Stochastic K-player Blotto games

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Abstract. We consider the following game. K players compete in N simultaneous contests. Each player i has a limited resource X_i and must decide how to allocate it to N contests. In each contest, if player i allocates more resources than player j, player i has a higher chance to win the contest than player j.

We find a unique symmetric Nash equilibrium. Moreover, if individual resources are private information, there exists a unique monotonic symmetric Bayesian equilibrium.

1. Introduction

Consider elections where each candidate has a campaign budget. The candidates have to allocate their campaign budgets (for media advertisement) among several states. If two candidates allocate different amounts for the same state, the candidate with higher amount has higher chance to win this state. Different states can have different values (number of votes) and different candidates can have different campaign budgets. Each candidate wants to maximize a number of states she wins (the value of those states). This election situation is typical for USA, Russia, France, and other countries.

How should candidates allocate their budgets? How different the behavior of candidates with different budgets should be? Will all candidates compete for all states? These and similar questions are analyzed in this paper.

We consider K players competing in N simultaneous contests with different values. Each player has her fixed budget and has to allocate it among N contests. The main assumption of this paper is that the winner of each contest is determined stochastically. In other words, we will assume that each contest is a lottery where higher wager means higher chance to win.

Given our main assumption we find a unique symmetric equilibrium of the game. It turns out that the players allocate their budgets in the same proportion in the symmetric Nash equilibrium. Moreover, the individual equilibrium strategy depends on the contests' values and the individual budget, but it is independent from the budgets of all other players. The equilibrium has a monotonic property: a player with higher budget has higher chance to win each contest. It is interesting to note that each player competes in each contest in the symmetric Nash equilibrium.

We consider also a situation when individual budgets are private information. It turns out that there exists a unique monotonic Bayesian equilibrium. Each player believes that all players allocate their budgets in the same proportion as it is in the case of complete information, and allocates her own budget in the same way. This is the main result of the paper. Since in many elections candidates indeed do not know budgets of other candidates, our result suggests a campaign budget allocation in these cases.

There are many other applications of our model, such as R&D, arm races, military conflicts, simultaneous rent-seeking activities, and so on.

There are two directions in the literature which are closed to the topic of this paper. The first one is different Colonel Blotto games, see Borel (1921), Blackett (1958), Laslier and Picard (2002), Kvasov (2006), and Weinstein (2006) among other. The main difference between our approach and this literature is that our contest winners are determined stochastically (which is almost always the case in the applications) and their contest winners are chosen deterministically: the player with the highest spending in a contest is the winner of this contest. We consider a general case: K players, different budgets, and different values in different contests. We find a unique Nash equilibrium in pure strategies in the general case. All papers on Colonel Blotto games have two competing players and analyze different mixed-strategy equilibria. Borel (1921), Blackett (1958), Weinstein (2006) consider identical budgets, three contests with the same values. Laslier and Picard (2002) consider N contests with the same values. Kvasov (2006) introduces a cost function of the budget and considers identical values and budgets; identical budgets and different values; identical values and different budgets. Our main assumption makes payoff functions continuous and allows obtaining a unique prediction. Classic Colonel Blotto games have multiplicity of equilibria. Our paper is the first which considers a situation when budgets are private information.

The second direction is a contest literature. This literature considers one contest (see Tullock, 1980; Nitzan, 1994, among other) or a sequence of contests (Rosen, 1986; Matros, 2006, among other) when the winner is determined stochastically. The only paper in which players have fixed resources, Matros (2006), considers elimination tournaments. This paper is a natural extension of the contest literature.

The rest of the paper is organized as follows. We consider the model and results in Section 2. Section 3 concludes.

2. The Model

There are K political candidates. Each candidate k has a campaign budget X^k which she can spend for her election campaign. We will use upper index for the candidates. There are N states. The value of state i is W_i for all players. We will use lower index for the states. Suppose that all state values and budgets are common knowledge.

Each candidate k has to allocate her campaign budget X^k across all N states. The candidates are competing in all N states simultaneously. We assume that candidate k wins state i with probability $\frac{x_i^k}{\sum_{j=1}^K x_i^j}$, where x_i^j is the campaign budget allocation of candidate j for state i. Nobody wins state i, if $x_i^k = 0$ for $k = 1, ..., K$.

All candidates submit their campaign budget allocations simultaneously. A pure strategy of candidate k is a N-dimension vector $\left(x_1^k, ..., x_N^k\right)$, such that $\sum_{j=1}^N x_j^k = X^k$.

The maximization problem of player k is

$$\max_{x_1^k,...,x_N^k} \frac{x_1^k}{\sum_{j=1}^K x_1^j} W_1 + ... + \frac{x_N^k}{\sum_{j=1}^K x_N^j} W_N, \tag{1}$$

$$s.t. \sum_{j=1}^N x_j^k = X^k. \tag{2}$$

The Lagrangian for the maximization problem (1)-(2) is

$$L\left(x_1^k, ..., x_N^k; \lambda^k\right) = \frac{x_1^k}{\sum_{j=1}^K x_1^j} W_1 + ... + \frac{x_N^k}{\sum_{j=1}^K x_N^j} W_N - \lambda^k \left(\sum_{j=1}^N x_j^k - X^k\right).$$

Therefore

$$\frac{\partial L}{\partial x_l^k} = \frac{\sum_{j \neq k} x_l^j}{\left[\sum_{j=1}^K x_l^j\right]^2} W_l - \lambda^k = 0, \text{ for any } l = 1, ..., N,$$

or

$$\frac{\sum_{j \neq k} x_1^j}{\left[\sum_{j=1}^K x_1^j\right]^2} W_1 = ... = \frac{\sum_{j \neq k} x_N^j}{\left[\sum_{j=1}^K x_N^j\right]^2} W_N = \lambda^k, \tag{3}$$

and

$$\sum_{j=1}^N x_j^k = X^k.$$

Dividing expressions (3) for player k on the same expressions for player i gives

$$\frac{\sum_{j \neq k} x_1^j}{\sum_{j \neq i} x_1^j} = ... = \frac{\sum_{j \neq k} x_N^j}{\sum_{j \neq i} x_N^j} = \frac{\lambda^k}{\lambda^i}. \tag{4}$$

Hence,

$$\sum_{j\neq k} x_l^j = \frac{\lambda^k}{\lambda^i} \sum_{j\neq i} x_l^j, \text{ for } l = 1, ..., N, \tag{5}$$

or

$$X_l - x_l^k = \frac{\lambda^k}{\lambda^i} \left(X_l - x_l^i \right), \tag{6}$$

where

$$X_l = \sum_{j=1}^{K} x_l^j$$

is the total campaign spending of all candidates on state l. Now we can find the total campaign spending of all candidates on each state in a symmetric equilibrium.

Note that

$$\sum_{j\neq k} X^j = \sum_{l=1}^{N} \sum_{j\neq k} x_l^j = \sum_{l=1}^{N} \frac{\lambda^k}{\lambda^i} \sum_{j\neq i} x_l^j = \frac{\lambda^k}{\lambda^i} \sum_{j\neq i} X^j$$

and

$$\frac{\sum_{j\neq k} X^j}{\sum_{j\neq i} X^j} = \frac{X - X^k}{X - X^i} = \frac{\lambda^k}{\lambda^i}, \tag{7}$$

where

$$X = \sum_{j=1}^{N} X_j = \sum_{k=1}^{K} X^k$$

is the total campaign spending on all states, or the total budget of all candidates.

Adding expressions (3) for all players gives

$$\frac{\sum_{k=1}^{K} \sum_{j\neq k} x_1^j}{\left[\sum_{j=1}^{K} x_1^j \right]^2} W_1 = ... = \frac{\sum_{k=1}^{K} \sum_{j\neq k} x_N^j}{\left[\sum_{j=1}^{K} x_N^j \right]^2} W_N = \sum_{k=1}^{K} \lambda^k,$$

or

$$\frac{W_1}{\sum_{j=1}^{K} x_1^j} = ... = \frac{W_N}{\sum_{j=1}^{K} x_N^j} = \frac{1}{K-1} \sum_{k=1}^{K} \lambda^k. \tag{8}$$

Therefore,

$$W_l \left(\sum_{j=1}^{K} x_i^j \right) = W_i \left(\sum_{j=1}^{K} x_l^j \right), \text{ for any } i, l = 1, ..., N, \tag{9}$$

or

$$W_l X_i = W_i X_l, \text{ for any } i, l = 1, ..., N.$$

Hence,

$$X = \sum_{j=1}^{N} X_j = \frac{X_1}{W_1} \sum_{j=1}^{N} W_j = ... = \frac{X_N}{W_N} \sum_{j=1}^{N} W_j.$$

In the symmetric equilibrium, the total campaign spending of all candidates on state i is

$$X_i = \frac{W_i}{\sum_{i=1}^{N} W_i} X = \frac{W_i}{\sum_{i=1}^{N} W_i} \sum_{k=1}^{K} X^k. \tag{10}$$

From (6) , (10) , and (7)

$$\frac{W_l}{\sum_{i=1}^{n} W_i} \sum_{k=1}^{K} X^k - x_l^k = \frac{X - X^k}{X - X^i} \left(\frac{W_l}{\sum_{i=1}^{N} W_i} \sum_{k=1}^{K} X^k - x_l^i \right),$$

or

$$x_l^k = \frac{X^k - X^i}{X - X^i} \frac{W_l}{\sum_{i=1}^{N} W_i} \sum_{k=1}^{K} X^k + \frac{X - X^k}{X - X^i} x_l^i.$$

Note that

$$X_l = \sum_{k=1}^{K} x_l^k =$$

$$\sum_{k=1}^{K} \left(\frac{X^k - X^i}{X - X^i} \frac{W_l}{\sum_{i=1}^{N} W_i} X + \frac{X - X^k}{X - X^i} x_l^i \right) =$$

$$\frac{1}{X - X^i} \left(\frac{W_l}{\sum_{i=1}^{N} W_i} X \sum_{k=1}^{K} \left(X^k - X^i \right) + x_l^i \sum_{k=1}^{K} \left(X - X^k \right) \right) =$$

$$\frac{1}{X - X^i} \left(X_l \sum_{k=1}^{K} \left(X^k - X^i \right) + (K-1) X x_l^i \right).$$

Therefore

$$x_l^i = \frac{X_l}{(K-1) X} \left[\left(X - X^i \right) - \sum_{k=1}^{K} \left(X^k - X^i \right) \right] =$$

$$x_l^i = \frac{X_l X^i}{X} = \frac{W_l}{\sum_{j=1}^{N} W_j} X^i,$$

and we obtain the following result

Theorem 1. *In the symmetric Nash equilibrium*

$$\left(x_1^k, ..., x_N^k \right) = \left(\left(\frac{W_1}{\sum_{j=1}^{N} W_j} \right), ..., \left(\frac{W_N}{\sum_{j=1}^{N} W_j} \right) \right) X^k, \tag{11}$$

for any candidate $k = 1, ..., K$.

First, note that the symmetric equilibrium is unique by the construction.

Second, all candidates compete for all states in the symmetric equilibrium. The intuition for this result is straightforward: since even a (very) small amount gives a positive chance to win a state, each candidate tries to win each state.

Third, the unique symmetric equilibrium has a monotonic property: a candidate with higher budget has higher chance to win each state.

Finally, Theorem 1 shows that the equilibrium budget allocation of candidate k depends on states' values, $(W_1, ..., W_N)$, and her own budget, X^k, and is independent from budgets of all other candidates. It means that if a candidate *does not know* the budgets of all other candidates, but *believes* that those budgets will be allocated according to (11), she will allocate her budget according to (11) too.

Theorem 2. *Suppose that individual budgets are private information. Then there exists a monotonic symmetric Bayesian equilibrium where candidate k with budget X^k allocates her budget in the following way*

$$\left(x_1^k, ..., x_N^k\right) = \left(\left(\frac{W_1}{\sum_{j=1}^N W_j}\right), ..., \left(\frac{W_N}{\sum_{j=1}^N W_j}\right)\right) X^k, \tag{12}$$

and believes that each candidate allocates the budget according to (12).

Theorem 2 describes a monotonic symmetric Bayesian equilibrium. Note that the players do not need to know the distribution function of individual budgets. It is enough to have "the right" beliefs which are consistent in the equilibrium. An interesting feature of the equilibrium is the same proportion allocation of the budget for all players. This feature gives the monotonic property: higher budget gives higher chance to win each contest.

3. Conclusion

We consider a stochastic K-player Blotto game where players have different budgets and different contests have different values. We show that there exists a unique symmetric Nash equilibrium. In this equilibrium, each player allocates her resources in the same proportion and competes in each contest. Therefore, the symmetric equilibrium has a monotonic property.

We show that if individual budgets are private information, there exists a monotonic symmetric Bayesian equilibrium. Our analysis demonstrates that the equilibrium allocation (12) is robust: it does not matter either players know anything about budgets of other players. It will be interesting to test our predictions in the experimental lab.

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