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On one–dimensional stochastic control problems: Applications to investment models[1]

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Abstract

The paper provides a systematic way for finding a partial differential equation that characterize directly the optimal control, in the framework of one–dimensional stochastic control problems of Mayer, with no constraints on the controls. The results obtained are applied to some significative models in financial economics.

Keywords: dynamic programming, stochastic control, quasilinear parabolic equation, investment problems.

JLE classification: C61.

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1 Introduction

The paper provides a systematic way for finding a partial differential equation (PDE) that can be applied directly to the optimal control in one–dimensional stochastic control problems of Mayer, where there are no constraints on the controls or, more generally, where the optimal control is interior to the control region. This new PDE is obtained from the optimality conditions of the stochastic maximum principle, and it is equivalent to the Hamilton–Jacobi–Bellman (HJB) equation.

Though the initial idea of obtaining a system of PDEs for the optimal control appears in [1] in connection with deterministic control problems, the main antecedents of this paper are: [11] and [10] in deterministic differential games; [5], in stochastic control problems, where the diffusion parameter of the state process is independent of the control variables; [4] in the Merton model; and [12] in a model of optimal liquidation in illiquid markets. In all these papers the use of the PDE for the optimal control has been proven useful. The objective of this paper is to extend the approach to the one–dimensional stochastic control problem of Mayer, where there is no running payoff functional, but the diffusion term of the state process depends on the control variable.

The paper is organized as follows. In Section 2 we present the control problem, as well as some definitions and notations. In Section 3 we obtain necessary optimality conditions in the form of PDEs that the adjoint feedback function and the optimal control must satisfy. The relationship between the new PDEs and the HJB equation is shown in Section 4, and a sufficient optimality condition is given in terms of a verification theorem in [3]. Section 5 contains applications of the theory to linear models in the dynamics. In particular, the existence of a solution is shown for the Merton model with deterministic coefficients, for a class of utility functions having a bounded relative risk tolerance index. In Section 6 models with a multiplicative structure in the dynamics are introduced. It turns out that some simple assumptions on the data allows us to solve a wide range of models of this type, from which we include an extension of Merton's model to situations where the investor's decisions may influence the evolution of the stochastic price process of the risky asset. The paper ends with some conclusions in Section 7.

2 The control problem

In this section the framework for the stochastic control problem to be considered is presented. First we shall introduce some useful notations. Throughout this paper, given a differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, we will denote by h_y the *partial derivative* of h with respect to the variable y and, if $n = 1$, by h' the derivative of h with respect to a variable other than time, and by $\dot{h}$ the derivative with respect to the time–variable t. The notation is analogous for the partial derivatives of second order. We will denote *total derivation* by $\partial/\partial x$. Vectors $v \in \mathbb{R}^n$ are row vectors and v^i is the ith component; finally, $^\top$ denotes transposition.

Let $[0, T]$ be a time interval with $0 < T < \infty$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P})$ be a complete filtered probability space. Assume that on this space an ℓ–dimensional Brownian motion $\{w(t)\}_{t\in[0,T]}$ is defined. Let E denote expectation under the probability measure P.

The state space is $\mathbb{R}$ and the control region is some convex subset $U \subseteq \mathbb{R}$. A U–valued *control process* $\{u(s)\}$ defined on $[t, T] \times \Omega$ is an $\mathcal{F}_s$–progressively measurable map $(r, \omega) \to u(r, \omega)$ from $[t, s] \times \Omega$ into U, that is, $\mathcal{B}_s \times \mathcal{F}_s$–measurable for each $s \in [t, T]$, where $\mathcal{B}_s$ denotes the Borel σ–field on $[t, s]$. For simplicity, we will denote $u(t)$ as $u(t, \omega)$.

The state process $X \in \mathbb{R}$ obeys the controlled stochastic differential equation (SDE)

$$dX(s) = f(s, X(s), u(s))\, ds + \sigma(s, X(s), u(s))\, dw(s), \quad s \geq t, \tag{1}$$

with initial condition $X(t) = x$. An important feature of the above equation is that the drift, f, and the noise coefficient, σ, are dependent on the control variable, u. Here, σ is a vector with ℓ components.

Definition 2.1 (Admissible control). A control $\{u(t)\}_{t\in[0,T]}$ is called admissible if

(i) for every (t, x) the SDE (1) with initial condition $X(t) = x$ admits a pathwise unique strong solution;

(ii) there exists some function $\phi : [0, T] \times \mathbb{R} \longrightarrow U$ such that u is in relative feedback to ϕ, i.e. $u(s) = \phi(s, X(s))$ for every $s \in [0, T]$.

Let $\mathcal{U}(t, x)$ denote the set of admissible controls corresponding to the initial condition $(t, x) \in [0, T] \times \mathbb{R}$.

Given the initial data $(t, x) \in [0, T] \times \mathbb{R}$, the criterion to be maximized is

$$e^{-\delta(T-t)} \mathrm{E}_{tx} \left\{ S(T, X(T; t, x)) \right\}, \tag{2}$$

in the class of controls $u \in \mathcal{U}(t, x)$, where E_{tx} denotes conditional expectation with respect to the initial condition (t, x). The constant $\delta \geq 0$ is the discount factor. The functions

$f:[0,T]\times\mathbb{R}\times U\longrightarrow\mathbb{R}$, $\sigma:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}^{1\times\ell}$, $S:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$, are all assumed to be continuous. They are also of class C^2 with respect to (x,u) and of class C^1 with respect to t.

The *value function* is defined as $V(t,x)=\sup_{u\in\mathcal{U}(t,x)}J(t,x;u)$. An admissible control $\widehat{u}\in\mathcal{U}$ is *optimal* if $V(t,x)=J(t,x;\widehat{u})$ for every initial condition (t,x).

In the specification of the problem we have supposed that the dimension of both the control and the state variable is one. However, the case with $n>1$ control variables can be reduced to the scalar case.

The classical method for determining feedback solutions in a control problem is based on finding the value function through the HJB equation and the optimal control from that. It is well known that if V is of class $C^{1,2}$, then it satisfies the HJB equation

$$\begin{aligned}&V_t(t,x)+\max_{u\in U}G(t,x,u,V_x(t,x),V_{xx}(t,x))=\delta V(t,x),\quad\forall(t,x)\in[0,T)\times\mathbb{R},\\&V(T,x)=S(T,x),\quad\forall x\in\mathbb{R},\end{aligned}$$

and the maximizing argument is optimal if it is admissible. Here

$$G(t,x,u,p,P)=f(t,x,u)p+\frac{1}{2}\sigma(t,x,u)\sigma^\top(t,x,u)P \tag{3}$$

denotes the *generalized Hamiltonian*. We will denote

$$\overline{u}(t,x,p,P)\in\operatorname{argmax}_{u\in U}G(t,x,u,p,P).$$

3 Necessary conditions

In this section we deduce a PDE that an optimal control must satisfy as an alternative to the HJB equation. Our derivation depends on the application of the stochastic maximum principle (MP hereafter). Conditions can be found in [13] on functions f, σ and S to allow for the application of the stochastic MP. We will take these conditions for granted in the derivation of the quasilinear PDE as a necessary condition for optimality.

Let (X,u) be an optimal control pair, with $u(t)=\phi(t,X(t))$. Applying the stochastic maximum principle, there are unique square integrable processes p and q that satisfy the backward adjoint equation

$$dp(s)=-H_x(s,X(s),\phi(s,X(s)),p(s),q(s))ds+q(s)dw(s),\quad s\in[t,T], \tag{4}$$

$$p(T)=S_x(T,X(T)), \tag{5}$$

where $H(t,x,u,p,q)=f(t,x,u)p+\sigma(t,x,u)q^\top$ is the *stochastic Hamiltonian.*

Definition 3.1 (Adjoint feedback) *A function $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ is called an adjoint feedback if it expresses the adjoint process p in terms of the state variable X, $p(s) = \gamma(s, X(s))$.*

To facilitate the exposition of our results, we impose the following throughout the paper: At the optimal $\widehat{\phi}$

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad f_u(t,x,\widehat{\phi}) \neq 0 \quad \text{and} \quad \sigma_u \sigma^\top (t,x,\widehat{\phi}) \neq 0.$$

These conditions imply that both γ and γ_x are different from zero, as can easily be seen in the proof of the next result. Thus, the function F defined on $[0,T] \times \mathbb{R} \times U$ by

$$F(t,x,u) = -\frac{f_u}{\sigma_u \sigma^\top}(t,x,u), \tag{6}$$

makes sense.

In the next proposition we show that, under suitable conditions, the adjoint feedback must satisfy a second order quasilinear PDE. The special structure is due to the supposition that the state variable is one–dimensional.

Proposition 3.1 *Suppose that γ is an adjoint feedback, continuous on $[0,T] \times \mathbb{R}$, of class $C^{1,2}$ on $[0,T) \times \mathbb{R}$ and that $G(t,x,u,p,P)$ is of class C^1 with respect to all the variables. Then, the adjoint feedback γ, almost everywhere, satisfies the PDE*

$$\gamma_t + \frac{\partial}{\partial x} \max_{u \in U} G\big(t,x,u,\gamma,\gamma_x\big) = 0, \tag{7}$$

with terminal condition

$$\gamma(T,x) = S_x(T,x). \tag{8}$$

Proof. We omit the arguments in several parts of the proof. Applying Itô's formula to γ we get

$$dp(s) = \left(\gamma_s + f\gamma_x + \frac{1}{2}\sigma\sigma^\top \gamma_{xx}\right) ds + \sigma \gamma_x dw(s), \quad s \in [t,T], \tag{9}$$

and equating the volatility terms of (4) and (9)

$$q = \sigma \gamma_x.$$

Next, equating the drift terms of (4) and (9),

$$\begin{aligned} \gamma_t + f\gamma_x + \frac{1}{2}\sigma\sigma^\top \gamma_{xx} &= -H_x \\ &= -f_x \gamma - \sigma_x q^\top \\ &= -f_x \gamma - \sigma_x \sigma^\top \gamma_x. \end{aligned} \tag{10}$$

Now we consider $\overline{G}(t,x,\gamma,\gamma_x) = \max_{u\in U} G(t,x,u,\gamma,\gamma_x)$, the maximum of G with respect to $u \in U$. By Danskin's Theorem, $\overline{G}$ is almost everywhere differentiable with respect to x, and the derivative at points where it exists is

$$\frac{\partial}{\partial x}\overline{G}(t,x,\gamma,\gamma_x) = f_x\gamma + f\gamma_x + \sigma_x\sigma^\top\gamma_x + \frac{1}{2}\sigma\sigma^\top\gamma_{xx}\Big|_{u=\overline{u}(t,x,\gamma,\gamma_x)}.$$

Thus, (10) can be rewritten as

$$\gamma_t + \frac{\partial}{\partial x}\overline{G}(t,x,\gamma,\gamma_x) = 0, \tag{11}$$

which is the PDE stated in the proposition. Finally, the terminal condition is a consequence of the MP.□ □

Remark 3.1 One aspect that may make PDE (7) impractical is that it depends on the maximizer $\overline{u}$ which is, in general, not known. To get an explicit PDE we will impose the condition that the optimal feedback must be interior to U. Notice that, even in this case, the PDE for the adjoint feedback continues to be non–explicit. However, the PDE for ϕ will always be explicit—see Remark 3.2 below—and holds for any one–dimensional control problem of Mayer type fulfilling the conditions imposed in this paper —i.e. smoothness and interiority of the optimal control—.

The next theorem shows the PDE that the optimal ϕ satisfies.

Theorem 3.1 *Suppose that γ is an adjoint feedback and $\phi \in \mathcal{U}$ is the unique admissible interior optimal Markov control of the problem* (1), (2), *continuous on* $[0,T]\times\mathbb{R}$, *and of class* $C^{1,2}$ *on* $[0,T)\times\mathbb{R}$. *Then, ϕ satisfies the quasilinear PDE of second order*

$$\begin{aligned}\frac{\partial}{\partial t}F(t,x,\phi) + \frac{\partial}{\partial x}\,G(t,x,\phi,F(t,x,\phi),F^2(t,x,\phi))&\\ +\frac{\partial^2}{\partial x^2}\,G(t,x,\phi,1,F(t,x,\phi)) = 0,&\end{aligned} \tag{12}$$

with terminal condition

$$S_x(T,x)F(T,x,\phi(T,x)) = S_{xx}(T,x). \tag{13}$$

Proof. Since the argument maximizing H is interior to U, the MP implies

$$H_u(s,X(s),\phi(s,X(s)),p(s),q(s)) = 0, \quad \forall s\in[t,T], \quad \text{P–a.s.}, \tag{14}$$

that is

$$f_u p + \sigma_u q^\top = 0.$$

Hence, since $q = \sigma\gamma_x$ was shown in the proof of Proposition 3.1 and recalling the definition of F in (6), the following equality holds:

$$\gamma(t,x)F(t,x,\phi(t,x)) = \gamma_x(t,x). \tag{15}$$

On the other hand, by the MP $\phi(t,x) = \overline{u}(t,x,\gamma(t,x),\gamma_x(t,x))$. We omit the arguments in some parts of the proof, when no confusion arises. We will go through the proof in the following steps.

1.- Divide the PDE (7), $\gamma_t + (\partial/\partial x)G = 0$, by γ and notice that

$$\frac{\partial}{\partial x}\frac{G}{\gamma} = \frac{1}{\gamma}\frac{\partial}{\partial x}G - \frac{\gamma_x}{\gamma^2}G$$

to get

$$\frac{\gamma_t}{\gamma} + \frac{\gamma_x}{\gamma^2}G + \frac{\partial}{\partial x}\frac{G}{\gamma} = 0. \tag{16}$$

2.- By definitions of G and (15)

$$\begin{aligned}\frac{\gamma_x}{\gamma^2}G(t,x,\phi,\gamma,\gamma_x) &= (fF + \frac{1}{2}\sigma\sigma^\top F^2)|_{(t,x,\phi)} = G(t,x,\phi,F,F^2),\\ \frac{1}{\gamma}G(t,x,\phi,\gamma,\gamma_x) &= (f + \frac{1}{2}\sigma\sigma^\top F)|_{(t,x,\phi)} = G(t,x,\phi,1,F).\end{aligned}$$

3.- Take the derivative of (16) with respect to x and then substitute the expressions in step 2 to find

$$\frac{\partial}{\partial x}\frac{\gamma_t}{\gamma} + \frac{\partial}{\partial x}G(t,x,\phi,F,F^2) + \frac{\partial^2}{\partial x^2}G(t,x,\phi,1,F) = 0.$$

4.- Finally, notice that $(\partial/\partial x)(\gamma_t/\gamma) = (\partial/\partial t)(\gamma_x/\gamma) = F_t$ because γ is of class $C^{1,2}$. Using this fact in the equation above, we get (12). The final condition is a consequence of the MP and the expression for F given in (15).□ □

Remark 3.2 Notice that F is always an explicit expression of (t,x,u), thus equation (12) depends only on t, x, and $\phi(t,x)$, once

$$F(t,x,\phi) = -\frac{f_u(t,x,\phi)}{(\sigma_u\sigma^\top)(t,x,\phi)}$$

is substituted throughout in (12). Then, a PDE only involving the unknown control $\phi(t,x)$ is obtained.

4 Value function and sufficient conditions

Before proceeding to establish sufficient conditions for optimality in this section, in the following definition we give a weak notion of a solution of the PDE (12). The reason is that for our purposes, it suffices to consider $C^{1,1}$ solutions.

Definition 4.1 A function ϕ is a $C^{1,1}$ of the Cauchy problem (12), (13), if it satisfies the integral equation

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} F^{\phi}(t,z)dz + J^{\phi}(t,x_2) - J^{\phi}(t,x_1) = 0, \quad x_1, x_2 \in \mathbb{R}, \tag{17}$$

and the same final condition

$$S_x(T,x)F(T,x,\phi(T,x)) = S_{xx}(T,x). \tag{18}$$

We have used the short–hand

$$F^{\phi}(t,x) = F(t,x,\phi(t,x)),$$
$$J^{\phi}(t,x) = G(t,x,\phi,F^{\phi},(F^{\phi})^2) + \frac{\partial}{\partial x}G(t,x,\phi,1,F^{\phi}).$$

In this section we show that a solution $\widehat{\phi}$ of class $\mathcal{C}^{1,1}$ of (17), (18) maximizing the generalized Hamiltonian for all (t,x) is a solution of the stochastic control problem (1), (2). We also find the connection between the optimal control, the adjoint feedback, and the value function.

A proposition gives the adjoint feedback in terms of a solution to (17), (18). To obtain an explicit expression for γ is important, since it is the derivative with respect to x of the value function, and consequently has the interpretation of a "shadow price". Furthermore, once γ is known, it is straightforward to obtain the value function, as will be shown in Theorem 4.1 below.

Proposition 4.1 *Let $\widehat{\phi}$ be an admissible control of class $\mathcal{C}^{1,1}$ satisfying* (17), (18)*. Then, for any $t \in [0,T]$ and $x \in \mathbb{R}$ and any $\alpha \in \mathbb{R}$, the adjoint feedback γ is of class $C^{1,2}$ and is given by*

$$\gamma(t,x) = S_x(T,\alpha)e^{\int_t^T J^{\widehat{\phi}}(s,\alpha)\,ds}e^{\int_\alpha^x F^{\widehat{\phi}}(t,z)\,dz}. \tag{19}$$

Proof. It is clear that γ, as given by (19), has the required smoothness. Taking the derivative in (19) with respect to t we get

$$\gamma_t(t,x) = \gamma(t,x)\Big(\frac{\partial}{\partial t}\int_\alpha^x F^{\widehat{\phi}}(t,z)\,dz - J^{\widehat{\phi}}(t,\alpha)\Big). \tag{20}$$

Since F satisfies (17), then selecting $x_1 = \alpha$ and $x_2 = x$

$$\frac{\partial}{\partial t}\int_\alpha^x F^{\widehat{\phi}}(t,z)dz - J^{\widehat{\phi}}(t,\alpha) = -J^{\widehat{\phi}}(t,x).$$

Substituting into (20), we have

$$\gamma_t(t,x) + \gamma(t,x)\, J^{\widehat{\phi}}(t,x) = 0. \tag{21}$$

Then, using the identities in steps 1 and 2 of the proof of Theorem 3.1

$$\begin{aligned}\gamma\, J^{\widehat{\phi}} &= \gamma G(t,x,\phi,F^{\phi},(F^{\phi})^2) + \gamma \frac{\partial}{\partial x} G(t,x,\phi,1,F^{\phi}) \\ &= \frac{\partial}{\partial x} G(t,x,\overline{u}(t,x,\gamma,\gamma_x),\gamma,\gamma_x).\end{aligned}$$

Thus, by (21), the expression defined in (19) satisfies (7). The final condition (8) follows from (13):

$$\gamma(T,x) = S_x(T,\alpha) e^{\int_\alpha^x F^{\widehat{\phi}}(T,z)\,dz} = S_x(T,\alpha) e^{\ln|S_x(T,x)/S_x(T,\alpha)|} = S_x(T,x).$$

Finally, the independence of γ with respect to the constant α is deduced by verifying that the derivative of γ with respect to α is zero. This is clear if $S_x(t,\alpha) = 0$, so suppose that $S_x(T,\alpha) \neq 0$. Then

$$\begin{aligned}\frac{\partial}{\partial \alpha}\gamma(t,x) &= \gamma(t,x)\Big(\frac{S_{xx}(T,\alpha)}{S_x(T,\alpha)} - F^{\widehat{\phi}}(t,\alpha) + \int_t^T J_x^{\widehat{\phi}}(s,\alpha)\,ds\Big) \\ &= \gamma(t,x)\Big(\int_t^T F_t^{\widehat{\phi}}(s,\alpha)ds + \int_t^T J_x^{\widehat{\phi}}(s,\alpha)\,ds\Big) = 0,\end{aligned}$$

where the second equality holds by (13) and the last equality is implied by (12).□ □

Given a solution $\widehat{\phi}$ of (17), (18), Proposition 4.1 shows that an adjoint feedback γ exists. From this information we construct the value function V. To simplify the notation, for $u \in U$, we define

$$\mathcal{G}^u(t,x) = G(t,x,u,\gamma(t,x),\gamma_x(t,x)).$$

Theorem 4.1 (Value function) *Let $\widehat{\phi}$ be an admissible control solution of (17), (18), such that*

$$\forall (t,x) \in [0,T]\times\mathbb{R},\ \forall u \in U, \quad \mathcal{G}^{\widehat{\phi}}(t,x) \geq \mathcal{G}^u(t,x). \tag{22}$$

Then for an arbitrary constant α, W given by

$$W(t,x) = e^{-\delta(T-t)}\Big(\int_\alpha^x \gamma(t,z)dz + \int_t^T \mathcal{G}^{\widehat{\phi}}(s,\alpha)ds + S(T,\alpha)\Big) \tag{23}$$

is a $C^{1,3}$ solution of the HJB equation and satisfies $W(T,x) = S(T,x)$. Moreover, if for all x

$$|\gamma(t,x)| \leq C(1+|x|^k) \tag{24}$$

for some constants C and $k > -1$, then $W = V$ is the value function, and $\widehat{\phi}$ is an optimal control.

Proof. It is obvious that W, defined in (23), is a function of class $\mathcal{C}^{1,3}$, with $W_x = e^{-\delta(T-t)}\gamma(t,x)$ and $W_{xx} = e^{-\delta(T-t)}\gamma_x(t,x)$, since we know $\gamma \in C^{1,2}$ by Proposition 4.1. Integrating (11) with respect to x and interchanging the order of the integration and derivation operations, we have

$$\frac{\partial}{\partial t}\int_\alpha^x \gamma(t,z)dz + \mathcal{G}^{\widehat{\phi}}(t,x) - \mathcal{G}^{\widehat{\phi}}(t,\alpha) = 0.$$

Taking the derivative with respect to t in (23)

$$e^{\delta(T-t)}(-\delta W(t,x) + W_t(t,x)) = \frac{\partial}{\partial t}\int_\alpha^x \gamma(t,z)dz - \mathcal{G}^{\widehat{\phi}}(t,\alpha) = -\mathcal{G}^{\widehat{\phi}}(t,x).$$

Hence, by definition of $\mathcal{G}^{\widehat{\phi}}(t,x)$

$$W_t(t,x) + W_x(t,x)f(t,x,\widehat{\phi}(t,x)) + \frac{1}{2}(\sigma\sigma^\top)(t,x,\widehat{\phi}(t,x))W_{xx}(t,x) = \delta W(t,x). \tag{25}$$

On the other hand, by assumption (22)

$$\begin{aligned} &W_x(t,x)f(t,x,\widehat{\phi}(t,x)) + \frac{1}{2}W_{xx}(t,x)(\sigma\sigma^\top)(t,x,\widehat{\phi}(t,x)) \\ &\qquad\geq G(t,x,u,W_x(t,x),W_{xx}(t,x)\sigma(t,x,u)), \quad \forall u \in U. \end{aligned} \tag{26}$$

In consequence, (25) and (26) imply that W satisfies the HJB equation. The final condition also holds, since

$$W(T,x) = \int_\alpha^x \gamma(T,z)dz + S(T,\alpha) = \int_\alpha^x S_x(T,z)\,dz + S(T,\alpha) = S(T,x),$$

due to (5). To complete the proof of the first part of the theorem, it is immediate to check that (23) does not depend on α, as this was done in Proposition 4.1.

Finally, if γ satisfies (24), then by (23) W is polynomially bounded. Hence, to make W the value function and $\widehat{\phi}$ truly optimal, it suffices to apply the verification theorem in [3].□
□

Remark 4.1 Condition (22) automatically holds when $\widehat{\phi}$ is interior to the control set U and for all $(t,x) \in [0,T]\times\mathbb{R}$ the function $G(t,x,u,\gamma,\gamma_x)$ is concave with respect to u, since the equality

$$G_u(t,x,\widehat{\phi},\gamma(t,x),\gamma_x(t,x)) = H_u(t,x,\widehat{\phi},\gamma(t,x),\sigma(t,x,\widehat{\phi})\gamma_x(t,x)) = 0$$

is satisfied trivially, by the stochastic maximum principle, and this means that $\widehat{\phi}$ is a critical point of the concave function $u \mapsto G(\ldots,u,\ldots)$, hence $\widehat{\phi}$ is a global maximum of $G(\ldots,u,\ldots)$.

5 Application to models with linear dynamics

We now show the form of equation (12) in the next two examples.

5.1 General problem

Consider a control problem with linear drift

$$f(t,x,u) = a(t)x + b(t)u$$

and linear diffusion coefficient

$$\sigma(t,x,u) = (c_1(t)x + d_1(t)u, \ldots, c_\ell(t)x + d_\ell(t)u),$$

where the time–dependent vectors $c(t) = (c_i(t))_{i=1}^{\ell}$ and $d(t) = (d_i(t))_{i=1}^{\ell}$ are differentiable, with $b(t) \neq 0$ and $d(t)d^\top(t) > 0$ for every $t \in [0,T]$. In the following we drop the time dependence from the notation. The definition of F in (6) gives

$$F(x,u) = -\frac{b}{cd^\top x + dd^\top u}, \quad \text{or} \quad u = \overline{u}(t,x,F) = -\left(\frac{b}{dd^\top}\right)F^{-1} - \left(\frac{cd^\top}{dd^\top}\right)x.$$

Obviously, for this particular class of models it is always possible to find $\overline{u}$. Hence, the PDE (7) satisfied by γ can be explicitly found (we omit the arguments)

$$\gamma_t + \frac{\partial}{\partial x}\left(\left(a - b\frac{cd^\top}{dd^\top}\right)x\gamma + \frac{1}{2}\left(cc^\top - \frac{(cd^\top)^2}{dd^\top}\right)x^2\gamma_x - \frac{1}{2}\frac{b^2}{dd^\top}\frac{\gamma^2}{\gamma_x}\right) = 0, \tag{27}$$

because

$$G(t,x,u,p,P) = (ax+bu)p + \frac{1}{2}(cc^\top x^2 + 2cd^\top xu + dd^\top u^2)P.$$

Moreover, an explicit PDE for $F = \gamma_x/\gamma$ arises from (12)

$$F_t + \frac{1}{2}\frac{\partial}{\partial x}\left(2A(t)xF + C(t)x^2F^2\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(-B(t)F^{-1} + C(t)x^2F\right) = 0, \tag{28}$$

where

$$A(t) = a(t) - b(t)\frac{cd^\top(t)}{dd^\top(t)}, \quad B(t) = \frac{b^2(t)}{dd^\top(t)}, \quad C(t) = cc^\top(t) - \frac{(cd^\top(t))^2}{dd^\top(t)}.$$

Substituting F in (28) with its expression in terms of ϕ

$$F(t,x,\phi) = -\frac{b(t)}{c(t)d^\top(t)x + dd^\top(t)\phi},$$

an explicit PDE for ϕ is obtained. The PDE for ϕ will be shown for the Merton model in the next section, in order to save space.

We shall now make a few remarks on the above PDEs.

1. Equation (27) has already been obtained in the financial literature in the case $c(t) = 0$ for all $t \in [0, T]$ and $\ell = 1$. See Section 5.2 below, where we study Merton's model with deterministic coefficients. It is also known that under these conditions, $c = 0$ and $\ell = 1$, the hodograph transform $X(t, \gamma(t, x)) = x$ (that is, for each t, $X(t, \cdot)$ is the inverse of function γ) linearizes (27) to the PDE

$$X_t - \Big(a - \frac{bc}{d}\Big)X - \Big(a - \frac{bc}{d} - \frac{b^2}{d^2}\Big)\gamma X_\gamma + \frac{b^2}{2d^2}\gamma^2 X_{\gamma\gamma} = 0,$$

with final condition $X(T, \gamma) = (S')^{-1}(\gamma)$, see [7] or [4]. Under suitable conditions, assuring that X is a global C^2 diffeomorphism for each t, the problem can be solved fairly well, and explicit expressions for the optimal control and the value function can be recovered. The same results can also be obtained by the martingale approach, see [6]. However, the hodograph transform does not linearize the PDE for γ in the general case with non–null vector c and $\ell > 1$ Brownian motions. An interesting question is to find a suitable transformation (if any) that works in this case.

2. For any constant ρ, $F = -\rho/x$ is a (stationary) solution of (28). This solution is consistent with a parametric family of objective functions S, of HARA (Hyperbolic Absolute Risk Aversion) type, $S(T, x) = \frac{A(T)}{1-\rho} x^{1-\rho}$, when $\rho > 0$, $\rho \neq 1$ or $S(T, x) = A(T) \ln x$ if $\rho = 1$, in both cases with $A(T) > 0$. Notice that F satisfies the final condition (13) and S is strictly concave. From Theorem 3.1, the linear control (in variable x)

$$\phi(t, x) = \left(d(t)d^\top(t)\right)^{-1} \left(\rho^{-1} b(t) + c(t)d^\top(t)\right) x$$

is a solution of (12) and satisfies the final condition (13), therefore it is a candidate for an optimal control of this family of problems. In the following section we give sufficient conditions for a solution of the PDE for the control, based in Theorem 4.1, to be actually an optimal control.

3. Knowledge of the PDE (12) allows us to address the inverse or integrability problem, which consists of recovering the utility function S from a given optimal investment control u. We analyze this problem here, for linear controls. The problem consists in determining an increasing and strictly concave function S such that $\phi(t, x) = m(t)x + n(t)$, with given smooth functions m and n, is the solution of the control problem, for suitable functions a, b, c and d, as well as of the dimension of the Brownian vector, ℓ. By (6)

$$\widehat{F}(t, x) = F(t, x, \phi(t, x)) = -\frac{b(t)}{c(t)d^\top(t)x + d(t)d^\top(t)(m(t)x + n(t))} \qquad (29)$$

must be a solution to (28), and the final condition (13) holds.

Let $\alpha(t) = c(t)d^\top(t) + d(t)d^\top(t)m(t)$, $\beta(t) = d(t)d^\top(t)n(t)$, and recall the definition of functions $A(t)$ and $C(t)$ given above. Substituting (29) into (28) we get the conditions (we omit the time argument)

$$\dot{b}\alpha^2 - b\alpha\dot{\alpha} = 0, \tag{30}$$

$$2\dot{b}\alpha\beta - b\dot{\alpha}\beta - \alpha b\dot{\beta} + \alpha bA\beta - b^2C\beta = 0, \tag{31}$$

$$\big(\dot{b}\beta - b\dot{\beta} + bA\beta + bC\beta\big)\beta = 0. \tag{32}$$

As above, we suppose $b(t) \neq 0$ and $d(t)d^\top(t) > 0$ for all t, and consider functions c, d such that $\alpha(t) \neq 0$ for all t. We distinguish two cases.

(a) $n(t) \neq 0$ for all t. Then, from (30) $\alpha(t) = kb(t)$ for some constant k. Hence

$$m(t) = \frac{kb(t) - c(t)d^\top(t)}{d(t)d^\top(t)} \tag{33}$$

must hold. On the other hand, equation (32) reduces to

$$\dot{\beta}(t) = \beta(t)\Big(\frac{\dot{b}(t)}{b(t)} + A(t) + C(t)\Big),$$

hence $\beta(t) = -\beta(T)\exp\{\int_t^T(\dot{b}(s)/b(s) + A(s) + C(s))ds\}$ and

$$\begin{aligned} n(t) &= \frac{-\beta(T)}{dd^\top(t)} e^{\int_t^T(\dot{b}(s)/b(s)+A(s)+C(s))ds} \\ &= \frac{-\beta(T)b(T)}{dd^\top(t)b(t)} e^{\int_t^T(A(s)+C(s))ds}. \end{aligned} \tag{34}$$

There are multiple selections of functions a, b, c and d such that both (33) and (34) hold. Plugging $\alpha(t) = kb(t)$ into (31) leads to

$$b\dot{b}k\beta - kb^2\dot{\beta} + kb^2A\beta - C\beta b^2 = 0$$

so by using (32) one obtains $-(1+k)C(t)b^2(t)\beta(t) = 0$ for all t. This implies $k = -1$ or $C(t) = 0$, because $b(t) \neq 0$, $\beta(t) \neq 0$ for all t. Notice that $C(t) = 0$ holds if $\ell = 1$.

To determine function S, we use the final condition (13). From the identity

$$-S'(x)\frac{b(T)}{\alpha(T)x + \beta(T)} = S''(x),$$

it is not difficult to find, with $k = \alpha(T)/b(T)$, that

$$
\begin{aligned}
S(x) &= \frac{-K}{(1-k)b(T)}(\alpha(T)x + \beta(T))^{-(1-k)/k}, \quad \text{if } k \neq 1, \\
&= \frac{K}{b(T)} \ln\left(\alpha(T)x + \beta(T)\right), \quad \text{if } k = 1,
\end{aligned} \tag{35}
$$

where the constants K and k are both strictly positive. In this way, S is strictly increasing. Moreover, choosing $b(T) < 0$, it is also strictly concave. The case with $\alpha(T) = 0$, $\beta(T) \neq 0$ is also possible, with $S(x) = -K(\beta(T)/b(T)) \exp\{(-b(T)/\beta(T))x\}$, $K > 0$ and $b(T)/\beta(T) > 0$.

(b) $n(t) = 0$ for some $t \in [0, T]$. Then n must actually be identically null because $\beta(t) = 0$ for all t. This is a consequence of supposing $b(t) \neq 0$, $\alpha(t) \neq 0$ and $d(t)d^\top(t) > 0$ for all t. Thus, the only constraint that appears is $\alpha(t) = kb(t)$ for some constant k. The form of S can be recovered from (35) with $\beta(T) = 0$ and $\alpha(T) \neq 0$.

5.2 Merton's Model with deterministic coefficients

The problem in Section 5.1 above encompasses, in particular, a variant of the investment model of Merton, [9], where there is no running utility from consumption, and where the objective is to maximize the utility of terminal wealth. That is, an investor wants to maximize the expected utility S of the final wealth at a fixed date T. Along the time interval $[0, T]$ the decision agent invests in two assets, one of them a risky asset whose price, P^1, evolves according to the SDE

$$
dP^1(t) = (b(t) + r(t))P^1(t)\,dt + \sigma(t)P^1(t)\,dw(t), \quad P^1(0) \text{ known},
$$

and the other is a bond account, P^0, which is driven by

$$
dP^0(t) = r(t)P^0(t)\,dt, \quad P^0(0) = 1,
$$

where r, b and σ are positive, deterministic functions of time.

Let $u(s)$ be the amount of wealth invested in the risky asset at time s, and let $X(s) - u(s)$ be the amount invested in the bond, where $X(s)$ is the accumulated wealth until time s. Then, $X(s)$ satisfies the SDE

$$
\begin{aligned}
dX(s) &= (r(s)X(s) + b(s)u(s))\,dt + \sigma(s)u(s)\,dw(s), \qquad t \leq s \leq T, \\
X(t) &= x, \qquad x \geq 0.
\end{aligned} \tag{36}
$$

Given wealth's level $X(t) = x$ at date $t \in [0, T]$, the problem is to choose an investment policy u solving the problem

$$\max_{u \in \mathcal{U}(t,x)} e^{-\delta(T-t)} \mathrm{E}_x \left\{ S(X(T)) \,|\, X(t) = x \right\} = \max_{u \in \mathcal{U}(t,x)} e^{-\delta(T-t)} \mathrm{E}_x S(X(T)),$$

subject to (36), where S is a strictly increasing and strictly concave utility function. This model has been profusely studied in the literature. Merton's model is obtained from the example in Section 5.1 by selecting $a = r$, $c = 0$ and $d = \sigma$, with $\ell = 1$. It is easy to compute (12) and the final condition (13)

$$\begin{aligned} &F_t + r(t) \frac{\partial}{\partial x} (xF) - \frac{1}{2} b(t) \theta(t) \frac{\partial^2}{\partial x^2} \left(\frac{1}{F} \right) = 0, \\ &F(T, x, \phi(T, x)) = \frac{S''(x)}{S'(x)}, \end{aligned}$$

where $\theta(t) = b(t)/\sigma(t)^2$. Then, the PDE for $\phi = -\theta(t)/F$ is

$$\frac{\partial}{\partial t} \left(-\frac{\theta(t)}{\phi(t,x)} \right) - r(t)\theta(t) \frac{\partial}{\partial x} \left(\frac{x}{\phi(t,x)} \right) + \frac{1}{2} b(t) \frac{\partial^2}{\partial x^2} \phi(t,x) = 0.$$

Taking the derivatives we get the Cauchy problem in $(t, x) \in [0, T] \times [0, \infty)$

$$\begin{aligned} &\phi_t - r(t)(\phi - x\phi_x) - \frac{\dot{\theta}(t)}{\theta(t)} \phi + \frac{1}{2} \sigma(t)^2 \phi^2 \phi_{xx} = 0 \quad t < T, x > 0, \\ &\phi(T, x) = \theta(T) R(x), \quad x > 0 \\ &\phi(t, 0) = 0, \quad t < T. \end{aligned} \tag{37}$$

Here, $R(x) = -S'(x)/S''(x)$ is the absolute risk tolerance index of the decision agent (the inverse of the absolute risk aversion index).

The equation found in [4] is a particular case when constant coefficients are considered. These authors use the equation to study asymptotic properties of the optimal investment control. The solution found in Section 5.1 for HARA utilities is $\widehat{\phi}(t, x) = \rho^{-1} \theta(t)\, x$ with $\rho > 0$, which is of course well known in the literature.

Notice that unless R satisfies $R(0) = 0$, the initial condition is not compatible with the boundary condition at $x = 0$ and the existence of a smooth solution is problematic. Thus, we impose $R(0) = 0$ in the following theorem, and show the existence of a solution to the Cauchy problem. We allow for utility functions S with unbounded absolute risk tolerance, but which exhibit bounded relative risk tolerance.

Theorem 5.1 *Assume that functions r, b, σ, θ and $\dot{\theta}$ are of class C^2 and bounded, with $d > 0$, $\theta > 0$, and that function R is of class C^2 and satisfies $R(0) = 0$, $R(x) > 0$ for*

$x > 0$, $\sup_{x\in[0,\infty)} R(x)/x < \infty$ and $\lim_{x\to 0^+} R(x)/x$ exist. Then, there is a solution of the Cauchy problem (37) of class $C^{1,2}$. Moreover, ϕ is globally Lipschitz in x and satisfies for all $t \in [0,T]$

$$\theta(t)x\left(\inf_{x\in[0,\infty]}\frac{R(x)}{x}\right) \le \phi(t,x) \le \theta(t)x\left(\sup_{x\in[0,\infty]}\frac{R(x)}{x}\right).$$

Proof. Let $v = \phi/x$ and $\tau = T - t$. Then, the Cauchy problem for v is (we omit the arguments in v)

$$\begin{aligned}
&v_\tau - r(T-\tau)xv_x + \frac{\dot\theta(T-\tau)}{\theta(T-\tau)}v - \frac{1}{2}\sigma(T-\tau)^2xv^2\big(2v_x + xv_{xx}\big) = 0,\\
&v(0,x) = \theta(T)\lim_{x\to 0^+}\frac{R(x)}{x}.
\end{aligned}$$

We rewrite the PDE in divergence form as follows

$$v_\tau - \frac{1}{2}\sigma(T-\tau)^2\frac{\partial}{\partial x}(x^2v^2v_x) = r(T-\tau)xv_x + \frac{\dot\theta(T-\tau)}{\theta(T-\tau)}v - \sigma^2(T-\tau)x^2vv_x^2.$$

The equation fulfills all the requirements of Theorem 8.1 in Chap. V in [8] (including compatibility between the final and the boundary condition at $x = 0$), except for the uniform parabolic condition in the second order term. However, the solution never vanishes, thus the equation is truly parabolic in $[0,T]\times(0,\infty)$. This can be seen as follows (we adopt here a device used in [2]). Consider a compact interval $[1/n, n]$ and the solution v_n in $[0,T]\times[1/n,n]$ satisfying $v_n(0,1/n) = \theta(T)nR(1/n)$, $v_n(0,n) = \theta(T)R(n)/n$. Such a solution exists and is of class $C^{1,2}$, with Hölder regularity on the derivatives, see [8]. Let $m_n(\tau) = \min_{x\in[1/n,n]} v_n(\tau,x)$. By Danskin's Theorem, function m_n is almost everywhere differentiable, and at points of differentiability $\dot m_n(\tau) = v_{n,\tau}(\tau,\xi(\tau))$, where $v_n(\tau,\xi(\tau)) = m_n(\tau)$. Moreover, $v_{n,x}(\tau,\xi(\tau)) = 0$ and $v_{n,xx}(\tau,\xi(\tau)) \ge 0$. Hence, $d^2(T-\tau)(\partial/\partial x)(x^2v_n^2v_{n,x}) \ge 0$. This information, used in the equation for v_n, gives

$$\dot m_n(\tau) \ge \frac{\dot\theta(T-\tau)}{\theta(T-\tau)}m_n(\tau) \quad \text{a.e. } \tau.$$

Hence, $m_n(\tau) \ge m_n(0)\theta(T-\tau)/\theta(T)$. Thus, we have the estimate

$$v_n(\tau,x) \ge m_n(\tau) \ge \theta(T-\tau)\inf_{x\in[1/n,n]}\frac{R(x)}{x} > 0.$$

Indeed, for $m < n$ and $x \in [1/m, m]$

$$v_n(\tau,x) \ge \theta(T-\tau)\inf_{x\in[1/m,m]}\frac{R(x)}{x} > r_m > 0 \tag{38}$$

for some constant r_m independent of n. The solution obtained by the method of [8] is the limit of the sequence $\{v_n(\tau, x)\}_{n\geq 1}$ by a diagonal argument. By (38), the limit also satisfies $v(\tau, x) > 0$ for any t, $x > 0$. Now, $\phi(t, x) = xv(T - t, x)$ is a solution of the Cauchy problem (37).

For the second part of the theorem, consider $M_n(\tau) = \max_{x\in[1/n,n]} v(\tau, x)$. A similar computation as done above gives $M_n(\tau) \leq \theta(T - \tau) \sup_{x\in[1/n,n]} R(x)/x$. Then,

$$\sup_{x\in[1/n,n]} v_{n,x}(\tau, x) \leq M_n(\tau, x) \leq \theta(T - \tau) \sup_{x\in[0,\infty]} \frac{R(x)}{x}.$$

Hence, the limit ϕ satisfies for all $t \in [0, T]$

$$\theta(T - \tau)x \left(\inf_{x\in[0,\infty]} \frac{R(x)}{x} \right) \leq \phi(t, x) \leq \theta(T - \tau)x \left(\sup_{x\in[0,\infty]} \frac{R(x)}{x} \right).$$

□ □ Next result establish existence of a solution to the Merton model when the relative risk aversion index of the agent is bounded by one.

Theorem 5.2 *Assume that S is increasing and that the conditions of the previous theorem hold. Furthermore, suppose that*

$$L := \sup_{x\in[0,\infty)} \frac{R(x)}{x} > 1. \tag{39}$$

Then, the solution of the Cauchy problem (37) *is the optimal control of the Merton model.*

Proof. To apply Theorem 4.1 it must also be shown that ϕ is admissible and that γ is polynomially bounded in x. The first claim follows since ϕ is lipschitz in x, hence a unique strong solution of the SDE exists. For the second claim, notice that the dependence of γ with respect to x comes, according to (19), from the term

$$e^{\int_\alpha^x F^{\widehat{\phi}}(t,z)\,dz} = e^{-\theta(t)\int_\alpha^x \frac{1}{\widehat{\phi}(t,z)}\,dz}.$$

Since $\theta(t) > 0$ and by the previous theorem $\widehat{\phi}(t, x) \leq L\theta(t)x$, we have that for any constant $\alpha > 0$

$$e^{\int_\alpha^x F^{\widehat{\phi}}(t,z)\,dz} \leq \left(\frac{x}{\alpha}\right)^{-\frac{1}{L}}.$$

Thus, for t fixed, $\gamma(t, x)$ satisfies the bound (24) with $k = -1/L$ since (39) assures $k > -1$. Finally, we check that $G(t, x, u, \gamma, \gamma_x)$ is concave in u. By (19) γ is positive. Hence, it is decreasing in $x > 0$ since the derivative

$$\gamma_x(t, x) = F(t, x, \phi(t, x))\gamma(t, x) = -\frac{b(t)}{\sigma(t)^2\phi(t, x)}\gamma(t, x) < 0,$$

because $b(t) > 0$ and $\widehat{\phi}(t,x) > 0$ for all t and $x > 0$. Hence $G_{uu} = \sigma(t)^2\gamma_x < 0$. Then, $\widehat{\phi}$ is optimal by Remark 4.1. □ □

The solution is illustrated in Figure 1 for a Merton model subject to a business cycle in the interest rate, which is reflected in $r(t) = 0.05 + 0.02\sin(\pi t/2)$. The variance parameter is $\sigma = 0.2$, the excess return is $b = 0.02$ and the time horizon $T = 2$. The utility function is $S(x) = x^{1-\rho}/(1-\rho) + x^{1-\beta}/(1-\beta)$ with $\rho = 0.9$ and $\beta = 3$. The absolute risk tolerance index, $R(x) = 10x\frac{x^{2.1}+1}{9x^{2.1}+30}$, fits the requirements of Theorem 5.2. In this model, poor people have relative risk aversion index of approximately 3, whilst for rich people it is approximately 0.9. Thus, rich people are more willing to invest in risky assets than poor people, and the solution shows convexity in wealth. For computing the solution, the routine pdepe implemented in MATLAB has been used.

6 Application to Separable Models

In this section we exploit the PDE (12) in problems that show a particular structure, which we call separable. Suppose $\ell = 1$, and that functions $S(x)$, $f(x,u) = f_0(x)f_1(u)$ and $\sigma(x,u) = \sigma_0(x)\sigma_1(u)$ are of class C^2 and independent of time. Assume that the products $f_0 f_1'$ and $\sigma_0\sigma_1'$ are different from zero. We also assume that for any constant value ν in the control region U, the SDE

$$dX = f_0(X)f_1(\nu)\,ds + \sigma_0(X)\sigma_1(\nu)\,dw(s)$$

admits a unique strong solution, for any initial condition (t,x).

We impose the following structural conditions on the data.

(i) there exists a constant k such that

$$f_0(x)S'(x) = k\sigma_0^2(x)S''(x), \quad \forall x \in \mathbb{R};$$

(ii) there exists λ, interior to the control region U, such that

$$kf_1'(\lambda) = -\sigma_1(\lambda)\sigma_1'(\lambda);$$

(iii) the following product does not depend on x

$$a \overset{\text{not.}}{=} \left(f_0'(x) + \frac{1}{k}\frac{f_0^2(x)}{\sigma_0^2(x)}\right)\left(f_1(\lambda) + \frac{1}{2}\frac{\sigma_1^2(\lambda)}{k}\right).$$

Although these conditions may seem stringent, they are fulfilled in some interesting and common models, such as Merton's model, as will be shown below. Our claim is the following.

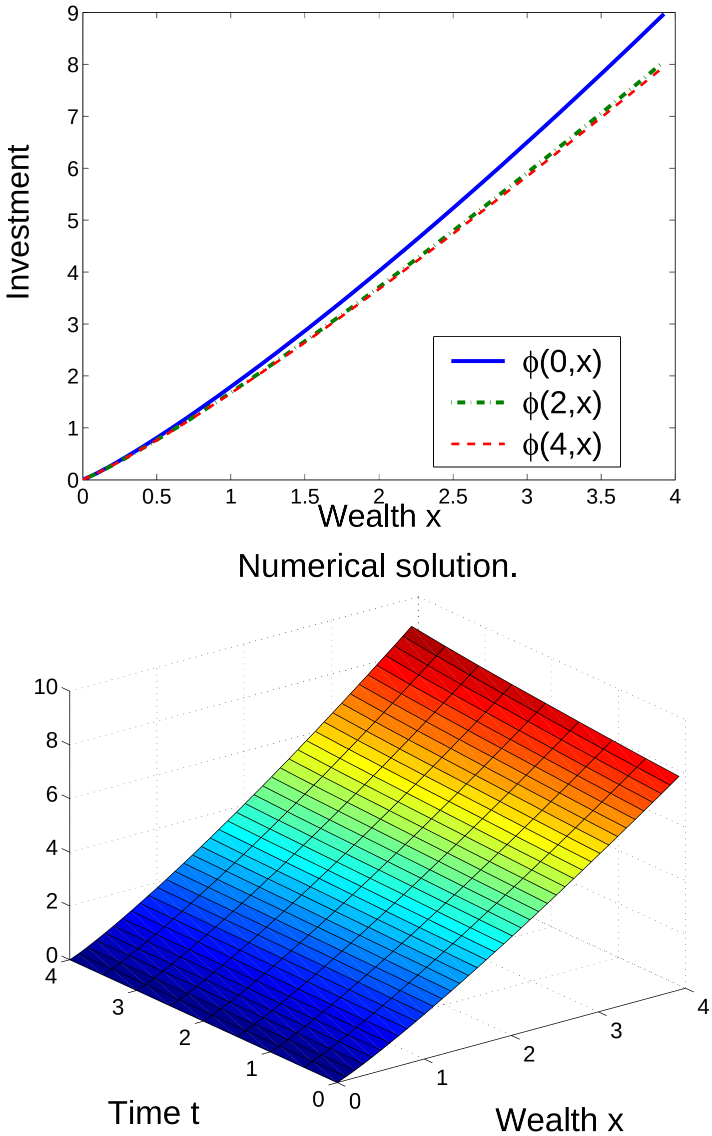

Figure 1: Solution profile and solution surface in the Merton model with relative risk tolerance index $R(x)/x = 10(x^{2.1} + 1)/(9x^{2.1} + 30)$.

Proposition 6.1 *Suppose that for a separable model, assumptions (i)–(iii) hold. If G is concave with respect to u, and S is polynomially bounded, then the optimal control is constant, $\widehat{\phi}(t,x) = \lambda$, with λ defined in (i)–(ii). Moreover, the value function is given by*

$$\begin{aligned} V(t,x) &= e^{-\delta(T-t)}\Big(e^{a(T-t)}S(x) + (1-e^{a(T-t)})\Big(S(\alpha) - \frac{f_0(\alpha)}{m(\alpha)}S'(\alpha)\Big)\Big), \\ &\qquad \textit{if } a \neq 0, \\ &= e^{-\delta(T-t)}\Big(S(x) + (T-t)f_0(\alpha)S_0'(\alpha)\Big(f_1(\lambda) + \frac{1}{2}\frac{\sigma_1^2(\lambda)}{k}\Big)\Big), \\ &\qquad \textit{if } a = 0, \end{aligned}$$

where a was defined in (ii) above, α is an arbitrary constant, and $m(\alpha) = f_0'(\alpha) + (1/k)(f_0^2(\alpha)/\sigma_0^2(\alpha))$.

Proof. With k defined in (ii), we have

$$F(x,u) = \left(\frac{-f_1'(u)}{\sigma_1(u)\sigma_1'(u)}\right)\left(\frac{f_0(x)}{\sigma_0^2(x)}\right), \quad \text{hence} \quad F(x,\widehat{\phi}) = \frac{1}{k}\frac{f_0(x)}{\sigma_0^2(x)},$$

which is independent of t. On the other hand

$$\begin{aligned} G(t,x,\widehat{\phi},1,F) &= f_0(x)\left(f_1(\lambda) + \frac{1}{2}\frac{\sigma_1^2(\lambda)}{k}\right), \\ G(t,x,\widehat{\phi},F,F^2) &= \frac{f_0^2(x)}{k\sigma_0^2(x)}\left(f_1(\lambda) + \frac{1}{2}\frac{\sigma_1^2(\lambda)}{k}\right). \end{aligned}$$

Thus, the PDE (17) for F is fulfilled because

$$J^{\widehat{\phi}}(t,x) = \left(\frac{f_0^2(x)}{k\sigma_0^2(x)} + f_0'(x)\right)\left(f_1(\lambda) + \frac{1}{2}\frac{\sigma_1^2(\lambda)}{k}\right) = a$$

by (iii). Thus, the constant control $\widehat{\phi} \equiv \lambda$ satisfies the equation (17), since $J^{\widehat{\phi}}$ is also constant. Regarding the final condition, it is given by $S'(x)F(T,x,\widehat{\phi}) = S''(x)$, which is simply (i).

To find the value function and show the optimality of $\widehat{\phi}$, we will use Theorem 4.1, hence we find first the adjoint feedback. As has just been shown, $J^{\widehat{\phi}}(t,x) = a$ for all (t,x), hence function F evaluated at the optimal control is $F^{\widehat{\phi}}(t,z) = \frac{1}{k}\frac{f_0(z)}{\sigma_0^2(z)} = \frac{S''(z)}{S'(z)} = (\ln S')'(z)$ for all (t,z), where it has been used (i) in the second equality. In accordance with (19), the adjoint variable is

$$\gamma(t,x) = S'(\alpha)e^{\int_\alpha^x (\ln S')'(z)dz + a(T-t)} = e^{a(T-t)}S'(x).$$

The generalized Hamiltonian evaluated at the optimal control is

$$\mathcal{G}^{\widehat{\phi}}(t,x) = e^{a(T-t)}\left(f_0(x)f_1(\lambda)S'(x) + \frac{1}{2}\sigma_0^2(x)\sigma_1^2(\lambda)S''(x)\right)$$

and using this, it is straightforward to find the value function when $a \neq 0$ by means of (23), once conditions (i)–(iii) are used. The case $a = 0$ is obtained by taking limits as $a \to 0$. Since V is polynomially bounded, Theorem 4.1 applies, showing that $\widehat{\phi} = \lambda$ is optimal. $\square$
$\square$

This result provides a solution to the HJB equation

$$V_t(t,x) + \max_{u \in \mathbb{R}} \left\{ f_0(x) f_1(u) V_x(t,x) + \frac{1}{2}\sigma_0^2(x)\sigma_1^2(u) V_{xx}(t,x) \right\} = \delta V(t,x),$$

$$V(T,x) = S_0(x),$$

and a (constant) maximizing control, under conditions (i)–(iii). At first sight, it is not apparent what the solution of the HJB equation is; it is even difficult to obtain the explicit form of this non–linear equation, since the maximization cannot be carried out explicitly.

Let us illustrate the use of conditions (i)–(iii) above in some specific models.

6.1 Logarithm utility function

Consider the problem of maximizing $e^{-\delta(T-t)} \mathrm{E}_{tx} \{\ln X(T)\}$ subject to

$$dX = bu\, X \ln X\, ds + \beta u\, X \sqrt{\ln X}\, dw(s),$$

with initial condition $X(t) = x > 1$, constants $b > 0$, $\beta > 0$, and control region $U = [0, \infty)$. Let us check that (i)–(iii) are fulfilled for suitable constants k and λ. Here, $f_0(x) = x \ln x$ and $\sigma_0(x) = x\sqrt{\ln x}$, thus (i) holds if and only if $k = -1$, and then (ii) gives $\lambda = b/\beta^2 > 0$; finally, it is easy to see that (iii) is automatically satisfied for $a = b^2/(2\beta^2)$. Thus, the constant control $\widehat{\phi} = b/\beta^2$ is a solution of the PDE and satisfies the final condition. Notice that, under $\widehat{\phi}$, the process $Y = \ln X$ has the dynamics

$$dY = \frac{b^2}{2\beta^2} Y ds + \frac{b}{\beta}\sqrt{Y} dw(s),$$

with $Y(t) = \ln x > 0$. This process is positive with probability one, so it admits a well–defined solution. Hence, $X > 1$ with probability one and the process X is well defined. The generalized Hamiltonian G is concave with respect to u because

$$G_{uu} = \beta^2 x^2 e^{-a(T-t)} \ln x = -\beta^2 e^{-a(T-t)} \ln x < 0.$$

By Proposition 6.1, $V(t,x) = \exp\left((b^2/(2\beta^2) - \delta)(T-t)\right) \ln x$, since $S(\alpha) - S'(\alpha)(f_0(\alpha)/m(\alpha)) = 0$.

6.2 Merton's Model for a large investor

Consider again the model of Merton introduced in Example 5.2, but now considering $u(s)$ not as the total wealth invested in the risky asset, but the proportion of wealth invested. Both problems are identical, although the meaning of the control variable is different. Consider also the case where the coefficients are constant. The model is separable, with $f_0(x) = \sigma_0(x) = x$, $f_1(u) = r + bu$ and $\sigma_1(u) = \sigma u$. Assumption (i) holds with $k = -\rho^{-1}$ for HARA utilities $S(x) = A(T)x^{1-\rho}/(1-\rho)$, where $A(T) > 0$, $\rho > 0$, $\rho \neq 1$, or $S(x) = A(T)\ln x$ (this case could be embedded in the above with $\rho = 1$), which were already considered in Section 5.2. Using (ii) we get the constant control: $\widehat{\phi} = \lambda = \rho^{-1}(b/\sigma^2)$—that obviously agrees with our findings in Section 5.2—and (iii) gives

$$a = (1-\rho)(r + b\lambda - (1/2)\,\rho\sigma^2\lambda^2) = (1-\rho)(r + (1/2)\rho^{-1}(b^2/\sigma^2)).$$

According to Proposition 6.1, the value function is

$$V(t,x) = e^{((a-\delta)(T-t))}S(x) = e^{((a-\delta)(T-t))}A(T)\frac{x^{1-\rho}}{1-\rho}.$$

Note that G is concave with respect to u because $G_{uu} = \sigma^2 x^2 V_{xx} < 0$.

Nothing new in the above, of course, but consider the following variation of the problem, which could be applicable in financial economics. Suppose that the investor is large, in the sense that his/her investment decisions influence the evolution of the market price of the asset. We are thinking of large financial institutions whose performance (benefits or losses) affect the global financial market. Well known examples of this possibility were the crashes due to the "Hedge Fund Crisis" of 1998 and some others, more recent cases. Nevertheless, our model is only academic, with a view to illustrating our results.

Suppose that the price of the risky asset is given by

$$dP^1(t) = (b(u) + r)P^1(t)\,dt + \sigma(u)P^1(t)\,dw(t), \quad P^1(0) \text{ known.}$$

We observe that the investment decisions of the investor influence the price dynamics through functions b and σ, which we consider to be of class C^2 and positive. The riskless return r is supposed to be constant. Then, the wealth evolves as

$$dX(s) = X(s)\big(r + u(s)b(u(s))\big)\,dt + X(s)u(s)\sigma(u(s))\,dw(s), \qquad t \le s \le T.$$

Thus, we identify a separable dynamics, with

$$\begin{aligned} f_0(x) &= \sigma_0(x) = x, \\ f_1(u) &= r + ub(u), \\ \sigma_1(u) &= u\sigma(u). \end{aligned}$$

Consider again a HARA utility function. We have already tested that condition (i) holds with $k = -\rho^{-1}$. For (ii), let us be more specific. Choose $b(u) = b\exp(-\beta u)$ and $\sigma(u) = \sigma$ with b, and σ positive constants, and let β be non–negative, satisfying $-\rho^{-1}b\exp(-\beta)(1-\beta)+\sigma^2 > 0$. Notice that $\beta = 0$ gives the Merton model for a small investor, where the above inequality means that the optimal proportion of wealth invested in the risky asset is not bigger than one, which is a desirable feature–it says that no borrowing is allowed–. Wealth dynamics becomes

$$dX(s) = X(s)(r + bu(s)e^{-\beta u(s)})\,dt + X(s)\sigma u(s)\,dw(s), \qquad t \le s \le T.$$

Condition (ii) gives the equation

$$-\rho^{-1}b\exp(-\beta\lambda)(1-\beta\lambda) + \sigma^2\lambda = 0 \tag{40}$$

for the determination of λ. The function $h(u) = -\rho^{-1}b\exp(-\beta u)(1-\beta u)+\sigma^2 u$ is continuous and by assumption, $h(1) = -\rho^{-1}b\exp(-\beta)(1-\beta)+\sigma^2 > 0$. Since $h(0) = -\rho^{-1}b < 0$ and h is strictly monotone increasing in $[0,1]$, equation (40) admits a unique solution, $\widehat{\phi} = \lambda \in (0,1)$, which is the constant optimal control. Finally, (iii) is satisfied since the left hand side factor defining a is independent of x. Thus, large investors with HARA utility may behave as in the typical Merton's model where the investment decisions do not affect the price, that is, the optimal investment rule for large investors is also proportional to total wealth as for small investors.

The value function of the problem can also be found

$$V(t,x) = \exp((a-\delta)(T-t))A(T)(1-\rho)^{-1}x^{1-\rho},$$

where $a = (1-\rho)\left(r + b\lambda\exp(-\beta\lambda) - (1/2)\rho\sigma^2\lambda^2\right)$. Finally, the generalized Hamiltonian G is concave with respect to u in the relevant range of values $[0,1]$, since the value function is strictly increasing and strictly concave. This can be verified testing that the second order derivative

$$G_{uu} = b\beta e^{-\beta u}(\beta u - 2)xV_x + \sigma^2 x^2 V_{xx} < 0,$$

for all $u \in [0,1]$, selecting $\beta < 2$.

Of course, conditions (i)–(iii) apply to much more general pairs of functions b, σ. From an economic point of view, our selection implies that by buying the risky asset, the investor diminishes the mean market price of the risky stock, having no effect on its volatility. For general b and σ, condition (ii) is $k(\lambda b(\lambda))' + (\lambda\sigma(\lambda))(\lambda\sigma(\lambda))' = 0$ for some $\lambda \in [0,1]$, that should be tested for the specific model at hand. Obviously, the concavity of the generalized Hamiltonian can also be studied as in the parametric example considered.

We have selected the particular pair b, σ above as an illustration of the result, and to show the difficulties associated in solving the problem by means of the HJB equation. Whilst with our methods the problem has been solved fairly well, finding the solution with the HJB equation is difficult, as the maximization condition

$$bxe^{-\beta u}(1-\beta u)V_x + x^2\sigma^2 u V_{xx} = 0 \tag{41}$$

cannot be solved explicitly to express u in terms of V_x, V_{xx}. Thus, the PDE (7) for the adjoint feedback γ cannot be explicitly given, since

$$\frac{\gamma_x}{\gamma} = F = -\frac{be^{-\beta u}(1-\beta u)}{\sigma^2 x u}$$

cannot be solved for $u = \overline{u}(x, \gamma, \gamma_x)$.

7 Conclusions

This paper proposes a systematic method to find a PDE for the optimal control to study one–dimensional stochastic models of Mayer type. These models are usual in financial economics, as the mean variance portfolio problem or the Merton model without consumption. The PDE for the optimal control is a Euler companion to the usual HJB equation. Whereas this equation is fully non–linear, the former is of quasilinear type. This fact allows us to show the existence of the optimal control in the Merton model with time varying (but deterministic) coefficients, even for utility functions with an unbounded absolute risk tolerance index (but with at most linear growth). We also provide sufficient conditions in terms of the PDE found, similar to the verification theorems in [3]. The connection between the optimal control and the value function is explicitly found through the adjoint feedback γ. Finally, the theory is applied to a family of problems that shows multiplicative separability in the state and the control variable, showing how the new PDE helps in determining the (a priori hidden) solution in the form of a constant control. The classical Merton's model belongs to this class, but also the problem which considers a large investor whose decision can influence the price of the stock. Surprisingly enough, it is shown that the optimal investment rule of this problem with non–linear dynamics is proportional to the wealth level, as in the traditional model, when the investor shows HARA preferences.

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