UNIVERSITÀ DELLA CALABRIA

Dipartimento di Economia e Statistica
Ponte Pietro Bucci, Cubo 0/C
87036 Arcavacata di Rende (Cosenza)
Italy
http://www.ecostat.unical.it/

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THE FISHER INFORMATION MATRIX IN DOUBLY CENSORED DATA FROM THE DAGUM DISTRIBUTION

Filippo Domma
Dipartimento di Economia e Statistica
Università della Calabria
Ponte Pietro Bucci, Cubo 0/C
Tel.: +39 0984 492427
Fax: +39 0984 492421
e-mail: f.domma@unical.it

Sabrina Giordano
Dipartimento di Economia e Statistica
Università della Calabria
Ponte Pietro Bucci, Cubo 0/C
Tel.: +39 0984 492426
Fax: +39 0984 492421
e-mail: sabrina.giordano@unical.it

Mariangela Zenga
Dipartimento di Metodi Quantitativi
Università degli Studi di Milano-Bicocca
Piazza Ateneo Nuovo,1 - 20146 Milano - Italy
Tel.: +39 02 6448.3158
Fax: +39 02 6448.3105
e-mail: mariangela.zenga@unimib.it

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Filippo Domma **and** **Sabrina Giordano**

Department of Economics and Statistics
University of Calabria
Via Pietro Bucci, Cubo 0C
87036 Arcavacata di Rende (CS) - Italy -
f.domma@unical.it ; sabrina.giordano@unical.it

Mariangela Zenga

Dipartimento di Metodi Quantitativi
Universitá degli Studi di Milano-Bicocca
Piazza Ateneo Nuovo,1
20146 Milano - Italy
mariangela.zenga@unimib.it

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Abstract

In this note, we provide the mathematical tools for computing the entries of the Fisher information matrix in case of the observations are doubly censored from a Dagum distribution.

Key words: Order Statistics, Maximum Likelihood Estimator, Fisher Information Matrix.

1 Introduction

The Dagum model (Dagum 1977, 1980) has been successfully used in studies on income and wage as well as wealth distributions (see Kleiber and Kotz (2003) and Kleiber (2007) for an excellent survey on the genesis and on empirical applications of the Dagum model). In this context, its features have been extensively analyzed by many authors. Recently, Quintano and D'Agostino (2006) proposed to model the income distribution in terms of individual characteristics using Dagum distributions with heterogeneous model parameters. Domma (2007) studied the asymptotic distribution of the maximum likelihood estimators of the parameters of the right-truncated Dagum model. In recent years, some authors have used this model or its transformation in different fields. For example, Domma and Perri (2009) studied various features of the log-Dagum distribution and applied this distribution in a financial framework to model daily returns of some stocks. Domma et al. (2009) proposed a further possibility to use the Dagum distribution in the reliability theory. In particular, they studied the reversed hazard rate, the mean residual life, the mean waiting time function, the variance of random variables residual life and reversed residual life and their monotonic properties.

In this note, we determine the Fisher information matrix in doubly censored data from the Dagum distribution. The organization of this note is as follows: in *Section 2*, we describe the main features of the Dagum model. *Section 3* contains the likelihood function in presence of doubly censored data. The Fisher Information Matrix is derived in *Section 4*.

2 The Dagum's distribution

The random variable T, continuous and non negative, is Dagum distributed if its cumulative distribution function (*cdf*) is

$$F_T(t;\boldsymbol{\theta}) = \left(1+\lambda t^{-\delta}\right)^{-\beta} \tag{1}$$

and the probability density function (*pdf*) is

$$f_T(t;\boldsymbol{\theta}) = \beta\lambda\delta t^{-\delta-1}\left(1+\lambda t^{-\delta}\right)^{-\beta-1} \tag{2}$$

where $\boldsymbol{\theta} = (\beta,\lambda,\delta)$, and $\beta > 0$, $\lambda > 0$ and $\delta > 0$. The parameter λ is a scale parameter, while β and δ are shape parameters. It is well-known that the Dagum distribution has positive asymmetry, it is unimodal for $\beta\delta > 1$ and zero-modal for $\beta\delta \leq 1$. Moreover, it is easy to verify that the *q-th* quantile of the Dagum distribution is $t(q) = \lambda^{\frac{1}{\delta}} \cdot (q^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\delta}}$, whereas the *r-th* moment is:

$$E(T^r|\beta,\lambda,\delta) = \beta\lambda^{\frac{r}{\delta}} B\left(\beta+\frac{r}{\delta}, 1-\frac{r}{\delta}\right) \tag{3}$$

for $\delta > r$, where $B(.,.)$ is the mathematical function Beta. The probability density function of the *k-th* order statistic, $T_{(k)}$, of a random sample of size n by (1) is

$$f_{T_{(k)}}(t_{(k)};\boldsymbol{\theta}) = C_k\beta\lambda\delta t_{(k)}^{-\delta-1}\left(1+\lambda t_{(k)}^{-\delta}\right)^{-(k\beta+1)}\sum_{j=0}^{n-k} C_{k,j}\left(1+\lambda t_{(k)}^{-\delta}\right)^{-\beta(k+j)-1} \tag{4}$$

where $C_k = \frac{n!}{(k-1)!(n-k)!}$ and $C_{n-k,j} = \begin{pmatrix} n-k \\ j \end{pmatrix} = \frac{(n-k)!}{j!(n-k-j)!}$.

3 Maximum Likelihood Estimation

In this section, MLEs of the parameters β, λ and δ of the Dagum distribution are derived in presence of doubly censored observations. To this purpose, suppose that the r smallest and $n-m$ largest observations are censored from a sample of size n by (1). The log-likelihood function, $\ell(\boldsymbol{\theta})$, of the remaining $m-r$ order statistics $t_{(r+1)}$, $t_{(r+2)}$, ..., $t_{(m)}$ is

$$\begin{aligned}
\ell(\boldsymbol{\theta}) &= \ln\left(\frac{n!}{r!(n-m)!}\right) - r\beta\ln\left(1+\lambda t_{(r)}^{-\delta}\right) + \\
&+ \sum_{i=r+1}^{m}\left\{\ln(\beta\lambda\delta) - (\delta+1)\ln(t_{(i)}) - (\beta+1)\ln\left(1+\lambda t_{(i)}^{-\delta}\right)\right\} \\
&+ (n-m)\ln\left\{1-\left[1+\lambda t_{(m+1)}^{-\delta}\right]^{-\beta}\right\}.
\end{aligned}$$

The maximum likelihood equations are obtained by equating to zero the partial derivatives of $\ell(\boldsymbol{\theta})$ with respect to β, λ and δ. These are given by

$$\begin{cases} \frac{\partial\ell(\boldsymbol{\theta})}{\partial\beta} = -\frac{r}{\beta}\ln F_T(t_{(r)};\boldsymbol{\theta}) + \frac{1}{\beta}\sum_{i=r+1}^{m}\left\{1+\ln F_T(t_{(i)};\boldsymbol{\theta})\right\} - \frac{(n-m)F_T(t_{(m+1)};\boldsymbol{\theta})\ln\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]}{\beta\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]} = 0 \\ \frac{\partial\ell(\boldsymbol{\theta})}{\partial\lambda} = -\frac{r\beta t_{(r)}^{-\delta}}{\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{-\frac{1}{\beta}}} + \sum_{i=r+1}^{m}\left\{\frac{1}{\lambda} - \frac{(\beta+1)t_{(i)}^{-\delta}}{\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{-\frac{1}{\beta}}}\right\} + \frac{\beta(n-m)t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}}{S_T(t_{(m+1)};\boldsymbol{\theta})} = 0 \\ \frac{\partial\ell(\boldsymbol{\theta})}{\partial\delta} = \frac{r\beta\lambda\ln\left(t_{(r)}\right)t_{(r)}^{-\delta}}{\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{-\frac{1}{\beta}}} + \sum_{i=r+1}^{m}\left\{\frac{1}{\delta} - \ln(t_{(i)}) + \frac{\lambda(\beta+1)\ln(t_{(i)})t_{(i)}^{-\delta}}{\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{-\frac{1}{\beta}}}\right\} - \frac{\beta\lambda(n-m)\ln\left(t_{(m+1)}\right)t_{(m+1)}^{-\delta}}{S_T(t_{(m+1)};\boldsymbol{\theta})\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{-1-\frac{1}{\beta}}} = 0, \end{cases}$$

where $S_T(t_{(m+1)};\boldsymbol{\theta}) = 1 - F_T(t_{(m+1)};\boldsymbol{\theta})$ is the survival function. The system does not admit any explicit solution, therefore the ML estimates $\hat{\boldsymbol{\theta}}_n = (\hat{\beta}_n, \hat{\lambda}_n, \hat{\delta}_n)$ can be obtained only by means of numerical procedures. Under the usual regularity conditions, the well-known asymptotic properties of the maximum likelihood method ensure that $\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right)$, where $\boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \left[\mathbf{I}\left(\boldsymbol{\theta}\right)\right]^{-1}$ is the asymptotic variance-covariance matrix and $\mathbf{I}\left(\boldsymbol{\theta}\right)$ is the Fisher Information Matrix, whose entries will be calculated in next section.

4 Fisher Information Matrix

In order to derive the entries of the Fisher Information Matrix, we preliminarily obtain the second partial derivatives of the log-likelihood function

$$\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial\beta^2} = -\frac{(m-r)}{\beta^2} - \frac{(n-m)F_T(t_{(m+1)};\boldsymbol{\theta})\left[\ln F_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}{\beta^2 S_T(t_{(m+1)};\boldsymbol{\theta})^2}$$

$$\begin{aligned} \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial\lambda^2} &= r\beta t_{(r)}^{-2\delta}\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}} - \frac{(m-r)}{\lambda^2} + (\beta+1)\sum_{i=r+1}^{m} t_{(i)}^{-2\delta}\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}} + \\ & -\beta(n-m)\left\{(\beta+1)\frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{2}{\beta}}}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2} - \frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{2+\frac{2}{\beta}}}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}\right\} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta^2} &= -r\beta\lambda t_{(r)}^{-\delta}\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}}\left[\ln\left(t_{(r)}\right)\right]^2 - \frac{(m-r)}{\delta^2} - \lambda(\beta+1)\sum_{i=r+1}^{m} t_{(i)}^{-\delta}\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}}\left[\ln\left(t_{(i)}\right)\right]^2 \\
&\quad -\beta\lambda(n-m)\left\{\lambda(\beta+1)\frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{2}{\beta}}\left[\ln\left(t_{(m+1)}\right)\right]^2}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]}\right. \\
&\quad \left. -\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\left[\ln\left(t_{(m+1)}\right)\right]^2}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]} + \beta\lambda\frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{2+\frac{2}{\beta}}\left[\ln\left(t_{(m+1)}\right)\right]^2}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \partial \lambda} &= -r\beta t_{(r)}^{-\delta}\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{\frac{1}{\beta}} - \sum_{i=r+1}^{m} t_{(i)}^{-\delta}\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{\frac{1}{\beta}} \\
&\quad +(n-m)\left\{\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\ln\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}\right. \\
&\quad \left. +\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\ln\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \partial \delta} &= r\lambda t_{(r)}^{-\delta}\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{\frac{1}{\beta}}\ln\left(t_{(r)}\right) + \lambda\sum_{i=r+1}^{m} t_{(i)}^{-\delta}\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{\frac{1}{\beta}}\ln\left(t_{(i)}\right) \\
&\quad -\lambda(n-m)\left\{\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\ln\left(t_{(m+1)}\right)\ln\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}\right. \\
&\quad \left. +\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\ln\left(t_{(m+1)}\right)\ln\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \delta} &= r\beta t_{(r)}^{-\delta}\left[F_T(t_{(r)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}}\ln\left(t_{(r)}\right) + (\beta+1)\sum_{i=r+1}^{m} t_{(i)}^{-\delta}\left[F_T(t_{(i)};\boldsymbol{\theta})\right]^{\frac{2}{\beta}}\ln\left(t_{(i)}\right) \\
&\quad -\beta(n-m)\left\{\frac{t_{(m+1)}^{-\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{1}{\beta}}\ln\left(t_{(m+1)}\right)}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]} - \lambda(\beta+1)\right. \\
&\quad \left. \frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{1+\frac{2}{\beta}}\ln\left(t_{(m+1)}\right)}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]} - \beta\lambda\frac{t_{(m+1)}^{-2\delta}\left[F_T(t_{(m+1)};\boldsymbol{\theta})\right]^{2+\frac{2}{\beta}}\ln\left(t_{(m+1)}\right)}{\left[S_T(t_{(m+1)};\boldsymbol{\theta})\right]^2}\right\}.
\end{aligned}
$$

To compute the entries of the Fisher information matrix, we must calculate the following expectations

$$
\begin{aligned}
E_{(k)}^{(1)} &= E\left\{\frac{\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\}, \quad E_{(k)}^{(2)} = E\left\{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{\frac{2}{\beta}}\right\} \\
E_{(k)}^{(3)} &= E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{2}{\beta}}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\}, \quad E_{(k)}^{(4)} = E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2+\frac{2}{\beta}}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\} \\
E_{(k)}^{(5)} &= E\left\{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]^{2}\right\}, \quad E_{(k)}^{(6)} = E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]^{2}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\} \\
E_{(k)}^{(7)} &= E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln\left(T_{(k)}\right)\right]^{2}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\}, \quad E_{(k)}^{(8)} = E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2+\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]^{2}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\} \\
E_{(k)}^{(9)} &= E\left\{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{\frac{1}{\beta}}\right\}, \quad E_{(k)}^{(10)} = E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\} \\
E_{(k)}^{(11)} &= E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\}, \quad E_{(k)}^{(12)} = E\left\{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{\frac{1}{\beta}}\left[\ln\left(T_{(k)}\right)\right]\right\} \\
E_{(k)}^{(13)} &= E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln\left(T_{(k)}\right)\right]\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\} \\
E_{(k)}^{(14)} &= E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln\left(T_{(k)}\right)\right]\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\} \\
E_{(k)}^{(15)} &= E\left\{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]\right\}, \quad E_{(k)}^{(16)} = E\left\{\frac{T_{(k)}^{-\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{1}{\beta}}\left[\ln\left(T_{(k)}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\} \\
E_{(k)}^{(17)} &= E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]}\right\} \quad E_{(k)}^{(18)} = E\left\{\frac{T_{(k)}^{-2\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{1+\frac{2}{\beta}}\left[\ln\left(T_{(k)}\right)\right]}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{2}}\right\}.
\end{aligned}
$$

We highlight that the expectations $E_{(k)}^{(1)}, ..., E_{(k)}^{(18)}$ are a particular case of the below expectation

$$
E_{i_1,i_2,i_3,i_4,i_5,i_6} = E\left\{\frac{T_{(k)}^{-i_1\delta}\left[F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{i_2+\frac{i_3}{\beta}}\left[\ln\left(T_{(k)}\right)\right]^{i_4}\left[\ln F\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{i_5}}{\left[S\left(T_{(k)};\boldsymbol{\theta}\right)\right]^{i_6}}\right\}
$$

with $i_1, i_2, i_3, i_4, i_5, i_6$ positive integers and for $k = r, r+1, ..., m, m+1$. Now, recalling that the *pdf* of the *k-th* order statistic, $T_{(k)}$, of a random sample of size n from a Dagum distribution is

$$
f_{T_{(k)}}(t_{(k)}, \boldsymbol{\theta}) = C_k \beta\lambda\delta t_{(k)}^{-\delta-1}\left(1+\lambda t_{(k)}^{-\delta}\right)^{-(k\beta+1)}\left[1-\left(1+\lambda t_{(k)}^{-\delta}\right)^{-\beta}\right]^{n-k},
$$

where $C_k = \frac{n!}{(k-1)!(n-k)!}$, it is simple to prove that $E_{i_1,i_2,i_3,i_4,i_5,i_6}$ can be rewritten as follows

$$E_{i_1,i_2,i_3,i_4,i_5,i_6} = C_k \sum_{j=0}^{n-k-i_6} \left(\begin{array}{c} n-k-i_6 \\ j \end{array} \right) \frac{(-1)^j}{\left(k+i_2+\frac{i_3}{\beta}+j\right)} E\left\{T^{-i_1\delta}\left[\ln\left(T\right)\right]^{i_4}\left[\ln F\left(T;\boldsymbol{\theta}\right)\right]^{i_5}|\beta_1,\lambda,\delta\right\}$$

where $E(.|\beta_1,\lambda,\delta)$ is the expectation of a Dagum distribution with parameters $\beta_1 = \beta\left(k+i_2+\frac{i_3}{\beta}+j\right)$, λ and δ. Now, putting $y = \left(1+\lambda t^{-\delta}\right)^{-1}$ in $E(.|\beta_1,\lambda,\delta)$, after some algebra, we have

$$E\left\{T^{-i_1\delta}\left[\ln\left(T\right)\right]^{i_4}\left[\ln F\left(T;\boldsymbol{\theta}\right)\right]^{i_5}|\beta_1,\lambda,\delta\right\} = \frac{\beta^{i_5}\beta_1}{\lambda^{i_1}\delta^{i_4}}\int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1}\left[\ln\frac{\lambda y}{1-y}\right]^{i_4}\left[\ln(y)\right]^{i_5}dy. \quad (5)$$

Putting $i_4 = i_5 = 0$ and $\beta_1 = \beta\left(k+i_2+\frac{i_3}{\beta}+j\right)$ in (5), we obtain

$$E\left\{T^{-i_1\delta}|\beta_1,\lambda,\delta\right\} = \frac{\beta_1}{\lambda^{i_1}}B\left(\beta_1-i_1,i_1+1\right). \quad (6)$$

For $k = r, r+1, .., m, m+1$, by (6), we can easily calculate $E_{(k)}^{(2)}$, $E_{(k)}^{(3)}$, $E_{(k)}^{(4)}$ and $E_{(k)}^{(9)}$:

$$E_{(k)}^{(2)} = E_{2,0,2,0,0,0} = \frac{\beta}{\lambda^2}C_k\sum_{j=0}^{n-k}\left(\begin{array}{c} n-k \\ j \end{array}\right)(-1)^j B\left(\beta(k+j),3\right),$$

$$E_{(k)}^{(3)} = E_{2,1,2,0,0,2} = \frac{\beta}{\lambda^2}C_k\sum_{j=0}^{n-k-2}\left(\begin{array}{c} n-k-2 \\ j \end{array}\right)(-1)^j B\left(\beta(k+1+j),3\right),$$

$$E_{(k)}^{(4)} = E_{2,2,2,0,0,2} = \frac{\beta}{\lambda^2}C_k\sum_{j=0}^{n-k-2}\left(\begin{array}{c} n-k-2 \\ j \end{array}\right)(-1)^j B\left(\beta(k+2+j),3\right),$$

$$E_{(k)}^{(9)} = E_{1,0,1,0,0,0} = \frac{\beta}{\lambda}C_k\sum_{j=0}^{n-k}\left(\begin{array}{c} n-k \\ j \end{array}\right)(-1)^j B\left(\beta(k+j),2\right).$$

Putting $i_5 = 0$ and $i_1 = i_4 = 1$ in (5), after some algebra, we get

$$\begin{aligned} E\left\{T^{-\delta}\ln(T)|\beta_1,\lambda,\delta\right\} &= \frac{\beta_1}{\lambda\delta}\left\{\ln(\lambda)B\left(\beta_1-2,2\right)+B\left(\beta_1-2,2\right)\left[\Psi\left(\beta_1-1\right)-\Psi\left(\beta_1+1\right)\right]-\right. \\ &\left. B\left(2,\beta_1-2\right)\left[\Psi\left(2\right)-\Psi\left(\beta_1+1\right)\right]\right\}; \end{aligned}$$

where $\Psi\left(.\right)$ is the digamma function. Using the latter expectation, we can to calculate $E_{(k)}^{(12)}$, $E_{(k)}^{(15)}$ and $E_{(k)}^{(16)}$

$$\begin{aligned} E_{(k)}^{(12)} = E_{1,0,1,1,0,0} &= \frac{\beta}{\lambda\delta}C_k\sum_{j=0}^{n-k}\left(\begin{array}{c} n-k \\ j \end{array}\right)(-1)^j\left\{\ln(\lambda)B\left(\beta(k+j)-1,2\right)+\right. \\ & B\left(\beta(k+j),2\right)\left[\Psi\left(\beta(k+j)\right)-\Psi\left(\beta(k+j)+2\right)\right]- \\ & \left. B\left(2,\beta(k+j)\right)\left[\Psi\left(2\right)-\Psi\left(\beta(k+j)+2\right)\right]\right\}, \end{aligned}$$

$$E_{(k)}^{(15)} = E_{1,0,2,1,0,0} = \frac{\beta}{\lambda\delta} C_k \sum_{j=0}^{n-k} \left(\begin{array}{c} n-k \\ j \end{array} \right) (-1)^j \left\{ \ln(\lambda) B\left(\beta(k+j), 2\right) + B\left(\beta(k+j)+1, 2\right) \left[\Psi\left(\beta(k+j)+1\right) - \Psi\left(\beta(k+j)+3\right) \right] - B\left(2, \beta(k+j)+1\right) \left[\Psi\left(2\right) - \Psi\left(\beta(k+j)+3\right) \right] \right\},$$

$$E_{(k)}^{(16)} = E_{1,1,1,1,0,1} = \frac{\beta}{\lambda\delta} C_k \sum_{j=0}^{n-k-1} \left(\begin{array}{c} n-k-1 \\ j \end{array} \right) (-1)^j \left\{ \ln(\lambda) B\left(\beta(k+1+j)-1, 2\right) + B\left(\beta(k+1+j), 2\right) \left[\Psi\left(\beta(k+1+j)\right) - \Psi\left(\beta(k+1+j)+2\right) \right] - B\left(2, \beta(k+1+j)\right) \left[\Psi\left(2\right) - \Psi\left(\beta(k+1+j)+2\right) \right] \right\}.$$

Putting $i_5 = 0$ and $i_1 = 2$ and $i_4 = 1$ in (5), after some algebra, it is simple to deduce that

$$E\left\{T^{-2\delta} \ln(T) | \beta_1, \lambda, \delta\right\} = \frac{\beta_1}{\lambda^2\delta} \left\{ \ln(\lambda) B\left(\beta_1 - 2, 3\right) + B\left(\beta_1 - 2, 1\right) \left[2B\left(\beta_1 - 2, 2\right) \left\{ \Psi\left(\beta_1 - 2\right) - \Psi\left(\beta_1\right) \right\} - B\left(\beta_1 - 2, 3\right) \right] \right\}.$$

Appliying the latter expectation, we calculate $E_{(k)}^{(17)}$ and $E_{(k)}^{(18)}$

$$E_{(k)}^{(17)} = E_{2,1,2,1,0,1} = \frac{\beta}{\lambda^2\delta} C_k \sum_{j=0}^{n-k-1} \left(\begin{array}{c} n-k-1 \\ j \end{array} \right) (-1)^j \left\{ \ln(\lambda) B\left(\beta(k+1+j), 3\right) + B\left(\beta(k+1+j), 1\right) \left[2B\left(\beta(k+1+j), 2\right) \left\{ \Psi\left(\beta(k+1+j)\right) - \Psi\left(\beta(k+1+j)+2\right) \right\} - B\left(\beta(k+1+j), 3\right) \right] \right\},$$

$$E_{(k)}^{(18)} = E_{2,1,2,1,0,2} = \frac{\beta}{\lambda^2\delta} C_k \sum_{j=0}^{n-k-2} \left(\begin{array}{c} n-k-2 \\ j \end{array} \right) (-1)^j \left\{ \ln(\lambda) B\left(\beta(k+2+j), 3\right) + B\left(\beta(k+2+j), 1\right) \left[2B\left(\beta(k+2+j), 2\right) \left\{ \Psi\left(\beta(k+2+j)\right) - \Psi\left(\beta(k+2+j)+2\right) \right\} - B\left(\beta(k+2+j), 3\right) \right] \right\}.$$

Putting $i_4 = 0$ and $i_5 = 1$ in (5), we have

$$E\left\{T^{-i_1\delta} \ln F(T) | \beta_1, \lambda, \delta\right\} = \frac{\beta\beta_1}{\lambda^{i_1}} \int_0^1 y^{\beta_1-(i_1+1)} (1-y)^{i_1} \ln(y) dy. \tag{7}$$

By (7), we determine $E_{(k)}^{(10)}$ and $E_{(k)}^{(11)}$

$$E_{(k)}^{(10)} = E_{1,1,1,0,1,2} = \frac{\beta^2}{\lambda} C_k \sum_{j=0}^{n-k-2} \left(\begin{array}{c} n-k-2 \\ j \end{array} \right) (-1)^j B\left(\beta(k+1+j), 2\right) \left\{ \Psi\left(\beta(k+1+j)\right) - \Psi\left(\beta(k+1+j)+2\right) \right\},$$

$$E^{(11)}_{(k)} = E_{1,1,1,0,1,1} = \frac{\beta^2}{\lambda} C_k \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} (-1)^j$$
$$B\left(\beta(k+1+j), 2\right) \left\{ \Psi\left(\beta(k+1+j)\right) - \Psi\left(\beta(k+1+j)+2\right) \right\}.$$

Setting $i_4 = 0$ and $i_5 = 2$ in (5), we obtain

$$E\left\{ T^{-i_1\delta} \left[\ln F(T)\right]^2 |\beta_1, \lambda, \delta \right\} = \frac{\beta\beta_1}{\lambda^{i_1}} \int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \left[\ln(y)\right]^2 dy. \quad (8)$$

integrating by part (8), the expectation $E^{(1)}_{(k)}$ is given by

$$E^{(1)}_{(k)} = E_{0,1,0,0,2,1} = 2C_k \sum_{j=0}^{n-k} \binom{n-k}{j} \frac{(-1)^j}{(k+1+j)^3}.$$

Putting $i_4 = i_5 = 1$ in (5), we have

$$E\left\{ T^{-i_1\delta} \left[\ln(T)\right] \left[\ln F(T)\right] |\beta_1, \lambda, \delta \right\} = \frac{\beta\beta_1}{\lambda^{i_1}\delta} \int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \left[\ln \frac{\lambda y}{1-y}\right] \ln(y) dy. \quad (9)$$

By the latter expectation, we can determine $E^{(13)}_{(k)}$ and $E^{(14)}_{(k)}$

$$E^{(13)}_{(k)} = E_{1,1,1,1,1,2} = \frac{\beta^2}{\lambda\delta} C_k \sum_{j=0}^{n-k-2} \binom{n-k-2}{j} \left\{ \ln(\lambda) I_1\left(\beta(k+1+j)\right) + \right.$$
$$\left. I_3\left(\beta(k+1+j)\right) - I_5\left(\beta(k+1+j)\right) \right\},$$

$$E^{(14)}_{(k)} = E_{1,1,1,1,1,1} = \frac{\beta^2}{\lambda\delta} C_k \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \left\{ \ln(\lambda) I_1\left(\beta(k+1+j)\right) + \right.$$
$$\left. I_3\left(\beta(k+1+j)\right) - I_5\left(\beta(k+1+j)\right) \right\},$$

where the intregral $I_1(.)$, $I_3(.)$ and $I_5(.)$ are given in the Appendix.

Setting $i_4 = 2$ and $i_5 =$ in (5), we have

$$E\left\{ T^{-i_1\delta} \left[\ln(T)\right]^2 |\beta_1, \lambda, \delta \right\} = \frac{\beta_1}{\lambda^{i_1}\delta^{i_4}} \left\{ \left[\ln(\lambda)\right]^2 B\left(\beta_1 - i_1, i_1+1\right) + \right.$$
$$2\ln(\lambda) \int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \ln(y) dy -$$
$$2\ln(\lambda) \int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \ln(1-y) dy +$$
$$\int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \left[\ln(y)\right]^2 dy -$$
$$\int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \ln(y)\ln(1-y) dy +$$
$$\left. \int_0^1 y^{\beta_1-(i_1+1)}(1-y)^{i_1} \left[\ln(1-y)\right]^2 dy \right\}.$$

By above expectation, $E_{(k)}^{(5)}$, $E_{(k)}^{(6)}$, $E_{(k)}^{(7)}$ and $E_{(k)}^{(8)}$ are

$$\begin{aligned} E_{(k)}^{(5)} = E_{1,0,2,2,0,0} &= \frac{\beta}{\lambda\delta^2} C_k \sum_{j=0}^{n-k} \binom{n-k}{j} \left\{ [\ln(\lambda)]^2 B\left(\beta(k+j)+1, 2\right) + \right. \\ & 2\ln(\lambda) I_1\left(\beta(k+j)+1\right) - 2\ln(\lambda) I_2\left(\beta(k+j)+1\right) + \\ & \left. I_3\left(\beta(k+j)+1\right) - 2 I_5\left(\beta(k+j)+1\right) + I_4\left(\beta(k+j)+1\right) \right\}, \end{aligned}$$

$$\begin{aligned} E_{(k)}^{(6)} = E_{2,1,2,2,0,1} &= \frac{\beta}{\lambda^2\delta^2} C_k \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \left\{ [\ln(\lambda)]^2 B\left(\beta(k+1+j), 3\right) + \right. \\ & 2\ln(\lambda) I_6\left(\beta(k+1+j)\right) - 2\ln(\lambda) I_7\left(\beta(k+1+j)\right) + \\ & \left. I_9\left(\beta(k+1+j)\right) - 2 I_8\left(\beta(k+1+j)\right) + I_{10}\left(\beta(k+1+j)\right) \right\}, \end{aligned}$$

$$\begin{aligned} E_{(k)}^{(7)} = E_{1,1,1,2,0,1} &= \frac{\beta}{\lambda\delta^2} C_k \sum_{j=0}^{n-k-1} \binom{n-k-1}{j} \left\{ [\ln(\lambda)]^2 B\left(\beta(k+1+j), 2\right) + \right. \\ & 2\ln(\lambda) I_1\left(\beta(k+1+j)\right) - 2\ln(\lambda) I_2\left(\beta(k+1+j)\right) + \\ & \left. I_3\left(\beta(k+1+j)\right) - 2 I_5\left(\beta(k+1+j)\right) + I_4\left(\beta(k+1+j)\right) \right\}, \end{aligned}$$

$$\begin{aligned} E_{(k)}^{(8)} = E_{2,2,2,2,0,2} &= \frac{\beta}{\lambda^2\delta^2} C_k \sum_{j=0}^{n-k-2} \binom{n-k-2}{j} \left\{ [\ln(\lambda)]^2 B\left(\beta(k+2+j), 3\right) + \right. \\ & 2\ln(\lambda) I_6\left(\beta(k+2+j)\right) - 2\ln(\lambda) I_7\left(\beta(k+2+j)\right) + \\ & \left. I_9\left(\beta(k+21+j)\right) - 2 I_8\left(\beta(k+21+j)\right) + I_{10}\left(\beta(k+2+j)\right) \right\}, \end{aligned}$$

where the intregrals $I_2(.)$, $I_4(.)$, $I_6(.)$, $I_7(.)$, $I_8(.)$, $I_9(.)$, and $I_{10}(.)$ are reported in the Appendix.

Now, using the expectations $E_{(k)}^{(1)}, \ldots, E_{(k)}^{(18)}$, it is simple to calculate the entries of the Fisher information matrix

$$I_{\beta\beta}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta^2}\right] = \frac{m-r}{\beta^2} + \frac{(n-m)}{\beta^2} E_{(m+1)}^{(1)}$$

$$\begin{aligned} I_{\lambda\lambda}(\boldsymbol{\theta}) &= -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda^2}\right] = -r\beta E_{(r)}^{(2)} + \frac{m-r}{\lambda^2} - (\beta+1) \sum_{i=r+1}^{m} E_{(i)}^{(2)} + \\ & \beta(n-m) \left\{ (\beta+1) E_{(m+1)}^{(3)} - E_{(m+1)}^{(4)} \right\} \end{aligned}$$

$$
\begin{aligned}
I_{\delta\delta}(\boldsymbol{\theta}) &= -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta^2}\right] = r\beta\lambda E^{(5)}_{(r)} + \frac{m-r}{\delta^2} + (\beta+1)\lambda \sum_{i=r+1}^{m} E^{(5)}_{(i)} + \\
&\quad \beta\lambda(n-m)\left\{(\beta+1)\,\lambda E^{(6)}_{(m+1)} - E^{(7)}_{(m+1)} + \beta\lambda E^{(8)}_{(m+1)}\right\}
\end{aligned}
$$

$$
I_{\beta\lambda}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \lambda}\right] = rE^{(9)}_{(r)} + \sum_{i=r+1}^{m} E^{(9)}_{(i)} - (n-m)\left\{E^{(10)}_{(m+1)} + E^{(11)}_{(m+1)}\right\}
$$

$$
I_{\beta\delta}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \delta}\right] = -r\lambda E^{(12)}_{(r)} - \lambda \sum_{i=r+1}^{m} E^{(12)}_{(i)} + \lambda(n-m)\left\{E^{(13)}_{(m+1)} + E^{(14)}_{(m+1)}\right\}
$$

$$
\begin{aligned}
I_{\lambda\delta}(\boldsymbol{\theta}) &= -E\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \delta}\right] = -r\beta E^{(15)}_{(r)} - (\beta+1) \sum_{i=r+1}^{m} E^{(15)}_{(i)} + \\
&\quad \beta(n-m)\left\{E^{(16)}_{(m+1)} - \lambda(\beta+1) + E^{(17)}_{(m+1)} - \beta\lambda E^{(18)}_{(m+1)}\right\}.
\end{aligned}
$$

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5 Appendix

In this section, for $p > 0$, we report the following expressions

$$I_1(p+1) = \int_0^1 z^p(1-z)\ln(z)dz = B\left(p+1,2\right)\left[\Psi(p+1)-\Psi(p+3)\right]$$

$$I_2(p+1) = \int_0^1 z^p(1-z)\ln(1-z)dz = B\left(2,p+1\right)\left[\Psi(2)-\Psi(p+3)\right]$$

$$\begin{aligned} I_3(p+1) &= \int_0^1 z^p(1-z)\left[\ln(z)\right]^2 dz = B\left(p+1,2\right)\left\{\left[\Psi(p+1)-\Psi(p+3)\right]^2 + \right. \\ &\left. +\Psi'(p+1)-\Psi'(p+3)\right\} \end{aligned}$$

$$\begin{aligned} I_4(p+1) &= \int_0^1 z^p(1-z)\left[\ln(1-z)\right]^2 dz = B\left(2,p+1\right)\left\{\left[\Psi(2)-\Psi(p+3)\right]^2 + \right. \\ &\left. +\Psi'(2)-\Psi'(p+3)\right\} \end{aligned}$$

$$I_5(p+1) = \int_0^1 z^p(1-z)\ln(z)\ln(1-z)dz = I(p+1) - I(p+2)$$

where

$$I(q+1) = \int_0^1 z^q \ln(z)\ln(1-z)dz = B(q+1,1)\left\{B(q+1,1)\left[\Psi(q+2)-\Psi(2)\right]-\Psi'(q+2)\right\}$$

for any $q > 0$, where $\Psi\left(.\right)$ and $\Psi'\left(.\right)$ are the digamma and trigamma functions, respectively. Moreover, using the integration by part, we have

$$I_6(p+1) = \int_0^1 z^p(1-z)^2\ln(z)dz = B(p+1,1)\left\{2I_1(p+1) - B(p+1,3)\right\}$$

$$I_7(p+1) = \int_0^1 z^p(1-z)^2\ln(1-z)dz = B(p+1,1)\left\{2I_2(p+2) + B(p+2,2)\right\}$$

$$
\begin{aligned}
I_8(p+1) &= \int\limits_0^1 z^p(1-z)^2\ln(z)\ln(1-z)dz = B(p+1,1)\left\{2I_5(p+2)-\right. \\
&\left. B(p+1,1)\left[2I_2(p+2)-B(p+2,2)\right]+I_1(p+2)\right\}
\end{aligned}
$$

$$
I_9(p+1) = \int\limits_0^1 z^p(1-z)^2\left[\ln(z)\right]^2 dz = \frac{2}{(p+1)^3}-\frac{4}{(p+2)^3}+\frac{2}{(p+3)^3}
$$

$$
I_{10}(p+1) = \int\limits_0^1 z^p(1-z)^2\left[\ln(1-z)\right]^2 dz = 2B(p+1,1)\left\{I_4(p+2)+I_2(p+2)\right\}.
$$