All Nash Equilibria of the Multi-Unit Vickrey Auction*

Andreas Blume†
Paul Heidhues‡
Jonathan Lafky†
Johannes Münster§
and
Meixia Zhang†

Abstract

This paper completely characterizes the set of Nash equilibria of the Vickrey auction for multiple identical units when buyers have non-increasing marginal valuations and there at least three potential buyers. There are two types of equilibria: In the first class of equilibria there are positive bids below the maximum valuation. In this class, above a threshold value all bidders bid truthfully on all units. One of the bidders bids at the threshold for any unit for which his valuation is below the threshold; the other bidders bid zero in this range. In the second class of equilibria there are as many bids at or above the maximum valuation as there are units. The allocation of these bids is arbitrary across bidders. All the remaining bids equal zero. With any positive reserve price equilibrium becomes unique: Bidders bid truthfully on all units for which their valuation exceeds the reserve price. *Journal of Economic Literature* Classification Numbers: C72, D44.

KEYWORDS: Vickrey auction, Multi-unit auction, ex-post equilibrium, reserve price, uniqueness.

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1 Introduction

This paper completely characterizes the set of Nash equilibria of the Vickrey auction for multiple identical units when buyers have non-increasing marginal valuations and there are at least three potential buyers. Equilibria fall into two classes: In one class there is positive probability that there are positive bids below the maximum valuation. In this class, there is a threshold for valuations such that all bidders bid truthfully on any unit for which they have a valuation exceeding the threshold. Furthermore, there is a distinct bidder who bids the threshold value on any unit for which his valuation is below the threshold. The remaining bidders bid zero on any unit for which their valuation is below the threshold. In the other class of equilibria, there are no positive bids below the highest valuation. In this class, each bidder bids at or above the highest valuation on some number of units and bids zero on the remaining units in such a manner that the total number of positive bids across all bidders equals the number of units that are for sale. In any equilibrium, except the conventional equilibrium in dominant strategies, there is positive probability that a bidder wins a unit at a price of zero. In this sense all of these equilibria are collusive.

We also observe that all equilibria of the Vickrey auction are *ex-post* equilibria, i.e. bidders have no incentive to change their behavior even after all private information is revealed and therefore suffer no regret. Indeed, the entire set of equilibria within the first class remain equilibria for *any* change of the distribution function of bidders' valuations, including changes that affect the support of the distribution of bidders' valuations.

With any positive reserve price equilibrium becomes unique: Bidders bid truthfully on all units for which their valuation exceeds the reserve price. From this perspective, our result can be interpreted as providing an alternative foundation for the focus on the truthful-bidding equilibrium.

Vickrey [1961] introduced the second-price sealed-bid auction for both the single- and the multiple-object case. With private values, there is a unique equilibrium in undominated strategies: Bidders bid their valuations. Milgrom [1981] notes the existence of other (asymmetric) equilibria in the single-unit case. For two bidders, Plum [1992] describes yet more equilibria in the single-unit case. Blume and Heidhues [2004] characterize all equilibria of the single-unit Vickrey auction with independent private values and three or more bidders. Blume and Heidhues [2001] also cover the two-bidder case. Tan and Yilankaya [forthcoming] show the existence of asymmetric equilibria with participation costs that are undominated. In contrast to these papers, here we consider the more complex multi-unit case.

2 Setup

There are n bidders indexed by $i = 1, \ldots, n$, and m identical units $j = 1, \ldots, m$. Bidder i's vector of valuations is denoted $\mathbf{v}^i = \left(v_1^i, \ldots, v_m^i\right)$, where v_j^i represents the marginal valuations of the j-th unit. Thus the value of obtaining j units is $\sum_{k=1}^{j} v_k^i$. We assume marginal valuations to be non-increasing, i.e. $v_j^i \geq v_{j+1}^i$, for all $j = 1, \ldots, m-1$. Each bidder i's valuation vector is independently drawn from some full support distribution F_i on the set $V := \{\mathbf{v}^i \in [0, v^h]^m | v_j^i \geq v_{j+1}^i, \forall j = 1, \ldots, m-1\}$.

The auctions rules are as follows: Each bidder i submits a bid vector $\mathbf{b}^i = (b_1^i, \ldots, b_m^i) \in B := \{\mathbf{b}^i \in \mathbb{R}_+^m | b_j^i \geq b_{j+1}^i, \forall j = 1, \ldots, m-1\}$ independently from and simultaneously with the other bidders. Restricting bid vectors to belong to the set B is without loss of generality. It simply expresses that bids in any bid vector are automatically ranked from highest to lowest, and permits us to talk about "a bidder's bid on his first, second, ... unit." The auctioneer collects all bidders' bids and ranks them from the highest to the lowest bid, breaking ties by choosing with equal probability among all possible rankings among tying bids. Each bidder receives a unit for each of his bids that is among the m highest ranked bids. If bidder i wins k_i units, then he pays the k_i highest losing bids among his rivals. Formally, define $\mathbf{c}^{-i}$ as the vector consisting of the m highest bids submitted by bidders other than bidder i, ordered so that $c_1^{-i} \geq \ldots \geq c_m^{-i}$. A bidder who gains k_i units pays $\sum_{k=1}^{k_i} c_{m-k_i+k}^{-i}$.

A bid function $\mathbf{b}^i(\cdot) : V \to B$ for bidder i maps valuation vectors into bid vectors. We denote a strategy profile by $\mathbf{b}(\cdot)$.

3 Examples

In order to illustrate the panoply of equilibria that are possible with only two bidders, to understand the role of limiting attention to the case with three or more bidders, and to get intuition for the proof of our characterization result, here we briefly discuss a few simple examples.

In Figure 1, we represent the essential aspects of one (type of) equilibrium in an auction with two bidders and two items for sale. Some features of this example survive when we restrict attention to three or more bidders, others do not. The two panels on top represent bidder one's bid function. Bidder two's bid function is shown in the bottom two panels. In this equilibrium bidder i's bid on his first (second) unit depends only on the higher (lower) of his two valuations. This feature, that a bidder's bid on his jth unit is independent of his valuation for his other units will be a general characteristic of all equilibria in which there is a positive probability of positive bids below the maximum valuation.

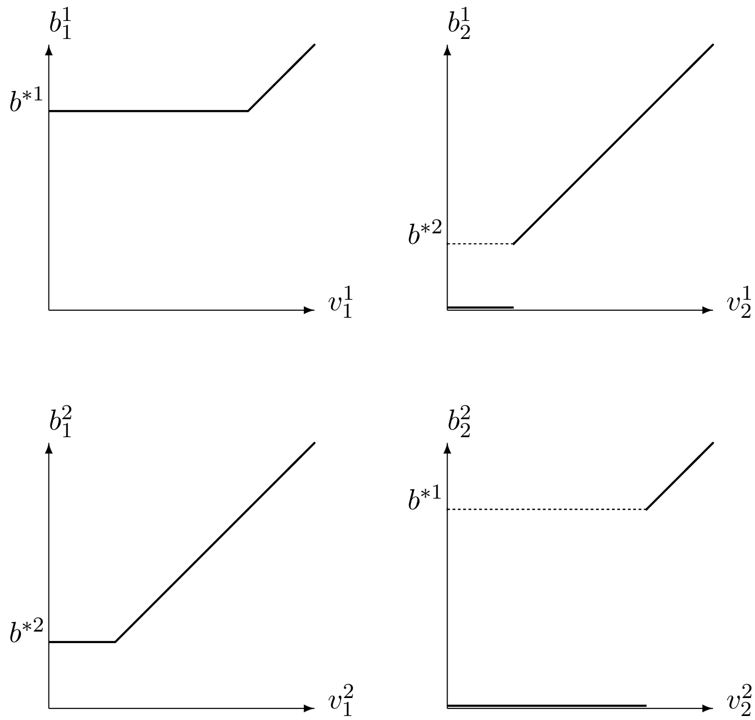

FIGURE 1

Bidder 1 bids b^{*1} on his first unit provided his high valuation satisfies $v_1^1 \leq b^{*1}$. Otherwise, he bids truthfully on his first unit. Since bidder 1 bids above his valuation on his first unit whenever he does not bid truthfully on that unit, we refer to him as a *high bidder* on his first unit. Bidder 1 bids zero on his second unit whenever his low valuation satisfies $v_2^1 \leq b^{*2}$. Otherwise, he bids truthfully on his second unit. Since bidder 1 bids below his valuation on his second unit whenever he does not bid truthfully on that unit, we refer to him as a *low bidder* on his second unit.

In the example, each bidder is a high bidder on his first unit and a low bidder on his second unit. We will find that having multiple high bidders is possible with more than two bidders only if the probability of positive bids below the maximum valuation is zero.

Observe also that in the example the critical value b^{*i} at which bidder i switches from bidding above his valuation to bidding truthfully on his first unit, differs across the two bidders. Again, this cannot occur with three or more bidders. With three or more bidders, if there is positive probability of positive bids below the maximum valuation, an equilibrium with distinct threshold values (b^{*1} and b^{*2}) as in FIGURE 1 is ruled out. To understand this, notice that a third bidder facing bidding behavior by the other two bidders as in FIGURE 1 would want to bid truthfully on his first unit for valuations above b^{*2}. But with two bidders bidding truthfully on their first unit above b^{*1}, it is no longer optimal for bidder 1 to maintain his postulated bidding strategy on the first unit.

This, in a nutshell, is the contagion effect that drives much of our result: If some bidder in a putative equilibrium puts in positive bids below the maximum valuation on some unit, e.g. bidder 1 bids near b^{*2} on his second unit, then for any other bidder, say bidder 2, who competes for that unit, those bids become potential prices. This disciplines this bidder's bidding behavior on that unit (viz. bidder 2 does not overbid on his first unit for valuations above b^{*2}). With three or more bidders in the auction, this discipline extends to at least two bidders. As a consequence the discipline extends to other units. In the example, with three bidders, it is no longer optimal for bidder 1 to overbid on his first unit for valuations for that unit in $[b^{*2}, b^{*1}]$.

A further possibility for equilibria in the two-bidder case, which generalizes the example of FIGURE 1, is that bidders have multiple gaps in their bid function, bidding truthfully outside the gaps, and adopting complementary roles of high and low bidders over the gaps. Here bidder one's gaps in his bid function for his first unit match bidder two's gaps in his bid function for his second unit, and vice versa. Any number and (matching) placement of gaps is possible. With three or more bidders all equilibria of this form disappear. The reason is that if two bidders have bid functions with gaps of this form, the third bidder has an incentive to bid inside these gaps. As a consequence, the bid functions with gaps are no longer optimal.

4 Results

We have two principal results that jointly characterize the entire set of Nash equilibria of the multi-unit Vickrey auction with three or more bidders. Our first result describes all Nash equilibria in which there is positive probability of positive bids below the maximum valuation; i.e., there exists at least one bidder i for whom $b_j^i(\mathbf{v}^i) \in (0, v^h)$ for at least one item j. We show that in this class of equilibria, there is a critical value b^* such that every bidder i bids truthfully on any unit j for which $v_j^i \geq b^*$. Furthermore, there is a single *high bidder* $\hat{\imath}$ who bids b^* on any unit j for which $v_j^{\hat{\imath}} \leq b^*$. The remaining bidders will be referred to as *low bidders.* Any low bidder i bids zero on any unit j for which $v_j^i \leq b^*$. It is important to emphasize that the high bidder is unique and that the critical value b^* is the same for all units. All proofs are in the Appendix.

Proposition 1 *Let the number of bidders satisfy $n \geq 3$. Suppose that $b_j^i\left(\mathbf{v}^i\right) \in \left(0, v^h\right)$ with positive probability for some bidder i and unit j. Then $\mathbf{b}\left(\cdot\right)$ is a Nash equilibrium if and only if* *there is a bidder* $\hat{\imath}$ *and some* $b^* \in \left(0, v^h\right)$ *such that*

$$b_j^{\hat{\imath}}\left(\mathbf{v}^{\hat{\imath}}\right) = \begin{cases} v_j^{\hat{\imath}} & \text{if } v_j^{\hat{\imath}} \geq b^* \\ b^* & \text{otherwise} \end{cases}$$

for all $j = 1, ..., m$ *and almost all* $\mathbf{v}^{\hat{\imath}}$, *and*

$$b_j^{i}\left(\mathbf{v}^{i}\right) = \begin{cases} v_j^{i} & \text{if } v_j^{i} \geq b^* \\ 0 & \text{otherwise} \end{cases}$$

for all $i \neq \hat{\imath}$, *all* $j = 1, ..., m$ *and almost all* $\mathbf{v}^i$.

To check that these strategies are equilibria it suffices to verify that no player can gain by deviating and playing his dominant strategy. First, consider the low bidders. For valuations above b^*, these bidders bid their valuation anyhow. In case the valuation for some unit j is below b^*, these bidders bid zero for this unit and do not obtain it. Raising such bids to their true valuation $v_j^i < b^*$, does not increase the probability of obtaining the object, since the mth highest bid is at b^* or above. Thus low bidders are playing a best response to the strategy of the high bidder. Similarly, suppose the high bidder has a valuation for object j below b^*. In equilibrium, with probability one, he only obtains the object if the $m - j + 1$-th highest rival bid is zero, and thus pays a price of zero. Thus, lowering his bid to his valuation neither affects the probability with which he obtains the object nor the price. Again therefore, he plays a best response.

For the converse result, the key observation is that if a bidder bids at or near some interior value b^* with positive probability, this induces a contagion process with the result that all bidders bid their true value above b^* for all units. Suppose bidder 1 bids at or near b^*. Then this bid must sometimes win as otherwise there would be a bidder who would obtain the object also when he has valuations below b^* for all units. This bidder would gain by switching to bidding his valuation on all units. Now consider bidder 2 and hold the behavior of all bidders other than bidder 1 and 2 fixed. There exists at least one unit for which bidder 2 competes directly with bidder 1's bid b^*, in the sense that by bidding slightly above b^* rather than below, bidder 2 increases the probability of obtaining that unit. As this is true for all bidders other than 1, all bidders sometimes bid at or above b^*. Hence there are both potentially many bids above as well as below b^*, which induces bidder 1 to bid at or above b^* for many units, which in turn induces other bidders to bid at or above b^* for many units. This contagion process continues until all bidders bid their valuation for all values above b^*.

We are left to consider the case in which no bidder bids in the interior.

Proposition 2 *Let the number of bidders satisfy $n \geq 3$. Suppose that in equilibrium $b_j^i\left(\mathbf{v}^i\right) \in \left(0, v^h\right)$ with probability zero for all bidders i and units j. Then either $b_j^i\left(\mathbf{v}^i\right) = 0$ for almost all $\mathbf{v}^i$, or $b_j^i\left(\mathbf{v}^i\right) \geq v^h$ for almost all $\mathbf{v}^i$. Furthermore,*

$$\#\left\{(i,j) \middle| b_j^i\left(\mathbf{v}^i\right) \geq v^h \text{ for almost all } v^i\right\} = m.$$

To see that the above strategy profile is an equilibrium, observe that any bidder who submits a positive bid for some unit, obtains that unit for free. Thus, submitting any positive bid on these units is part of a best response. Furthermore, the only way a bidder could increase the number of objects he obtains with positive probability is to bid at or above the highest possible valuation for some additional unit(s). For each unit he would obtain over and above the ones he gets in equilibrium, his payment increases by at least the highest possible valuation v^h. Thus deviating is unprofitable. The converse statement is established in the Appendix.

We conclude with considering the robustness of the Nash equilibria of the Vickrey auction. Four types of robustness are considered, robustness against varying the type distribution on a fixed payoff-type space, robustness against removing bidders, robustness against adding bidders with a larger set of payoff types, and robustness against introducing a positive reserve price. We find that the last of the four tests is the most stringent.

Remark 1 *All equilibria of the Vickrey auction are ex-post equilibria.*

A Bayesian Nash equilibrium in a Bayesian game is an *ex-post* equilibrium if players' strategies remain optimal even if all private information is made public. This condition clearly holds for all equilibria in Propositions 1 and 2, and thus for all equilibria of the Vickrey auction. In an *ex-post* equilibrium agents will never have to face regret. Furthermore, these equilibria are robust in the sense that they are invariant to changes of the distribution of player's private information (on a given type space).

The asymmetric equilibria are not, however, *ex post* in the sense of Holzman and Monderer (2004). They require what they refer to as "*ex-post* equilibria in Vickrey-Clarke Groves mechanisms" to remain equilibria when an arbitrary subset of players is excluded from playing. In the asymmetric equilibria of Proposition 1, if the high bidder is excluded, the remaining low bidders who bid zero would have a positive probability of obtaining the object when bidding zero. Thus, once the high bidder is taken out, low bidders gain from bidding their true valuations. As Holzman and Monderer show, their notion of "*ex post* equilibria in Vickrey-Clarke Groves mechanisms" requires player to use symmetric strategies. In our setting, this simplifies to the unique weakly dominant strategy profile. If, however, the high bidder is known to be active, our asymmetric equilibria of Proposition 1 are robust to adding or excluding low bidders.

Remark 2 *All equilibria of Proposition 1 are robust to enlarging the type space by including bidders with higher valuations than* v^h.

The equilibria of Proposition 1 are not only robust to changing the distribution over a given payoff-type space but also allow arbitrary extensions of the type space to bidders with possibly higher valuations.[1] This is not the case with the equilibria of Proposition 2. Indeed, only if there is a single high bidder who bids at v^h for all units is it possible to prescribe equilibrium bidding behavior for the new types (i.e. bid their valuation) without changing the bidding behavior of existing types.

Suppose the auctioneer sets a positive reserve price $r > 0$, such that for any unit a bidder obtains, his bid has to be at least as high as the reserve price and the reserve price is the minimum price for any unit. Without loss of generality, we can identify bids below the reserve price, or not bidding, with bidding zero. If m' is the number of bids at or above r, the auctioneer hands out $\mu = \min\{m', m\}$ units to the bidders with the μ highest bids. A bidder who gains k_i units pays $\sum_{k=1}^{k_i} \max\left\{c_{m-k_i+k}^{-i}, r\right\}$.

A positive reserve price below the maximum valuation has the same contagious effect as having bids with positive probability in that range. Bidders with valuations below the reserve price will refrain from bidding above the reserve price, for fear of winning an item at a price above their valuation for that item. As a consequence, any bidder will put in a bid above the reserve price for any item for which his valuation exceeds the reserve price. Bidders with valuations between those bids and the reserve price will want to bid in that range. With three or more bidders, this eliminates any potential gaps in the bid function above the reserve price, and therefore bidders bid truthfully for any unit with a valuation above the reserve price. The details of the proof are virtually identical to that of Proposition 1 and therefore omitted.

Corollary 1 *With a positive reserve price* $r > 0$*, equilibrium is unique: Bidders refrain from bidding on any unit for which their valuation is less than the reserve price. Otherwise they bid their valuation.*

[1] With free disposal, zero is a natural lower bound on possible types.

5 Appendix: Proofs

We begin by establishing the contagion property that is central to obtaining the characterization result.

Lemma 1 *Suppose that for some bidder i, some unit j, some $c \in (0, v^h)$ and all $\epsilon > 0$ there is positive probability that $b^i_j \in [c, c+\epsilon)$. Then $b^l_k(\mathbf{v}^l) = v^l_k$ for (almost) all $v^l_k \in (c, v^h)$, all $l = 1, \dots, n$, and all $k = 1, \dots, m$; i.e., every bidder bids truthfully on every unit for which his valuation exceeds c.*

Proof: Since the argument is somewhat lengthy and for readability the proof will be presented in a series of steps. The property that is established in each step will be highlighted by italics.

Step 1. Whenever $b^i_j \geq c$, bidder i wins at least j units with positive probability. Otherwise, there is probability one that at least $m - j + 1$ bids made by bidders other than i are above c. But this cannot be because then one of these bidders, say bidder l, with positive probability would win $k_l > 0$ units and pay more than his value for his k_lth unit.

Step 2. If bidder $l_1 \neq i$ has $m - j + 1$ or more valuations above c, then he has at least one bid above c, by Step 1. If conditional on bidder l_1 having one bid above c there is positive probability that all bids $b^i_j \geq c$ remain winning bids with positive probability for bidder i, and bidder l_1 has $m - j + 1$ or more valuations above c, then bidder l_1 has at least two bids above c. If on the other hand conditional on bidder l_1 having one bid above c, for some $\eta > 0$ all bids $b^i_j \in [c, c+\eta)$ become losing bids with probability one, then there is probability one that there are at least $m - j$ other bids above c. Suppose these $m - j$ bids are made by bidders other than l_1. This cannot be because then one bidder other than bidder l_1, say bidder l, with positive probability would win k_l units and pay more than his value for his k_lth unit. Thus, again if bidder l_1 has $m - j + 1$ or more valuations above c, he will have at least two bids above c. Iterating this argument, we conclude that *if bidder l_1 has $m - j + 1$ or more valuations above c, he will have at least $m - j + 1$ bids above c. Furthermore, from Step 1 it follows that if a bidder $l_1 \neq i$ has less than $m - j + 1$ valuations above c, he will have no more than $m - j$ bids above c.*

Step 3. One implication of *Step 2* is that there is positive probability that there are at least $m + 1$ bids at or above c made by bidders i and l_1.

A bidder $l_2 \neq i, l_1$ cannot bid with probability one at or above v^h for his first unit. If such a bid wins in the event of $m+1$ or more positive bids by i and l_1, then with positive probability l_2 pays more than his value for his first unit. If such a bid loses, then a bidder other than bidder l_2 would pay more than his value for at least one of his units.

Exchanging the roles of bidders in the above argument, it follows that with positive probability all bidders other than bidder i bid below v^h on their first unit. Hence, with positive probability bidder i's bid on his last unit exceeds the highest bid of bidders other than i on their first unit. The probability of bidder i's bid on his mth unit being equal to or exceeding v^h, however, equals zero. Otherwise the bids of the other bidders on their first unit have to be bounded away from v^h. In that case, denote the supremum of first-unit bids by bidders other than bidder i by $\bar{b}$. Note (using *Step 2*) that $v^h > \bar{b} > c > 0$, and hence it follows from *Step 1* that a bid at or above $\bar{b}$ wins at least one unit with positive probability. Then, if there is a bidder $l_1 \neq i$ whose distribution of first-unit bids has a mass point at $\bar{b}$, the other bidders have an incentive to bid above $\bar{b}$ on their first unit with positive probability. If there is no such bidder, and wlog $\bar{b}$ is the supremum of first-unit bids by bidder $l_1 \neq i$, then there is a bidder $l_2 \neq i, l_1$ who with positive probability has an incentive to bid in the interval $[\bar{b}, v^h)$, where we use the requirement that his first-unit bids are bounded away from v^h. If such bids are in the interior of this interval with positive probability, we have a contradiction because this violates the definition of $\bar{b}$. Otherwise, we must have a mass point at $\bar{b}$, which would take us back to the earlier case.

Since bidder i's bid on his mth unit is less than v^h with positive probability, there is a bidder other than bidder i who bids with positive probability in the interval $(0, v^h)$. Thus, from the foregoing argument, with the role of bidders exchanged, it follows that *with positive probability all bidders bid below v^h on their first unit. Hence, with positive probability all bidders bid above zero on their last unit.*

Step 4. Consider three distinct bidders l_1, l_2 and l_3. Suppose bidder l_1 bids with positive probability at or above v^h on his first unit. Then the bids of bidder l_2 on his last unit must be bounded away from v^h. Denote the corresponding least upper bound by $\bar{b}_m$. From *Step 3*, we know that $\bar{b}_m \in (0, v^h)$. Then bidder l_3 has an incentive to bid with positive probability at or above $\bar{b}_m$ on his first unit. If such bids with positive probability are below v^h, we have a contradiction because then bidder l_2 would have an incentive to bid with positive probability above $\bar{b}_m$ on his last unit in violation of the definition of $\bar{b}_m$. Continue then with the case where both bidders l_1 and l_3 with positive probability bid at or above v^h on their first unit. Then the bids of bidder l_2 on his $m-1$th unit must be bounded away from v^h. Denote the corresponding least upper bound by $\overline{b}_{m-1}$ and note that $\overline{b}_{m-1} \in [\overline{b}_m, v^h)$. Then bidder l_3 has an incentive to bid with positive probability at or above $\overline{b}_{m-1}$ on his first two units. If in this case bidder l_3's bids on his second unit are below v^h with positive probability, we have a contradiction because then bidder l_2 would have an incentive to bid above $\overline{b}_{m-1}$ on his $m-1$th unit. Continue then with the case where bidder l_1 bids with positive probability at or above v^h on his first unit and bidder l_3 bids with positive probability at or above v^h on his first two units. Iterating this argument, we find that bidder l_2's bid on his first unit must be bounded away from v^h. Denote the corresponding least upper bound by $\overline{b}_1$ and note that $0 < \overline{b}_m \leq \overline{b}_{m-1} \leq \ldots \leq \overline{b}_1 < v^h$. Then bidder l_3 has an incentive to bid with positive probability at or above $\overline{b}_1$ on all of his units. If in this case bidder l_3's bids on his mth unit are below v^h with positive probability, we have a contradiction because then bidder l_2 would have an incentive to bid above $\overline{b}_1$ on his first unit in violation of the definition of $\overline{b}_1$. But at the same time, it is impossible for bidder l_3 to bid with positive probability at or above v_h on all of his units because then with positive probability there would be $m+1$ bids at or above v^h and therefore at least one bidder with positive probability would win an item at a price above his valuation for that item. Since the choice of the three bidders was arbitrary, it follows that *the probability that any bidder bids at or above v^h on any of his units is zero.* Combined with the earlier observation that all bidders bid with positive probability in $(0, v^h)$ on their last unit, this implies that *all bidders bid with positive probability in $(0, v^h)$ on* all *of their units.*

Step 5. Pick an arbitrary pair of distinct bidders l_1 and l_2. Suppose bidder l_1's bids on his last unit are bounded away from v^h. Denote the corresponding least upper bound by $\overline{h}$. From *Step 4*, $0 < \overline{h} < v^h$ and from *Step 1*, bids at or above $\overline{h}$ win at least one unit with positive probability. Hence bidder l_2 must bid with positive probability in $[\overline{h}, v^h)$ on his first unit, as he cannot be bidding at or above v^h. This in turn implies that bidder l_1 has an incentive to bid with positive probability in $(\overline{h}, v^h)$ on his last unit, contrary to the definition of $\overline{h}$. Since the choice of bidders was arbitrary, it follows that *for all $\epsilon > 0$, there is positive probability that* all *bids of* all *bidders are in the interval $(v^h - \epsilon, v^h)$.*

Step 6. Suppose there is an interval $(e, e') \subseteq (c, v^h)$ in which no bidder bids on his first unit with positive probability. Then there is a maximal such interval $(\underline{e}, \overline{e}) \subseteq (c, v^h)$ by exactly the same argument as in Lemma A3 of Blume and Heidhues [2004]. From *Step 5*, we know that $\overline{e} < v^h$ and that with positive probability every bidder bids in $(\overline{e}, v^h)$ on all of his units.

Denote the infimum of positive probability bids on any bidder's last unit in the interval $[\overline{e}, v^h]$ by $\underline{b}_m$. Suppose that $\overline{e} < \underline{b}_m$. Without loss of generality, suppose that $\overline{e}$ is the infimum of bidder l_1's first unit bids in the interval $[\overline{e}, v^h]$ and that $\underline{b}_m$ is the infimum of bidder l_2's mth-unit bids in the same interval. Then, if bidder l_3 has all of his valuations in the interval $(\overline{e}, \underline{b}_m)$, he will have all of his bids in the interval $(\overline{e}, \underline{b}_m]$; this follows from *Step 2*. If in this case he bids with positive probability in $(\overline{e}, \underline{b}_m)$ on his mth unit, we have the desired contradiction. Otherwise, bidder l_3 with positive probability bids at $\underline{b}_m$ on his last unit. But then, if bidder l_2 has all of his valuations in the interval $(\overline{e}, \underline{b}_m)$ he will have all of his bids in the same interval, and we have again reached a contradiction. *We conclude that we must have* $\overline{e} = \underline{b}_m$.

Then take a bidder all of whose valuations are in $(\underline{e}, \overline{e})$ *to derive a contradiction to the assumption that no one bids with positive probability in* $(\underline{e}, \overline{e})$ *on his first unit. Distinguish the cases where* $\underline{e} = c$ *and* $\underline{e} > c$ *:*

Consider the case where $\underline{e} = c$. Without loss of generality, suppose that $\overline{e}$ is the infimum of bidder i_1's mth-unit bids in the interval $[\overline{e}, v^h]$. Then by *Step 2* a bidder i_2 other than i and i_1 who has all of his valuations in the interval $(c, \overline{e})$ has an incentive to bid on his first unit in the interval $(c, \overline{e}]$. If such bids are in the interval $(c, \overline{e})$, we have the desired contradiction. Therefore, suppose the distribution of bidder i_2's bids on his first unit when all of his valuations are in the interval $(c, \overline{e})$ has a mass point at $\overline{e}$. *Step 1* implies that the distribution of bidder i_1's bids on his mth unit cannot have a mass point at $\overline{e}$; otherwise bidder i_2 with positive probability would win a unit at a price above his valuation for that unit.

There are two cases to consider: First, if $i_1 = i$, then for all bidders other than bidder i their distributions of first-unit bids must have mass points at $\overline{e}$. Second, if $i_1 \neq i$, then since i_1's distribution of bids on his mth unit does not have a mass point at $\overline{e}$, for any $\epsilon > 0$, there is positive probability that bidder i bids in the interval $(\overline{e}, \overline{e} + \epsilon)$ on all of his units. But this implies that $\overline{e}$ is the infimum of bidder i's bids on his last unit in the interval $[\overline{e}, v^h]$. Thus, in either case all bidders other than bidder i must have a mass point at $\overline{e}$ in their distributions of first-unit bids. But then, consider the case where bidder i has $m-1$ valuations above $\overline{e}$ and his mth valuation below $\overline{e}$. Since with positive probability all of his rivals bid at $\overline{e}$ on their first unit and there are at least two such rivals, i has to bid for $m-1$ units above $\overline{e}$ to ensure that he receives those units whenever their prices are below his values for those units. At the same time, *Step 2* implies that he will bid below $\overline{e}$ for his last unit. Let the remaining bidders all have valuations below $\overline{e}$ such that they bid at $\overline{e}$ on their first unit. In this case there is a bidder with valuation on his first unit below $\overline{e}$, who wins one unit and pays $\overline{e}$. Since this case has positive probability, this gives the desired contradiction.

Consider the case where $\underline{e} > c$. Without loss of generality, suppose that $\underline{e}$ is the supremum of bidder i_1's 1st-unit bids in the interval $[c, \underline{e}]$. Similarly, without loss of generality suppose that $\overline{e}$ is the infimum of bidder i_2's mth-unit bids in the interval $[\overline{e}, v^h]$. Then a bidder i_3 other than i_1 and i_2 who has all of his valuations in the interval $(\underline{e}, \overline{e})$ has an incentive to bid on his first unit in the interval $[\underline{e}, \overline{e}]$. If such bids are in the interval $(\underline{e}, \overline{e})$, we have the desired contradiction.

Therefore, suppose the distribution of bidder i_3's bids on his first unit has a mass point at $\overline{e}$. Note that the distribution of bidder i_2's bids on his mth unit cannot have a mass point at $\overline{e}$; otherwise bidder i_3 with positive probability would win a unit at a price above his valuation for that unit. There are two cases to consider:

First, consider $i_2 = i_1$. Then for all bidders other than bidder i_1, their distributions of first-unit bids must have mass points at either $\overline{e}$ and\or $\underline{e}$. If there is no mass point at $\underline{e}$, the argument for the case $\underline{e} = c$ applies. If on the other hand, there is a bidder i_4 other than i_1 whose distribution of first-unit bids has a mass point at $\underline{e}$, then there is a bidder (other than i_4 and i_1) who with positive probability prefers to bid in $(\underline{e}, \overline{e})$ on his last unit. But this implies that bidder i_4 sometimes wants to outbid this bidder with his first unit bid when all his valuations are in the interval $(\underline{e}, \overline{e})$. Since, by assumption he is not bidding in the interior of this interval, his distribution of first-unit bids must also have a mass point at $\overline{e}$. Hence, all bidders other than i_1 must have distributions of first-unit bids with mass points at $\overline{e}$.

Second, if $i_2 \neq i_1$, then since i_2's distribution of bids on his mth unit does not have a mass point at $\overline{e}$, for any $\epsilon > 0$, there is positive probability that bidder i_1 bids in the interval $(\overline{e}, \overline{e} + \epsilon)$ on all of his units. But this implies that $\overline{e}$ is the infimum of bidder i_1's bids on his last unit in the interval $[\overline{e}, v^h]$, which takes us back to the previous case. Thus, in either case all bidders other than bidder i_1 must have a mass point at $\overline{e}$ in their distributions of first-unit bids.

Now suppose there is no bidder, like i_3 above, whose distribution of first-unit bids has a mass-point at $\overline{e}$. From the foregoing argument, it is without loss of generality to focus on the case where $i_2 = i_1$. Then, for all bidders other than i_1, their distribution of first-unit bids must have a mass point at $\underline{e}$. Then there are at least two bidders, which are different from i_1, who for a positive-probability set of values prefers to bid in the interval $(\underline{e}, \overline{e}]$ on their mth unit. If such bids with positive probability are in $(\underline{e}, \overline{e})$, then we get a contradiction because at least one bidder other than i_1 would have to have a mass point at $\overline{e}$ in his distribution of first-unit bids. Hence, both bidders' distribution of mth-unit bids must have a mass point at $\overline{e}$, which is impossible because then there would be positive probability that a bidder wins a unit at a price above his valuation for this unit.

Hence, we may conclude that in every interval above c there is some bidder who bids in this interval with positive probability on his first unit.

Step 7. Suppose there is an interval $(d, d') \subset (c, v_h)$ and a bidder l_1 who does not bid with positive probability in (d, d') on his first unit. Then, by *Step 2,* bidder l_1 must bid truthfully on his mth unit over this interval. Then, using *Step 2* once more, any bidder $l_2 \neq l_1$ must bid truthfully on his first unit over the same interval. Hence, any bidder $l_3 \neq l_1, l_2$ must bid truthfully on both his first and his mth unit over (d, d'). But then bidder l_1 must bid truthfully on his first unit over the interval (d, d'), which leads to a contradiction. *Therefore every bidder bids with positive probability in every interval* $(d, d') \subset (c, v_h)$ *on his first unit. Hence, every bidder bids truthfully on his* m*th unit over the interval* (c, v^h)*. This implies that every bidder bids truthfully on his first unit over the interval* (c, v^h).
Step 8. Suppose that for some bidder l and unit k, we have $v_k^l \in (c, v^h)$. Consider two bidders l_1 and l_2 different from bidder l. From *Step 7,* for any $\epsilon > 0$, there is positive probability that all of bidder l_1's bids are in the interval $(v_k^l - \epsilon, v_k^l)$ and that all of bidder l_2's bids are in the interval $(v_k^l, v_k^l + \epsilon)$. Thus, if bidder l were to bid below $v_k^l - \epsilon$, he would run the risk of not winning his kth unit when it is available at a price below his valuation for that unit. Similarly, if bidder l were to bid above $v_k^l + \epsilon$, he would run the risk of winning a kth unit at a price exceeding his valuation for that unit. *Therefore, for any valuation* $v_k^l \in (c, v^h)$*, it is uniquely optimal for bidder* l *to bid truthfully on his* k*th unit.* □

We are ready to prove Proposition 1.

Proof of Proposition 1: We have established that all such strategy profiles are Nash equilibria in the text. We are left to show that no other equilibrium exists in which some bidder bids in $(0, v^h)$ with positive probability.

Suppose that $b_j^i\left(\mathbf{v}^i\right) \in \left(0, v^h\right)$ with positive probability for some bidder i and unit j. Let

$$b^* := \inf\left\{b \in (0, v^h) | \exists i, j \text{ s.t. } \forall \epsilon > 0, \Pr\left\{b_j^i \in [b, b+\epsilon)\right\} > 0\right\}.$$

For almost all $v_j^i > b^*$, bidders bid truthfully by Lemma 1, i.e. $b_j^i(\mathbf{v}^i) = v_j^i$. If $b^* = 0$, we are done.

Thus consider the case where $b^* > 0$. Whenever a bidder has his valuation for a unit in $(0, b^*)$, then he bids in $[0, b^*]$ for this unit; otherwise, by Lemma 2 there would be positive probability of this bidder winning a unit at a price above his valuation for that unit. Trivially, such bids cannot be in $(0, b^*)$.

Suppose there are two distinct bidders l_1 and l_2 (and possibly others) who with positive probability submit a bid b^* on their first unit. Then there exists a bidder $l_3 \neq l_1, l_2$ who bids, with positive probability, for exactly $m-1$ units above b^* and for his last unit below b^*. This implies that with positive probability l_1 wins exactly one unit for a price b^* when his valuation for this unit is in $(0, b^*)$. Thus, there is at most one bidder who bids on his first unit at b^* with positive probability.

If no bidder were to bid with positive probability at b^* on his first unit, then all bidders would bid with positive probability at zero on their first unit. Hence, such bids would win with positive probability. But it that case, bidders can gain from deviating to bidding their valuation. Thus, there is exactly one bidder, say bidder i, who bids on his first unit at b^* with positive probability.

The remaining bidders must bid zero on all of the units for which their valuation is less than b^*, for otherwise they run the risk of winning those units at prices above their valuations for those units. As a consequence bidder i bids b^* on all units for which his valuation is in the set $(0, b^*)$. □

Proof of Proposition 2: We have argued in the main text that these strategy profiles are equilibria. It remains to show that there is no other equilibrium in which no bidder bids in $(0, v^h)$ with positive probability.

Suppose that, with positive probability, the number of bids at or above v^h is smaller than m. Then there is a bidder who bids zero for some unit and wins this unit with positive probability less than one. This bidder can raise his payoff by switching to always bidding his value.

Suppose that, with positive probability, the number of bids at or above v^h is greater than m. Then there exist a bidder who, with positive probability, wins one unit for a price greater or equal v^h. This bidder can raise his payoff by switching to always bidding his value.

This implies that the number of bids at or above v^h is equal to m with probability one. Since bids are independent across bidders, it follows that if bidder i bids at or above v^h for his j - th unit with positive probability, he must bid at or above v^h for his j - th unit with probability one. □

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