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Tobit Model with Covariate Dependent Thresholds

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Abstract

Tobit models are extended to allow threshold values which depend on individuals' characteristics. In such models, the parameters are subject to as many inequality constraints as the number of observations, and the maximum likelihood estimation which requires the numerical maximisation of the likelihood is often difficult to be implemented. Using a Bayesian approach, a Gibbs sampler algorithm is proposed and, further, the convergence to the posterior distribution is accelerated by introducing an additional scale transformation step. The procedure is illustrated using the simulated data, wage data and prime rate changes data.

Key words: Bayesian analysis, Censored regression model, Markov chain Monte Carlo, Sample selection model, Tobit model, Unknown censoring threshold.

1 Introduction

A censored regression model has been very popular and well-known as a standard Tobit (Type I Tobit) model in economics since it was first introduced by Tobin (1958) to analyze the relationship between household income and household expenditures on a durable good where there are some households with zero expenditures (see e.g., Amemiya 1984 for a survey). The standard Type I Tobit model is given by

$$y_i = \begin{cases} y_i^*, & \text{if } y_i^* \geq d, \\ \text{n.a.}, & \text{otherwise,} \end{cases} \quad i = 1, 2, \ldots, n, \tag{1}$$

$$y_i^* = \boldsymbol{x}_i' \boldsymbol{\alpha} + \epsilon_i, \quad \epsilon_i \sim \text{ i.i.d. } \mathcal{N}(0, \tau^2), \tag{2}$$

where y_i is a dependent variable, y_i^* is a latent dependent variable, d is a censoring limit, $\boldsymbol{x}_i$ is a $K \times 1$ covariate vector and $\boldsymbol{\alpha}$ is a corresponding $K \times 1$ regression coefficient vector. We observe a response variable y_i when it is greater than or equal

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to a threshold d. The threshold is assumed to be a known constant, and often set equal to zero for convenience. Bayesian estimation method of such Tobit model was first proposed by Chib (1992). Chib (1992) developed Gibbs sampling procedure using the idea of data augmentation, which is widely used in the literature, and compared the efficacy of the different Monte Carlo methods.

When d is unknown, it will be absorbed into the constant term of the regression, and other coefficients are estimated properly. However, as discussed in Zuehlke (2003), the individual threshold values may be of great interest and need to be estimated separately. It is more natural to extend the standard Tobit model such that the deterministic thresholds can vary with individuals depending on their characteristics. In such a model with covariate dependent threshold, the i-th response variable y_i is observed if it is greater than or equal to a threshold $d_i = \boldsymbol{w}_i'\boldsymbol{\delta}$ where $\boldsymbol{w}_i$ and $\boldsymbol{\delta}$ are a $J \times 1$ covariate vector and a corresponding coefficient vector respectively.

The alternative extension of the standard Tobit model is known as a sample selection model or a generalized Tobit (Type II Tobit) model in the literature,

$$y_i = \begin{cases} y_i^*, & \text{if } z_i^* \geq 0, \\ \text{n.a.}, & \text{otherwise}, \end{cases} \qquad i = 1, 2, \ldots, n, \tag{3}$$

$$z_i^* = \boldsymbol{w}_i'\boldsymbol{\theta} + \xi_i, \tag{4}$$

$$y_i^* = \boldsymbol{x}_i'\boldsymbol{\beta} + \eta_i, \tag{5}$$

where $(\xi_i, \eta_i)' \sim$ i.i.d. $\mathcal{N}(\mathbf{0}, \Sigma)$, $(\boldsymbol{w}_i, \boldsymbol{x}_i)$ are independent variable vectors, $(\boldsymbol{\theta}, \boldsymbol{\beta})$ are corresponding coefficient vectors and the $(1, 1)$ element of Σ is set equal to 1 for the identification. The sample rule is determined by a latent random variable z_i^*, and we observe the response variable y_i when $z_i^* \geq 0$. The latent variable z_i^* is allowed to be correlated with the response variable y_i^*. When the correlation coefficient, ρ, between (z_i^*, y_i^*) is not equal to zero, a sample selection model is considered a Tobit model with a stochastic threshold model.

However, the correlation coefficient is often estimated to be almost one in empirical studies (see e.g. Table 7 in Section 5.1). If ρ is equal to one, the generalized Tobit model reduces to the special case of the Tobit model with covariate dependent thresholds which we just mentioned above. Whether we should use a standard Tobit model, a sample selection model or a Tobit model with covariate dependent thresholds has been an important issue in the empirical studies (see Cragg 1971; Lin and Schmidt 1984; Melenberg and Soest 1996). In this paper, we take Bayesian approach to deal with such a problem using a marginal likelihood (Chib 1995) and DIC (Deviance Information Criterion, see Spiegelhalter, Best, Carlin, and van der Linde 2002) as a model selection criterion.

The purpose of this paper is three-fold. First we propose a Markov chain Monte

Carlo (MCMC) estimation method for a Tobit model with the thresholds which depend on individuals' characteristics. Since the parameters are subject to as many inequality constraints as the number of observations, the numerical maximisation of the likelihood is often difficult to be implemented. Using Bayesian approach, we describe a Gibbs sampler algorithm to estimate parameters, and, further, show that the speed of convergence to the posterior distribution can be accelerated by adding one more step to the simple Gibbs sampler.

Second, we extend our proposed model to a friction model introduced by Rosett (1959) (see also Maddala 1983) with covariate dependent thresholds, and a two-limit Tobit model with covariate dependent thresholds.

Third, we illustrate our proposed estimation methods using both simulated data and real data. The real data examples include the popular example of hourly wage of married women (discussed in Mroz 1987) and prime rate changes in Japan. For wage data, the model comparison is conducted using a marginal likelihood and DIC.

The rest of the paper is organised as follows. In Section 2, we propose an efficient Gibbs sampler for a Tobit model with covariate dependent thresholds and, in Section 3, extensions to friction model and two-limit Tobit model are discussed. Section 4 illustrates our proposed estimation method using simulated data. Empirical studies are shown in Section 5 using wage data and prime rate changes data. Section 6 concludes the paper.

2 Tobit model with covariate dependent thresholds

2.1 Gibbs sampler

We first describe Gibbs sampler for a Tobit (standard Tobit Type 1) model (see e.g., Chib 1995). The prior distributions of $(\boldsymbol{\alpha}, \tau^2)$ are assumed to be distributed conditionally multivariate normal and inverse gamma respectively,

$$\boldsymbol{\alpha}|\tau^2 \sim \mathcal{N}(\boldsymbol{a}_0, \tau^2 A_0), \quad \tau^2 \sim \mathcal{IG}\left(\frac{n_0}{2}, \frac{S_0}{2}\right), \tag{6}$$

where $\boldsymbol{a}_0$ is a $K \times 1$ known constant vector, A_0 is a $K \times K$ known constant matrix, and n_0, S_0 are known positive constants. To implement Markov chain Monte Carlo, we use a data augmentation method by sampling unobserved latent response variable y_i^*. Using $\boldsymbol{y}^*$, model (1)–(2) reduces to an ordinary linear regression model, $\boldsymbol{y}^* = X\boldsymbol{\alpha} + \boldsymbol{\epsilon}$, where $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)'$, $X' = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)' \sim \mathcal{N}(0, \tau^2 I_n)$. Given $\boldsymbol{y}^*$, the conditional posterior distributions of $(\boldsymbol{\alpha}, \tau^2)$ are

$$\boldsymbol{\alpha}|\tau^2, \boldsymbol{y}^* \sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1), \quad \tau^2|\boldsymbol{y}^* \sim \mathcal{IG}\left(\frac{n_1}{2}, \frac{S_1}{2}\right),$$

where $A_1^{-1} = A_0^{-1} + X'X$, and $\boldsymbol{a}_1 = A_1(A_0^{-1}\boldsymbol{a}_0 + X'\boldsymbol{y}^*)$, $n_1 = n_0 + n$, and $S_1 = \boldsymbol{y}^{*'}\boldsymbol{y}^* + \boldsymbol{a}_0' A_0^{-1}\boldsymbol{a}_0 + S_0 - \boldsymbol{a}_1' A_1^{-1}\boldsymbol{a}_1$ (see Appendix A1). Let $\boldsymbol{y}_o = (y_{o,1}, y_{o,2}, \dots, y_{o,m})'$ and $\boldsymbol{y}_c^* = (y_{c,1}^*, y_{c,2}^*, \dots, y_{c,n-m}^*)'$ denote $m \times 1$ and $(n-m) \times 1$ vectors of observed (uncensored) and censored dependent variables respectively (see Appendix A1). Then, we can sample from posterior distribution using Gibbs sampler in two blocks:

1. Initialise $\boldsymbol{\alpha}$ and τ^2.
2. Sample $\boldsymbol{y}_c^* | \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}_o$.
   Generate $y_{c,i}^* | \boldsymbol{\alpha}, \tau^2 \sim \mathcal{TN}_{(-\infty,d)}(\boldsymbol{x}_i'\boldsymbol{\alpha}, \tau^2)$, $i = 1, 2, \dots, n-m$, for censored observations, where $\mathcal{TN}_{(a,b)}(\mu, \sigma^2)$ denotes a normal distribution $\mathcal{N}(\mu, \sigma^2)$ truncated on the interval (a, b).
3. Sample $(\boldsymbol{\alpha}, \tau^2) | \boldsymbol{y}_c^*, \boldsymbol{y}_o$
   (a) Sample $\tau^2 | \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{IG}(n_1/2, S_1/2)$,
   (b) Sample $\boldsymbol{\alpha} | \tau^2, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1)$.
4. Go to 2.

Next, we extend it to the model with covariate dependent thresholds model. By adding another block to the above sampler, we can derive the Gibbs sampler for the Tobit model with covariate dependent thresholds. In the standard Tobit model (1)–(2), the threshold d is assumed to be known and constant. However, it is usually unknown and may vary with the individual characteristics. Thus we extend it to allow unknown but covariate dependent thresholds as follows.

$$y_i = \begin{cases} y_i^*, & \text{if } y_i^* \geq \boldsymbol{w}_i'\boldsymbol{\delta}, \\ \text{n.a.}, & \text{otherwise}, \end{cases} \quad i = 1, 2, \dots, n, \tag{7}$$

$$y_i^* = \boldsymbol{x}_i'\boldsymbol{\alpha} + \epsilon_i, \quad \epsilon_i \sim \text{ i.i.d. } \mathcal{N}(0, \tau^2), \tag{8}$$

where $(\boldsymbol{w}_i, \boldsymbol{x}_i)$ are $J \times 1$ and $K \times 1$ covariate vectors and $(\boldsymbol{\delta}, \boldsymbol{\alpha})$ are corresponding $J \times 1$ and $K \times 1$ regression coefficient vectors. The known constant threshold d in (1) is replaced by the unknown but covariate dependent threshold, $\boldsymbol{w}_i'\boldsymbol{\delta}$.

To conduct Bayesian analysis of the proposed Tobit model (7)–(8), we assume that prior distributions of $(\boldsymbol{\alpha}, \tau^2)$ are given by (6) and that a prior distribution of $\boldsymbol{\delta}$ is $\boldsymbol{\delta} | \tau^2 \sim \mathcal{N}(\boldsymbol{d}_0, \tau^2 D_0)$. Then, the conditional posterior distributions of $(\boldsymbol{\alpha}, \tau^2, \boldsymbol{\delta})$ are

$$\begin{aligned} \boldsymbol{\alpha} | \boldsymbol{\delta}, \tau^2, \boldsymbol{y}^* &\sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1), \quad \tau^2 | \boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{y}^* \sim \mathcal{IG}\left(\frac{n_1}{2}, \frac{S_1}{2}\right), \\ \boldsymbol{\delta} | \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* &\sim \mathcal{TN}_{R_o \cap R_c}(\boldsymbol{d}_0, \tau^2 D_0), \end{aligned}$$

where $n_1 = n_0 + n + J$, $S_1 = \boldsymbol{y}^{*'}\boldsymbol{y}^* + \boldsymbol{a}_0' A_0^{-1}\boldsymbol{a}_0 - \boldsymbol{a}_1' A_1^{-1}\boldsymbol{a}_1 + S_0 + (\boldsymbol{\delta} - \boldsymbol{d}_0)' D_0^{-1}(\boldsymbol{\delta} - \boldsymbol{d}_0)$, $A_1^{-1} = A_0^{-1} + X'X$, $\boldsymbol{a}_1 = A_1(A_0^{-1}\boldsymbol{a}_0 + X'\boldsymbol{y}^*)$, $R_o = \{\boldsymbol{\delta} | \ \boldsymbol{w}_i'\boldsymbol{\delta} \leq y_i \text{ for uncensored } i\}$,

$R_c = \{\boldsymbol{\delta}|\ \boldsymbol{w}_i'\boldsymbol{\delta} > y_i^*$ for censored $i\}$(see Appendix A2). The Gibbs sampler is implemented in three blocks as follows.

1. Initialise $\boldsymbol{\delta}, \boldsymbol{\alpha}$ and τ^2 where $\boldsymbol{\delta} \in R_o$.

2. Sample $\boldsymbol{y}_c^*|\boldsymbol{\alpha}, \tau^2, \boldsymbol{\delta}, \boldsymbol{y}_o$. Generate $y_{c,i}^*|\boldsymbol{\delta}, \boldsymbol{\alpha}, \tau^2 \sim \mathcal{TN}_{(-\infty, \boldsymbol{w}_i'\boldsymbol{\delta})}(\boldsymbol{x}_i'\boldsymbol{\alpha}, \tau^2)$, $i = 1, 2, \ldots, n-m$, for censored observations.

3. Sample $(\boldsymbol{\alpha}, \tau^2)|\boldsymbol{\delta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o$

   (a) Sample $\tau^2|\boldsymbol{\delta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{IG}(n_1/2, S_1/2)$,

   (b) Sample $\boldsymbol{\alpha}|\tau^2, \boldsymbol{\delta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \ \sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1)$.

4. Sample $\boldsymbol{\delta}|\boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* \sim \mathcal{TN}_{R_o \cap R_c}(\boldsymbol{d}_0, \tau^2 D_0)$.

5. Go to 2.

Steps 2 and 3 are similar to those in the simple Tobit model. To sample from the conditional posterior distribution of $\boldsymbol{\delta}$ in Step 4, we generate one component δ_j of $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_J)'$ at a time given other components $\boldsymbol{\delta}_{-j} = (\delta_1, \ldots, \delta_{j-1}, \delta_{j+1} \ldots, \delta_J)'$. Since $\boldsymbol{\delta}$ should lie in the region $R_o \cap R_c$, the δ_j is subject to the constraint $L_j \le \delta_j \le U_j$ where $\boldsymbol{w}_{i,-j} = (w_{i1}, \ldots, w_{i,j-1}, w_{i,j+1}, \ldots, w_{iJ})'$,

$$\begin{aligned} L_j &= \max_i L_{ij}, \quad L_{ij} = \begin{cases} w_{ij}^{-1}(y_i - \boldsymbol{w}_{i,-j}'\boldsymbol{\delta}_{-j}) & \text{if } w_{ij} < 0 \text{ for uncensored } i \ , \\ w_{ij}^{-1}(y_i^* - \boldsymbol{w}_{i,-j}'\boldsymbol{\delta}_{-j}) & \text{if } w_{ij} > 0 \text{ for censored } i \ , \\ -\infty, & \text{otherwise,} \end{cases} \\ U_j &= \min_i U_{ij}, \quad U_{ij} = \begin{cases} w_{ij}^{-1}(y_i - \boldsymbol{w}_{i,-j}'\boldsymbol{\delta}_{-j}) & \text{if } w_{ij} > 0 \text{ for uncensored } i \ , \\ w_{ij}^{-1}(y_i^* - \boldsymbol{w}_{i,-j}'\boldsymbol{\delta}_{-j}) & \text{if } w_{ij} < 0 \text{ for censored } i \ , \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\boldsymbol{d}_{0,-j} = (d_{01}, \ldots, d_{0,j-1}, d_{0,j+1}, \ldots, d_{0J})'$ and let $D_{0,j,j} D_{0,j,-j}, D_{0,-j,-j}$ denote a prior variance of d_{0j}, a prior covariance vector of d_{0j} and $\boldsymbol{d}_{0,-j}$, and a prior covariance matrix of $\boldsymbol{d}_{0,-j}$ respectively. Then, we sample δ_j, for $j = 1, 2, \ldots, J$, using the conditional truncated normal posterior distribution,

$$\begin{aligned} \delta_j|\boldsymbol{\delta}_{-j}, \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* &\sim \mathcal{TN}_{(L_j, U_j)}(m_j, s_j^2\tau^2), \\ m_j &= d_{0j} + D_{0,j,-j} D_{0,-j,-j}^{-1}(\boldsymbol{\delta}_{-j} - \boldsymbol{d}_{0,-j}), \\ s_j^2 &= D_{0,j,j} - D_{0,j,-j} D_{0,-j,-j}^{-1} D_{0,j,-j}'. \end{aligned}$$

Note that this reduces to $\mathcal{TN}_{(L_j, U_j)}(d_{0j}, \tau^2 D_{0,j,j})$ for a diagonal D_0.

2.2 Acceleration of the Gibbs sampler

The above sampling scheme for $\boldsymbol{\delta}$ is to sample one component at a time which is known to be an inefficient method, and the obtained samples may exhibit high autocorrelations (see Sections 4.1 and 5.1).

To reduce such high sample autocorrelations, we consider a generalised Gibbs move via the scale group $\Gamma = \{g > 0 : g(\boldsymbol{\delta}, \boldsymbol{\alpha}, \tau, \boldsymbol{y}_c^*) = (g\boldsymbol{\delta}, g\boldsymbol{\alpha}, g\tau, g\boldsymbol{y}_c^*)\}$ (see Appendix B). Let $\boldsymbol{x}_{o,i}$ denote the independent variable vector of the i-th observed response $y_{o,i}$ and $X_o' = (\boldsymbol{x}_{o,1}, \ldots, \boldsymbol{x}_{o,m})$. Noting that $\lambda_L < g^{-1} < \lambda_U$ where

$$\begin{aligned}
\lambda_L &= \max_i L_{\lambda,i}, \quad L_{\lambda,i} = \begin{cases} \max\left(0, \frac{\boldsymbol{w}_{o,i}'\boldsymbol{\delta}}{y_{o,i}}\right), & \text{if } y_{o,i} > 0, \\ 0, & \text{otherwise,} \end{cases} \\
\lambda_U &= \min_i U_{\lambda,i}, \quad U_{\lambda,i} = \begin{cases} \frac{\boldsymbol{w}_{o,i}'\boldsymbol{\delta}}{y_{o,i}}, & \text{if } y_{o,i} < 0, \\ \infty, & \text{otherwise,} \end{cases}
\end{aligned}$$

the acceleration steps are given as follows. When $\lambda_U < \infty$,

5. (a) Generate $g^{-1} \sim \mathcal{U}(\lambda_L, \lambda_U)$ where $\mathcal{U}(a, b)$ denotes a uniform distribution on the interval (a, b).

   (b) Accept g with probability

   $$\min\left[1, g^{-(n_0+m-1)} \exp\left\{-\frac{-2q(g^{-1}-1) + r(g^{-2}-1)}{2\tau^2}\right\}\right],$$

   where $q = \boldsymbol{\alpha}' X_o' \boldsymbol{y}_o + \boldsymbol{\alpha}' A_0^{-1} \boldsymbol{a}_0 + \boldsymbol{d}_0' D_0^{-1} \boldsymbol{\delta}$, $r = \boldsymbol{y}_o' \boldsymbol{y}_o + \boldsymbol{a}_0' A_0^{-1} \boldsymbol{a}_0 + \boldsymbol{d}_0' D_0^{-1} \boldsymbol{d}_0 + S_0$. If rejected, set $g = 1$.

   (c) Let $g\boldsymbol{\delta} \to \boldsymbol{\delta}$, $g\boldsymbol{\alpha} \to \boldsymbol{\alpha}$, $g\tau \to \tau$ and $g\boldsymbol{y}_c^* \to \boldsymbol{y}_c^*$.

6. Go to 2.

When $\lambda_U = \infty$, we replace Step 5 (a) (b) by

5. (a)$'$ Generate $g^{-1} \sim \mathcal{TN}_{(\lambda_L,\infty)}(1, \sigma_\lambda^2)$ where σ_λ^2 is some specified constant (e.g., $\sigma_\lambda^2 = 0.5^2$).

   (b)$'$ Accept g with probability

   $$\min\left[1, g^{-(n_0+m-1)} \exp\left\{-\frac{-2q(g^{-1}-1) + r(g^{-2}-1)}{2\tau^2} + \frac{(g^{-1}-1)^2)}{2\sigma_\lambda^2}\right\}\right].$$

   If rejected, set $g = 1$.

We illustrate how effective this acceleration step is to improve the speed of the convergence to the target posterior distribution in Sections 4.1 and 5.1.

3 Extensions

In this section, we describe two useful econometric models which can be obtained as extensions of our basic covariate dependent threshold model (7)–(8).

3.1 Friction model

First extension is a friction model introduced by Rosett (1959) (see also Maddala 1983). The model has been used, for example, to analyze the nominal and real wage rigidity (Christofides and Li 2005), the prime rate change (Forbes and Mayne 1989), and the price change of a product or brand over time (Desarbo, Rao, Steckel, Wind, and Colombo 1987). The friction model with covariate dependent thresholds is given by

$$y_i = \begin{cases} y_i^* - \boldsymbol{w}_i'\boldsymbol{\delta}, & \text{if } y_i^* - \boldsymbol{w}_i'\boldsymbol{\delta} < c, \\ c, & \text{if } \boldsymbol{w}_i'\boldsymbol{\delta} + c \le y_i^* \le \boldsymbol{v}_i'\boldsymbol{\zeta} + c, \\ y_i^* - \boldsymbol{v}_i'\boldsymbol{\zeta}, & \text{if } y_i^* - \boldsymbol{v}_i'\boldsymbol{\zeta} > c, \end{cases} \quad i = 1, 2, \ldots, n, \tag{9}$$

$$y_i^* = \boldsymbol{x}_i'\boldsymbol{\alpha} + \epsilon_i, \quad \epsilon_i \sim \text{ i.i.d. } \mathcal{N}(0, \tau^2). \tag{10}$$

where c is a known constant. Since c is set equal to 0 in most applications, we focus on the friction model with $c = 0$. Let $R_{\boldsymbol{\delta}}$ and $R_{\boldsymbol{\zeta}}$ denote truncation regions defined by

$$R_{\boldsymbol{\delta}} = R_{\boldsymbol{\delta},o} \cap R_{\boldsymbol{\delta},c}, \qquad R_{\boldsymbol{\zeta}} = R_{\boldsymbol{\zeta},o} \cap R_{\boldsymbol{\zeta},c},$$

$$R_{\boldsymbol{\delta},o} = \left\{\boldsymbol{\delta} \mid \boldsymbol{w}_i'\boldsymbol{\delta} \le \boldsymbol{v}_i'\boldsymbol{\zeta} \text{ for uncensored } i\right\}, \qquad R_{\boldsymbol{\zeta},o} = \left\{\boldsymbol{\zeta} \mid \boldsymbol{w}_i'\boldsymbol{\delta} \le \boldsymbol{v}_i'\boldsymbol{\zeta} \text{ for uncensored } i\right\},$$

$$R_{\boldsymbol{\delta},c} = \left\{\boldsymbol{\delta} \mid \boldsymbol{w}_i'\boldsymbol{\delta} \le y_i^* \text{ for censored } i\right\}, \qquad R_{\boldsymbol{\zeta},c} = \left\{\boldsymbol{\zeta} \mid y_i^* \le \boldsymbol{v}_i'\boldsymbol{\zeta} \text{ for censored } i\right\}.$$

Assuming that the normal prior distribution for $\boldsymbol{\zeta}$, $\boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{z}_0, \tau^2 Z_0)$ and the same prior distributions for other parameters as in the previous section (normal for $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, inverse gamma for τ^2), we implement Gibbs sampler in six blocks:

1. Initialise $\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{\alpha}$ and τ^2 where $\boldsymbol{\delta} \in R_{\delta,o}$ and $\boldsymbol{\zeta} \in R_{\zeta,o}$.
2. Sample $\boldsymbol{y}_c^*|\boldsymbol{\alpha}, \tau^2, \boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_o$. Generate $y_{c,i}^*|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{\alpha}, \tau^2 \sim \mathcal{TN}_{[\boldsymbol{w}_i'\boldsymbol{\delta}, \boldsymbol{v}_i'\boldsymbol{\zeta}]}(\boldsymbol{x}_i'\boldsymbol{\alpha}, \tau^2)$, $i = 1, 2, \ldots, n-m$, for censored observations.
3. Sample $(\boldsymbol{\alpha}, \tau^2)|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o$
   (a) Sample $\tau^2|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{IG}(n_1/2, S_1/2)$ where $n_1 = n_0 + 2J + n$ and $S_1 = \boldsymbol{y}^{*\prime}\boldsymbol{y}^* + \boldsymbol{a}_0'\boldsymbol{A}_0^{-1}\boldsymbol{a}_0 - \boldsymbol{a}_1'\boldsymbol{A}_1^{-1}\boldsymbol{a}_1 + S_0 + (\boldsymbol{\delta} - \boldsymbol{d}_0)'\boldsymbol{D}_0^{-1}(\boldsymbol{\delta} - \boldsymbol{d}_0) + (\boldsymbol{\zeta} - \boldsymbol{z}_0)'\boldsymbol{Z}_0^{-1}(\boldsymbol{\zeta} - \boldsymbol{z}_0)$.
   (b) Sample $\boldsymbol{\alpha}|\tau^2, \boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1)$ where $\boldsymbol{A}_1^{-1} = \boldsymbol{A}_0^{-1} + \boldsymbol{X}'\boldsymbol{X}$ and $\boldsymbol{a}_1 = \boldsymbol{A}_1(\boldsymbol{A}_0^{-1}\boldsymbol{a}_0 + \boldsymbol{X}'\boldsymbol{y}^*)$.

4. Sample $\boldsymbol{\delta}|\boldsymbol{\zeta}, \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* \sim \mathcal{TN}_{R_\delta}(\boldsymbol{d}_1, \tau^2 D_1)$ where $\boldsymbol{D}_1^{-1} = \boldsymbol{D}_0^{-1} + \sum_{i \in \mathcal{O}^-} \boldsymbol{w}_i \boldsymbol{w}_i'$, $\boldsymbol{d}_1 = \boldsymbol{D}_1 \left\{ \boldsymbol{D}_0^{-1} \boldsymbol{d}_0 + \sum_{i \in \mathcal{O}^-} (\boldsymbol{x}_i' \boldsymbol{\alpha} - y_i) \boldsymbol{w}_i \right\}$ and $\mathcal{O}^- = \{i \mid y_i < 0\}$.

5. Sample $\boldsymbol{\zeta}|\boldsymbol{\delta}, \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* \sim \mathcal{TN}_{R_\zeta}(\boldsymbol{z}_1, \tau^2 Z_1)$. where $\boldsymbol{Z}_1^{-1} = \boldsymbol{Z}_0^{-1} + \sum_{i \in \mathcal{O}^+} \boldsymbol{v}_i \boldsymbol{v}_i'$, $\boldsymbol{z}_1 = \boldsymbol{Z}_1 \left\{ \boldsymbol{Z}_0^{-1} \boldsymbol{z}_0 + \sum_{i \in \mathcal{O}^+} (\boldsymbol{x}_i' \boldsymbol{\alpha} - y_i) \boldsymbol{v}_i \right\}$ and $\mathcal{O}^+ = \{i \mid y_i > 0\}$.

6. Go to 2.

3.2 Two-limit Tobit model

The second extension is a two-limit Tobit model in which the dependent variable is doubly censored with known limits (see e.g., Maddala 1983). The model has been applied in the various empirical studies, such as the estimation of electric vehicle demand share equation (Hill 1987), and the stock price exchange movements with limits (Charemza and Majerowska 2000). The extended two-limit model is given by

$$y_i = \begin{cases} y_i^*, & \text{if } \boldsymbol{w}_i' \boldsymbol{\delta} \le y_i^* \le \boldsymbol{v}_i' \boldsymbol{\zeta}, \\ \text{n.a.}, & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, n, \tag{11}$$

$$y_i^* = \boldsymbol{x}_i' \boldsymbol{\alpha} + \epsilon_i, \quad \epsilon_i \sim \text{ i.i.d. } \mathcal{N}(0, \tau^2), \tag{12}$$

where y_i is observed only when it falls within the interval, and the upper and lower limits depend on the individual's characteristics.

Assuming the same prior distributions as those in the previous friction model (normal for $\boldsymbol{\alpha}$, $\boldsymbol{\delta}$, $\boldsymbol{\zeta}$, inverse gamma for τ^2), we obtain the Gibbs sampling algorithm as follows.

1. Initialise $\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{\alpha}$ and τ^2 where $\boldsymbol{\delta} \in R_{\delta,o}$ and $\boldsymbol{\zeta} \in R_{\zeta,o}$.

2. Sample $\boldsymbol{y}_c^*|\boldsymbol{\alpha}, \tau^2, \boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_o$. Generate $y_{c,i}^*|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{\alpha}, \tau^2 \sim \mathcal{TN}_{(-\infty, \boldsymbol{w}_i' \boldsymbol{\delta}) \cup (\boldsymbol{v}_i' \boldsymbol{\zeta}, \infty)}(\boldsymbol{x}_i' \boldsymbol{\alpha}, \tau^2)$, $i = 1, 2, \ldots, n - m$, for censored observations.

3. Sample $(\boldsymbol{\alpha}, \tau^2)|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o$

   (a) Sample $\tau^2|\boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{IG}(n_1/2, S_1/2)$,

   (b) Sample $\boldsymbol{\alpha}|\tau^2, \boldsymbol{\delta}, \boldsymbol{\zeta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{N}(\boldsymbol{a}_1, \tau^2 A_1)$.

4. Sample $\boldsymbol{\delta}|\boldsymbol{\zeta}, \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* \sim \mathcal{TN}_{R_\delta}(\boldsymbol{d}_0, \tau^2 D_0)$.

5. Sample $\boldsymbol{\zeta}|\boldsymbol{\delta}, \boldsymbol{\alpha}, \tau^2, \boldsymbol{y}^* \sim \mathcal{TN}_{R_\zeta}(\boldsymbol{z}_0, \tau^2 Z_0)$.

6. Go to 2.

where n_1, S_1, $\boldsymbol{a}_1$, $\boldsymbol{A}_1$ are the same with those for the friction model (see Section 3.1), and

$$R_\delta = R_{\delta,o} \cap R_{\delta,lc} \cap R_{\delta,rc}, \qquad R_\zeta = R_{\zeta,o} \cap R_{\zeta,lc} \cap R_{\zeta,rc},$$
$$R_{\delta,o} = \{\boldsymbol{\delta}|\ \boldsymbol{w}_i'\boldsymbol{\delta} \le y_i \text{ for uncensored } i\}, \qquad R_{\delta,lc} = \{\boldsymbol{\delta}|\ \boldsymbol{w}_i'\boldsymbol{\delta} > y_i^* \text{ for left censored } i\},$$
$$R_{\delta,rc} = \{\boldsymbol{\delta}|\ \boldsymbol{w}_i'\boldsymbol{\delta} < \boldsymbol{v}_i'\boldsymbol{\zeta} \text{ for right censored } i\}, \quad R_{\zeta,o} = \{\boldsymbol{\zeta}|\ \boldsymbol{v}_i'\boldsymbol{\zeta} \ge y_i \text{ for uncensored } i\},$$
$$R_{\zeta,lc} = \{\boldsymbol{\zeta}|\ \boldsymbol{w}_i'\boldsymbol{\delta} < \boldsymbol{v}_i'\boldsymbol{\zeta} \text{ for left censored } i\}, \quad R_{\zeta,rc} = \{\boldsymbol{\zeta}|\ \boldsymbol{v}_i'\boldsymbol{\zeta} < y_i^* \text{ for right censored } i\}.$$

4 Illustrative examples

In this section, we consider two examples, (i) Tobit model with covariate dependent thresholds and (ii) extended friction model, to illustrate our estimation procedure using simulated data.

4.1 Tobit model with covariate dependent thresholds

First, we consider Tobit model with covariate dependent thresholds given by (7) and (8). Let true values of the model be $\boldsymbol{\delta} = (1, 5, 10)'$, $\boldsymbol{\alpha} = (2, 1, 1)'$, and $\tau^2 = 0.6$. All covariates are generated using a standard normal distribution, *i.e.*, $x_{ij} \sim$ i.i.d. $\mathcal{N}(0, 1)$. The number of generated observations is 500, and 53 percent of them were censored. We assumed the prior distribution for $\boldsymbol{\delta}$, $\boldsymbol{\alpha}$ and τ as follows:

$$\boldsymbol{\delta}|\tau^2 \sim \mathcal{N}(0, 10\tau^2 I_3), \quad \boldsymbol{\alpha}|\tau^2 \sim \mathcal{N}(0, 10\tau^2 I_3), \quad \tau^2 \sim \mathcal{IG}\,(0.1, 0.1)\,.$$

In the Gibbs sampling from the posterior distribution, the initial 2,000 variates are discarded as the burn-in period and the subsequent $10,000$ values are retained.

Table 1: Posterior means, standard deviations, 95% credible intervals and inefficiency factors obtained from simple Gibbs sampler

	True	Mean	Stdev	95% Interval	Inef
δ_1	1.0	0.968	0.062	(0.868, 1.088)	6.6
δ_2	5.0	4.933	0.204	(4.641, 5.448)	219.8
δ_3	10.0	9.758	0.448	(9.124, 10.904)	219.5
α_1	2.0	2.022	0.049	(1.926, 2.119)	1.6
α_2	1.0	0.974	0.051	(0.874, 1.073)	3.3
α_3	1.0	1.066	0.048	(0.972, 1.161)	3.5
τ^2	0.6	0.624	0.058	(0.520, 0.750)	3.4

Table 1 shows the true parameter values, posterior means, standard deviations, 95

% credible intervals, and inefficiency factors for the simple Gibbs sampler. The inefficiency factor is defined as $1 + 2\sum_{s=1}^{\infty} \rho_s$ where ρ_s is the sample autocorrelation at lag s, and are computed to measure how well the MCMC chain mixes (see e.g. Chib 2001). It is the ratio of the numerical variance of the posterior sample mean to the variance of the sample mean from uncorrelated draws. The inverse of inefficiency factor is also known as relative numerical efficiency (Geweke 1992). When the inefficiency factor is equal to m, we need to draw MCMC samples m times as many as uncorrelated samples.

The posterior means are close to the true values, and all true values are contained in the 95% credible intervals. The inefficiency factors are $1 \sim 220$ which seem to increase as the value of δ_i becomes large. This is probably because the posterior distribution for large δ_i's becomes more sensitive to the linear inequality constraints.

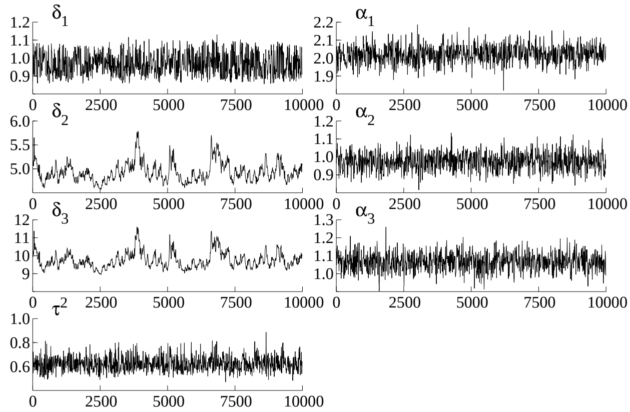
Table 2: Posterior means, standard deviations, 95% credible intervals and inefficiency factors obtained from accelerated Gibbs sampler

	True	Mean	Stdev	95% Interval	Inef
δ_1	1.0	0.968	0.063	(0.868, 1.089)	4.9
δ_2	5.0	4.928	0.191	(4.619, 5.363)	44.1
δ_3	10.0	9.747	0.418	(9.080, 10.699)	46.2
α_1	2.0	2.021	0.050	(1.923, 2.118)	2.4
α_2	1.0	0.974	0.050	(0.879, 1.073)	1.7
α_3	1.0	1.065	0.048	(0.971, 1.160)	1.9
τ^2	0.6	0.625	0.058	(0.521, 0.746)	3.9

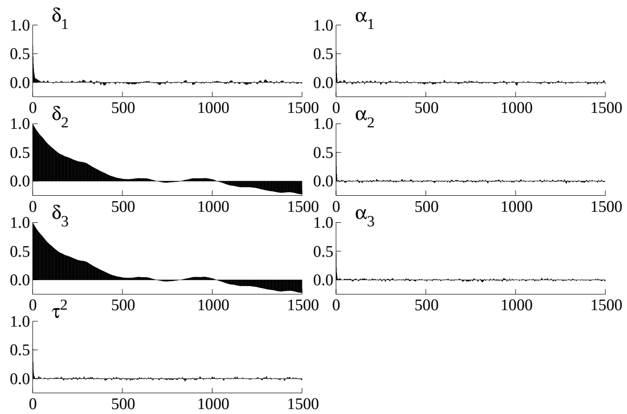
In Table 2, the corresponding summary statistics for the accelerated Gibbs sampler are shown. The inefficiency factors of δ_i's are $4 \sim 47$ suggesting that such an acceleration step can improve the mixing property of the Markov chain.

Figure 1 shows the sample paths, sample autocorrelations functions, and estimated marginal posterior distributions from the simple Gibbs sampler. While the sample autocorrelations vanish quickly for $\boldsymbol{\alpha}$ and τ^2, those of $\boldsymbol{\delta}$ do not vanish until 1500 lags indicating the slow convergence of the distribution of the MCMC samples to the posterior distribution.

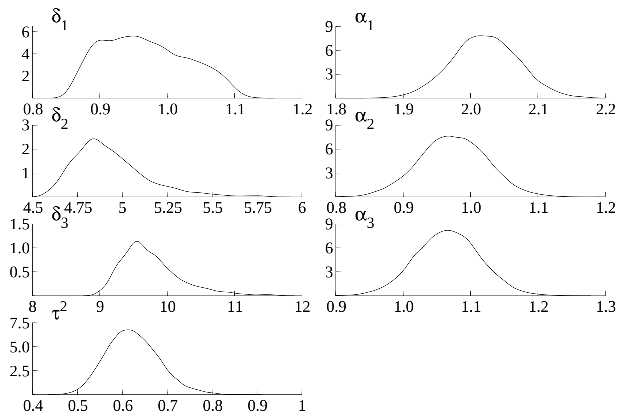
On the other hand, as shown in Figure 2, the sample paths from accelerated Gibbs sampler indicate that the Markov chains are mixing very well and all sample autocorrelations decay very quickly. It also shows that the acceleration step is effective to improve the speed of the convergence to the target distribution in the MCMC implementation.



(a) Sample paths.

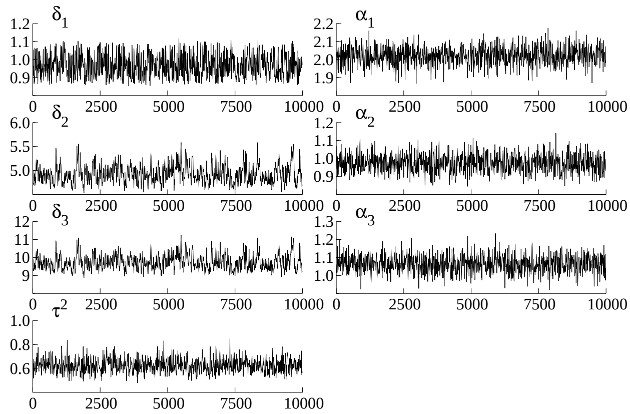


(b) Sample autocorrelations.

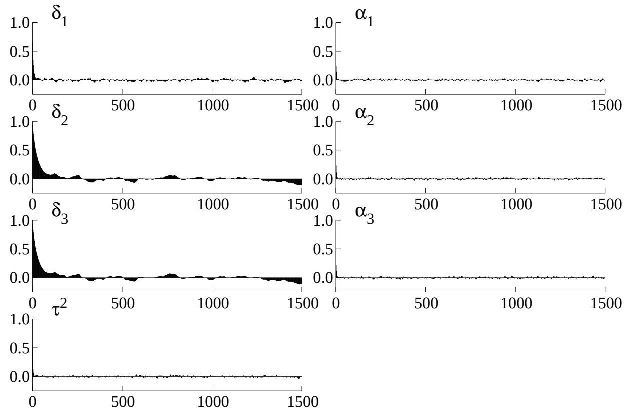


(c) Marginal posterior distributions.

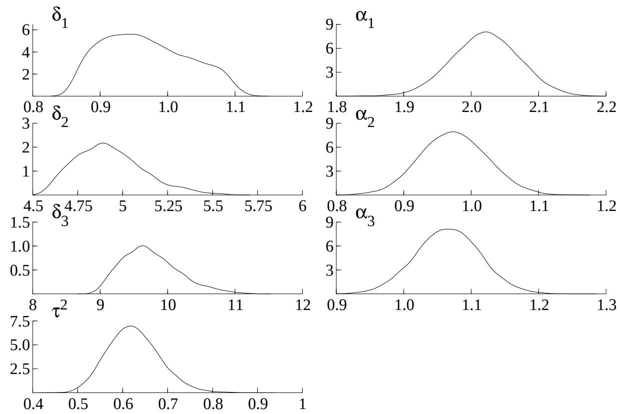
Figure 1: Tobit model. Simple Gibbs sampler.



(a) Sample paths.



(b) Sample autocorrelations.



(c) Marginal posterior distributions.

Figure 2: Tobit model. Accelerated Gibbs sampler.

4.2 Friction model

Next we illustrate our procedure for the friction model defined by (9) and (10). True values are $\boldsymbol{\delta} = (-1, -0.4)'$, $\boldsymbol{\zeta} = (0.5, 0.2)'$, $\boldsymbol{\alpha} = (2, 0.5)'$, and $\tau^2 = 0.49$. Covariates are generated using a standard normal distribution except for the second covariates of $\boldsymbol{w}_i$ and $\boldsymbol{v}_i$ (*i.e.*, w_{i2} and v_{i2}) where they are obtained by taking the absolute value of standard normal random variables. We generated five hundred observations, in which 34.8 percent were censored. The 37 percent were positive, while 28.2 percent were negative. Prior distributions are assumed as follows.

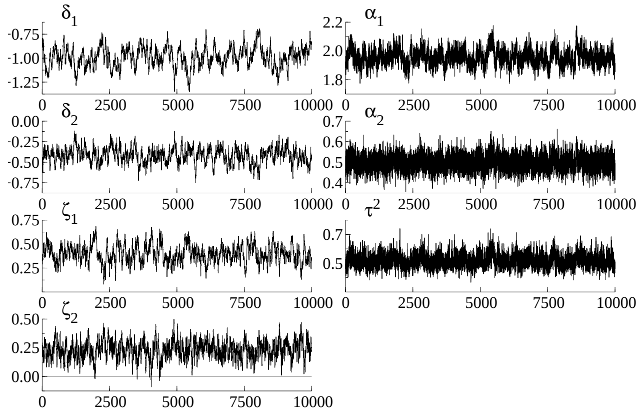
$$\begin{aligned} &\boldsymbol{\delta} \mid \tau^2 \sim \mathcal{N}\left(0, 10\tau^2 I_2\right), \quad \boldsymbol{\zeta} \mid \tau^2 \sim \mathcal{N}\left(0, 10\tau^2 I_2\right), \\ &\boldsymbol{\alpha} \mid \tau^2 \sim \mathcal{N}\left(0, 10\tau^2 I_2\right), \quad \tau^2 \sim \mathcal{IG}\left(0.1, 0.1\right). \end{aligned} \tag{13}$$

The proposed Gibbs sampler is implemented where we discard first $4,000$ samples and the subsequent $10,000$ samples are recorded to conduct Bayesian inference. Table 3 summarizes MCMC outputs: true parameter values, posterior means, standard deviations, 95 % credible intervals, and inefficiency factors. The posterior means are close to the true values, and all true values are contained in the 95% credible intervals. The inefficiency factors for $\boldsymbol{\zeta}$, $\boldsymbol{\alpha}$ and τ are not so large ($7 \sim 49$) while those for $\boldsymbol{\delta}$ are larger ($80 \sim 110$), suggesting that we may need a similar acceleration step to the Gibbs sampler as in Section 4.1.

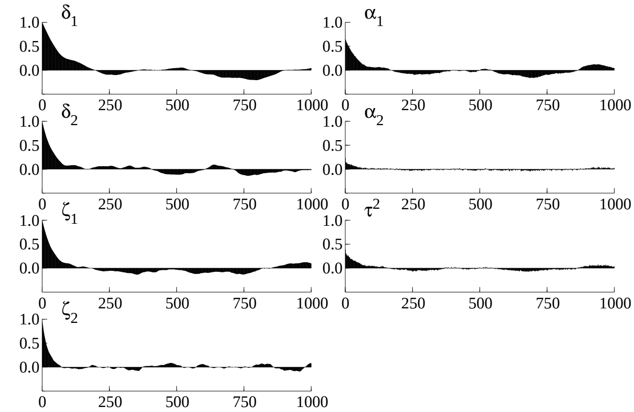
Table 3: Friction model: posterior means, standard deviations, 95% credible intervals, and inefficiency factors

	True	Mean	Stdev	95% Interval	Inef
δ_1	-1.0	-0.990	0.105	$(-1.195,\ -0.788)$	106.8
δ_2	-0.4	-0.418	0.093	$(-0.606,\ -0.247)$	81.9
ζ_1	0.5	0.399	0.093	(0.222, 0.590)	48.6
ζ_2	0.2	0.233	0.077	(0.077, 0.381)	36.2
α_1	2.0	1.957	0.056	(1.851, 2.071)	34.9
α_2	0.5	0.499	0.038	(0.425, 0.574)	7.2
τ^2	0.49	0.519	0.047	(0.434, 0.618)	17.8

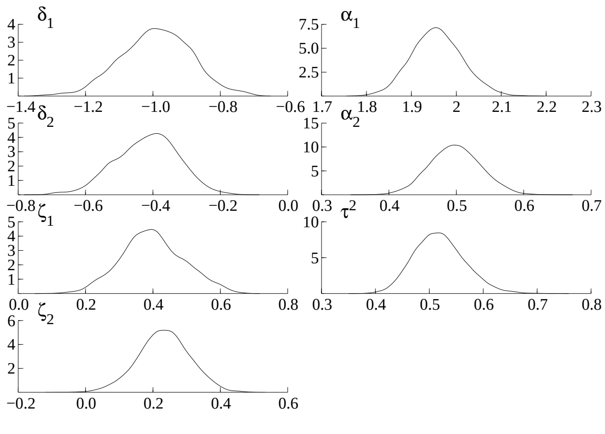
Figures 3 shows the sample paths, sample autocorrelations functions, and estimated marginal posterior distributions. The sample paths indicate that the Markov chains are mixing well and all sample autocorrelations seem to decay fairly quickly to zero. The estimated marginal posterior distributions have true values around their mode as expected.



(a) Sample paths.



(b) Sample autocorrelations.



(c) Marginal posterior distributions.

Figure 3: Friction model.

5 Empirical studies

5.1 Wage function of married women

First we apply the Tobit model with covariate dependent thresholds to the wage function of married women using a simple and accelerated Gibbs sampler. We consider the popular labor supply data by Mroz (1987) on 753 married white women in 1975 and compare three candidate models:

Model $M1$: Type 1 Tobit model with a fixed threshold

Model $M2$: Type 1 Tobit model with covariate dependent thresholds

Model $M3$: Sample selection (Type 2 Tobit) model

using marginal likelihoods and Deviance Information Criteria. The ages of those women in the dataset are between thirty and sixty, and 428 individuals worked during the year (hence 43.2% of wage data are censored). Table 4 shows variables considered in the following analysis.

Table 4: Female labour supply data from Mroz (1987)

Variable name	Description
wage	wife's average hourly earnings (in 10 dollars)
education	years of schooling (standardized)
age	wife's age (standardized)
income	family income (standardized)
kids	1 if there are children under 18, else 0
experience	actual years of wife's previous labour market experience (standardized)
city	1 if live in large urban area, else 0

The dependent variable is wife's average hourly earnings (in 10 dollars) and independent variables for the regression equation are education (years of schooling), experience (actual years of wife's previous labour), squared experience, and city (1 if live in large urban area, else 0). The education and experience variables are standardized such that their means and variances are equal to 0 and 1 respectively. The independent variables for the threshold equation and selection equation are education, age (wife's age), squared age, income (family income) and kids (1 if live in large urban area, else 0) The age and income variables are also standardized.

Model M1 (Type 1 Tobit model with a fixed threshold). The initial 10,000 variates are discarded and the subsequent $50,000$ values are recorded to conduct an inference. The summary statistics are given in Table 5. The inefficiency factors are very low, which implies the fast convergence of the distribution of MCMC samples to the posterior distribution. The 95% credible intervals do not include zero for variables, education, experience, and experience^2. The posterior probability of positive (negative) effects is greater than 0.95 for education and experience (for experience^2).

Table 5: $M1$: Type 1 Tobit model with a fixed threshold

Posterior means, standard deviations, 95% credible intervals
and inefficiency factors

Variable	Mean	Stdev	95% Interval	Inef
const	−6.546	1.045	(−8.601, −4.503)	1.2
education	0.643	0.081	(0.484, 0.803)	1.0
experience	2.167	0.240	(1.705, 2.647)	2.3
experience^2	−0.593	0.138	(−0.866, −0.324)	1.8
city	−0.084	0.379	(−0.822, 0.661)	0.8
τ^2	20.013	1.483	(17.293, 23.108)	4.2

Model M2 (Type 1 Tobit model with covariate dependent thresholds). Using the generalised Gibbs sampler discussed in Section 2, the initial 20,000 variates are discarded and the subsequent $100,000$ values are recorded where the acceptance rate of MH algorithm in the acceleration step was 57.6%. The summary statistics are given in Table 6. The inefficiency factors are based on the Gibbs sampler with the acceleration step (the factors without the acceleration step are given in brackets). The inefficiencies for regression equations are relatively small, while those for the threshold equation are still large in the range of $150 \sim 840$ even with the acceleration step.

In the estimated threshold equation, the 95% credible intervals do not include zero for age^2 and income, indicating that the individual threshold of reserved wage varies with these characteristics. The married women with higher family income tend to have higher reservation wage, while those with high education seem to have negative effects on reservation wage. For the regression equation, summary statistics are similar to those obtained in the standard Tobit model (Table 5).

Model M3 (Type 2 Tobit model). The initial 10,000 variates are discarded and the subsequent $50,000$ values are recorded. The summary statistics are given in Table 7. The inefficiency factors are relatively small.

Table 6: $M2$: Type 1 Tobit model with covariate dependent thresholds

Posterior means, standard deviations, 95% credible intervals and inefficiency factors [inefficiency factors obtained from the simple Gibbs sampler]

Threshold equation

Variable	Mean	Stdev	95% Interval	Inef	[Inef]
const	0.685	0.408	(−0.203, 1.385)	834.8	[1713.5]
education	−0.039	0.028	(−0.090, 0.015)	819.1	[1748.8]
age	0.114	0.056	(−0.003, 0.213)	166.5	[504.1]
age^2	0.132	0.079	(0.015, 0.237)	153.2	[180.0]
income	0.205	0.150	(0.072, 0.368)	403.0	[1084.5]
kids	−0.027	1.008	(−0.295, 0.304)	434.1	[727.6]

Regression equation

Variable	Mean	Stdev	95% Interval	Inef	[Inef]
const	−6.179	0.078	(−8.174, −4.221)	5.9	[7.1]
education	0.623	0.230	(0.471, 0.778)	5.6	[7.0]
experience	2.054	0.133	(1.610, 2.513)	3.0	[3.5]
experience^2	−0.553	0.133	(−0.817, −0.295)	1.9	[2.1]
city	−0.030	0.365	(−0.749, 0.686)	0.6	[1.0]
τ^2	18.42	1.363	(15.927, 21.266)	6.8	[6.1]

In the selection equation, The 95% credible intervals do not include zero for variables, education and income. The posterior probability that education (family income) has positive (negative) effect on the participation in the labour market is greater than 0.95.

For the regression equation, summary statistics are somewhat similar to those obtained in the standard Tobit model and a Tobit model with covariate dependent thresholds (Tables 5 & 6). The correlation coefficient ρ is found to be very high, suggesting that models $M1$ or $M2$ may be preferred.

We compared three models using the marginal likelihoods and the DIC. The marginal likelihoods and their standard errors are computed using the method of Chib (1995), and the DIC and their standard errors are based on 20 iterations of 5000 samples. The results are shown in Table 8. Using the marginal likelihood as a criterion of model selection, the model $M1$ has the largest value and selected as the best model. On the other hand, using the DIC, the model $M2$ attains the smallest value and selected as the best. This difference might be because the DIC sometimes tends to choose less parsimonious models as AIC (Akaike Information Criterion) as discussed in Spiegelhalter et al. (2002). However, we note that our Tobit model with covariate dependent thresholds is a good alternative model to the standard Tobit and Type 2 Tobit models.

Table 7: $M3$: Sample selection (Type 2 Tobit) model

Posterior means, standard deviations, 95% credible intervals and inefficiency factors

Selection equation

Variable	Mean	Stdev	95% Interval	Inef
const	−1.618	0.246	(−2.100, −1.137)	14.8
education	0.145	0.019	(0.107, 0.182)	13.9
age	−0.014	0.028	(−0.071, 0.038)	61.7
age^2	−0.000	0.026	(−0.052, 0.052)	37.2
income	−0.067	0.029	(−0.121, −0.008)	53.4
kids	−0.077	0.066	(−0.213, 0.048)	48.5

Regression equation

Variable	Mean	Stdev	95% Interval	Inef
const	−6.320	0.931	(−8.164, −4.505)	3.8
education	0.619	0.073	(0.476, 0.764)	2.4
experience	0.398	0.123	(0.159, 0.642)	57.6
experience^2	−0.078	0.072	(−0.225, 0.061)	25.1
city	0.040	0.194	(−0.343, 0.422)	41.0
σ^2	17.64	1.396	(15.035, 20.511)	83.5
ρ	0.991	0.004	(0.982, 0.996)	99.6

Table 8: Log marginal likelihood and DIC

Model	Log Marginal Likelihood	DIC
$M1$	−1489.5 (0.002)	2937.4 (0.1)
$M2$	−1516.9 (0.03)	2901.7 (0.8)
$M3$	−1536.1 (0.88)	2984.8 (0.7)

5.2 Prime rate changes

The second example is an application of the friction model with covariate dependent thresholds to the study on prime rate changes in Japan. The prime rate is the bank's lending rate for most favorable customers, and, because of its sticky movement, a friction model is sometimes applied to analyze prime rate changes (see e.g., Forbes and Mayne 1989). In this example, we use the monthly short-term prime rate data (principal banks) from July 1985 through June 2006 reported by Bank of Japan (see Figure 4).

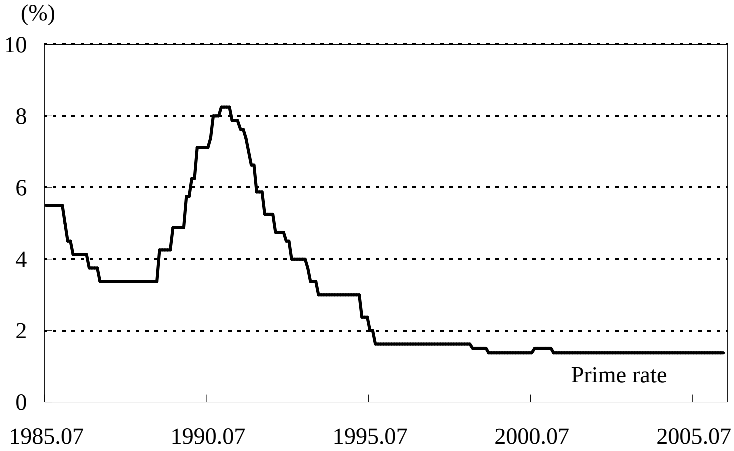


Figure 4: Monthly prime rate data in Japan.

Table 9: Variables for the friction model of prime rate changes

Variable name	Description
$dprime_t$	difference of the prime rates at months t and $t-1$
$dcall_t$	difference of the average call rates at months $t-1$ and $t-2$ (standardized)
$damount_t$	difference of the outstanding amounts in the call money market at the end of months $t-1$ and $t-2$ (standardized)
iip_t	index of the industrial productions for durable consumer goods shipment at month t (standardized)
$sales_t$	commercial sales value ratio of the wholesale at month t: ratio to the same month of the previous year (standardized)
$unemploy_t$	number of unemployed persons at month t (standardized)

Table 9 summarizes the variables we considered. The dependent variable is the difference of the prime rates at times t and $t-1$. There are two hundred fifty observations in which 9 and 24 observations show positive and negative prime rate changes, respectively. The sample mean, standard deviation, minimum and maximum are −0.017, 0.173, −0.750, and 0.875.

The independent variables for the regression equation are differences of the average call rates and amounts (reported by Bank of Japan) in the call money market at months $t-1$ and $t-2$, and independent variables for the threshold equations are selected from those related to the business cycle: index of industrial production for durable consumer goods shipment, commercial sales value ratio of the wholesale, and the number of unemployed persons at month t. First two variables are reported by Ministry of Economy, Trade and Industry, while the number of unemployed persons is reported by Ministry of Internal Affairs and Communications.

Table 10: Friction model of prime rate changes:
Posterior means, standard deviations, 95% credible intervals
and inefficiency factors

Threshold equations

	Mean	Stdev	95% Interval	Inef
Lower threshold				
δ_1 (const)	−1.746	0.441	(−2.846, −1.090)	359.9
δ_2 (iip)	−0.150	0.160	(−0.495, 0.140)	47.3
δ_3 (sales)	−0.187	0.180	(−0.571, 0.143)	45.0
δ_4 (unemploy)	−0.511	0.206	(−0.993, −0.183)	148.3
Upper threshold				
ζ_1 (const)	2.848	0.821	(1.659, 4.845)	358.5
ζ_2 (iip)	−0.537	0.301	(−1.228, −0.052)	158.6
ζ_3 (sales)	−0.809	0.373	(−1.689, −0.235)	220.1
ζ_4 (unemploy)	−0.456	0.296	(−1.124, 0.054)	126.3

Regression equation

	Mean	Stdev	95% Interval	Inef
α_1 (dcall)	0.327	0.118	(0.132, 0.598)	137.8
α_2 (damount)	0.180	0.129	(−0.049, 0.462)	57.6
τ^2	1.152	0.558	(0.484, 2.619)	348.2

To estimate parameters of the proposed friction model, we generated $100,000$ MCMC samples after discarding $40,000$ samples, The results are found in Table 10.

In the regression equation, the posterior probability that the call rate changes have a positive effect on prime rate changes is greater than 0.95. This result is quite natural because the call rate is used as one of instruments for the central bank's policy in order to control interest rates.

Among independent variables for the threshold equations, the unemployment rate has a negative effect on the lower threshold, while other variables (iip and sales) have negative effects on the upper threshold, because their 95% credible intervals do not include 0. The lower unemployment rate would increase the lower threshold, while the higher index of industrial production or the higher commercial sales value ratio would decrease the upper threshold. Thus, these results indicate that banks would change the prime rates more often when the economy is expanding than when it is declining.

6 Conclusion

Bayesian analysis of a Tobit model with covariate dependent thresholds is described using the MCMC estimation method, and the acceleration step based on Liu and Sabatti (2000) is introduced to improve the mixing property of the MCMC samples. Two important extensions of the proposed models are also discussed.

Numerical examples using simulated data are given to illustrate the proposed estimation methods and their efficiencies are investigated. In empirical studies, using the labor supply data on the married women, the Tobit model with covariate dependent thresholds is estimated and model comparisons are conducted based on the marginal likelihood and Deviance Information Criterion. Further, the friction model with covariate dependent thresholds is estimated using the prime rate data in Japan.

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Appendix A

A1 Posterior probability densities for a standard Tobit model

Joint posterior probability density of $(\boldsymbol{y}_c^*, \boldsymbol{\alpha}, \tau^2)$.

$$\begin{aligned}\pi(\boldsymbol{y}_c^*, \boldsymbol{\alpha}, \tau^2|\boldsymbol{y}_o) &\propto (\tau^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{y}^* - X\boldsymbol{\alpha})'(\boldsymbol{y}^* - X\boldsymbol{\alpha})\right\} \prod_i I(y_{c,i}^* \le d) \\ &\times (\tau^2)^{-\frac{K}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{\alpha} - \boldsymbol{a}_0)' A_0^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}_0)\right\} \times (\tau^2)^{-(\frac{n_0}{2}+1)} \exp\left\{-\frac{S_0}{2\tau^2}\right\}\end{aligned}$$

Conditional posterior probability density of $(\boldsymbol{\alpha}, \tau^2)$.

$$\begin{aligned}\pi(\boldsymbol{\alpha}, \tau^2|\boldsymbol{y}_c^*, \boldsymbol{y}_o) &= \pi(\tau^2|\boldsymbol{y}_c^*, \boldsymbol{y}_o)\pi(\boldsymbol{\alpha}|\tau^2, \boldsymbol{y}_c^*, \boldsymbol{y}_o) \\ &\propto (\tau^2)^{-(\frac{n_1}{2}+1)} \exp\left\{-\frac{S_1}{2\tau^2}\right\} \times (\tau^2)^{-\frac{K}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{\alpha} - \boldsymbol{a}_1)' A_1^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}_1)\right\}\end{aligned}$$

where $n_1 = n_0 + n, S_1 = \boldsymbol{y}^{*'}\boldsymbol{y}^* + S_0 + \boldsymbol{a}_0' A_0^{-1}\boldsymbol{a}_0 - \boldsymbol{a}_1' A_1^{-1}\boldsymbol{a}_1,\ A_1^{-1} = A_0^{-1} + X'X, \boldsymbol{a}_1 = A_1(A_0^{-1}\boldsymbol{a}_0 + X'\boldsymbol{y}^*)$.
Conditional posterior probability distribution of $\boldsymbol{y}^*$. $y_i^* \sim \mathcal{TN}_{(-\infty,d)}(\boldsymbol{x}_i'\boldsymbol{\alpha}, \tau^2)$ for censored observation.

A2 Joint posterior density of $(\boldsymbol{y}_c^*, \boldsymbol{\alpha}, \tau^2)$.

$$\begin{aligned}\pi(\boldsymbol{y}_c^*, \boldsymbol{\alpha}, \tau^2|\boldsymbol{y}_o) &\propto (\tau^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{y}^* - X\boldsymbol{\alpha})'(\boldsymbol{y}^* - X\boldsymbol{\alpha})\right\} \prod_i I(y_{c,i}^* \le \boldsymbol{w}_i'\boldsymbol{\delta}) \prod_j I(y_{o,j}^* \ge \boldsymbol{w}_j'\boldsymbol{\delta}) \\ &\times (\tau^2)^{-\frac{K}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{\alpha} - \boldsymbol{a}_0)' A_0^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}_0)\right\} \times (\tau^2)^{-\frac{J}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{\delta} - \boldsymbol{d}_0)' D_0^{-1}(\boldsymbol{\delta} - \boldsymbol{d}_0)\right\} \\ &\times (\tau^2)^{-(\frac{n_0}{2}+1)} \exp\left\{-\frac{S_0}{2\tau^2}\right\}\end{aligned}$$

Conditional posterior probability density of $(\boldsymbol{\alpha}, \tau^2)$.

$$\begin{aligned}&\pi(\boldsymbol{\alpha}, \tau^2|\boldsymbol{\delta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o) = \pi(\tau^2|\boldsymbol{\delta}, \boldsymbol{y}_c^*, \boldsymbol{y}_o) \times \pi(\boldsymbol{\alpha}|\boldsymbol{\delta}, \tau^2, \boldsymbol{y}_c^*, \boldsymbol{y}_o) \\ &\propto (\tau^2)^{-(\frac{n_1}{2}+1)} \exp\left\{-\frac{S_1}{2\tau^2}\right\} \times (\tau^2)^{-\frac{K}{2}} \exp\left\{-\frac{1}{2\tau^2}(\boldsymbol{\alpha} - \boldsymbol{a}_1)' A_1^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}_1)\right\}\end{aligned}$$

where $n_1 = n_0 + n + J, S_1 = \boldsymbol{y}^{*'}\boldsymbol{y}^* + S_0 + \boldsymbol{a}_0' A_0^{-1}\boldsymbol{a}_0 - \boldsymbol{a}_1' A_1^{-1}\boldsymbol{a}_1 + (\boldsymbol{\delta} - \boldsymbol{d}_0)' D_0^{-1}(\boldsymbol{\delta} - \boldsymbol{d}_0)$, $A_1^{-1} = A_0^{-1} + X'X, \boldsymbol{a}_1 = A_1(A_0^{-1}\boldsymbol{a}_0 + X'\boldsymbol{y}^*)$.
Conditional posterior probability distribution of $\boldsymbol{y}^*$. $y_i^*|\boldsymbol{\alpha}, \tau^2 \sim \mathcal{TN}_{(-\infty,\boldsymbol{w}_i'\boldsymbol{\delta})}(\boldsymbol{x}_i'\boldsymbol{\alpha}, \tau^2)$ for censored observation.
Conditional posterior probability distribution of $\boldsymbol{\delta}$. $\boldsymbol{\delta}|\boldsymbol{y}_c^*, \boldsymbol{y}_o \sim \mathcal{TN}_{R_o \cap R_c}(\boldsymbol{d}_0, \tau^2 D_0)$.

Appendix B Acceleration of the Gibbs sampler

To accelerate the convergence, we use a generalised Gibbs sampler introduced by Liu and Sabatti (2000). Consider a scale group $\Gamma = \{g > 0 : g(\boldsymbol{\varphi}) = (g\boldsymbol{\delta}, g\boldsymbol{\alpha}, g\tau, g\boldsymbol{y}_c^*)\}$ where $\boldsymbol{\varphi} = (\boldsymbol{\delta}, \boldsymbol{\alpha}, \tau, \boldsymbol{y}_c^*)$. The unimodular left-Harr measure $L(dg)$ for this scale group is $L(dg) = g^{-1}dg$ and the corresponding Jacobian is $J_g = g^{J+K+1+n-m}$. Let $(\boldsymbol{w}_{o,i}, \boldsymbol{x}_{o,i})$ denote the independent variable vectors of the i-th observed response $y_{o,i}$ and $X_o' = (\boldsymbol{x}_{o,1}, \ldots, \boldsymbol{x}_{o,m})$. By Theorem 1 of Liu and Sabatti (2000), the conditional probability density of g is given by

$$\begin{aligned}\pi(g|\boldsymbol{\varphi}, \boldsymbol{y}_o) &\propto \pi(g\boldsymbol{\delta}, g\boldsymbol{\alpha}, g\tau, g\boldsymbol{y}_c^*|\boldsymbol{y}_o) \times |J_g| \times L(dg)\\ &\propto g^{-(n_0+m-1)} \times g^{-2} \exp\left[-\frac{-2qg^{-1} + rg^{-2}}{2\tau^2}\right] I(\lambda_L \le g^{-1} \le \lambda_U),\end{aligned}$$

where $q = \boldsymbol{\alpha}' X_o' \boldsymbol{y}_o + \boldsymbol{\alpha}' A_0^{-1}\boldsymbol{a}_0 + \boldsymbol{d}_0' D_0^{-1}\boldsymbol{\delta}$ and $r = \boldsymbol{y}_o'\boldsymbol{y}_o + \boldsymbol{a}_0' A_0^{-1}\boldsymbol{a}_0 + \boldsymbol{d}_0' D_0^{-1}\boldsymbol{d}_0 + S_0$. Since this is not a well-known distribution, we conduct Metropolis-Hastings algorithm to sample from $\pi(g|\boldsymbol{\varphi}, \boldsymbol{y}_o)$. When $\lambda_U < \infty$, we generate a candidate $g'^{-1} \sim \mathcal{U}(\lambda_L, \lambda_U)$ given the current point g. We accept g' with probability

$$\min\left[1, \left(\frac{g'}{g}\right)^{-(n_0+m-1)} \exp\left\{-\frac{-2q(g'^{-1} - g^{-1}) + r(g'^{-2} - g^{-2})}{2\tau^2}\right\}\right].$$

We usually need to repeat the algorithm until it converges to $\pi(g|\boldsymbol{\varphi}, \boldsymbol{y}_o)$. However, by Theorem 2 of Liu and Sabatti (2000), we only need to conduct Metropolis-Hastings algorithm once using the initial value $g = 1$ since the Metropolis-Hastings transition kernel function satisfies $T_{\boldsymbol{\varphi}}(g, g') = T_{g_0^{-1}(\boldsymbol{\varphi})}(gg_0, g'g_0)$ for all $g, g', g_0 \in \Gamma$ where

$$\begin{aligned}&T_{\boldsymbol{\varphi}}(g, g')L(dg')\\ &= \frac{I(\lambda_L < g'^{-1} < \lambda_U)}{g'^2(\lambda_U - \lambda_L)} \min\left[1, \left(\frac{g'}{g}\right)^{-(n+m-1)} \exp\left\{-\frac{-2q(g'^{-1} - g^{-1}) + r(g'^{-2} - g^{-2})}{2\tau^2}\right\}\right] dg',\end{aligned}$$

where $g \neq g'$, and $T_{\boldsymbol{\varphi}}(g, g)L(dg) = 1 - \int T_{\boldsymbol{\varphi}}(g, g')L(dg')$. Thus, we generate a candidate g' and accept it with probability

$$\min\left[1, g'^{-(n_0+m-1)} \exp\left\{-\frac{-2q(g'^{-1} - 1) + r(g'^{-2} - 1)}{2\tau^2}\right\}\right].$$

Similarly, when $\lambda_U = \infty$, we generate a candidate $g'^{-1} \sim \mathcal{TN}_{(\lambda_L, \infty)}\left(1, \sigma_\lambda^2\right)$, for some σ_λ^2 and accept g' with probability

$$\min\left[1, g'^{-(n_0+m-1)} \exp\left\{-\frac{-2q(g'^{-1} - 1) + r(g'^{-2} - 1)}{2\tau^2} + \frac{(g'^{-1} - 1)^2}{2\sigma_\lambda^2}\right\}\right].$$

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