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Measuring polarization by reduced-form indices*

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Abstract

An abbreviated or reduced-form monotonic polarization index is an increasing function of the between-group term and a decreasing function of the within-group term of a population subgroup decomposable inequality index. The between-group term represents the „identification" component of polarization and the within-group term can be regarded as an inverse indicator of the „alienation" component of polarization. An ordering for ranking alternative distributions of income using such polarizations indices has been developed. Several polarization indices of the said type have been characterized using intuitively reasonable axioms. Finally, we also consider the dual problem of retrieving the inequality index from the specified form of a polarization index.

Keywords: Polarization, ordering, axioms, indices, characterization, duality.
JEL Classification: C43, D31, D63, O15.

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1. Introduction

A surge of interest has been observed in the measurement of polarization in the last decade because of its role in analyzing the evolution of the distribution of income, economic growth and social conflicts. Loosely speaking, polarization refers to clustering of incomes around local poles or subgroups in a distribution, where the individuals belonging to the same subgroup possess a feeling of identification among them and share a feeling of alienation against individuals in a different subgroup (see Esteban and Ray, 1994). That is, individuals belonging to the same subgroup identify themselves with the members of the subgroup in terms of income but in terms of the same characteristic they feel themselves as non-identical from members of the other subgroups. Since an increase in the 'identification' component increases homogeneity (equality) within a subgroup and higher 'alienation' leads to a greater heterogeneity (inequality) between subgroups, both 'identification' and 'alienation' are increasingly related to polarization. Thus, polarization involves an equity-like component (identification) and an inequity-like component (alienation). Evidently, a high level of polarization, as characterized by the presence of conflicting subgroups, may generate social conflicts, rebellions and tensions (see Pressman, 2001). Esteban and Ray (1994) developed an axiomatic characterization of an index of polarization in a quasi-additive framework by directly taking into account the above aspects[1,2] .

Zhang and Kanbur (2001) proposed an index of polarization, which incorporates the intuition behind the 'identification' and 'alienation' factors. Their index is given by the ratio between the between-group and within-group components of inequality, where for any partitioning of the population into disjoint subgroups, such as subgroups by age, sex, race, region, etc., between-group inequality is given by the level of inequality that arises due to variations in average levels of income among these subgroups. On the other hand, within-group inequality arises due to variations in incomes within each of the

[1] See also Esteban and Ray (1999), D'Ambrosio(2001), Gradin (2002), Duclos et al.(2004), Lasso de la Vega and Urrutia(2006) and Esteban et al. (2007).

[2] The Esteban - Ray (1994) notion of polarization contrasts with the concept of bi-polarization, which is measured by the dispersion of the distribution from the median towards the extreme points (see Foster and Wolfson, 1992, Wolfson ,1994,1997, Wang and Tsui, 2000 ,Chakravarty and Majumder, 2001,Chakravarty and D'Ambrosio,2009 and Lasso de la Vega et al. ,2009). For a recent discussion on alternative notions of polarization , see Chakravarty (2009).

subgroups. Thus, the between-group term can be taken as an indicator of alienation and the within-group component is inversely related to identification. A similar approach adopted by Rodriguez and Salas (2003) considered bi-partitioning of the population using the median and defined a bi-polarization index as the difference between the between-group and within-group terms of the Donaldson-Weymark (1980) S-Gini index of inequality (see also Silber et al., 2007). Such indices are 'reduced-form' or 'abbreviated' indices that can be used to characterize the trade-off between the alienation and identification components of polarization.

As Esteban and Ray (2005, p.27) noted the Zhang-Kanbur formulation is a 'direct translation of the intuition behind' the postulates that polarization is increasing in between-group inequality and decreasing in within-group inequality. Since the Zhang-Kanbur -Rodriguez-Salas approach enables us to understand the two main components of polarization, identification and alienation, in an intuitive way, our paper makes some analytical and rigorous investigation using the idea that polarization is related to between-group inequality and within-group inequality in increasing and decreasing ways respectively.

Now, polarization indices can give quite different results. Evidently, a particular index will rank income distributions in a complete manner. However, two different indices may rank two alternative income distributions in opposite directions. In view of this, it becomes worthwhile to develop necessary and sufficient conditions that make one distribution more or less polarized than another unambiguously. This is one objective of this paper. We can then say whether one income distribution has higher or lower polarization than another by all abbreviated polarization indices that satisfy certain conditions. In such a case it does not become necessary to calculate the values of the polarization indices to check polarization ranking of distributions. If the population is bi-partitioned using the median, then this notion of polarization ordering becomes close to the Wolfson (1994, 1997) concept of bi-polarization ordering.

Next, given the diversity of numerical indices it will be a worthwhile exercise to characterize alternative indices axiomatically for understanding which index becomes more appropriate in which situation. An axiomatic characterization gives us insight of the underlying index in a specific way through the axioms employed in the characterization exercise. This is the second objective of our paper. We characterize several polarization indices, including a generalization of the Rodriguez-Salas form. The structure of a normalized ratio form index parallels that of the Zhang-Kanbur index. We then show that the different sets of intuitively reasonable axioms considered in the characterization exercises are independent, that is, each set is minimal in the sense that none of its proper subset can characterize the index.

Finally, we show that it is also possible to start with a functional form of a polarization index and determine the inequality index which would generate the given polarization index. Specifically, we wish to determine a set of sufficient conditions on the form of a polarization index to guarantee that there exists an inequality index, which would produce the polarization index. This may be regarded as the dual of the characterization results for polarization indices.

Since subgroup decomposable inequality indices form the basis of our analysis, in the next section of the paper we make a discussion on such indices. The polarization ordering is discussed and analyzed in Section 3. The characterization theorems and a duality theorem are presented in Section 4. Finally, Section 5 concludes the paper.

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2. Background

For a population of size *n*, the vector $x=(x_1, x_2, .., x_n)$ represents the distribution of income, where each x_i is assumed to be drawn from the non-degenerate interval $[\upsilon, \infty)$ in the positive part R^1_{++} of the real line R^1. Here x_i stands for the income of person i of the population. For any $x_i \in [\upsilon, \infty)$, $x \in D^n = [\upsilon, \infty)^n$, the *n*-fold Cartesian product of $[\upsilon, \infty)$. The set of all possible income distributions is $D = \bigcup_{n \in N} D^n$, where N is the set of natural numbers. For all $n \in N$, for all $x = (x_1, x_2, ..., x_n) \in D^n$, $\sum_{i=1}^{n} (x_i / n)$, the mean of x, is denoted by $\lambda(x)$(or simply by λ). For all $n \in N$, 1^n denotes the *n*-coordinated vector of ones. The non-negative orthant of the n-dimensional Euclidean space R^n is denoted by R^n_+.An inequality index is a function $I : D \to R^1_+$.

An inequality index is said to be population subgroup decomposable if it satisfies the following axiom:

Subgroup Decomposability (SUD): For all $k \geq 2$ and for all $x^1, x^2, ..., x^k \in D$,

$$I(x) = \sum_{i=1}^{k} \omega_i\left(\underline{n}, \underline{\lambda}\right) \; I\left(x^i\right) + I\left(\lambda_1 1^{n_1}, \lambda_2 1^{n_2}, ..., \lambda_k 1^{n_k}\right), \qquad (1)$$

where n_i is the population size associated with the distribution x^i, $n = \sum_{i=1}^{k} n_i$, $\lambda_i = \lambda(x^i)$ =mean of the distribution x^i, $\underline{\lambda} = \left(\lambda_1, \lambda_2, ..., \lambda_k\right)$, $\underline{n} = (n_1, n_2, ..., n_k)$, $\omega_i(\underline{n}, \underline{\lambda})$ is the positive weight attached to inequality in x^i, assumed to depend on the vectors $\underline{n}$ and $\underline{\lambda}$, and $x = \left(x^1, x^2, ..., x^k\right)$. **SUD** shows that for any partitioning of the population, total inequality can be broken down into its within-group and between-group components. The between-group term gives the level of inequality that would arise if each income in a subgroup were replaced by the mean income of the subgroup and the within- group term is the weighted sum of inequalities in different subgroups (see Foster, 1985 and Chakravarty, 2009). Since for inequality and **SUD** to be well defined, we need $n, k \in \Gamma$ and $n_i \in \Gamma$ for all $1 \leq i \leq k$, we assume throughout the paper that $n \geq 4$, where $\Gamma = N \setminus \{1\}$.

Shorrocks (1980) has shown that a twice continuously differentiable inequality index $I : D \to R^1$ satisfying scale invariance (homogeneity of degree zero), subgroup decomposability, the Population Principle (invariance under replications of the population), symmetry (invariance under reordering of incomes), continuity and non-negativity (the non-negative index takes on the value zero if only if all the incomes are equal) must be of the following form:

$$I_c(x) = \begin{cases} \dfrac{1}{nc(c-1)} \sum_{i=1}^{n} \left[\left(\dfrac{x_i}{\lambda}\right)^c - 1\right], c \neq 0,1, \\ \dfrac{1}{n} \sum_{i=1}^{n} \log \dfrac{\lambda}{x_i}, c = 0, \\ \dfrac{1}{n} \sum_{i=1}^{n} \dfrac{x_i}{\lambda} \log \dfrac{x_i}{\lambda}, c = 1. \end{cases} \qquad (2)$$

The family I_c, which is popularly known as the generalized entropy family satisfies the Pigou-Dalton transfers principle, a postulate, which requires inequality to reduce under a transfer of income from a person to anyone who has a lower income such that the transfer does not change the relative positions of the donor and the recipient. The transfer decreases I_c by a larger amount the lower is the value of c. If $c = 0$, I_c coincides with the Theil (1972) mean logarithmic deviation I_{ML}. For $c = 1$, I_c becomes the Theil (1967) entropy index of inequality. For $c = 2$, I_c becomes half the squared coefficient of variation. The well-known Gini index of inequality becomes subgroup decomposable if subgroup income distributions are non-overlapping. Since our formulation of SUD does not depend on such a restriction, I_c does not contain the Gini index as a special case.

The absolute sister of the family I_c, that is, the class of subgroup decomposable indices that remains invariant under equal translation of all incomes is given by:

$$I_\theta(x) = \frac{1}{n}\sum_{i=1}^{n}\left[e^{\theta(x_i-\lambda)} - 1\right],\ \theta \neq 0,$$

$$I_V(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \lambda^2. \qquad (3)$$

The variance I_V and the exponential index I_θ, for all real non-zero values of θ, satisfy the Pigou-Dalton transfers principle (see Chakravarty and Tyagarupananda, 2009).

The weight attached to the inequality of subgroup i in the decomposition of the family I_c is given by $\omega_i(\underline{n},\underline{\lambda}) = (n_i/n)\big/(\lambda/\lambda_i)^c$. The corresponding weights in the decomposition of I_θ and I_V are given by $\omega_i(\underline{n},\underline{\lambda}) = \left(n_i e^{\theta\lambda_i}\right)\big/\left(ne^{\theta\lambda}\right)$ and $\{n_i/n\}$ respectively. Evidently, the sum of these weights across subgroups becomes unity only for the two Theil indices and the variance.

If there is a progressive transfer of income between two persons in a subgroup then inequality within the subgroup decreases without affecting between-group inequality. But polarization increases because of higher homogeneity/identification of individuals within a subgroup. Of two subgroups, a proportionate (an absolute) reduction in all incomes of the one with lower mean keeps the subgroup relative (absolute) inequality unchanged but reduces its mean income further. Likewise, a proportionate (an absolute) increase in the incomes of the other subgroup increases its mean but keeps relative (absolute) inequality unaltered. This in turn implies that *BI* increases. In other words, a greater distancing between subgroup means, keeping within-group inequality unchanged, increases between-group inequality making the subgroups more heterogeneous. A sufficient condition that ensures fulfillment of this requirement is that the decomposition coefficient $\omega_i(\underline{n},\underline{\lambda})$ is independent of the subgroup means. The only subgroup decomposable indices for which this condition holds are the Theil mean logarithmic deviation index I_{ML}, which corresponds to $c = 0$ in (2)[2], and the variance. We denote the set $\{I_{ML}, I_V\}$ of these two indices by SD. For further analysis, we restrict our attention to the set SD. Note that the members of SD are onto functions so that they vary continuously over the entire non-negative part of the real line. (It may be mentioned here that the Esteban – Ray (2005) discussion on the Kanbur-Zhang index is based on the functional form I_{ML}.)

3. The Polarization Ordering

Following our discussion in Section 1, we define a polarization index P as a real valued function of income distributions of arbitrary number of subgroups of a population, partitioned with respect to some homogeneous characteristic. Formally,

Definition 1: By a polarization index we mean a continuous function $P : \Omega \to R^1$, where $\Omega = \bigcup_{k \in \Gamma} \left(\prod_{n_i \in \Gamma, 1 \le i \le k} D^{n_i} \right)$.

For any $x = (x^1, x^2, ..., x^k) \in \Omega$, $k \in \Gamma$, the real number $P(x)$ indicates the level of polarization associated with x.

Often economic indicators abbreviate the entire income distribution in terms of two or more characteristics of the distribution. For instance, a ‘reduced-form’ welfare function expresses social welfare as an increasing function of efficiency (mean income) and a decreasing function of inequality (see Ebert, 1987; Amiel and Cowell, 2003 and Chakravarty, 2009, 2009a). Likewise, we have

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[2] Buourguignon (1979) developed a characterization of I_{ML} using $\omega_i(\underline{n},\underline{\lambda}) = n_i / n$.

Definition 2: A polarization index P is called abbreviated or reduced-form if for all $x=\left(x^1,x^2,...,x^k\right)\in\Omega, k\in\Gamma$, $P(x)$ can be expressed as $P(x)=f\left(BI(x),WI(x)\right)$, where $I\in SD$ is arbitrary and the real valued function f defined on R_+^2 is continuous.

We refer to the function f considered above as a characteristic function. Clearly, the polarization index defined above will be a relative or an absolute index according as we choose I_{ML} or I_V as the inequality index.

Since the characteristics 'identification' and 'alienation' are regarded as being intrinsic to the concept of polarization, in order to take them into account correctly we assume that the function f is monotonic, that is, it is increasing in BI and decreasing in WI. An abbreviated polarization index with a monotonic characteristic function will be called feasible.

In order to develop a polarization ordering of the income distributions, consider the distributions $x=\left(x^1,x^2,...,x^k\right), y=\left(y^1,y^2,...,y^k\right) \in\prod_{i=1}^{k}D^{n_i}$, where $k\geq 2$, $n_i\geq 2$, $1\leq i\leq k$, are arbitrary. Then we say that x is more polarized than y, what we write $x\succ_P y$, if $P(x)>P(y)$ for all feasible polarization indices $P:\prod_{i=1}^{k}D^{n_i}\rightarrow R^1$. Our definition of $\succ_P$ is general in the sense that we do not assume equality of the total income of the distributions.

As we have noted in the previous section, given $y=\left(y^1,y^2,...,y^k\right)\in\prod_{i=1}^{k}D^{n_i}$, we can generate $x=\left(x^1,x^2,...,x^k\right)\in\prod_{i=1}^{k}D^{n_i}$, which is more polarized than y, by one of the following three polarization increasing transformations: (i) decreasing WI (keeping BI unchanged), (ii) increasing BI (keeping WI unchanged), and (iii) decreasing WI and increasing BI. We can write these three conditions more compactly as $BI(x)\geq BI(y)$ and $WI(x)\leq WI(y)$ with strict inequality in at least one case. The following theorem demonstrates equivalence of this with $x\succ_P y$.

Theorem 1: Let $x=\left(x^1,x^2,...,x^k\right), y=\left(y^1,y^2,...,y^k\right) \in \prod_{i=1}^{k} D^{n_i}$, where $k \geq 2$, $n_i \geq 2$, $1 \leq i \leq k$, are arbitrary. Then the following conditions are equivalent:

(i) $x \succ_P y$.

(ii) $BI(x) \geq BI(y)$ and $WI(x) \leq WI(y)$ for any inequality index I in SD, with strict inequality in at least one case.

What Theorem 1 says is the following: if condition (ii) holds then we can unambiguously say that distribution x is regarded as more polarized than distribution y by all reduced-form polarization indices that are increasing in BI and decreasing in WI. Note that we do not require equality of the mean incomes of the distributions for this result to hold.

Proof of Theorem 1: Suppose $x \succ_P y$ holds. Consider the polarization index $P_\varepsilon(x)=BI(x)-\varepsilon WI(x)$, where $\varepsilon > 0$ is arbitrary. By definition $P_\varepsilon(x)$ is a feasible index. Now, $P_\varepsilon(x) > P_\varepsilon(y)$ implies that $BI(x)-BI(y) > \varepsilon(WI(x)-WI(y))$. Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$, we get $BI(x) \geq BI(y)$.

Next, consider the feasible index $P'_\varepsilon(x)=\varepsilon BI(x)-WI(x)$, where $\varepsilon > 0$ is arbitrary. Then $P'_\varepsilon(x) > P'_\varepsilon(y)$ implies that $WI(x)-WI(y) < \varepsilon(BI(x)-BI(y))$. Again because of arbitrariness of $\varepsilon > 0$, we let $\varepsilon \to 0$ and find that $WI(x) \leq WI(y)$.

Now, at least one of the inequalities $BI(x) \geq BI(y)$ and $WI(x) \leq WI(y)$ has to be strict. This is because if $BI(x)=BI(y)$ and $WI(x)=WI(y)$, then $P(x)=f(BI(x),WI(x))= f(BI(y),WI(y))$, that is, $P(x)=P(y)$, which contradicts the assumption $x \succ_P y$.

The proof of the converse follows from the defining condition of the feasible polarization index, that is, increasingness in the first argument and decreasingness in the second argument. Δ

The polarization ordering defined in the theorem is a quasi-ordering-it is transitive but not complete. To see this, consider the bi-partitioned distributions $x=((1,3,5),(2,6))$ and $y=((1,3,5),(2,4))$. Let us choose I_V as the index of inequality and denote its between and within-group components by BI_V and WI_V respectively. Then $BI_V(x)=(6/25), BI_V(y)=0$. Also $WI_V(x)=(16/5)$, $WI_V(y)$ =2. Thus, we have

$BI_V(x) > BI_V(y)$ and $WI_V(x) > WI_V(y)$. This shows that the distributions x and y are not comparable with respect to $\succ_P$ and hence $\succ_P$ is not a complete ordering. Next, suppose that for three distributions x, y and z, partitioned with respect to the same characteristic into equal number of subgroups, we have $x \succ_P y$ and $y \succ_P z$. Then it is easy to check that $x \succ_P z$ holds, which demonstrates transitivity of $\succ_P$.

Now, to see that inequality ordering of income distributions is different from polarization ordering, consider the bi-partitioned distributions $y = ((a,c),(b,d))$ and $x = ((a,c-\varepsilon),(b+\varepsilon,d))$, where $a < b < c < d$ and $0 < \varepsilon < (c-b)/2$. Then it is easy to see that $BI_V(y) < BI_V(x)$ but $WI_V(y) > WI_V(x)$. Hence for all feasible polarization indices P, we have $P(y) < P(x)$. But by the Pigou-Dalton transfers principle, $I_V(y) > I_V(x)$. Next, let us consider the income distribution $x = (x^1, x^2, ..., x^k) \in \prod_{i=1}^{k} D^{n_i}$ and generate the distribution $y = (y^1, y^2, ..., y^k)$ from x by the following transformation: $y^i = x^i$ for all $i \neq j$ and y^j is obtained from x^j by a progressive transfer of income between two persons in subgroup j. By construction, $BI(x) = BI(y)$ and $WI(y) < WI(x)$, where $I \in SD$. This in turn implies that for any feasible polarization index P, $P(y) > P(x)$. But the inequality ordering here is $I(x) > I(y)$. Thus, in these two cases polarization and inequality rank the distributions in completely opposite ways. The intuitive reasoning behind this is that while each of the two components BI and WI is related to inequality in an increasing manner, for polarization the former has an increasing relationship but for the latter the relationship is a decreasing one.

Now, to relate $\succ_P$ with the bi-polarization ordering, which relies on the increased spread and increased bipolarity axioms, suppose that the distributions are partitioned into two subgroups with incomes below and above the median. The increased spread axiom says that polarization should go up under increments (reductions) in incomes above (below) the median. The increased bipolarity axiom, which requires bi-polarization to increase under a progressive transfer of income on the either side of the median, is a bunching or clustering principle. It is shown that an unambiguous ranking of two income distributions by all relative, symmetric, population replication invariant bi-polarization indices that satisfy the increased spread and increased bipolarity axioms can be achieved through comparison of their relative bipolarization curves. A relative bi-polarization curve shows the deviations of the total incomes of different population proportions from the corresponding totals that they would enjoy under the hypothetical distribution where everybody has the median income. (See Foster and Wolfson, 1992, Wolfson, 1997, 1999, Chakravarty et al., 2007 and Chakravarty, 2009.) Note that here alienation refers to increase in the distance between the groups below and above the median and hence is similar in spirit to the increased spread axiom. Likewise, the increased bipolarity axiom possesses the same flavor as the identification criterion. Thus, the two notions of polarization ordering are essentially the same when the two population subgroups are formed using the median[3].

4. **The Characterization Theorems**

A characterization result requires specification of a set of axioms which seem to be appropriate for a polarization index in a particular framework. These axioms become helpful in understanding the underlying polarization index in an intuitive way.

We begin by specifying the following axioms.

$(A1)$ For all $x=(x^1,x^2,...,x^k)\in\Omega$, $k\in\Gamma$ and for any non-negative α, $f(BI(x)+\alpha,WI(x))-f(BI(x),WI(x))=\psi(BI(x),WI(x))g(\alpha)$ for some continuous functions $\psi:R_+^2\to R_{++}^1$ and $g:R_+^1\to R_+^1$, where g is increasing, $g(0)=0$ and $I\in SD$.

$(A2)$ For all $x=(x^1,x^2,...,x^k)\in\Omega$, $k\in\Gamma$ and for any non-negative β, $f(BI(x),WI(x))-f(BI(x),WI(x)+\beta)=\varphi(BI(x),WI(x))h(\beta)$ for some continuous functions $\varphi:R_+^2\to R_{++}^1$ and $h:R_+^1\to R_+^1$, where h is increasing, $h(0)=0$ and $I\in SD$.

Axiom $(A1)$ says that increment in polarization resulting from an increase in *BI* by the amount α can be decomposed into two continuous factors, one a non-negative function of α alone and the other a positive valued function of *BI* and *WI*. Increasingness of the

[3] In a recent contribution, Bossert and Schworm (2008) showed that the two-group approach can be interpreted in terms of treating polarization as an aggregate of inverse welfare measures of the two groups under consideration.See also Chakravarty et al. (2007) for a related discussion.

function g reflects the view that polarization is increasing in *BI*. The assumption $g(0)=0$ ensures that if there is no change in *BI*, there will be no change in the value of the polarization index (assuming that *WI* remains unaltered). Many economic indicators satisfy this type of axiom. For instance, for the two-person welfare function $W(x_1, x_2) = 1-e^{-x_1-x_2}$, the change $W(x_1+c, x_2)-W(x_1, x_2)$, where $c>0$, can be expressed as the product $\left(e^{-x_1-x_2}\right)\left(1-e^{-c}\right)$, so that here we have $\psi(x_1, x_2)=\left(e^{-x_1-x_2}\right)>0$ and $g(c)=\left(1-e^{-c}\right)$. Observe that both ψ and g are continuous, g is increasing and $g(0)=0$. Axiom $(A2)$ can be interpreted similarly.

Often we may need to assume that a polarization index is normalized, that is, for a perfectly equal distribution the value of the polarization index is zero. Formally,

$(A3)$ For arbitrary $k \in \Gamma$, if $x=\left(x^1, x^2, ..., x^k\right) \in \Omega$ is of the form $x^i = c1^{n_i}$, where $n_i \in \Gamma$ for all $1 \le i \le k$ and $c>0$ is a scalar, then for any $I \in SD$, $f\left(BI(x), WI(x)\right)=0$.

Since for a perfectly equal distribution x, $BI(x)=WI(x)=0$, we may restate axiom $(A3)$ as $f(0,0)=0$.

The following theorem can now be stated.

Theorem 2: Assume that the characteristic function is continuously differentiable. Assume also that the right partial derivative of the characteristic function at zero with respect to each argument exists and is positive for the first argument and negative for the second argument. Then a feasible polarization index $P:\Omega \to R^1$ with such a characteristic function satisfies axioms $(A1)$, $(A2)$ and $(A3)$ if and only if it is of one of the following forms for some arbitrary positive constants c_1 and c_2:

$(i) P_1(x)=c_1 BI(x)-c_2 WI(x)$,

$(ii) P_2(x)=\dfrac{c_1}{\log a}\left(a^{BI(x)}-1\right)-c_2 WI(x),\ a>1$,

$(iii) P_3(x)=\left(a^{BI(x)}-1\right)\left(\dfrac{c_1}{\log a}+\rho WI(x)\right)-c_2 WI(x),\ 0<a<1, -c_2 \le \rho \le 0$,

$(iv) P_4(x)=c_1 BI(x)-\dfrac{c_2}{\log b}\left(b^{WI(x)}-1\right), b>1$,

$(v) P_5(x) = c_1 BI(x) - \left(b^{WI(x)} - 1\right)\left(\frac{c_2}{\log b} + \sigma BI(x)\right), 0 < b < 1, -c_1 \le \sigma \le 0,$

$(vi) P_6(x) = \frac{c_1}{\log a}\left(a^{BI(x)} - 1\right) - \frac{c_2}{\log b}\left(b^{WI(x)} - 1\right),\ a > 1, b > 1,$

$(vii) P_7(x) = \frac{c_1}{\log a}\left(a^{BI(x)} - 1\right) - \frac{c_2}{\log b}\left(b^{WI(x)} - 1\right) + \eta\left(a^{BI(x)} - 1\right)\left(b^{WI(x)} - 1\right),\ a > 1, 0 < b < 1,$

$0 \le \eta \log a \le c_1,$

$(viii) P_8(x) = \frac{c_1}{\log a}\left(a^{BI(x)} - 1\right) - \frac{c_2}{\log b}\left(b^{WI(x)} - 1\right) + \eta\left(a^{BI(x)} - 1\right)\left(b^{WI(x)} - 1\right),\ 0 < a < 1, b > 1,$

$-c_2 \le \eta \log b \le 0,$

$(ix)\quad P_9(x) = \frac{c_1}{\log a}\left(a^{BI(x)} - 1\right) - \frac{c_2}{\log b}\left(b^{WI(x)} - 1\right) + \eta\left(a^{BI(x)} - 1\right)\left(b^{WI(x)} - 1\right),\ 0 < a, b < 1,$

$\frac{c_1}{\log a} \le \eta \le -\frac{c_2}{\log b},$

where $x = \left(x^1, x^2, ..., x^k\right) \in \Omega,\ k \in \Gamma$ and $I \in SD$ are arbitrary.

Here the only assumptions we make about f are its continuous differentiability and existence of partial derivatives at the end point 0. Many economic indicators satisfy these assumptions. It is known that if the partial derivatives exist at the end point 0, then they are right partial derivatives (Rudin, 1987, p.104).

Proof of Theorem 2: Since the components of the two inequality indices considered are onto functions, we can restate axioms $(A1)$ and $(A2)$ as follows:

$$f(s+\alpha, t) - f(s,t) = \psi(s,t) g(\alpha), \tag{4}$$

$$f(s,t) - f(s, t+\beta) = \varphi(s,t) h(\beta), \tag{5}$$

where $s, t, \alpha, \beta \ge 0$ are arbitrary. Putting $s = 0$ in (4) and assuming positivity of α we get

$$f(\alpha, t) - f(0,t) = \psi(0,t) g(\alpha). \tag{6}$$

By assumption $\psi(0,t) > 0$ and $g(\alpha) > g(0) = 0$. From (4) and (6) it then follows that

$$\frac{f(s+\alpha, t) - f(s,t)}{f(\alpha, t) - f(0,t)} = \frac{\psi(s,t)}{\psi(0,t)}, \tag{7}$$

for all $s, t \ge 0$.

For a fixed $t \in R_+^1$, define $f_t : R_+^1 \to R^1$ by $f_t(s) = f(s,t)$, where $s \geq 0$. Then continuous differentiability of f implies that f_t is also continuously differentiable. We rewrite (7) in terms of f_t as follows:

$$\frac{f_t(s+\alpha) - f_t(s)}{f_t(\alpha) - f_t(0)} = \frac{\psi(s,t)}{\psi(0,t)}. \tag{8}$$

Note that the right hand side of (8) is independent of α. So we can divide the denominator and numerator of the left hand side of (8) by α and take the limit of the resulting expressions as $\alpha \to 0$. Then (8) becomes

$$\frac{f_t'(s)}{f_t'(0)} = \frac{\psi(s,t)}{\psi(0,t)}, \tag{9}$$

where f_t' stands for the derivative of f_t. By assumption the right hand side of (9) is positive. This along with positivity of $f_t'(0)$ (by assumption) implies that $f_t'(s) > 0$ for all $s \geq 0$. From this it follows that $\dfrac{\partial f(s,t)}{\partial s} > 0$ for all $s, t \geq 0$.

Because of independence of the right hand side of (8) of α, the derivative of the left hand side of (8) with respect to α is zero. This gives $(f_t(\alpha) - f_t(0)) f_t'(s+\alpha) = (f_t(s+\alpha) - f_t(s)) f_t'(\alpha)$, from which it follows that

$$\frac{f_t(s+\alpha) - f_t(s)}{f_t(\alpha) - f_t(0)} = \frac{f_t'(s+\alpha)}{f_t'(\alpha)}. \tag{10}$$

Equations (8), (9) and (10) jointly imply that $\dfrac{f_t'(s+\alpha)}{f_t'(\alpha)} = \dfrac{f_t'(s)}{f_t'(0)}$, which gives $f_t'(s+\alpha) = (f_t'(s) f_t'(\alpha)) / f_t'(0)$. Define the function $\mu_t : R_+^1 \to R^1$ by $\mu_t(s) = f_t'(s) / f_t'(0)$. Then the previous equation becomes

$$\mu_t(s+\alpha) = \mu_t(s)\eta_t(\alpha) \tag{11}$$

for all $s, \alpha \geq 0$. Since f is continuously differentiable, μ_t is continuous. The general nontrivial solution to the functional equation (11) is given by $\mu_t(s) = (a(t))^s$ for some continuous function $a : R_+^1 \to R_{++}^1$, where $s \geq 0$ is arbitrary (Aczel, 1966, p.41).

Letting $f_t'(0)=w(t)$, we can now write f_t' as $f_t'(s)=(a(t))^s w(t)$ for some continuously differentiable maps $a,w: R_+^1 \to R_{++}^1$. Integrating f_t' we get

$$f_t(s)=\begin{cases}\dfrac{(a(t))^s w(t)}{\log a(t)}+w_1(t),\ a(t)\neq 1,\\ sw(t)+w_1(t),\ a(t)=1,\end{cases} \tag{12}$$

where $s \geq 0$ is arbitrary and $w_1: R_+^1 \to R^1$ is continuously differentiable. We rewrite (12) more explicitly as

$$f(s,t)=\begin{cases}\dfrac{(a(t))^s w(t)}{\log a(t)}+w_1(t),\ a(t)\neq 1,\\ sw(t)+w_1(t),\ a(t)=1.\end{cases} \tag{13}$$

where $s,t \geq 0$ are arbitrary.

We now show that $a(t)$ is a constant for all $t \geq 0$. First, note that there is nothing to prove if $a(t)=1$ for all $t \geq 0$. If $a(t)\neq 1$ for some $t \geq 0$, then consider the set $B=\{t \geq 0 : a(t)\neq 1\}$, which is assumed to be non-empty. Now, (4) along with the first equation in (13) implies that for all $t \in B$ and for all $s \geq 0$,

$$\frac{(a(t))^{s+\alpha} w(t)}{\log a(t)}-\frac{(a(t))^s w(t)}{\log a(t)}=\psi(s,t)g(\alpha). \tag{14}$$

Putting $s=0$ in (14) we get $\dfrac{((a(t))^\alpha -1)w(t)}{\log a(t)}= \psi(0,t)g(\alpha)$, which gives

$$((a(t))^\alpha -1)=\phi(t)g(\alpha), \tag{15}$$

where $\phi(t)=(\psi(0,t)\log a(t))/w(t)$ and $t \in B$ is arbitrary. Since by assumption $a(t)\neq 1$ for all $t \in B$, the right hand side of (15) is non-zero for all $\alpha > 0$. Substituting $\alpha = 1$ and 2 in (15) we get $((a(t))-1)=\phi(t)g(1)$ and $((a(t))^2 -1)=\phi(t)g(2)$ respectively. Dividing the right (left) hand side of the second equation by the corresponding side of the first equation, we get $((a(t))+1)=g(2)/g(1)$, which implies that for all $t \in B$, $a(t)= -1+g(2)/g(1)=c$, a positive constant. But $a(t)=1$ for all nonnegative $t \in B^c$, the complement of B. Since $a(t)$ is a continuous map on its domain and B is a non-empty set, B^c must be empty. Thus, $a(t)=c$, a positive constant not equal to one, for all $t \geq 0$. Hence in either case $a(t)$ is a constant. In the sequel we will write a in place of $a(t)$.

Therefore, equation (13) now can be written as

$$f(s,t) = \begin{cases} \dfrac{a^s w(t)}{\log a} + w_1(t), 0 < a \neq 1, \\ sw(t) + w_1(t), a = 1, \end{cases} \tag{16}$$

where $s, t \geq 0$ are arbitrary, w, w_1 are continuously differentiable and w is positive valued.

Proceeding in a similar manner and making use of axiom $(A2)$ we get

$$f(s,t) = \begin{cases} \dfrac{b^t \gamma(s)}{\log b} + \gamma_1(s), 0 < b \neq 1, \\ t\gamma(s) + \gamma_1(s), b = 1, \end{cases} \tag{17}$$

for some continuously differentiable maps $\gamma, \gamma_1 : R_+^1 \to R$, γ being negative valued. We can also show that $\dfrac{\partial f(s,t)}{\partial t} < 0$ for all $s, t \geq 0$.

Now, for comparing (16) and (17) we need to consider various cases.

Case I: $f(s,t) = sw(t) + w_1(t) = t\gamma(s) + \gamma_1(s)$. (18)

By axiom $(A3)$, $w_1(0) = \gamma_1(0) = 0$. Putting $s = 0$ in (18), we get $w_1(t) = t\gamma(0)$. Likewise, for $t = 0$, we have $sw(0) = \gamma_1(s)$. Substituting these expressions for w_1 and γ_1 in (18), we get $sw(t) + t\gamma(0) = t\gamma(s) + sw(0)$, from which it follows that $s(w(t) - w(0)) = t(\gamma(s) - \gamma(0))$. Since this holds for all $s, t \geq 0$, there exists a constant θ such that $w(t) = w(0) + \theta t$ and $\gamma(s) = \gamma(0) + \theta s$. Hence $f(s,t) = s(w(0) + \theta t) + t\gamma(0)$. Differentiating this form of f partially with respect to s and t, we get $\dfrac{\partial f(s,t)}{\partial s} = (w(0) + \theta t) > 0$ and $\dfrac{\partial f(s,t)}{\partial t} = (\gamma(0) + \theta s) < 0$. Now, if $\theta > 0$, then negativity of $\dfrac{\partial f(s,t)}{\partial t}$ cannot hold for all $s \geq 0$. On the other hand, if $\theta < 0$, then positivity of $\dfrac{\partial f(s,t)}{\partial s}$ cannot hold for all sufficiently large positive t. Hence the only possibility is that $\theta = 0$.

Consequently, $f(s,t) = sw(0) + t\gamma(0) = c_1 s - c_2 t$, where $c_1 = w(0) > 0$ and $c_2 = -\gamma(0) > 0$ (by positivity and negativity of partial derivatives of f with respect to s and t respectively, as shown earlier).

Case II: $f(s,t) = \dfrac{a^s w(t)}{\log a} + w_1(t) = t\gamma(s) + \gamma_1(s), 0 < a \neq 1.$ (19)

By axiom $(A3)$,

$$\frac{w(0)}{\log a} + w_1(0) = \gamma_1(0) = 0. \tag{20}$$

Putting $s = 0$ in (19) and using the information $\gamma_1(0) = 0$ from (20) in the resulting expression we get $f(0,t) = \dfrac{w(t)}{\log a} + w_1(t) = t\gamma(0)$. Substituting the expression for $w_1(t)$ obtained from this equation into (19) we have

$$f(s,t) = \frac{(a^s - 1)w(t)}{\log a} + t\gamma(0). \tag{21}$$

Similarly, putting $t = 0$ in (19) we find $\dfrac{a^s w(0)}{\log a} + w_1(0) = \gamma_1(s)$, which, in view of $w_1(0) = -w(0)/\log a$ (obtained from (20)) gives $\gamma_1(s) = \dfrac{(a^s - 1)w(0)}{\log a}$. Substituting this value of $\gamma_1(s)$ into (19) we get

$$f(s,t) = \frac{(a^s - 1)w(0)}{\log a} + t\gamma(s). \tag{22}$$

Equating the functional forms of f given by (21) and (22) we then have $\dfrac{(a^s - 1)(w(t) - w(0))}{\log a} = t(\gamma(s) - \gamma(0))$, from which it follows that for all $s,t > 0$, $\left(\dfrac{\gamma(s) - \gamma(0)}{(a^s - 1)/\log a}\right) = \dfrac{(w(t) - w(0))}{t}$ =constant= θ (say). This gives $\gamma(s) = \gamma(0) + \theta\dfrac{(a^s - 1)}{\log a}$ for all $s,t \geq 0,$ and $w(t) = w(0) + \theta t$. Substitution of the functional form of $\gamma(s)$ into (22) yields

$$f(s,t)=\frac{(a^s-1)(w(0)+\theta t)}{\log a}+t\gamma(0)\ . \qquad (23)$$

Now, $\frac{\partial f(s,t)}{\partial s}=a^s(w(0)+\theta t)>0$ for all $s,t\ge 0$. For $s=0$ this implies that

$$(w(0)+\theta t)>0 \qquad (24)$$

holds for all $t\ge 0$. Hence $\theta\ge 0$, otherwise for a sufficiently high value of t, $(w(0)+\theta t)$ will be negative.

Also

$$\frac{\partial f(s,t)}{\partial t}=\theta\frac{(a^s-1)}{\log a}+\gamma(0)<0 \qquad (25)$$

for all $s,t\ge 0$.

Sub-case I: $a>1$. Then $\frac{(a^s-1)}{\log a}$ is increasing and unbounded in $s\ge 0$. So if $\theta>0$, then choosing $s>0$ sufficiently large, we can make the left hand side of the inequality in (25) positive, which is a contradiction. So the only possibility is that $\theta=0$. Plugging $\theta=0$ into (23) we get $f(s,t)=\frac{(w(0))(a^s-1)}{\log a}+t\gamma(0)$, which, in view of our earlier notation, can be rewritten as $f(s,t)=\frac{c_1(a^s-1)}{\log a}-c_2t$ with $c_1=w(0)>0$ and $c_2=-\gamma(0)>0$.

Sub-case II: $0<a<1$. In this case also (24) holds so that $\theta\ge 0$. We rewrite the inequality in (25) as $\theta<\frac{\gamma(0)\log a}{(1-a^s)}$ for all $s>0$, which implies that $\theta\le\gamma(0)\log a$. Using our earlier notation, we have $f(s,t)=(a^s-1)\left(\frac{c_1}{\log a}+\rho t\right)-c_2t$, where, $c_1=w(0)>0$, $c_2=-\gamma(0)>0$ and $\rho=\theta/\log a$. Also $0\ge\rho=\theta/\log a\ge\gamma(0)=-c_2$.

Case III: $f(s,t)=sw(t)+w_1(t)=\frac{b^t\gamma(s)}{\log b}+\gamma_1(s),\ 0<b\ne 1.$

Solution in this case is similar to that of Case II and (by symmetry) is given by

$$f(s,t)=\begin{cases} c_1 s - c_2\dfrac{(b^t-1)}{\log b}, b>1, \\ c_1 s - (b^t-1)\left(\dfrac{c_2}{\log b}+\sigma t\right), 0<b<1, \end{cases}$$

where $c_1, c_2 > 0$ are same as before and $\sigma(-c_1 \le \sigma \le 0)$ is a constant.

Case IV: $f(s,t)=\dfrac{a^s w(t)}{\log a}+w_1(t)=\dfrac{b^t\gamma(s)}{\log b}+\gamma_1(s), 0<a,b\ne 1,$ (26)

for all $s,t \ge 0$.

Applying axiom $(A3)$ to (26) we get

$$\frac{w(0)}{\log a}+w_1(0)=0 \text{ and } \frac{\gamma(0)}{\log b}+\gamma_1(0)=0. \tag{27}$$

Putting $s=0$ in (26) we get $\dfrac{w(t)}{\log a}+w_1(t)=\dfrac{b^t\gamma(0)}{\log b}+\gamma_1(0)$, which in view of the second equation in (27) can be rewritten as $\dfrac{w(t)}{\log a}+w_1(t)=\dfrac{(b^t-1)\gamma(0)}{\log b}$. Substituting the value of $w_1(t)$ obtained from this equation into the first expression for $f(s,t)$ in (26) we have

$$f(s,t)=\frac{(a^s-1)w(t)}{\log a}+\frac{(b^t-1)\gamma(0)}{\log b}. \tag{28}$$

Next, put $t=0$ in (26) to get $\dfrac{a^s w(0)}{\log a}+w_1(0)=\dfrac{\gamma(s)}{\log b}+\gamma_1(s)$. We solve these two equations to get $\gamma_1(s)=\dfrac{a^s w(0)}{\log a}+w_1(0)-\dfrac{\gamma(s)}{\log b}$, which in view of $w_1(0)=-\dfrac{w(0)}{\log a}$ (from the first equation in (27)) gives $\gamma_1(s)=\dfrac{(a^s-1)w(0)}{\log a}-\dfrac{\gamma(s)}{\log b}$. Substitution of this form of $\gamma_1(s)$ into the second expression for $f(s,t)$ in (26) yields

$$f(s,t)=\frac{(a^s-1)w(0)}{\log a}+\frac{(b^t-1)\gamma(s)}{\log b}. \tag{29}$$

Equating (28) and (29) and simplifying we get

$$\frac{(a^s-1)(w(t)-w(0))}{\log a}=\frac{(b^t-1)(\gamma(s)-\gamma(0))}{\log b}, \tag{30}$$

for all $s,t \geq 0$. As in the earlier cases $\gamma(s) = \gamma(0) + \theta \frac{(a^s - 1)}{\log a}$ and $w(t) = w(0) + \theta \frac{(b^t - 1)}{\log b}$ for some constant θ. Substituting this form of $\gamma(s)$ into (29) we get

$$f(s,t) = \frac{(a^s - 1)w(0)}{\log a} + \frac{(b^t - 1)}{\log b}\left(\gamma(0) + \theta\left(\frac{a^s - 1}{\log a}\right)\right). \tag{31}$$

Now, $\frac{\partial f(s,t)}{\partial s} > 0$ implies that

$$\frac{\theta(b^t - 1)}{\log b} + w(0) > 0 \tag{32}$$

for all $t \geq 0$. On the other hand, $\frac{\partial f(s,t)}{\partial t} < 0$ implies that

$$\frac{\theta(a^s - 1)}{\log a} + \gamma(0) < 0, \tag{33}$$

for all $s \geq 0$.

Again various sub-cases come under consideration.

Sub-case I: $a > 1, b > 1$. Applying the same logic as in the case II, we get $\theta = 0$. So the general solution in this case is $f(s,t) = \frac{c_1}{\log a}(a^s - 1) - \frac{c_2}{\log b}(b^t - 1)$, where $c_1 = w(0), c_2 = -\gamma(0) > 0$ are same as in Case I.

Sub-case II: $a > 1, 0 < b < 1$. Considering (33) and noting that $\frac{(a^s - 1)}{\log a}$ is positive and unbounded above we conclude that $\theta \leq 0$. From (32) we get $\theta > \frac{w(0)\log b}{(1 - b^t)}$ for all $t > 0$, which implies that $\theta \geq w(0)\log b$. Thus, the general solution given by (31) becomes $f(s,t) = \frac{c_1}{\log a}(a^s - 1) - \frac{c_2}{\log b}(b^t - 1) + \eta(a^s - 1)(b^t - 1)$, where $c_1 = w(0) > 0$, $c_2 = -\gamma(0) > 0$ and $\eta = \frac{\theta}{\log a \log b}$, with $0 \leq \eta \log a \leq c_1$.

Sub-case III: $0 < a < 1, b > 1$. Here using (32) we conclude that $\theta \geq 0$. Moreover, from (33), $\theta < \frac{\gamma(0)\log a}{(1-a^s)}$ for all $s > 0$, which implies that $\theta \leq \gamma(0)\log a$. Thus, $0 \leq \theta \leq \gamma(0)\log a$. Consequently, $f(s,t) = \frac{c_1}{\log a}(a^s - 1) - \frac{c_2}{\log b}(b^t - 1) + \eta(a^s - 1)(b^t - 1)$, where $c_1 = w(0)$ and $c_2 = -\gamma(0)$ are positive and $-c_2 \leq \eta \log b \leq 0$ with $\eta = \frac{\theta}{\log a \log b}$.

Sub-case IV: $0 < a < 1, 0 < b < 1$. Applying the same logic as before we get $f(s,t) = \frac{c_1}{\log a}(a^s - 1) - \frac{c_2}{\log b}(b^t - 1) + \eta(a^s - 1)(b^t - 1)$, where $w(0)\log b \leq \theta \leq \gamma(0)\log a$, which implies that $\frac{c_1}{\log a} \leq \eta \leq -\frac{c_2}{\log b}$, with $\eta = \frac{\theta}{\log a \log b}$. This completes the necessity part of the proof. The sufficiency is easy to check. Δ

The constants c_1 and c_2 reflect importance of alienation and identification in the aggregation. They can be interpreted as scale parameters in the sense that, given other things, an increase in c_1 increases polarization. Likewise, ceteris paribus, if c_2 decreases then polarization increases. The other parameters can be interpreted similarly. For $c_1 = c_2 = 1$, P_1 becomes the Rodriguez-Salas index of polarization, if we subdivide the population into two non-overlapping groups using the median and use the Donaldson-Weymark S-Gini index $I_{\hat{\varepsilon}}(x) = 1 - \sum_{i=1}^{n}\left(i^{\hat{\varepsilon}} - (i-1)^{\hat{\varepsilon}}\right)\hat{x}_i \Big/ \lambda n^{\hat{\varepsilon}}$ as the index of inequality , where $\hat{\varepsilon} > 1$ is an inequality sensitivity parameter and $\hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ is that permutation of x such that $\hat{x}_1 \geq \hat{x}_2 \geq .... \geq \hat{x}_n$. For $\hat{\varepsilon} = 2$, $I_{\hat{\varepsilon}}$ becomes the Gini index. In the Rodriguez-Salas case for P_1 to increase under a progressive transfer on the same side of the median, it is necessary that $2 \leq \hat{\varepsilon} \leq 3$.

However, Rodriguez-Salas index regards all income distributions that have equal between-group and within-group components of inequality as equally polarized. Thus, a distribution x with $BI(x) = WI(x) = .3$ becomes equally polarized as the equal distribution y with $BI(y) = WI(y) = 0$. Therefore, in situations of the type where $BI = WI$, P_1 can avoid this problem if we make different choices of c_1 and c_2. The same remark applies to the choices of a_1 and a_2 in the normalized ratio form index $P_{a_1,a_2}(x)=\left(\frac{a_1^{BI(x)}}{a_2^{WI(x)}}-1\right)$, which is obtained as a particular case of P_7 as follows. If in P_7 we set $\frac{c_1}{\log a}=-\frac{c_2}{\log b}=\eta=1$, then on simplification we get $P_7(x)=a^{BI(x)}b^{WI(x)}-1$, which we can rewrite as $P_7(x)=\left(\frac{a_1^{BI(x)}}{a_2^{WI(x)}}-1\right)$, where $a_1=a>1$ and $1/b=a_2>1$. Therefore for suitable choices of the parameters we get the normalized ratio form index $\left(\frac{a_1^{BI(x)}}{a_2^{WI(x)}}-1\right)$ as a special case of P_7.

In order to demonstrate independence of the three axioms, we need to construct indicators of polarization that will fulfill any two of the three axioms but not the remaining one. The feasible characteristic function $f_1(s,t)=(s-t^2)$ satisfies axioms $(A1)$ and $(A3)$ but not axiom $(A2)$. Likewise, the feasible characteristic function $f_2(s,t)=(s^2-t)$ fulfills axioms $(A2)$ and $(A3)$ but not axiom $(A1)$. Finally, the feasible characteristic function $f_3(s,t)=(s-t-1)$ is a violator of axiom $(A3)$ but not of axioms $(A1)$ and $(A2)$. We can therefore state the following:

Remark 1: Axioms $(A1)$, $(A2)$ and $(A3)$ are independent.

For the index given by (i) the ratio c_2/c_1 is the marginal rate of substitution of alienation for identification along an iso-polarization contour. This ratio shows how WI can be traded off for BI along the contour. In fact, we can take this trade-off into account in a more general way. If BI is increased by δ, then for keeping the level of polarization unaltered it becomes necessary to increase WI by some amount $g_1(\delta)$, say. By a similar argument, if WI is increased by δ then a corresponding positive change in BI by $g_2(\delta)$, say, will be necessary to keep level of polarization constant (see also Chakravarty et al., 2009). Formally,

$(A4)$ For all $x = (x^1, x^2, ..., x^k) \in \Omega$, $k \in \Gamma$ and for any non-negative δ, $f(BI(x), WI(x)) = f(BI(x) + \delta, WI(x) + g_1(\delta)) = f(BI(x) + g_2(\delta), WI(x) + \delta)$ for some continuous functions $g_1, g_2 : R^1_+ \to R^1_+$.

Using axiom $(A4)$ we can develop a joint characterization of the normalized ratio form index P_{a_1,a_2} and the difference form index P_1. This is shown below.

Theorem 3: Assume that the characteristic function is continuously differentiable. Assume also that the right partial derivative of the characteristic function at zero with respect to the first argument exists and is positive. Then a feasible polarization index $P : \Omega \to R^1$ with such a characteristic function satisfies axioms $(A1)$, $(A3)$ and $(A4)$ if and only if it is of one of the following forms:

$(i) P_{c_1,c_2}(x) = c_1 BI(x) - c_2 WI(x)$ for some arbitrary constants $c_1, c_2 > 0$,

$(ii) P_{a_1,a_2}(x) = c\left(\frac{a_1^{BI(x)}}{a_2^{WI(x)}} - 1\right)$ for some arbitrary constants $c > 0$, $a_1, a_2 > 1$,

where $x = (x^1, x^2, ..., x^k) \in \Omega$, $k \in \Gamma$ and $I \in SD$ are arbitrary.

Proof: From the proof of Theorem 2 we know that axioms $(A1)$ and $(A3)$ force f to take one of the two forms given by (16). Now, suppose f is given by the second form in (16). Applying axiom $(A4)$ to this case we have

$$sw(t) + w_1(t) = (s + g_2(\delta))w(t + \delta) + w_1(t + \delta), \tag{34}$$

for all $s, t, \delta \geq 0$. Putting $s = 0$ in (34) we get $w_1(t) - w_1(t + \delta) = g_2(\delta)w(t + \delta)$, which when subtracted from (34), on simplification, gives $s(w(t + \delta) - w(t)) = 0$, from which we get $w(t + \delta) = w(t)$ for all $t, \delta \geq 0$. Thus, $w(t) =$ a constant$= c_1$, say. Substituting this value of $w(t)$ in the equation $w_1(t) - w_1(t + \delta) = g_2(\delta)w(t + \delta)$, we get $w_1(t) - w_1(t + \delta) = g_3(\delta)$ for all $t, \delta \geq 0$, where $g_3(\delta) = c_1 g_2(\delta)$. Note that by axiom $(A3)$, $w_1(0) = 0$. So, $g_3(\delta) = -w_1(\delta)$, which implies that $w_1(t + \delta) = w_1(t) + w_1(\delta)$ for all $t, \delta \geq 0$. The only continuous solution to this functional equation is $w_1(t) = q't$ for some $q' \in R^1$ (see Aczel, 1966, p.34). Hence in this case f is given by $f(s,t) = c_1 s + q't$. By increasingness of f

in s, $c_1 > 0$. Note also that $q' = f(0,1) < f(0,0) = 0$ (by axiom $(A3)$). So we rewrite the general solution as $f(s,t) = c_1 s - c_2 t$, where $c_1, c_2 > 0$.

Next, we take up the first form in (16). By axiom $(A4)$,

$$\frac{a^s w(t)}{\log a} + w_1(t) = \frac{a^{s+g_2(\delta)} w(t+\delta)}{\log a} + w_1(t+\delta) \tag{35}$$

for all $s, t, \delta \geq 0$. Putting $s = 0$ in both sides of (35) we have

$$\frac{w(t)}{\log a} + w_1(t) = \frac{a^{g_2(\delta)} w(t+\delta)}{\log a} + w_1(t+\delta). \tag{36}$$

Subtracting the left (right) hand side of (36) from the corresponding side of (35) and then rearranging the resulting expression we get

$$\frac{(a^s - 1)}{\log a}\left(a^{g_2(\delta)} w(t+\delta) - w(t)\right) = 0. \tag{37}$$

But $\frac{(a^s - 1)}{\log a} > 0$ for all $s > 0$. This shows that

$$\left(a^{g_2(\delta)} w(t+\delta) - w(t)\right) = 0 \tag{38}$$

for all $t, \delta \geq 0$.

Now, recall from (16) that $w(t) > 0$ for all $t \geq 0$. Therefore, from (38) we get

$$\frac{w(t+\delta)}{w(t)} = a^{-g_2(\delta)} \tag{39}$$

for all $t, \delta \geq 0$. Putting $t = 0$ in (39) we have

$$\frac{w(\delta)}{w(0)} = a^{-g_2(\delta)}. \tag{40}$$

From (39) and (40) it follows that

$$\frac{w(t+\delta)}{w(t)} = \frac{w(\delta)}{w(0)} \tag{41}$$

for all $t, \delta \geq 0$. As we have noted in the proof of Theorem 2, the general solution to this equation is given by $w(t) = c' \varsigma^t$ for some constants $c', \varsigma > 0$. A comparison of (36) and (38) gives $w_1(t) = w_1(t+\delta)$ for all $t, \delta \geq 0$, so that $w_1(t) =$ constant$= \xi$, say. Hence the complete solution in this case is $f(s,t)=\frac{a^s c' \varsigma^t}{\log a}+\xi$. By axiom $(A3)$, $\xi=-\frac{c'}{\log a}$. Consequently, $f(s,t)=\frac{c'}{\log a}\left(a^s\varsigma^t-1\right)$. Increasingness and decreasingness of f in its first and second arguments respectively require that $a>1$ and $\varsigma<1$. So the solution can be written as $f(s,t)=c\left(\frac{a_1^s}{a_2^t}-1\right)$, where $c>0$ and $a_1,a_2>1$ are constants. This completes the necessity part of the proof. The sufficiency is easy to check. Δ

Since the constants c_1 and c_2 in the above theorem are arbitrary, we can choose them to be equal to the corresponding constants in Theorem 2 and therefore use the same notation. The same remark applies for the constants a_1 and a_2.

To check independence of axioms $(A1)$, $(A3)$ and $(A4)$, consider the characteristic functions f_1, f_3 (as defined earlier) and $f_4(s,t)=\left(2^{s-t}+s-t-1\right)$. Then f_1 satisfies axioms $(A1)$ and $(A3)$ but not axiom $(A4)$, f_3 is a violator of axiom $(A3)$ but not of the other two, while f_4 fulfills all the axioms except $(A1)$. We therefore have

Remark 2: Axioms $(A1)$, $(A3)$ and $(A4)$ are independent.

We can also prove the following theorem.

Theorem 4: Assume that the characteristic function is continuously differentiable. Assume also that the right partial derivative of the characteristic function at zero with respect to the second argument exists and is negative. Then a feasible polarization index $P:\Omega\to R^1$ with such a characteristic function satisfies axioms $(A2)$, $(A3)$ and $(A4)$ if and only if it is of one of the following forms:

$(i)P_{c_1,c_2}(x)=c_1BI(x)-c_2WI(x)$ for some constants $c_1,c_2>0$,

$(ii)P_{a_1,a_2}(x)=c\left(\frac{a_1^{BI(x)}}{a_2^{WI(x)}}-1\right)$, for some constants $c>0$, $a_1,a_2>1$,

where $x=\left(x^1,x^2,...,x^k\right)\in\Omega$, $k\in\Gamma$ and $I\in SD$ are arbitrary.

The characteristic function f_2 meets axioms $(A2)$ and $(A3)$ but not $(A4)$. On the other hand f_3 violates axiom $(A3)$ but not the remaining two. Finally, f_4 fulfills all the axioms except $(A2)$. This enables us to state the following:

Remark 3: Axioms $(A2)$, $(A3)$ and $(A4)$ are independent.

The transformed ratio form index $\left(1+P_{a_1,a_2}\right)$ has a structure similar to the Zhang-Kanbur index $P_{ZK}(x)=BI(x)/WI(x)$. However, one minor problem with P_{ZK} is its discontinuity if $WI(x)=0$. The transformed index and hence P_{a_1,a_2} do not suffer from this shortcoming. However, the alienation and identification components of polarization are incorporated correctly in the formulation of P_{ZK}.

We now consider the dual problem of generating an inequality index from a specific polarization index. For this purpose we assume at the outset that for fixed $k\in\Gamma$ and $(n_1,n_2,....,n_k)\in\Gamma^k$, the polarization index $P:\prod_{i=1}^{k}D^{n_i}\to R^1$ satisfies the following axiom:

$(A5)$: For all $x=\left(x^1,x^2,...,x^k\right)\in\prod_{i=1}^{k}D^{n_i}$, $P(y)-P(x)=v_i(\underline{n},\underline{\lambda})g\left(x^i\right)$, where $y=\left(y^1,y^2,...,y^k\right)$ with $y^i=\lambda\left(x^i\right)1^{n_i}$ and $y^j=x^j$ for $j\neq i$; v_i is a positive real number, assumed to depend on the vector $(\underline{n},\underline{\lambda})$ and g is a non-negative valued function defined on $\bigcup_{i=1}^{k}D^{n_i}$.

The transformation that takes us from x to y makes the distribution y^i in subgroup i perfectly equal and leaves distributions in all other subgroups unchanged. Given positivity of v_i, axiom $(A5)$ states that the resulting change in polarization, as indicated by $P(y)-P(x)$, is non-negative (since g is non-negative). This is quite sensible. Assuming that x^i is unequal, a movement towards perfect equality makes the subgroup more homogeneous and because of closer identification of the individuals in the subgroup, polarization should not reduce. Since the transformation does not affect the distributions in all subgroups other than subgroup i, we are assuming that the change does not depend on unaffected subgroups' distributions. However, it is assumed to depend on x^i, the original distribution in subgroup i, and the vectors of population sizes of the subgroups and their mean incomes.

Theorem 5: If the continuous polarization index $P:\prod_{i=1}^{k} D^{n_i} \to R^1$ satisfies axiom $(A5)$, then there exists a corresponding subgroup decomposable continuous inequality index $I:\left(\prod_{i=1}^{k} D^{n_i}\right)\cup\left(\bigcup_{i=1}^{k} D^{n_i}\right)\to R^1$, which takes on the value zero for the perfectly equal distribution on $\bigcup_{i=1}^{k} D^{n_i}$.

Proof: Given $x=\left(x^1,x^2,...,x^k\right)\in\prod_{i=1}^{k} D^{n_i}$ and $\lambda_i=\lambda\left(x^i\right)$, define a sequence $\{y(i)\}$ as follows:

$y(0)=x,$

$y(1)=\left(\lambda_1 1^{n_1},x^2,...x^k\right),$

$y^j(2)=y^j(1)$ for $j\neq 2$, $y^2(2)=\lambda_2 1^{n_2}$,

$y^j(3)=y^j(2)$ for $j\neq 3$, $y^3(3)=\lambda_3 1^{n_3}$, and so on. Finally,

$y^j(k)=y^j(k-1)$ for $j\neq k$ and $y^k(k)=\lambda_k 1^{n_k}$.

Thus, for any $i, 1\leq i\leq k$, we have $y(i)=\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.....,\lambda_i 1^{n_i},x^{i+1},....,x^k\right)$. Note that for all i and j, $\lambda\left(y^j(i)\right)=\lambda\left(x^j\right)$, $\lambda(y(i))=\lambda(x)$ and $y(k)=\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)$.

It is given that for any $i, 1\leq i\leq k$, $P(y(i))-P(y(i-1))=v_i(\underline{n},\underline{\lambda})g\left(x^i\right)$. Summing over all i, we get $P(y(k))-P(y(0))=\sum_{i=1}^{k} v_i(\underline{n},\underline{\lambda})g\left(x^i\right)$. That is,

$$P\left(\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},....,\lambda_k 1^{n_k}\right)\right)-P(x)=\sum_{i=1}^{k} v_i(\underline{n},\underline{\lambda})g\left(x^i\right). \tag{42}$$

Now define $I:\left(\prod_{i=1}^{k} D^{n_i}\right)\cup\left(\bigcup_{i=1}^{k} D^{n_i}\right)\to R^1$ by the following relation:

$$I(x)=\begin{cases}\left(\frac{1}{c_1}+\frac{1}{c_2}\right)P\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},...,\lambda_k 1^{n_k}\right)-\frac{1}{c_2}P(x)\ for\ x=\left(x^1,x^2,..,x^k\right)\in\prod_{i=1}^{k}D^{n_i},\\ g(x)\ if\ x\in\bigcup_{i=1}^{k}D^{n_i},\end{cases}$$

where $c_1,c_2>0$ are arbitrary constants. Clearly, there is no ambiguity in the definition of I. By continuity of P, I is continuous. From the above definition it follows that $P\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)=c_1 I\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)$ and $g\left(x^i\right)=I\left(x^i\right)$, $1\le i\le k.$ Substituting this into (42) we get

$$P(x)=c_1 I\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)-c_2\sum_{i=1}^{k}\omega_i(\underline{n},\underline{\lambda})g\left(x^i\right), \tag{43}$$

where $\omega_i(\underline{n},\underline{\lambda})=v_i(\underline{n},\underline{\lambda})/c_2$. This in turn gives:

$I(x)=\frac{1}{c_1}P\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)+\frac{1}{c_2}\left\{P\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)-P(x)\right\}=$ $I\left(\lambda_1 1^{n_1},\lambda_2 1^{n_2},.......,\lambda_k 1^{n_k}\right)+\sum_{i=1}^{k}\omega_i(\underline{n},\underline{\lambda})I\left(x^i\right)$. Thus, I is subgroup decomposable. To show that I takes on the value zero for the perfectly equal distribution on $\bigcup_{i=1}^{k}D^{n_i}$; observe that $I\left(x^i\right)=(P(y)-P(x))/v_i(\underline{n},\underline{\lambda})$, which implies that $I\left(c1^{n_i}\right)=0$ for all $i, 1\le i\le k$ and for all $c>0$. Δ

Remark 4: From (43) we observe that P can be expressed as $(c_1 BI-c_2 WI)$ for some subgroup decomposable inequality index I that becomes zero for the perfectly equal distribution on $\bigcup_{i=1}^{k}D^{n_i}$, where $c_1,c_2>0$ are arbitrary constants.

Remark 5: Since $\left(\prod_{i=1}^{k}D^{n_i}\right)\bigcup\left(\bigcup_{i=1}^{k}D^{n_i}\right)$ is a closed subset of D and I is continuous, I can be continuously extended to D (Rudin, 1987, p.99). (Here we assume that D can be identified with $\bigcup_{\substack{m_j\in\Gamma,1\le j\le l,\\ l\in\Gamma}}\left(\prod_{j=1}^{l}D^{m_j}\right)\bigcup\left(\bigcup_{j=1}^{l}D^{m_j}\right)$.)

4. Conclusion

Polarization is concerned with clustering of incomes in subgroups of a population, where the partitioning of the population into subgroups is done in an unambiguous way. A reduced-form polarization index is one which abbreviates an income distribution in terms of 'alienation' and identification' components of polarization. The between-group term of a subgroup decomposable inequality index is taken as an indicator of alienation, whereas within –group inequality is regarded as an inverse indicator of identification. A criterion for ranking different income distributions by all reduced-form indices is developed under certain mild conditions. Some polarization indices have been characterized using alternative sets of independent axioms. Finally, the dual problem of generating an index of inequality from a given form of polarization index is investigated.

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