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THE DISCRETE NERLOVE-ARROW MODEL: EXPLICIT SOLUTIONS

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The discrete Nerlove-Arrow model: Explicit solutions

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Abstract

We use the optimality principle of dynamic programming to formulate a discrete version of the Nerlove-Arrow maximization problem. When the payoff function is concave we derive an explicit solution to the problem. If the time horizon is long enough there is a "transiently stationary" (turnpike) value for the optimal capital after which the capital must decay as the end of the time horizon approaches. If the time horizon is short the capital is left to decay after a first-period increase or decrease depending on the capital's initial value. Results are illustrated with the payoff function μK^{λ} where K is the capital and $0 < \lambda < 1, \mu > 0$. With this function, the solution is in closed form.

Keywords: Nerlove-Arrow, dynamic programming, optimization, turnpike.

JEL Classification: C61,O21.

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1 Introduction

In their seminal paper Nerlove and Arrow (1962) describe the eponymous model which tackles a firm's search for the stream of advertising expenditure used to purchase the "goodwill" that will maximize the present value profit.

The problem's generality is remarkable. Indeed, it rapidly became clear that goodwill can just as well be human capital of some sorts (Becker, 1962), health capital (Grossman, 1972), or a stock of durable goods leased to others (Weber, 2005). The "profit" in those cases is an individual's earnings, a population's well-being or rents collected. (See Kamien and Schwartz (1991), Sethi (1977) and Feichtinger, Hartl and Sethi (1994) for reviews; also De Souza and Yoneyama (1991) for an application in public health). In a general framework we will thus refer to a stock K of some unspecified capital instead of goodwill, and to a payoff function $\pi(K)$ instead of a profit function.

The model has been extended not just to diverse application areas, but also to account for stochastic effects (Raman, 2006), budgetary constraints (Sethi, 1977), or both (Marinelli, 2007). Further extensions entail interactions between several firms (see Karray and Zaccour (2007), Rubel and Zaccour (2007), Doraszelski and Markovich (2007), Grosset and Viscolani (2009) for recent papers on this topic).

One aspect of the solution that has attracted considerable attention is the so-called "turnpike" (McKenzie, 1976, 1982). This imaginative terminology graphically captures a common characteristic of the solution when the time horizon is long enough. Indeed, in this case the payoff is maximized by bringing the capital to a "transiently stationary" value (the turnpike) where it must stay a certain duration before decaying as the end of the time horizon approaches.

The Nerlove-Arrow problem is a difficult one, usually formulated in continuous time, and solved using advanced mathematical techniques from the calculus of variation and from optimal control, both deterministic and stochastic (Kamien and Schwartz, 1991). However, explicit solutions are rarely forthcoming. Insights are often provided in terms of “necessary conditions”. Alternatively, numerical methods are used which usually amount to a discretization of the problem.

Here we will formulate the original Nerlove-Arrow problem in a discrete framework, then solve it explicitly. The solution will shed light, in particular, on conditions for the existence of a turnpike.

Section 2 describes the discretized form of the problem. The optimality principle of dynamic programming gives rise to a simple non-autonomous iterative procedure that yields the optimal solution for any payoff function. Section 3 moves the algorithm further when the payoff function is concave. In this case the operators $\{G_m\}$ used in the iterative procedure are extremely simple functions determined by a finite sequence $\{c_m\}$ that is calculated explicitly on the basis of the model's specifications (payoff function, discount rate, unit price of capital and depreciation rate). Section 4, in the mostly self-contained Proposition 3, translates the iterative procedure into explicit expressions for the solution. Section 5 illustrates the results with the profit function μK^λ for which the solution is in closed form. Section 6 wraps things up with a brief discussion and concluding remarks.

2 Discretized Nerlove-Arrow model

2.1 Discretization

We let K be the capital and $\pi(K)$ be the payoff function. The continuous-time Nerlove-Arrow dynamic optimization model aims to find the expenditure z on capital that maximizes the present-value payoff over a time horizon $(0, T)$. With K and z being function of time t, the maximized payoff is

$$W(T) \stackrel{\text{def.}}{=} \max_z \left\{ \int_0^T e^{-rt}(\pi(K) - z)dt \right\} \tag{1}$$

with

$$z = \beta(\dot{K} + K\delta) \tag{2}$$

where r is the discount rate; β is the unit price of capital and δ is the rate at which capital depreciates; $K(0)$ is the initial value of K.

We now consider a time horizon of k discrete periods with an initial capital K_0 and $k-1$ subsequent unknown values $K_1, K_2, ..., K_{k-1}$. The discrete version of Eq. (2) yields the expenditures

$$z_m = \beta(K_{m+1} - K_m(1-\delta)),\ m = 0, 1, \ldots, k-2, \tag{3}$$

or

$$K_{m+1} = K_m(1-\delta) + z_m/\beta. \tag{4}$$

The discrete version of the maximization problem in Eq. (1) is then

$$W_k(K_0) = \max_z \left\{ \sum_{m=0}^{m=k-1} (\pi(K_m) - z_m)(1+r)^{-m} \right\}. \tag{5}$$

The last expenditure z_{k-1} must be equal to 0 since any positive z_{k-1} would lower $\pi(K_{k-1}) - z_{k-1}$, the last term of the sum in Eq. (5). Therefore

$$W_1(K_0) = \pi(K_0). \tag{6}$$

2.2 Dynamic programming formulation

We use the optimality principle of dynamic programming to write Eq. (5) as

$$
\begin{aligned}
W_k(K_0) &= \max_{z} \left\{ (\pi(K_0) - z_0) + (1+r)^{-1} \sum_{m=1}^{m=k-1} (\pi(K_m) - z_m)(1+r)^{-m+1} \right\} \\
&= \max_{z_0 \geq 0} \left\{ (\pi(K_0) - z_0) + (1+r)^{-1} W_{k-1}(K_0(1-\delta) + z_0/\beta) \right\} \\
&= \pi(K_0) + \beta K_0(1-\delta) + \max_{K_1 \geq K_0(1-\delta)} \left\{ (1+r)^{-1} W_{k-1}(K_1) - \beta K_1 \right\}, \qquad (7)
\end{aligned}
$$

where we have formulated the maximization problem by seeking the optimal K_m's rather than the optimal z_m's.

Working backwards we then have, for $p = 0, 1, \ldots, k-2$,

$$
\begin{aligned}
W_{k-p}(K_p) = \pi(K_p) + \beta K_p(1-\delta) + \\
\max_{K_{p+1} \geq K_p(1-\delta)} \left\{ (1+r)^{-1} W_{k-p-1}(K_{p+1}) - \beta K_{p+1} \right\} \qquad (8)
\end{aligned}
$$

and $W_1(K_{k-1}) = \pi(K_{k-1})$. We divide both sides of Eq. (8) by $1+r$ and then subtract βK_p to obtain

$$
\begin{aligned}
&\frac{W_{k-p}(K_p)}{1+r} - \beta K_p = \\
&\frac{\pi(K_p) - \beta K_p(\delta + r) + \max_{K_{p+1} \geq K_p(1-\delta)} \{(1+r)^{-1} W_{k-p-1}(K_{p+1}) - \beta K_{p+1}\}}{1+r}. \qquad (9)
\end{aligned}
$$

The expression on the left-hand side and the one in the curly braces on the right-hand side of Eq. (9) have the same form, at the orders $k-p$ for the former and $k-p-1$ for the latter. In order to exploit this fact we first define the function

$$
J^*(K) \stackrel{\text{def.}}{=} \frac{\pi(K) - \beta K(\delta + r)}{1+r} \qquad (10)
$$

which up to the multiplicative constant $1+r$ is the net profit function of Eq. (12) in Nerlove and Arrow (1962).

We define the sequence $H_m(K)$ $(m = 1, 2, ...)$ of functions on the left-hand side of Eq. (9):

$$H_1(K) \stackrel{\text{def.}}{=} \frac{W_1(K)}{1+r} - \beta K = \frac{\pi(K)}{1+r} - \beta K = J^*(K) + \frac{\beta K(\delta - 1)}{1+r} \tag{11}$$

and

$$H_m(K) = J^*(K) + \frac{1}{1+r} \times \begin{cases} \beta K(\delta - 1) & \text{if } m = 1 \\ \max\limits_{K' \geq K(1-\delta)} H_{m-1}(K') & \text{if } m \geq 2. \end{cases} \tag{12}$$

For the moment we assume that a finite max exists in Eq. (12). With these notations Eq. (9) becomes

$$H_{k-p}(K_p) = \frac{W_{k-p}(K_p)}{1+r} - \beta K_p \tag{13}$$

$$= J^*(K_p) + \frac{\max\limits_{K_{p+1} \geq K_p(1-\delta)} \{H_{k-p-1}(K_{p+1})\}}{1+r} \tag{14}$$

where the optimal K_{p+1} is the value of K_{p+1} at which the maximum is attained. When this maximum is attained at $K_{p+1} = K_p(1-\delta)$ we will say that K_{p+1} is "sticky" or that the capital is left to decay with none being purchased ($z_p = 0$).

If we define

$$G_m(K) \stackrel{\text{def.}}{=} \{K'; K' \text{ maximizes } H_m, K' \geq K(1-\delta)\}, \tag{15}$$

then Eq. (14) shows that for a given initial K_0, the optimal K_m's are given by the non-autonomous iterative process (operators changing with each iteration):

$$K_1 = G_{k-1}(K_0),\ K_2 = G_{k-2}(K_1), \ldots,\ K_{k-1} = G_1(K_{k-2}). \tag{16}$$

The solution thus hinges on the knowledge of the functions H_m of Eq. (12), which can be calculated numerically, but at a considerable computational cost. Indeed, calculating $H_m(K)$ requires the composition of m functions, each with a maximum that usually has to be found numerically.

The solution given here will rest on the idea that we only need to know the value(s) at which each H_m reaches a maximum. The problem is made simpler when H_m has a single maximum, as will be the case when the payoff function is concave.

3 Assumptions and preliminary results

3.1 Assumptions

For the remainder of the paper the payoff function π is assumed to be differentiable and concave with $\pi(0) = 0$. The derivative $\dot{\pi}$ then decreases while remaining non-negative. It must therefore converge to a limit $d \geq 0$ when $K \to \infty$.

Given the derivative

$$\dot{J}^*(K) = \frac{\dot{\pi}(K) - \beta(\delta + r)}{1 + r} \tag{17}$$

of J^* we next dispose of two trivial cases: β either small or large. Equation (17) shows that if $\beta < \frac{d}{r+\delta}$ then $\dot{J}^*(K)$ has a positive lower bound. Therefore $J^*(K)$ tends to ∞ for $K \to \infty$ and the problem does not have a bounded solution.

If $\frac{\dot{\pi}(0)}{r+\delta} < \beta$ then Eq. (17) shows that $\dot{J}^*(K) < 0$ for all K. The second term on the right-hand side of Eq. (12) is a non-increasing function of K and therefore H_m is decreasing on $[0, \infty)$. All optimal K_m's are therefore sticky: $K_m = K_0(1 - \delta)^m$, $m = 1, 2, \ldots, k - 1$.

To avoid trivialities we assume for the remainder of the paper that

$$\frac{d}{r + \delta} < \beta < \frac{\dot{\pi}(0)}{r + \delta}. \tag{18}$$

In a later section we will illustrate the results with the concave function $\pi(K) = \mu K^\lambda$ $(0 < \lambda < 1)$ for which $d = 0$ and $\dot{\pi}(0) = \infty$. The problem will therefore be non-trivial for any $\beta > 0$.

The max in Eq. (12) complicates the definition of the H_m's. However the iterative procedure of (16) requires only the values at which the H_m's reach a maximum. We will produce a sequence of functions $\{H_m^*\}$, closely related to the H_m's, with each H_m^* reaching a single maximum at some tractable c_m (with non-decreasing c_m's). We will show that each H_m^* coincides with H_m on $[c_{m-1}/(1-\delta), \infty)$ and that $c_m > c_{m-1}/(1-\delta)$. The functions H_m and H_m^* therefore reach a maximum at the same value c_m.

Each derivative $\dot{H}_m^*$ will be a decreasing function which makes the calculation of c_m, the zero of $\dot{H}_m^*$, a simple numerical matter. For some payoff functions, such as $\pi(K) = \mu K^\lambda$, the c_m's have a closed-form expression.

3.2 The functions $\{H_m^*\}_{m=1,2,\ldots}$

We define

$$H_m^*(K) \stackrel{\text{def.}}{=} J^*(K) + \frac{1}{1+r} \times \begin{cases} \beta K(\delta - 1) & \text{if } m = 1 \\ H_{m-1}^*(K(1-\delta)) & \text{if } m \geq 2. \end{cases} \tag{19}$$

The functions $H_1^*(K)$ and $H_1(K)$ of Eq. (12) are identical. For $m \geq 2$ the one difference is that the max of $H_{m-1}(K')$ is replaced by $H_{m-1}^*(K')$ at $K' = K(1-\delta)$ (as if the max were always attained at $K(1-\delta)$, i.e. all K_m's were sticky).

It is easy to see that

$$H_m^*(K) = \sum_{q=1}^{q=m} \frac{\pi(K(1-\delta)^{q-1})}{(1+r)^q} - \beta K, \; m = 1, 2, \ldots \tag{20}$$

The derivatives of these function are

$$\dot{H}_m^*(K) = \sum_{q=1}^{q=m} \frac{(1-\delta)^{q-1}\dot{\pi}(K(1-\delta)^{q-1})}{(1+r)^q} - \beta \tag{21}$$

$$= \frac{1}{1+r} \sum_{q=1}^{q=m} \rho^{q-1}\dot{\pi}(K(1-\delta)^{q-1}) - \beta, \tag{22}$$

where

$$\rho \stackrel{\text{def.}}{=} \frac{1-\delta}{1+r} < 1. \tag{23}$$

Because $\dot{\pi}$ is bounded and $\rho < 1$, the sequence of derivatives $\left\{\dot{H}_m^*\right\}$ converges to a limit $\dot{H}_\infty^*(K)$ for $m \to \infty$:

$$\dot{H}_\infty^*(K) \stackrel{\text{def.}}{=} \lim_{m\to\infty} \dot{H}_m^*(K) = \frac{1}{1+r}\sum_{q=1}^{q=\infty} \rho^{q-1}\dot{\pi}(K(1-\delta)^{q-1}) - \beta. \tag{24}$$

Because $\dot{\pi}$ is decreasing, each derivative $\dot{H}_m^*$ is also a decreasing function with a value at 0 equal to:

$$\dot{H}_m^*(0) = \frac{1}{1+r}\sum_{q=1}^{q=m} \rho^{q-1}\dot{\pi}(0) - \beta \tag{25}$$

$$= \frac{\dot{\pi}(0)(1-\rho^m)}{(1+r)(1-\rho)} - \beta \tag{26}$$

$$= \frac{\dot{\pi}(0)(1-\rho^m)}{\delta+r} - \beta,\ m = 1, 2, \ldots \tag{27}$$

We next investigate when the derivatives go from being positive to negative, i.e. circumstances under which H_m^* reaches a maximum.

3.3 Theoretical results on the H_m^*'s

The following proposition provides results on the behavior of the derivatives $\dot{H}_m^*$.

Proposition 1. *We assume that* (18) *holds. The derivative $\dot{J}^*$ is a decreasing function that is positive at 0 and reaches 0 at K^*, the root of*

$$\dot{J}^*(K^*) = 0 = \frac{\dot{\pi}(K^*) - \beta(\delta+r)}{1+r}. \tag{28}$$

We define

$$m^* \stackrel{\text{def.}}{=} \left\lceil \frac{\ln\left(1 - \frac{\beta(r+\delta)}{\dot{\pi}(0)}\right)}{\ln(\rho)} \right\rceil \geq 0, \tag{29}$$

where [•] is the integer part function.

If $m^* \geq 1$ then for any $m \leq m^*$ the derivative $\dot{H}^*_m$ is non-positive at $K = 0$ and decreases on $[0, \infty)$. This means that if we define

$$c_m \stackrel{def.}{=} \inf \left\{ K > 0; \dot{H}^*_m(K) < 0 \right\} \tag{30}$$

then $c_m = 0$ for $m \leq m^*$.

For any $m > m^*$ then $\dot{H}^*_m(K)$ is positive for $K = 0$ and drops below 0 at $c_m > 0$ which is now the root of

$$\dot{H}^*_m(c_m) = \frac{1}{1+r} \sum_{q=1}^{q=m} \rho^{q-1} \dot{\pi}(c_m(1-\delta)^{q-1}) - \beta = 0. \tag{31}$$

The c_m's increase for $m \to \infty$ and reach a limit c_∞ which is the root of $\dot{H}^*_\infty(K) = 0$ and is strictly larger than K^*, the root of $\dot{J}^*(K) = 0$. We can then define

$$p^* \stackrel{def.}{=} \max \{m; c_m \leq K^*\} \tag{32}$$

and we have

$$\begin{aligned} & 0 = c_1 = c_2 = \ldots = c_{m^*} \\ & < c_{m^*+1} < \frac{c_{m^*+1}}{1-\delta} < c_{m^*+2} < \frac{c_{m^*+2}}{1-\delta} \ldots < \frac{c_{p^*-1}}{1-\delta} < c_{p^*} \leq K^* \\ & < c_{p^*+1} < \begin{cases} \dfrac{c_{p^*}}{1-\delta}. \\ c_{p^*+2} < \ldots < c_\infty. \end{cases} \end{aligned} \tag{33}$$

Proof. See Appendix A.1. □

The next proposition provides the required result on the maximum of each H_m.

Proposition 2. *When (18) holds then for $m = 1, 2, \ldots, p^*$ we have:*

- *P_1: $H_m(K) = H^*_m(K)$ for $K \geq c_{m-1}/(1-\delta)$ ($c_{-1} \equiv 0$).*

- *P_2: The functions H_m increase on $[0, c_m]$ and decrease on (c_m, ∞).*

We also have

- *P_3: For $m \geq p^*+1$ the functions H_m increase on $[0, K^*]$ and decrease on (K^*, ∞).*

Proof. See Appendix A.2. □

We now provide the explicit solutions to the discretized Nerlove-Arrow problem.

4 Main result

Proposition 2 states that each function H_m $(m = 1, 2, \ldots, p^*)$ has a unique maximum at c_m. We redefine the subsequent c_m's $(m \geq p^* + 1)$ as being all equal to K^*, rather than to the maximum of each H_m^*. In this way H_m has a unique maximum at

$$c_m \overset{\text{re-def.}}{=} \begin{cases} c_m & \text{for } m = 1, 2, \ldots, p^* \\ K^* & \text{for } m \geq p^* + 1. \end{cases} \tag{34}$$

The functions G_m of Eq. (15) are

$$G_m(K) = \begin{cases} K(1-\delta) & \text{if } K > c_m/(1-\delta) \\ c_m & \text{otherwise} \end{cases} \tag{35}$$

where each G_m has the unique fixed point $c_m = G_m(c_m)$.

To simplify the presentation of the results we define the following partition of the positive axis:

$$I_0 \overset{\text{def.}}{=} \left[0, \frac{K^*}{1-\delta}\right) ; I_m \overset{\text{def.}}{=} \left[\frac{K^*}{(1-\delta)^m}, \frac{K^*}{(1-\delta)^{m+1}}\right), \; m = 1, 2, \ldots. \tag{36}$$

The next result uses the index j of the interval I_j that contains the initial K_0 to formulate the explicit solutions.

Proposition 3. *We consider the discrete Nerlove-Arrow model of* (5) *with a concave payoff function* $\pi(K)$ *whose derivative converges to* $d \geq 0$ *for* $K \to \infty$*. We assume that the unit price* β *of capital satisfies*

$$\frac{d}{r+\delta} < \beta < \frac{\dot{\pi}(0)}{r+\delta}. \tag{37}$$

The time horizon is k *and the initial stock is* K_0 *belonging to some* I_j*. The integer* p^* *is the largest integer* m *for which the root of* $\dot{H}_m^*(K) = 0$ *is no larger than* K^**, the root of* (28)*. We also recall the* c_m*'s redefined in* (34)*.*

The optimal values $K_1, K_2, \ldots, K_{k-1}$ *are obtained through the following iteration:*

$$K_m = G_{k-m}(K_{m-1}),\ m = 1, 2, \ldots, k-1, \tag{38}$$

where the G_m*'s are given by Eq.* (35)*.*

If we define

$$w \overset{def.}{=} k - 1 - p^* \tag{39}$$

we will say that the time horizon k *is "short" (or "long") when* $w \leq 0$ *(or* $w > 0$*). Explicit expressions for the* K_m*'s are obtained by considering two cases which depend on the value of* w *(Figure 1).*

Case C1: $w \leq 0$*, i.e. "short time horizon". The optimal values are*

$$K_1 = \begin{cases} K_0(1-\delta) & \text{if } K_0 \geq c_{k-1}/(1-\delta) \\ c_{k-1} & \text{if } K_0 < c_{k-1}/(1-\delta) \end{cases} \tag{40}$$

and

$$K_m = K_1(1-\delta)^{m-1}, m = 2, 3, \ldots, k-1. \tag{41}$$

Case C2: $w > 0$*, i.e. "long time horizon". We distinguish between two subcases, depending on the interval* I_j *that contains* K_0*.*

1. *Subcase C2a: $K_0 \in I_0$, i.e. "low K_0". The first w K_m's are equal to the "transiently stationary" (turnpike) value K^*:*

$$K_1 = K_2 = \ldots = K_w = K^*. \tag{42}$$

Then

$$K_{w+1} = c_{p^*} \tag{43}$$

with the last $p^* - 1$ K_m's being sticky ("exit period" of duration $p^* - 1$):

$$K_m = c_{p^*}(1-\delta)^{m-w-1},\ m = w+2, w+3, \ldots, k-1. \tag{44}$$

2. *Subcase C2b. $K_0 \in I_j$, $j \geq 1$, i.e. "high K_0". If $w < j$ then all K_m's are sticky:*

$$K_m = K_0(1-\delta)^m, m = 1, 2, \ldots, k-1. \tag{45}$$

If $w \geq j$ then only the first j K_m's are sticky

$$K_m = K_0(1-\delta)^m, m = 1, 2, \ldots, j. \tag{46}$$

When $w > j$ the next $w - j$ K_m's are equal to K^*:

$$K_m = K^*, m = j+1, j+2, \ldots, w. \tag{47}$$

Whether $w = j$ or not we have

$$K_{w+1} = c_{p^*}, \tag{48}$$

with the last $p^* - 1$ K_m's being sticky (exit period):

$$K_m = c_{p^*}(1-\delta)^{m-w-1}, m = w+2, w+3, \ldots, k-1. \tag{49}$$

Proof. See Appendix A.3. □

The optimal expenditures z_m are obtained through Eq. (3). In particular the "transiently stationary" expenditure z^* corresponds to K^* and is

$$z^* = \beta(K^* - K^*(1-\delta)) = \beta\delta K^*. \tag{50}$$

The maximized present-value payoff $W_k(K_0)$ is given in Eq. (5).

The solution as described in Eqs. (40)-(49) is consistent with what is known in the continuous framework (existence of a turnpike, etc). The fact that the solution depends on the time horizon is reflected in the iteration $K_m = G_{k-m}(K_{m-1})$ which shows that each K_m is a function of K_{m-1} that depends on the remaining duration $k-m$.

The results quantify precisely the fact that for a long enough time horizon the capital is brought down or up as quickly as possible to the transiently stationary value K^*. The capital is left to decay with no more expenditures as the end of the time horizon approaches.

For a short time horizon k and an initial value K_0 larger than $c_{k-1}/(1-\delta)$ the capital is left to decay. For an initial value smaller than $c_{k-1}/(1-\delta)$ the optimal capital jumps up to $K_1 = c_{k-1}$ if $K_0 < c_{k-1}$ and jumps down to the same $K_1 = c_{k-1}$ if $c_{k-1} < K_0 < c_{k-1}/(1-\delta)$. After this first period the capital is left to decay.

Proposition 3 shows that for a given time horizon k each optimal K_m is either sticky or one of three numbers: c_{k-1}, c_{p^*} or K^* which are simple to calculate numerically. In the example given below, they are in closed form.

5 Application

5.1 Concave payoff function

We consider the payoff function

$$\pi(K) = \mu K^{\lambda},\ 0 < \lambda < 1,\ \mu > 0 \tag{51}$$

which is concave. The derivative $\dot{\pi}(K) = \lambda\mu K^{\lambda-1}$ tends to $d = 0$ for $K \to \infty$. The fact that $d = 0$ and $\dot{\pi}(0) = \infty$ means that (37) holds for any positive β.

All quantities of interest can be expressed in closed form. Indeed, the stationary value K^* of Eq. (28) is

$$K^* = \left(\frac{\beta(\delta + r)}{\mu\lambda}\right)^{\dfrac{1}{\lambda - 1}}. \tag{52}$$

The integer p^* of Eq. (32) is

$$p^* = \left[\frac{\ln\left(\dfrac{\delta - 1 + (1-\delta)^{\lambda}}{\delta + r}\right)}{\ln\left(\dfrac{(1-\delta)^{\lambda}}{1+r}\right)}\right] \tag{53}$$

with $[\bullet]$ the integer part function. For $m \leq p^*$, each c_m of Eq. (31) is

$$c_m = \left(\frac{\beta(1+r)\left(1 - \dfrac{(1-\delta)^{\lambda}}{1+r}\right)}{\mu\lambda\left\{1 - \left(\dfrac{(1-\delta)^{\lambda}}{1+r}\right)^{m}\right\}}\right)^{\dfrac{1}{\lambda - 1}}. \tag{54}$$

The solutions plotted in Figure 2 and Figure 3 were obtained with

$$\beta = 0.4,\ \delta = 0.3,\ r = 0.25,\ \mu = 1,\ \lambda = 0.55 \tag{55}$$

from which

$$K^* = 7.66,\ p^* = 3,\ c_{p^*} = 6.36. \tag{56}$$

Figure 2 (or Figure 3) shows the optimal K_m's and z_m's for an initial value $K_0 = 3$ (or $K_0 = 13$) that is smaller (or larger) than K^*. In each figure panels a1 and b1 depict the solution for a long time horizon ($k = 8, w = k - p^* - 1 = 4 > 0$, Case C2). Panels a2 and b2 depict the solution for a short time horizon ($k = 3, w = k - p^* - 1 = -1 \leq 0$, Case C1).

5.2 Suboptimality analysis

In order to verify our results we perturbed the optimal z_m's and checked that the resulting payoff is indeed smaller than the optimal one. We did this with the example above ($K_0 = 3, k = 8$) by increasing every optimal z_m to 130% of its optimal value. The resulting payoff was 99.1 % of the optimal one. A decrease to 70 % of optimal values results in a payoff that is 98.8 % of the optimal one. When each optimal z_m was independently and randomly taken between 70 % and 130 % of its optimal value (uniform distribution), the resulting payoff was basically never less than 99 % of the optimal one. A "suboptimality sensitivity analysis" is beyond the scope of this paper, but these results suggest that at least in some cases the payoff is quite insensitive to departures from optimality.

5.3 Sensitivity analysis

Substantive insights are gained from explicit solutions. For example the effect of the depreciation rate δ on p^* of Eq. (53) sheds light on the durations of the transiently stationary period and of the exit period (during which the optimal stock decays at the rate δ). When δ increases from 0.0 to 1.0, then Eq. (53) shows that $p^* - 1$ drops from $+\infty$ to 0. This means that for a fixed k and a δ sufficiently small then $p^* \approx +\infty$ and the integer $w = k - p^* - 1$ is negative. We are in Case C1 with all K_m's sticky except possibly K_1 depending on the initial capital K_0. This result has a substantive economic interpretation. Indeed, when the depreciation rate δ is small enough then the payoff is maximized with a single expenditure at the first period if K_0 is smaller than $c_{k-1}/(1-\delta)$. The payoff is maximized without any expenditure if K_0 is larger than $c_{k-1}/(1-\delta)$.

With δ back at 0.3 and an interest rate r that increases from 0 to $+\infty$, the duration p^*-1 of the exit period drops from 3 to 0. For a fixed k the integer $w = k-p^*-1$ is thus an increasing function of δ with consequences that can be explored with Proposition 3.

5.4 Concave to linear payoff function

We recover the case of a linear payoff function by letting λ of (51) tend to 1. Then p^* of Eq. (53) tends to $+\infty$ and with $w \leq 0$ we are in Case C1. If $\mu < \beta(\delta + r)$ then K^* of Eq. (52) and the c_m's of Eq. (54) approach 0 when $\lambda \to 1$. This means all optimal K_m's are sticky: the unit cost β of capital is too high relatively to the marginal profit μ and the optimal strategy is to let capital decay with no new purchase.

If $\mu > \beta(\delta+r)$ then K^* of Eq. (52) and the c_m's of Eq. (54) tend to $+\infty$ when $\lambda \to 1$. A careful application of Proposition 3 in the Case C1 shows that $K_1 = c_{k-1} \to +\infty$ with other K_m's being sticky. The optimal overall payoff $W_k(K_0)$ is therefore unbounded when $\lambda \to 1$ (because the unit cost β is low enough). This trivial result can be derived from first principles by considering the iteration of (16) combined with the fact that J^* of Eq. (10) is itself linear when π is the linear function $\pi(K) = \mu K$.

6 Discussion

The derivatives $\dot{H}_m^*$ in Eq. (22) were decreasing only because every $\dot{\pi}(K(1-\delta)^{q-1})$ was decreasing, which hinged crucially on π being concave. However not all payoff functions are concave. It is no doubt possible to extend the results to a function that is concave only beyond some K^+ by restricting the initial K_0 to be larger than a minimum to be determined. It is unclear to what extent the approach used here could be generalized to other payoff functions.

Another extension is to include a budgetary constraint, as in Sethi (1977). Constraining the model by imposing a maximum total expenditure is a difficult problem. Having a maximum expenditure at each period can however be incorporated into Eq. (14). We do this by seeking an optimal K_{p+1} not in $[K_p(1-\delta), \infty)$, but rather in $[K_p(1-\delta), K_p(1-\delta)+\xi/\beta])$ where ξ is an upper bound to the expenditure at each period.

Finally we note that if the time step tends to 0 then the discrete solution approaches the solution to the equivalent continuous-time problem. We conjecture that the non-autonomous iterative procedure of Eq. (16) would then converge to a non-autonomous differential equation, which may or may not yield a known solution of the continuous-time maximization problem of Eq. (1).

A Appendix

A.1 Proof of Proposition 1

The proofs up to Eq. (31) are elementary and omitted. Subsequent results hinge on the fact that m^* of (29) is the largest value of m in Eq. (27) for which $\dot{H}^*_m(0)$ is negative. Equation (21) shows that

$$\dot{H}^*_m(K) - \dot{H}^*_{m-1}(K) = \frac{(1-\delta)^{m-1}\dot{\pi}(K(1-\delta)^{m-1})}{(1+r)^m} > 0 \tag{57}$$

which means that $\dot{H}^*_m(K)$ increases with m. The c_m's ($m > m^*$) then also increase with m. The c_m's converge to some c_∞, the root of $\dot{H}^*_\infty(K) = 0$. To show that c_∞ is strictly larger than K^*, it is enough to prove that $\dot{H}^*_\infty(K^*) > 0$:

$$\dot{H}^*_\infty(K^*) = \frac{1}{1+r}\sum_{q=1}^{q=\infty} \rho^{q-1}\dot{\pi}(K^*(1-\delta)^q) - \beta \tag{58}$$

$$> \frac{\dot{\pi}(K^*)}{1+r}\sum_{q=1}^{q=\infty} \rho^{q-1} - \beta \tag{59}$$

$$= \frac{\dot{\pi}(K^*)}{r+\delta} - \beta = 0 \tag{60}$$

where (59) comes from the fact that $\dot{\pi}(K^*)$ is smaller than $\dot{\pi}(K^*(1-\delta)^q)$ and (60) from the definition of K^* (Eq. (28)). The integer p^* defined in Eq. (28) is then the index of the last c_m no larger than K^*.

Differentiating both sides of Eq. (19) for $m \geq 2$ yields

$$\dot{H}^*_{m+1}(K) = \dot{J}^*(K) + \rho \dot{H}^*_m(K(1-\delta)). \tag{61}$$

Substituting $c_m/(1-\delta)$ for K in this equation yields for any $m > m^*$:

$$\begin{aligned} \dot{H}^*_{m+1}(c_m/(1-\delta)) &= \dot{J}^*(c_m/(1-\delta)) + \rho \dot{H}^*_m(c_m) \\ &= \dot{J}^*(c_m/(1-\delta)) \end{aligned} \tag{62}$$

since $\dot{H}^*_m(c_m) = 0$. We know that if $c_m/(1-\delta) < K^*$ then $\dot{J}^*(c_m/(1-\delta)) > 0$. Equation (62) shows that $\dot{H}^*_{m+1}(c_m/(1-\delta))$ is then also positive and therefore c_{m+1} (the value at which $\dot{H}^*_{m+1} = 0$) is necessarily larger than $c_m/(1-\delta)$.

Equation (61) used with $m = p^*$ and $K = c_{p^*+1}$ yields

$$\dot{H}^*_{p^*+1}(c_{p^*+1}) = \dot{J}^*(c_{p^*+1}) + \rho \dot{H}^*_{p^*}(c_{p^*+1}(1-\delta)) = 0. \tag{63}$$

The fact that $\dot{J}^*(c_{p^*+1}) < 0$ means that $\dot{H}^*_{p^*}(c_{p^*+1}(1-\delta)) > 0$ and therefore

$$K^* < c_{p^*+1} < \frac{c_{p^*}}{(1-\delta)} \tag{64}$$

which completes the proof of (33). (The separate inequalities in the braces of (33) reflect the fact that the value of $\dfrac{c_{p^*}}{1-\delta}$ relatively to c_{p^*+2} and to c_∞ is uncertain (but unimportant)).

A.2 Proof of Proposition 2

We will prove P_1 and P_2 by finite induction. The results are true at the order $m = 1$ because $H_1(K) = H_1^*(K)$ for all K.

Proof of P_1. We assume P_1 is true for an $m \leq p^* - 1$. To prove the result at the order $m+1$ we express $H_{m+1}(K)$ for $K \geq c_m/(1-\delta)$. In this case we have (with "IH" standing for induction hypothesis):

$$\begin{aligned}
H_{m+1}(K) &= J^*(K) + \frac{\max_{K' \geq K(1-\delta)} H_m(K')}{1+r} && \text{(definition)} && (65)\\
&= J^*(K) + \frac{\max_{K' \geq K(1-\delta)} H_m^*(K')}{1+r} && (\text{IH } P_1; c_m \geq c_{m-1}/(1-\delta)) && (66)\\
&= J^*(K) + \frac{H_m^*(K(1-\delta))}{1+r} && (H_m^* \text{ max at } c_m) && (67)\\
&= H_{m+1}^*(K) && \text{(definition (19))} && (68)
\end{aligned}$$

which proves P_1 at the order $m+1$ and up to p^*. This means that for any $m \leq p^* - 1$ the function H_{m+1} increases in $[c_m/(1-\delta), c_{m+1}]$ and decreases in $[c_{m+1}, +\infty)$.

Proof of P_2. We assume P_2 is true for an $m \leq p^* - 1$. Given that P_1 is proven we only need to show that H_{m+1} increases in $[0, c_m/(1-\delta))$. In this interval we have

$$\begin{aligned}
H_{m+1}(K) &= J^*(K) + \frac{\max_{K' \geq K(1-\delta)} H_m(K')}{1+r} && \text{(definition)} && (69)\\
&= J^*(K) + \frac{H_m(c_m)}{1+r} && (\text{IH } P_2). && (70)
\end{aligned}$$

Equation (70) shows that up to an additive constant the functions H_{m+1} and J^* coincide on $[0, c_m/(1-\delta))$. The fact that $c_m/(1-\delta) < K^*$ and that J^* is increasing on $[0, K^*]$ means that H_{m+1} is also increasing on $[0, c_m/(1-\delta))$, which completes the proof of P_2.

We first prove P_3 at the order $m = p^* + 1$. For $K \leq c_{p^*}/(1-\delta)$ we then have, as in Eqs. (69)-(70):

$$H_{p^*+1}(K) = J^*(K) + \frac{\max\limits_{K' \geq K(1-\delta)} H_{p^*}(K')}{1+r} \qquad \text{(definition)} \tag{71}$$

$$= J^*(K) + \frac{H^*_{p^*}(c_{p^*})}{1+r} \qquad (c_{p^*} \geq c_{p^*-1}/(1-\delta)). \tag{72}$$

The fact that $K^* < c_{p^*}/(1-\delta)$ means that $H_{p^*+1}(K)$ reaches a maximum at K^* and then decreases on $[K^*, c_{p^*}/(1-\delta))$. For $K > c_{p^*}/(1-\delta)$ we have, as in Eq. (68):

$$H_{p^*+1}(K) = J^*(K) + \frac{\max\limits_{K' \geq K(1-\delta)} H_{p^*}(K')}{1+r} \qquad \text{(definition)} \tag{73}$$

$$= J^*(K) + \frac{H^*_{p^*}(K(1-\delta))}{1+r} \qquad (c_{p^*} \geq c_{p^*-1}/(1-\delta)) \tag{74}$$

$$= H^*_{p^*+1}(K) \tag{75}$$

which decreases on $[c_{p^*}/(1-\delta), \infty)$ because $c_{p^*+1} < c_{p^*}/(1-\delta)$. This proves that $H_{p^*+1}(K)$ increases on $[0, K^*]$ and decreases on (K^*, ∞). An immediate induction carries the result over to H_m for any $m \geq p^* + 1$.

A.3 Proof of Proposition 3

The results are direct consequences of the definition of G_{k-1} in (35) and of the inequalities of (33) recalled below with the redefined c_m's:

$$\begin{aligned} & 0 = c_1 = c_2 = \ldots = c_{m^*} \\ & < c_{m^*+1} < \frac{c_{m^*+1}}{1-\delta} < c_{m^*+2} < \frac{c_{m^*+2}}{1-\delta} \ldots < \frac{c_{p^*-1}}{1-\delta} < c_{p^*} \leq K^* \\ & = c_{p^*+1} \begin{cases} < \dfrac{c_{p^*}}{1-\delta}. \\ = c_{p^*+2} = c_{p^*+2} \ldots. \end{cases} \end{aligned} \tag{76}$$

The expression for $K_1 = G_{k-1}(K_0)$ of Eq. (40) reflects the definition of G_{k-1}. Given that

$$K_1(1-\delta) \geq c_{k-1}(1-\delta) \geq c_{k-2} \tag{77}$$

we have

$$K_2 = G_{k-2}(K_1) = K_1(1-\delta) \tag{78}$$

which proves Eq. (41) for $m = 2$. The proof proceeds in a similar fashion for m's up to $k-1$.

To prove (42) we note that

$$1 \leq m \leq w \Rightarrow k-m \geq p^* + 1 \Rightarrow K_m = G_{k-m}(K_{m-1}) = K^* \tag{79}$$

since $K_0(1-\delta) < K^*$ and K^* is the fixed point of each G_{k-m}. We next have

$$K_{w+1} = G_{k-w-1}(K_w) = G_{p^*}(K^*) = c_{p^*} \tag{80}$$

because from (76) we know that $K^*(1-\delta) < c_{p^*}$. This proves Eq. (43).

We have

$$K_{w+2} = G_{k-w-2}(K_{w+1}) = G_{p^*-1}(c_{p^*}) = c_{p^*}(1-\delta) \tag{81}$$

where the last equality comes from the fact that $c_{p^*}(1-\delta) > c_{p^*-1}$. This proves Eq. (44) for $m = w+2$. A similar reasoning proves Eq. (44) for subsequent m's to $k-1$.

To prove Eq. (45) we recall that $c_m = K^*$ for $m \geq p^* + 1$ and therefore

$$c_{k-1} = K^* \Rightarrow K_1 = G_{k-1}(K_0) = K_0(1-\delta) \in I_{j-1} \tag{82}$$

$$c_{k-2} = K^* \Rightarrow K_2 = G_{k-2}(K_1) = K_1(1-\delta) \in I_{j-2} \tag{83}$$

continuing up to

$$c_{p^*+2} = K^* \Rightarrow K_{w-1} = G_{p^*+2}(K_{w-2}) = K_{w-2}(1-\delta) \in I_{j-w+1}, \tag{84}$$

$$c_{p^*+1} = K^* \Rightarrow K_w = G_{p^*+1}(K_{w-1}) = K_{w-1}(1-\delta) \in I_{j-w}, \tag{85}$$

$$c_{p^*} \leq K_w(1-\delta) \Rightarrow K_{w+1} = G_{p^*}(K_w) = K_w(1-\delta) \in I_{j-w-1}. \tag{86}$$

The last $p^* - 1$ K_m's are also sticky because when $m \geq w+2$ then for every iteration $K_m = G_{k-m}(K_{m-1})$ the quantity $K_{m-1}(1-\delta)$ is larger than the value c_{k-m} at which H_{k-m} reaches its maximum.

To prove (46) we note that $k-1 > j+p^*$ and therefore

$$c_{k-1} = K^* \Rightarrow K_1 = G_{k-1}(K_0) = K_0(1-\delta) \in I_{j-1} \tag{87}$$

$$c_{k-2} = K^* \Rightarrow K_2 = G_{k-2}(K_1) = K_1(1-\delta)^2 \in I_{j-2} \tag{88}$$

up to

$$c_{k-j-1} = K^* \Rightarrow K_{j-1} = G_{k-j+1}(K_{j-2}) = K_0(1-\delta)^{j-1} \in I_1, \tag{89}$$

$$c_{k-j} = K^* \Rightarrow K_j = G_{k-j}(K_{j-1}) = K_0(1-\delta)^j \tag{90}$$

where K_j of Eq. (90) is in the interval $[K^*, K^*/(1-\delta))$. When $w-j > 0$ the fact that $c_{k-j-r} = K^*$ for $1 \leq r \leq w-j$ means that

$$K_{j+1} = G_{k-j-1}(K_j) = K^* \tag{91}$$

$$K_{j+2} = G_{k-j-2}(K_{j+1}) = K^* \tag{92}$$

$$\dots$$

$$K_w = G_{k-w}(K_{w-1}) = K^* \tag{93}$$

which proves Eq. (47) when $w-j > 0$. Equations (48)-(49) are proven in the same way Eqs. (43)-(44) were.

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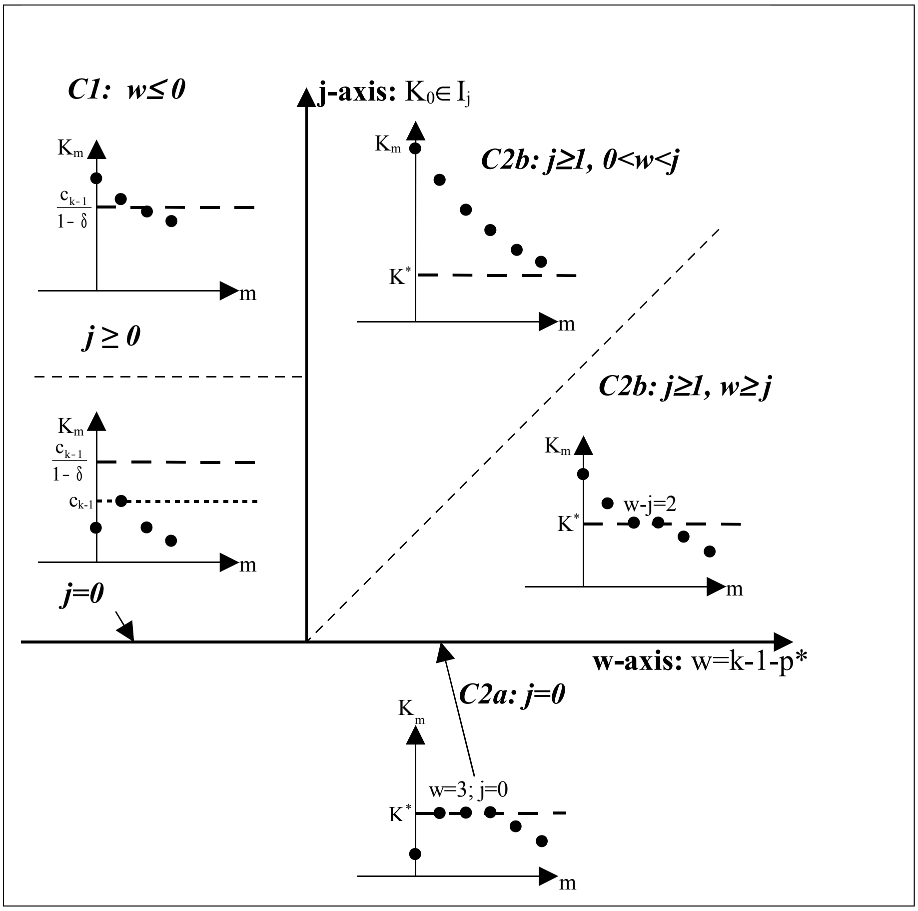

Figure 1: **Optimal K_m's in (w, j) state space**. In case C1, with a short time horizon $(w \leq 0)$ and $K_0 \geq c_{k-1}/(1-\delta)$, all K_m's are sticky (upper diagram on left side). If $K_0 < c_{k-1}/(1-\delta)$ (which implies $j = 0$) then the first optimal capital K_1 is c_{k-1} and subsequent ones are left to decay. In the figure (lower diagram on left side) this means a first period increase to c_{k-1} because $K_0 < c_{k-1}$. If $c_{k-1} < K_0 < c_{k-1}/(1-\delta)$ there would be a first period "non-sticky" decrease to c_{k-1} with subsequent optimal capitals left to decay. In case C2a (long time horizon $(w > 0)$ with an initial capital $K_0 \in I_0, (j = 0)$, the first optimal capital K_1 jumps to K^* where the capital remains w periods before decaying. In Case C2b (long time horizon $(w > 0)$ with an initial capital $K_0 \in I_j, (j \geq 1)$, the capital is left to decay a number of periods that depends on how large K_0 is: if $w < j$ (above the first diagonal) then the capital decays during the full $k-1$ periods. If $w \geq j$ (below the first diagonal) then the capital decays during the first j periods; it stays at the value K^* for $w - j$ periods; it then transits one period at the value c_{p^*} and decays over the last $p^* - 1$ periods.

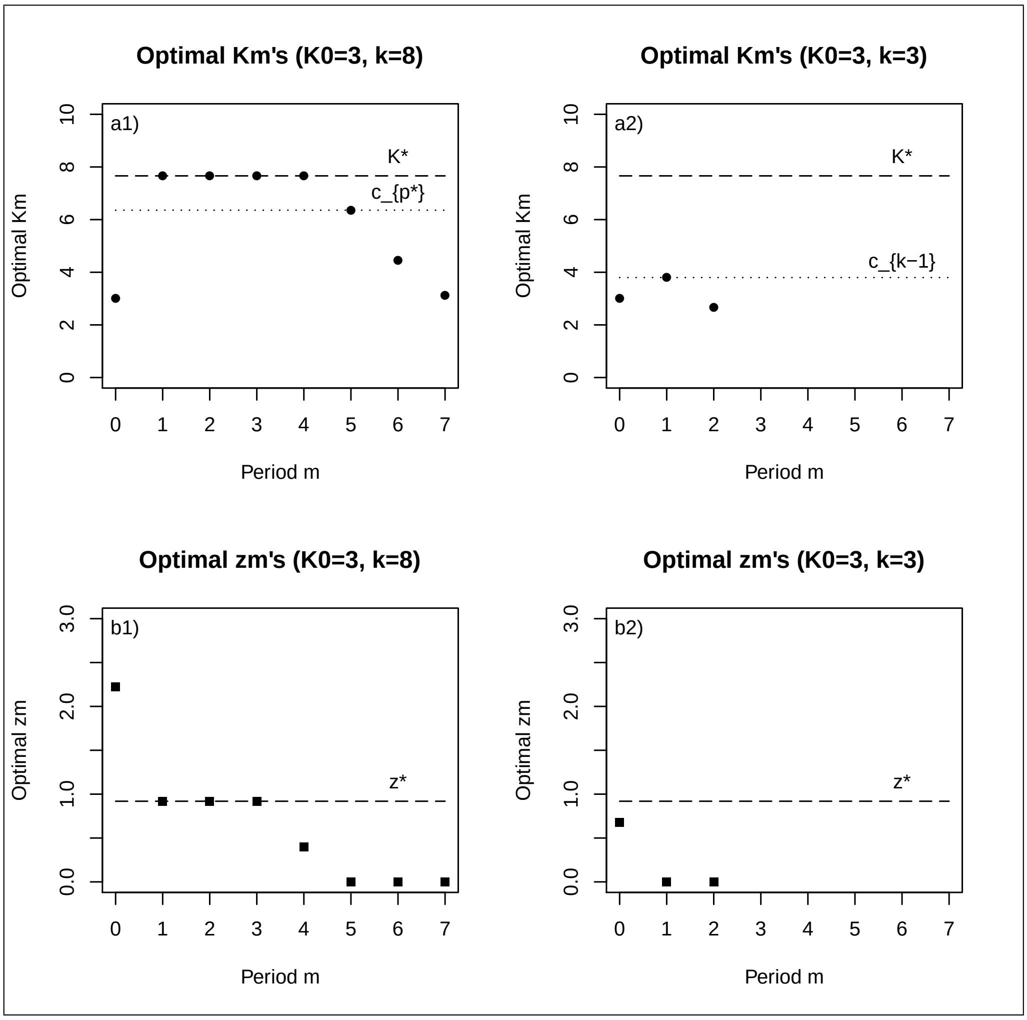

Figure 2: **Low initial capital** K_0 ($K_0 = 3 < K^*$). Panels a1 and b1 represent the K_m's and corresponding z_m's for a long time horizon $k = 8$ ($w > 0$). The optimal capital K_1 at the first period is equal the stationary value K^* (Panel a1). Optimal K_m's remain at K^* for $w = 4$ periods (Eq. (42)). After one period at the value c_{p^*} (Eq. (43)) the optimal values are left to decay with no more expenditures: the last three z_m's are 0 (Figure b1). Panels a2 and b2 are for a short time horizon $k = 3$ ($w < 0$). The first optimal capital K_1 is equal to $c_{k-1} = c_2$ (Figure a2). Subsequent values are left to decay (Eqs. (40)-(41)): there was an expenditure only during the first period (Panel b2).

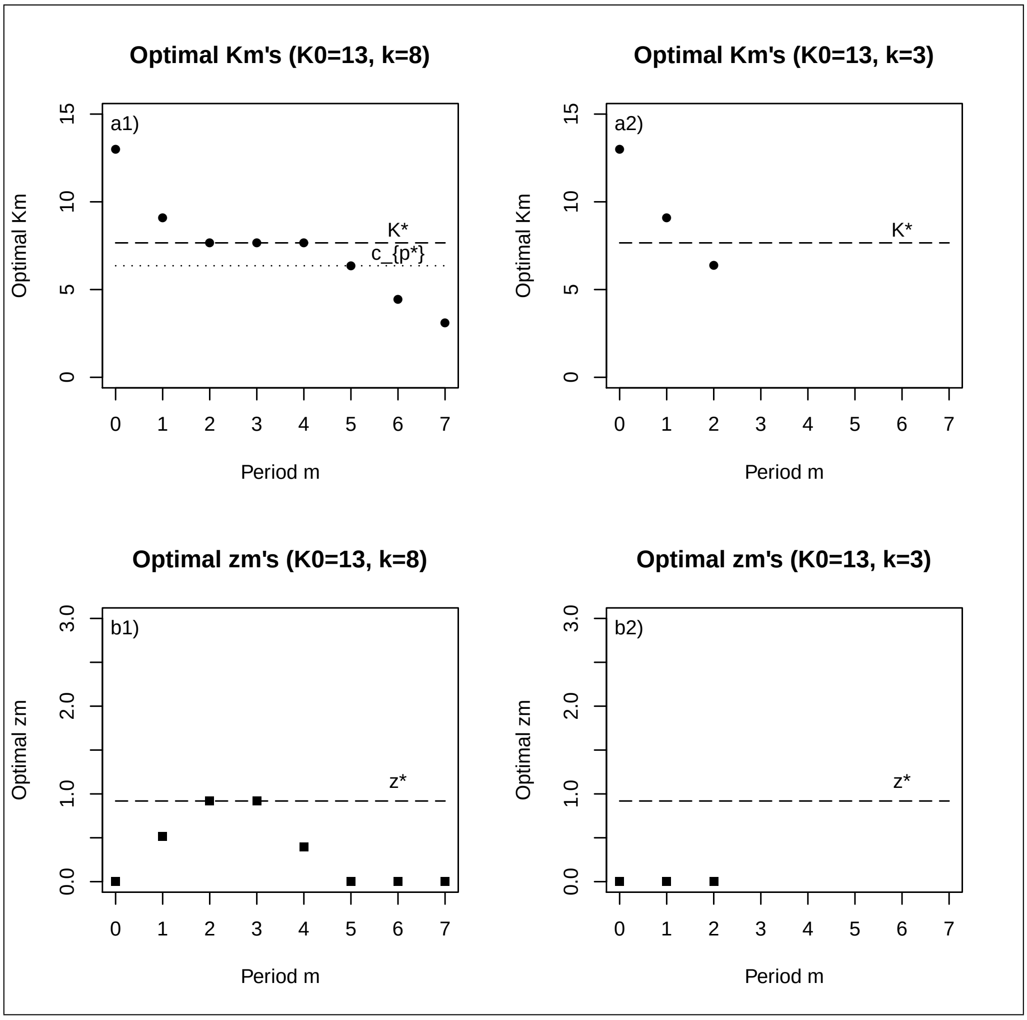

Figure 3: **High initial** K_0 ($K_0 = 13 > K^*$). Panels a1 and b1 represent the K_m's and corresponding z_m's for a long time horizon $k = 8$ ($w > 0$). The initial values K_m (here only K_1) decay with no expenditures until they reach the stationary value K^* (Panel a1, Eq. (46)). Optimal K_m's remain at K^* for $w - j = 3$ periods (Eq. (47)). After one period at the value c_{p^*} (Eq. (48)) the optimal values are left to decay with no more expenditures: the last three z_m's are 0 (Panel b1). Panels a2 and b2 are for a short time horizon $k = 3$ ($w = -1 \leq 0$). The capital is left to decay (Eqs. (40)-(41)) with no expenditure (Panel b2).