholes formula, calculate an implied volatility. However it is known that the standard Black-Scholes model does not generate the observed option prices for all possible strike prices and expiry dates for any single value of the volatility.

Recently a number of authors Dupire [Dup94], Derman and Kani [DK94], Rubinstein [Rub94] and Lagnado and Osher [LO97] have attempted to deal with this problem for a class of index options. They assume that the index level S follows a

diffusion process, governed by the SDE

$$dS = \mu S dt + \sigma(S,t)S dW_t \qquad (1.3)$$

σ is a strictly deterministic function, which may depend explicitly upon the index S as well as time t. It is the approach of Lagnado and Osher which we are concerned with in this article. We will first demonstrate how their calibration procedure arises from the theory of inverse problems via Tikhonov regularisation. Then we will develop an alternative strategy which avoids some technical implementation problems. Finally we will present a relatively fast algorithm for solving the partial differential equations which determine the volatility function and give some numerical examples.

We should note here that an alternative approach to determining the volatility function when σ is independent of time has recently been presented by Bouchouev. In this procedure the volatility is determined by solving a non linear integral equation. [Bou97]. A detailed discussion of this technique, which relies upon the construction of an approximation to the fundamental solution of a generalised Black-Scholes equation, may be found in [CCEH99]. An excellent general survey article on the state of the art for inverse problems arising in finance is the paper by Bouchouev and Isakov [BI99].

In the current work we assume that the market prices for a series of options with expiration dates $T_1, T_2, \ldots T_N$ are known. We also assume that for each expiration date T_i, options with strike prices $K_{i1}, K_{i2}, \ldots K_{iM_i}$ are traded. Given an option with expiry date T_i and strike K_{ij} we let the bid and ask prices for the option be denoted V_{ij}^b and V_{ij}^a respectively.

By use of the no arbitrage pricing process for an option written on an index following (1.3) we find that the price V of the option is given by

$$\frac{\partial V}{\partial t} + (r-D)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} = rV. \qquad (1.4)$$

Once we specify the functional form of σ in (1.4) together with appropriate final and boundary conditions, we can determine the theoretical price $V(S,t,K,T,\sigma)$ of an option on a process following (1.3). Market calibration then requires us to find a local volatility function such that the predicted value V falls between the corresponding bid ask spreads. In other words

$$V_{ij}^b \leq V(S_0, 0, K_{ij}, T_i, \sigma) \leq V_{ij}^a \qquad (1.5)$$

To satisfy (1.5) we try to minimise the functional

$$G(\sigma) = \sum_{i=1}^{N}\sum_{j=1}^{M_i}(V(S_0, 0, K_{ij}, T_i, \sigma) - V_{ij})^2 \qquad (1.6)$$

where $V_{ij} = \frac{1}{2}(V_{ij}^a + V_{ij}^b)$, with respect to σ.

Unfortunately however there is simply not enough market data available to uniquely determine the value of σ which minimises (1.6). Moreover the minimum value of a functional like (1.6) typically does not depend continuously on the data. Consequently the problem of determining σ in this way is *ill posed*. Ill posed problems are often dealt with using regularisation strategies. Recently, Lagnado and Osher [LO97] proposed a strategy for determining σ which is based upon the well

known Tikhonov regularisation. Before we discuss Lagnado and Osher's methodology we therefore present a brief introduction to some of the ideas and results in the theory of inverse problems.

2. Inverse Problems and Tikhonov Regularisation

Let us review the problem discussed in the introduction. We are interested in determining an unknown function $\sigma(S, t)$ from market data. In other words, given the option pricing PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} = rV \qquad (2.1)$$

and a space Ω containing the market data, determine the coefficient $\sigma(S, t)$. This is an example of an *inverse problem*. The general theory of inverse problems is discussed in Kirsch [Kir96]. For an overview of the relevant material from functional analysis, see Rudin [Rud73].

An inverse problem is simply the converse of a direct problem. For example, determining the zeroes of a polynomial is a direct problem. If instead we are asked to determine which polynomial has a given set of zeroes then we have an inverse problem. At first glance there appears to be some arbitrariness involved in deciding which of these two problems is the direct one and which is the inverse. However there is a fundamental distinction between the two.

To understand this distinction we observe that inverse problems may often be cast in the following form:

Definition 2.1. Let X and Y be linear spaces. The standard form for an inverse problem is as follows: Given $y \in Y$ and $K : X \to Y$, find $x \in X$ such that

$$Kx = y \qquad (2.2)$$

In applications K is frequently a *compact operator*. A compact operator is one which takes bounded sets to compact (i.e. closed and bounded) sets. Compact operators are as "nice" as one can reasonably hope a linear operator on an infinite dimensional space to be, duplicating many of the properties of linear operators (i.e. matrices) on finite dimensional vector spaces.

It is very important to note, for reasons discussed below, that in a real application we are unlikely to know y exactly. Rather, we will have information about it to some degree of precision. More precisely, if $y \in Y$, where Y is a normed linear space with norm $\| \cdot \|$, then we have an element $y^\epsilon \in Y$ with $\|y - y^\epsilon\| < \epsilon$. That is, we know y to within ϵ of its true value.

This fact has important consequences for the problem of solving (2.2). Inverse problems are typically *ill posed*. To understand what an ill posed problem is we need to know the definition of a *well posed* problem. (Historically the definition is due to Hadamard).

Definition 2.2 (well posedness)**.** Problem (2.2) is said to be well posed if
1. For every $y \in Y$, $\exists x \in X$ such that $Kx = y$
2. For every $y \in Y$ $\exists$ at most one $x \in X$ such that $Kx = y$, and
3. The solution x depends continuously on the data y

A problem is said to be ill posed if it is not well posed. Most inverse problems are ill posed. It is this fact which distinguishes the direct from the inverse problem. For the case where K is a compact operator, the inverse problem is *always* ill posed.

This is the content of the next result which we have taken from Kirsch ([Kir96] p13).

Theorem 2.3. *Let X, Y be normed spaces and $K : X \to Y$ be a compact linear operator with nullspace $\mathcal{N}(K) := \{x \in X : Kx = 0\}$. Let the dimension of $X/\mathcal{N}(K)$ be infinite. Then there exists a sequence $(x_n) \subset X$, such that $Kx_n \to 0$ but (x_n) does not converge. We can even choose (x_n) such that $\|x_n\| \to \infty$. In particular if K is one to one, the inverse $K^{-1} : Y \supset K(X) \to X$ is unbounded.*

This result tells us that inverse problems involving compact operators on infinite dimensional spaces are always ill posed. In particular part 3 of definition 2.2 fails.

The fact that the solution fails to depend continuously upon the data is the source of many of the difficulties inherent in solving an inverse problem. As mentioned above, typically we do not know the value of y in (2.2), rather we know some approximation to it, y^ϵ. Since the inverse problem is ill posed, even assuming that we are able to obtain a solution x^ϵ to the approximating problem

$$Kx^\epsilon = y^\epsilon, \tag{2.3}$$

there is no guarantee that the approximating solution x^ϵ is anywhere near the true solution to problem (2.2), since the solution does not depend continuously upon y. Moreover there is no a priori guarantee that the data y^ϵ is in the range of K!

One tries to ameliorate this problem by introducing a *regularisation strategy*.

Definition 2.4. A regularisation strategy for the equation $Kx = y$ is a family of linear and bounded operators

$$R_\alpha : Y \to X, \alpha > 0 \text{ such that}$$
$$\lim_{\alpha \to 0} R_\alpha Kx = x, \forall x \in X$$

i.e. the operators $R_\alpha K$ converge *pointwise* to the identity. The real number α is known as the *regularisation parameter*.

The content of this definition is that the operators R_α form a 'sequence' of approximations to the inverse K^{-1} of K. Pointwise convergence of the operators R_α is the best that we can expect. Indeed if the operator is compact, then the operators R_α are not uniformly bounded, the sequence $(R_\alpha Kx)$ does not converge uniformly and the operators $R_\alpha K$ do not converge to the identity in the operator norm topology. (See Theorem 2.2 of Kirsch [Kir96])

The definition of a regularisation strategy assumes that we know the data y for the equation $Kx = y$ exactly. In practice however we have data y^ϵ where $\|y - y^\epsilon\| \le \epsilon$. If we apply the regularisation strategy to the problem with perturbed data we obtain a family of approximate solutions

$$x^{\alpha,\epsilon} = R_\alpha y^\epsilon \tag{2.4}$$

to our real problem $Kx = y$. We can compute the error in the approximation by the triangle inequality.

$$\|x^{\alpha,\epsilon} - x\| \le \|R_\alpha y^\epsilon - R_\alpha y\| + \|R_\alpha y - x\| \tag{2.5}$$

$$\le \|R_\alpha\|\|y^\epsilon - y\| + \|R_\alpha Kx - x\| \tag{2.6}$$

which implies that

$$\|x^{\alpha,\epsilon} - x\| \le \epsilon\|R_\alpha\| + \|R_\alpha Kx - x\| \tag{2.7}$$

The inequality (2.7) shows that the error between the true solution and the approximate solution obtained from the regularisation strategy depends upon both

the error from the regularisation strategy applied to the real data y and the error in the data ϵ multiplied by the norm $\|R_\alpha\|$ (this norm is called the *condition number* for the regularisation strategy). We choose a regularisation strategy so as to minimise the right hand side of (2.7).

The philosophy behind this strategy is one of pragmatism. We cannot hope to determine the exact solution to our problem because of its very nature. Consequently we attempt to find an approximation which is as close to the true solution as possible. To achieve this we are moved to replace our problem with one which is "close" to the original, but does not possess the ill conditioning which the makes the original intractable. In other words: Don't try and solve the given problem, try and solve a different one whose solution is close to that of your problem. This is the essence of all regularisation methods.

There are many approaches to constructing regularisation strategies. Probably the best known is due to Tikhonov. (See [Tik63] for the original discussion). The origin of the so called *Tikhonov regularisation* scheme lies with the problem of dealing with an over determined linear system $Kx = y$, where K is a linear operator on a finite dimensional vector space. i.e. a matrix. Typically one tries to minimise the defect $\|Kx - y\|$ with respect to $x \in X$ for some norm on Y. We might try the same strategy on an infinite dimensional space, however if the operator K is compact then the problem is ill posed (see Lemma 2.10 of Kirsch [ibid]).

Tikhonov proposed the following alternative. Given the problem of solving (2.2) where $K : X \to Y$ is a bounded linear operator, we instead determine the minimum value of the *Tikhonov Functional*

$$J_\alpha(x) = \|Kx - y\|^2 + \alpha\|x\|^2, \text{ for } x \in X \tag{2.8}$$

Tikhonov proved the following result. See Kirsch, Theorem 2.11 for a straightforward proof. The positive real number α is known as the *regularisation parameter*.

Theorem 2.5. *Let* $K : X \to Y$ *be a linear and bounded operator between Hilbert spaces and let* $\alpha > 0$. *Then the functional (2.8) has a unique minimum* $x^\alpha \in X$, *which is the unique solution of the* normal *equation*

$$\alpha x^\alpha + K^*Kx^\alpha = K^*y \tag{2.9}$$

Here the operator $K^* : Y \to X$ *is the adjoint of the operator* K. *i.e. the operator such that*

$$(Kx, y)_Y = (x, K^*y)_X \tag{2.10}$$

for all $x \in X, y \in Y$. $(\cdot, \cdot)_X$ *and* $(\cdot, \cdot)_Y$ *are the inner products for the Hilbert spaces* X *and* Y *respectively.*

Given a linear space X and norms $\|\cdot\|$ and $\|\cdot\|_1$ on X we say that $\|\cdot\|_1$ is a *stronger norm* than $\|\cdot\|$ if there exists $c > 0$ such that for all $x \in X$ we have

$$\|x\| \leq c\|x\|_1.$$

An important fact is that if we replace the Tikhonov functional (2.8) with

$$J_\alpha^1(x) = \|Kx - y\|^2 + \alpha\|x\|_1^2 \tag{2.11}$$

in which $\|\cdot\|_1$ is a stronger norm defined on some subspace $X_1 \subset X$ then a result similar to theorem 2.5 holds.

With these details in mind we now return to the original problem, namely determining a volatility function $\sigma(S, t)$ from market data. In particular we discuss the regularisation strategy devised by Lagnado and Osher to determine this function.

3. A Regularisation Strategy for Determining Volatility

The aim of section 4 is to develop a regularisation strategy for the determination of asset volatility. Our regularisation strategy is an alternative to the procedure of Lagnado and Osher [LO97]. For this reason we will present here a brief discussion of Lagnado and Osher's technique.

In [LO97] a regularisation strategy based upon minimising the functional

$$F(\sigma) = \|\|\nabla\sigma\|\|_2^2 + \lambda G(\sigma), \tag{3.1}$$

where G is given by (1.6), is developed . Here, ∇ is the usual gradient operator from multivariable calculus.

To understand where this strategy comes from we must recall that the goal is to minimise the functional

$$G(\sigma) = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[V(S_0, 0; K_{ij}, T_i, \sigma) - V_{ij} \right]^2. \tag{3.2}$$

Since this problem is ill-posed, the discussion in section the use of a regularisation strategy. Lagnado and Osher introduce one based upon the Tikhonov functional (2.11). The standard Tikhonov approach, based upon (2.8) would, in this instance, be to minimise the functional

$$\tilde{F}_\alpha(\sigma) = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[V(S_0, 0; K_{ij}, T_i, \sigma) - V_{ij} \right]^2 + \alpha\|\sigma\|^2 \tag{3.3}$$

which is of course equivalent to minimising

$$\lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \left[V(S_0, 0; K_{ij}, T_i, \sigma) - V_{ij} \right]^2 + \|\sigma\|^2 \tag{3.4}$$

Lagnado and Osher have chosen a slightly different regularisation. Heuristically their strategy is easy to understand. Recall that the gradient of a function yields a vector which is normal to the surface described by the function. It therefore gives a vector which points in the direction in which the function value changes most rapidly. Therefore, the strategy involved with (3.1) is to minimise the functional $G(\sigma)$, while simultaneously mimimising the speed at which the function σ moves away from the minimum.

Technically this amounts to minimising in a stronger norm. We may consider a space of twice differentiable functions on $(0, \infty) \times (0, \infty)$ which are square integrable, for which

$$\int_0^\infty \int_0^\infty |\sigma(S, t)|^2 dS dt \leq c \int_0^\infty \int_0^\infty \left((\frac{\partial\sigma}{\partial S})^2 + (\frac{\partial\sigma}{\partial t})^2 \right) dS dt \tag{3.5}$$

for some $c > 0$. Such a set of functions form a *Sobolev space*[1], H^2, which is also a Hilbert space. On H^2 the norm $\|\|\nabla\sigma\|\|_2$ is stronger than the norm $\|\sigma\|_2$. By

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[1] In general, a Sobolev space is a Hilbert space of square integrable functions with certain smoothness conditions imposed. For a detailed discussion of such spaces refer to chapter one of [Wlo87].

the comments immediately following theorem 2.5, the functional (3.1) has a unique minimum $\sigma \in H^2$. This minimum will be taken as the solution to our problem.

Let us rewrite the right hand side of (3.1) as an integral:

$$F(\sigma) = \int_0^\infty \int_0^\infty \left((\frac{\partial \sigma}{\partial S})^2 + (\frac{\partial \sigma}{\partial t})^2 + \lambda \sum_{i=1}^N \sum_{j=1}^{M_i} (V(S,t,K_{ij},T_i,\sigma) - V_{ij})^2 h(S,t)\right) dS dt$$

(3.6)

where $h(S,t) = \delta(S - S_0)\delta(t)$, and $\delta(x - x_0)$ is the Dirac delta function. The problem is now to minimise the functional (3.6). In order to do this we apply the Euler-Lagrange equation from the calculus of variations to

$$L = (\frac{\partial \sigma}{\partial S})^2 + (\frac{\partial \sigma}{\partial t})^2 + \lambda \sum_{i=1}^N \sum_{j=1}^{M_i} (V(S,t,K_{ij},T_i,\sigma) - V_{ij})^2 \delta(S - S_0)\delta(t). \quad (3.7)$$

We should make a technical comment at this point. The application of the Euler-Lagrange equations in this context requires justification. The rationale behind the calculus of variations is to allow the functional we are minimising to vary over some space of suitably chosen functions. However, in order to bring the functional $G(\sigma)$ under the integral sign it was necessary to introduce the Dirac delta function. Consequently we are dealing with a functional which consists of the gradient of a smooth function with a tempered distribution added on to it. Whether the Euler Lagrange equations will yield the minimum for such a functional, or what such a minimum would be is far from clear, as there is no established theory to guide us in such a situation. (For a discussion of the calculus of variations and the conditions under which the Euler-Lagrange equations *are* known to yield a miniumum, see the book by Ewing, [Ewi85]).

Leaving this technical issue to one side, we formally apply the Euler-Lagrange equations, namely

$$\frac{d}{dS}\frac{\partial L}{\partial \sigma_S} + \frac{d}{dt}\frac{\partial L}{\partial \sigma_t} - \frac{\partial L}{\partial \sigma} = 0 \quad (3.8)$$

where σ_S is the partial derivative of σ with respect to S and σ_t is the partial derivative with respect to t. We have

$$\frac{d}{dS}\frac{\partial L}{\partial \sigma_S} = 2\frac{\partial^2 \sigma}{\partial S^2}, \quad \frac{d}{dt}\frac{\partial L}{\partial \sigma_t} = 2\frac{\partial^2 \sigma}{\partial S^2}$$

and

$$\frac{\partial L}{\partial \sigma} = 2\lambda \sum_{i=1}^N \sum_{j=1}^{M_i} (V(S,t,K_{ij},T_i,\sigma) - V_{ij})\frac{\partial V}{\partial \sigma}\delta(S - S_0)\delta(t) \quad (3.9)$$

Recall that the variational derivative of V with respect to σ is

$$\frac{\delta V^{ij}}{\delta \sigma} = \left[\frac{d}{d\epsilon}V(S,t,K_{ij},T_i,\sigma + \epsilon h)\right]_{\epsilon=0} \quad (3.10)$$

The references [Ewi85] and Olver [Olv93] contain discussions on the properties of the variational derivative.

In this case $h(S,t) = \delta(S - S_0)\delta(t)$. We may evaluate (3.10) by the chain rule to obtain

$$\frac{\delta V^{ij}}{\delta \sigma} = \frac{\partial}{\partial \sigma}V(S_0,0,K_{ij},T_i,\sigma)\delta(S - S_0)\delta(t) \quad (3.11)$$

From these calculations we finally obtain the Euler-Lagrange equations for the functional $F(\sigma)$

$$\frac{\partial^2 \sigma}{\partial S^2} + \frac{\partial^2 \sigma}{\partial t^2} - \lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\delta V^{ij}}{\delta \sigma}(S_0, 0, K_{ij}, T_i, \sigma)\left[V(S_0, 0, K_{ij}, T_i, \sigma) - V_{ij}\right] = 0$$

(3.12)

Thus to determine the function $\sigma(S, t)$ we must solve equation (3.12). To do this we require an efficient procedure for the evaluation of the variational derivative (3.11). Lagnado and Osher have produced a technique for computing (3.11), which they employ together with a gradient descent method to solve (3.12).

To determine the variational derivative, we need only compute $\partial V/\partial \sigma$, and this may be done in the following manner. We consider the partial differential operator

$$\mathcal{L}(\sigma) = \frac{\partial}{\partial t} + (r - q)S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(S, t)\frac{\partial^2}{\partial S^2} - r$$

(3.13)

Then if we pick some suitably smooth function h, we may formally differentiate the expression

$$\mathcal{L}(\sigma + \epsilon h)V(S, t, K_{ij}, T_i, \sigma + \epsilon h) = 0$$

(3.14)

with respect to ϵ. We obtain

$$h\frac{\partial}{\partial \sigma}\frac{\partial V}{\partial t} + h(r - q)S\frac{\partial}{\partial \sigma}\frac{\partial V}{\partial S} + \frac{1}{2}h\sigma^2 S^2\frac{\partial}{\partial \sigma}\frac{\partial^2 V}{\partial S^2} + h\sigma S^2\frac{\partial^2 V}{\partial S^2} - rh\frac{\partial V}{\partial \sigma} = 0 \quad (3.15)$$

When we reverse the order of differentiation this leads to the equation

$$\mathcal{L}(\sigma)\frac{\partial V}{\partial \sigma} = -\sigma^2(S, t)S^2\frac{\partial^2}{\partial S^2}V(S, t, K_{ij}, T_i, \sigma)$$

(3.16)

With suitable (homogeneous) initial and boundary conditions imposed, we may solve (3.16) for $\partial V/\partial \sigma$, which upon multiplication by the Dirac distributions, leads to the variational derivative in (3.12).

4. Alternative Regularisation Strategies

Lagnado and Osher's regularisation strategy calibrates the model for one fixed value of the index S_0, at one fixed point in time $t = 0$. There is consequently no guarantee that the value of σ determined by this procedure will be correct either for other values of the index or at future times. Nor is there any guarantee that σ will be everywhere positive.

We now present two alternatives to this strategy. In the first alternative, the aim is to take into account variation of the option price with changing value of the index over time. In other words attempt to minimise

$$H(\sigma) = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_0^\infty \int_0^{T_{cur}} (V(S, t, K_{ij}, T_i, \sigma) - V^{ij})^2 dSdt.$$

(4.1)

Here T_{cur} is the current time. In this case of course V^{ij} will be a set of empirical option prices taken over a range of values of S and t. In other words, we calibrate the volatility to the options's history.

This problem is also ill posed and following Lagnado and Osher we propose the regularisation strategy of minimising the Tikhonov functional

$$J_\lambda(\sigma) = |||\nabla\sigma|||^2 + \lambda H(\sigma).$$

(4.2)

This regularisation strategy has the technical advantage that it contains no terms involving the Dirac delta function. Consequently, the standard results from the calculus of variations apply.

The Euler-Lagrange equations for (4.2) are

$$\frac{\partial^2 \sigma}{\partial S^2} + \frac{\partial^2 \sigma}{\partial t^2} - \lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\partial V}{\partial \sigma}(S, t, K_{ij}, T_i, \sigma)(V(S, t, K_{ij}, T_i, \sigma) - V^{ij}) = 0 \quad (4.3)$$

We need to impose boundary conditions on (4.3). We set $\sigma(0, t) = 0$ and $\sigma(S, 0) = \sigma_0(S)$. $\sigma_0(S)$ is an unknown function of S which we must determine from the data. One possibility is to take the implied volatilities at at a set of points $\{(n\Delta S, 0)\}_{n=0}^{n=N}$ for a particular strike and fit a spline function to those values to obtain an approximation for the lower boundary.

The second, somewhat easier alternative, is to replace the elliptic PDE (4.3) with a second order ODE. In order to do this, we make the simplifying assumption that the volatility function σ is independent of time. Set the current time as $t = 0$. Then the obvious regularisation strategy is to minimise the Tikhonov functional

$$J_\lambda(\sigma) = \left(\frac{\partial \sigma}{\partial S}\right)^2 + \lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \int_0^\infty (V(S, 0, K_{ij}, T_i, \sigma) - V^{ij})^2 dS, \quad (4.4)$$

for which the Euler-Lagrange equations read

$$\frac{\partial^2 \sigma}{\partial S^2} - \lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\partial V}{\partial \sigma}(S, 0, K_{ij}, T_i, \sigma)(V(S, 0, K_{ij}, T_i, \sigma) - V^{ij}) = 0 \quad (4.5)$$

There remains the question of what the appropriate conditions are to impose on this problem to ensure a unique solution? For equation (4.5) the natural problem would seem to be a two point boundary value problem, since the alternative initial value problem requires knowledge of the first derivative of σ at some point. Consequently the problem which we propose is to solve equation (4.5) on the interval $[S_0, S_1]$ with the values of $\sigma(S)$ at the endpoints specified.

To determine the values of $\sigma(S_0)$ and $\sigma(S_1)$ there are various approaches which are possible. The simplest of these is to take the Black-Scholes implied volatility for options of a specified strike at the points $S = S_0$ and $S = S_1$.

5. Solving the Equation

In this section we will deal with the elliptic PDE (4.3). The equation (4.5) may be dealt with by standard numerical procedures for solving second order nonlinear ordinary differential equations. The book by Iserles [Ise96] contains an extensive discussion of numerical procedures for ODEs.

Equation (4.3) contains no terms involving the Dirac delta, consequently it does not have the same technical problems associated with it that the original Lagnado and Osher equation possess. Our purpose in this section is to present a method for solving this equation for $\sigma(S, t)$. In section 6 we present some numerical examples.

Given a set of data V^{ij} we wish to use (4.3) to determine the volatility $\sigma(S, t)$ which minimises the functional (4.2). To do this we solve the PDE (4.3) according to the following iterative procedure.

We replace the region $\{0 \leq S < \infty\} \times \{0 \leq t < \infty\}$ with the finite rectangle $\Omega = [0, S_{max}] \times [0, t_{max}]$. We will construct an approximation for $\sigma(S, t)$ at a set of points $(m\Delta S, n\Delta t)$ on Ω. First we choose a value for the regularisation parameter λ.

Algorithm for solving equation (4.3)

1) Introduce a function $\varphi_0(S, t)$. This will be the initial approximation to the true volatility σ.
2) At each point $(m\Delta S, n\Delta t)$ determine $\varphi_0(m\Delta S, n\Delta t)$.
3) Use the Black-Scholes formula as an approximation for

$$V(m\Delta S, n\Delta t, K_{ij}, T_i, \varphi_0(m\Delta S, n\Delta t))$$

and

$$\frac{\partial V}{\partial \sigma}(m\Delta S, n\Delta t, K_{ij}, T_i, \varphi_0(m\Delta S, n\Delta t))$$

That is, if the Black-Scholes price of a European call is $C(S, t, K, T, \sigma)$, then use

$$C(m\Delta S, n\Delta t, K_{ij}, T_i, \varphi_0(m\Delta S, n\Delta t))$$

as the approximation for V. Similarly, use

$$\frac{\partial C}{\partial \sigma}(m\Delta S, n\Delta t, K_{ij}, T_i, \varphi_0(m\Delta S, n\Delta t))$$

as the approximation for $\frac{\partial V}{\partial \sigma}$. The put case is analogous.
4) Use the values obtained in 3 to approximate

$$W(S, t, K_{ij}, T_i,) = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\partial V}{\partial \sigma}(S, t, K_{ij}, T_i, \sigma)(V(S, t, K_{ij}, T_i, \sigma) - V^{ij})$$

5) Solve the Poisson equation

$$\frac{\partial^2 \sigma}{\partial S^2} + \frac{\partial^2 \sigma}{\partial t^2} = \lambda W(S, t, K_{ij}, T_i),$$

where the value of W was obtained in step 4, subject to the appropriate boundary conditions.
6) Take the function obtained in step 5 and call this φ_1. Repeat the above procedure from step 3, using φ_1 in place of φ_0 to obtain the next approximation. Continue until the difference between successive iterates φ_k and φ_{k+1} is smaller than some desired tolerance.

End of algorithm

In step 3 we employ the Black-Scholes formula to obtain approximate values for V and $\partial V / \partial \sigma$. This greatly improves the speed of implementation. Other methods may be preferred. For example one could employ some analytical approximation to the solution of (1.4) such as that developed by Bouchouev. (See [Bou97] and [BI97] for the details). Lagnado and Osher determine V and $\partial V / \partial \sigma$ by solving the appropriate parabolic PDEs at every iteration, which is computationally very intensive. We have preferred speed of implementation over the additional accuracy provided by solving the PDEs. Fortunately however, the Black-Scholes formula is extremely robust. For many volatility functions φ_0, the actual option value $V(m\Delta S, n\Delta t, K_{ij}, T_i, \varphi_0(m\Delta S, n\Delta t))$ is very close to the price given by the Black-Scholes formula as described in step 3. Consequently, we expect that the loss of accuracy due to the use of the Black-Scholes formula will only be marginal.

6. Numerical Results

To test this procedure we attempted to recover known volatility functions from solutions of the corresponding Black Scholes equation. We first choose a volatility function $\sigma(S, t)$ and use it to generate a set of option prices V^{ij} by solving the Black Scholes equation (1.4). Next we employ the algorithm described in section 4 to see if we can recover the function $\sigma(S, t)$ from the artificial prices V^{ij}.

For our first test, we consider the volatility function $\sigma(S) = 0.2 + 0.1e^{-0.2S}$.

The interest rate $r = 0.05$ We considered only one time to option maturity $T = 10$. We take 11 strikes: $K_1 = 50, K_2 = 60, ...K_{11} = 160$. We set $\Delta S = 1.6$ and $\Delta t = 0.004$.

We present the L^2 norm of the difference between successive iterates φ_k and φ_{k+1} and the maximum difference between the true value of σ and the final approximation over the given domain.

1)Initial estimate: $\varphi_0 = 0.1$ Regularisation parameter: $\lambda = 1$. Process terminated after 6 iterations.

$$\max |\sigma(S) - \varphi_6(S)| = 0.00716727$$
$$\|\varphi_1 - \varphi_0\|_2 = 9.8041089894303202$$
$$\|\varphi_2 - \varphi_1\|_2 = 2.3566551888897506 \times 10^{-2}$$
$$\|\varphi_3 - \varphi_2\|_2 = 6.4383818723267293 \times 10^{-4}$$
$$\|\varphi_4 - \varphi_3\|_2 = 2.0466438806148549 \times 10^{-5}$$
$$\|\varphi_5 - \varphi_6\|_2 = 7.1647139139197532 \times 10^{-7}$$
$$\|\varphi_6 - \varphi_5\|_2 = 0.264708 \times 10^{-7}$$

2) Initial estimate: $\varphi_0 = 0.2$ Regularisation parameter: $\lambda = 1$. Process terminated after 4 iterations.

$$\max |\sigma(S) - \varphi_4(S)| = 0.00715878$$
$$\|\varphi_1 - \varphi_0\|_2 = 0.103717 4077597777$$
$$\|\varphi_2 - \varphi_1\|_2 = 3.0325555794476101 \times 10^{-5}$$
$$\|\varphi_3 - \varphi_2\|_2 = 6.3646217735724078 \times 10^{-7}$$
$$\|\varphi_4 - \varphi_3\|_2 = 0.168184 \times 10^{-7}$$

3) Initial estimate: $\varphi_0 = 0.3$ Regularisation parameter:$\lambda = 1$ Process terminated after 6 iterations.

$$\max |\sigma(S) - \varphi_6(S)| = 0.00715471$$
$$\|\varphi_1 - \varphi_0\|_2 = 9.5881790823710862$$
$$\|\varphi_2 - \varphi_1\|_2 = 7.3680277273244360 \times 10^{-2}$$
$$\|\varphi_3 - \varphi_2\|_2 = 1.3635998687818252 \times 10^{-3}$$
$$\|\varphi_4 - \varphi_3\|_2 = 3.8256836564473186 \times^{-5}$$
$$\|\varphi_5 - \varphi_4\|_2 = 1.2287543708140822 \times 10^{-6}$$
$$\|\varphi_6 - \varphi_5\|_2 = 0.430032 \times 10^{-7}$$

4) Initial estimate: $\varphi_0 = 0.1$. Regularisation parameter: $\lambda = 0.10$. Process terminated after 3 iterations.

$$\max |\sigma(S) - \varphi_3(S)| = 0.00769063$$
$$\|\varphi_1 - \varphi_0\|_2 = 9.5179392683130626$$

$$\|\varphi_2 - \varphi_1\|_2 = 1.9029430309230468 \times 10^{-4}$$

$$\|\varphi_3 - \varphi_2\|_2 = 0.501462 \times 10^{-7}$$

5) Initial estimate: $\varphi_0 = 0.1$ Regularisation parameter: $\lambda = 0.20$ Process terminated after 4 iterations.

$$\max |\sigma(S) - \varphi_4(S)| = 0.00152660$$

$$\|\varphi_1 - \varphi_0\|_2 = 9.5484992994518212$$

$$\|\varphi_2 - \varphi_1\|_2 = 7.8009681711020294 \times 10^{-4}$$

$$\|\varphi_3 - \varphi_2\|_2 = 8.2606991761690897 \times 10^{-7}$$

$$\|\varphi_4 - \varphi_3\|_2 = 0.104276 \times 10^{-8}$$

Clearly the results for this simple example are good. The true volatility function varies between 0.2 and 0.3. Thus 6 iterations was usually sufficient to recover the true volatility function correct to within 4% of the true value. The procedure was also quite fast. However we would expect that with more complicated volatility functions and more expiry times the complexity, and hence computation times will increase.

Next we consider the volatility function $\sigma(S) = 0.6(1 - e^{-0.03S})$. We take one maturity date, $T = 10/252$, let $0 \leq S \leq 160$ and pick eleven strikes: $K_1 = 50, K_2 = 60, \ldots K_{11} = 160$. The constant rate of interest is taken to be 0.05. The regularisation parameter is $\lambda = 0.3$.

The first problem is to determine the boundary conditions. We set $\sigma(0, t) = 0$ and $\sigma(160, t) = 0.5951 \approx 0.6(1 - e^{-4.8})$. The boundary conditions for $\sigma(S, 0)$ and $\sigma(S, T)$ are more difficult. In a real problem we would not know these boundary values.

There are a number of approaches to determining the upper and lower boundary conditions. The method we have employed here is to take a simple linear fit of the implied volatilities at the endpoints, for the strike $K = 60$. This yields the function $\widetilde{\sigma}(S) = 0.29662 + 0.003248S$.

For comparison we have computed the volatilities for two cases, namely:

$$\text{Case 1}: \sigma(S, 0) = \widetilde{\sigma}(S), \quad \sigma(S, T) = 0.6(1 - e^{-0.03S}), \tag{6.1}$$

$$\text{Case 2}: \sigma(S, 0) = \sigma(S, T) = \widetilde{\sigma}(S). \tag{6.2}$$

The algorithm terminated after 5 iterations for both cases. The results are presented in figures 1 and 2. In both graphs the continuous curve represents the true volatility function, and the dotted curve the reconstructed values. We see that with Case 1, the derived volatility function is indistinguishable from the true function, which is what we would expect, since we have supplied one of the true boundary conditions. The results for Case 2 are only fair. The reconstructed volatility function matches the true volatility in the middle of the interval, that is for, approximately, $35 \leq S \leq 90$, and at the endpoints. For values towards the ends of the interval however, the error becomes quite large.

Clearly then, the accuracy of the algorithm is determined by how close the boundary conditions we supply are to the true boundary conditions. If these cannot be found the algorithm will not yield accurate results.

Next we apply this methodology to some market data. The date we use is from the FTSE100 for an index option expiring on December 11, 1999. The data was

FIGURE 1. Volatility reconstruction. Case 1

FIGURE 2. Volatility reconstruction. Case 2

obtained through Datastream. We have a range of evenly spaced strikes between 5500 and 6900, over 67 days, from August 28, 1999 to November 30,1999.

In order to determine the volatility we require boundary conditions. In particular we require $\sigma(S,0), \sigma(S,T_{fin}), \sigma(0,t)$ and $\sigma(S_{max},t)$. We will set $\sigma(0,t) = 0$. To determine $\sigma(S,0)$ and $\sigma(S,T_{fin})$ we will use the integral equation method of Bouchouev [Bou97] in the form described in [CCEH99]. This allows us to take the values of the option across the strikes for a fixed time t and, by solving an integral equation, determine a volatility function $\sigma(S)$ for that particular time. We do this for the option values for August 28 and November 30, and set the functions so obtained to be $\sigma(S,0)$ and $\sigma(S,T_{fin})$ respectively.

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