Sequential Contests with Ability Revelation

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Abstract

We consider a two-stage elimination contest where before the competition in the second round takes place contestants' abilities are revealed. We derive a monotonic symmetric equilibrium and make a comparison of the expected effort with and without revelation of ability. Our main finding is that the revelation always increases the first-round effort but decreases the second-round effort. As our numerical examples show, the expected two-round total effort can, however, either increase or decrease with the revelation.

Keywords: Contests; Tournaments; Complete information; Incomplete information.

JEL classification: C73; D82; M51

1 Introduction

Contests or tournaments as a form of competition are prevalent in many social and economic settings. On the fields or in the stadiums, virtually all competitions are tournaments. Promotions within firms, elections, patent races, and many political campaigns can also be understood as contests of one form or another. Early literature on tournaments focuses on the rationales behind using them as a contractual arrangement and compare them

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with optimal contracts (e.g., Lazear and Rosen, 1981; Green and Stokey, 1983; Nalebuff and Stiglitz, 1983). More recently, researchers, perhaps influenced by the mechanism design approach, have shifted their attention to optimal contest designs (e.g., Gradstein and Konrad, 1999; Moldovanu and Sela, 2001, 2006; Matros, 2005). This more recent literature tackles issues such as the optimal prize structures and the optimal way of organizing contestants into rounds and sub-contests.

This paper considers another important aspect of contest design that is relevant to many contests in practice: whether it is optimal in a sequential contest to reveal information about contestants' abilities. Many sport competitions as well as career advancements in organizations are organized as sequential contests. In many instances of these competitions, the natural environment of the contests inevitably render contestants' abilities a public knowledge. For example, it is a common practice in sport tournaments that all matches are recorded by the coaches as a part of the preparation. As a result, teams are well informed of their rivals' abilities. In organizations, effort exerted by employees in the initial stage of their career typically reveals their abilities. When a group of employees are subsequently selected for a promotion and enter into another round of competition for higher places in the hierarchy, they would therefore have a pretty good idea of how capable their colleagues are. Note, however, that even though abilities are revealed in these later rounds of competition so that every contestant knows that the strongest member has a competitive edge over all others, the strongest member still has to exert effort because firms require individual performance at all levels in order to operate and audience expects the teams to perform in sport events. This observation will be captured as an important feature of our model.

It should be noted that we are not dealing with a general mechanism design problem in which the designer's objective is to maximize some effort parameters. Rather, we concentrate on the question of how, in a multi-round elimination contest, revelation of abilities affects the effort exerted by the contestants. We adopt the two-stage elimination contest of Moldovanu and Sela (2006) as a platform to study the effects of ability revelation. The ability parameters are private information in the initial round that are nevertheless *fully* revealed before the second-round, and this changes the information environment facing the advancing contestants.[1] The contestants

[1] There are many different kinds of feedback policies. One can easily think of, for example, full revelation or partial revelation with some statistics such as the mean or median

fully anticipate this, and that fact, in turn, affects their effort level not only in the last rounds but also in the first round.[2] Thus, the first-round contest is equivalent to an all-pay auction with incomplete information while the second-round contest an all-pay auction with complete information.

We derive a monotonic symmetric subgame perfect equilibrium of this contest. Then, we compare contests with and without ability revelation. Our result indicates that the first-round total effort is always higher with ability revelation while the second-round total effort is always lower. Two numerical examples demonstrate that the expected two-round total effort can be either higher or lower with ability revelation.

The exposition of the rest of the paper is as follows. Section 2 describes the design of the contest. Section 3 contains the derivation of the monotonic symmetric equilibrium of the two-stage contest with ability revelation. For comparison, we also reproduce the equilibrium of the contest without ability revelation in Moldovanu and Sela (2006). A thorough comparison of the two contests is made in Section 4. Section 5 concludes. Appendix A contains the order-statistical tools that are used in the paper. All the proofs are in Appendix B.

2 The Design of the Contest

The two-stage elimination contest is organized as follows. There are n risk-neutral contestants, equally divided into t groups each with $k = n/t$ members. In the first round of the contest, each contestant competes with his $k-1$ rivals within the group. In each of these t sub-contests, a winner is singled out to receive one of the t runner-up prizes W_1, and the remaining $k-1$ participants will be eliminated. However, the potential award from winning in the first round is more than W_1; a first-round winner has the

ability. This paper considers a full ability revelation where abilities of all contestants are revealed after the first round. This is an extreme kind of ability revelation. However, as we argued, the natural design of many contests provides this information to the contestants. The full revelation we focus on is therefore not an artifact but a common feature of many sequential contests in practice.

[2]While we only consider a two-stage contest, the change of the information structure after an ability revelation allows us to apply backward induction to the subsequent rounds of a contest with more than two rounds. To study the effect of the ability revelation, we therefore restrict our analysis to two-stage contests.

opportunity to compete in the second round for a main prize of W_2. Figure 1 provides a visual depiction of the contest structure.

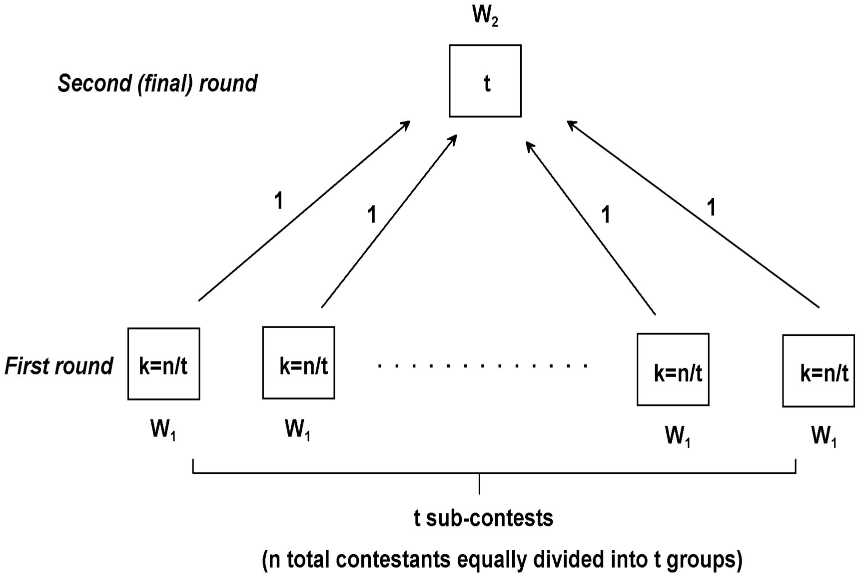

Figure 1: Contest Structure

Contestant i competes with his rivals by exerting costly effort at a constant marginal cost of c_i where c_i is his private information. The competition technology is deterministic: suppose x_i denotes the effort exerted by contestant i, then i wins a contest if and only if $x_i > x_j$ for all $j \neq i$ contestants in his group. In the case of a tie, a prize is assigned with equal probability among winners. Note that no matter winning or losing, the effort exerted is sunk.[3] The cost of effort can be interpreted as the ability of a contestant. A contestant with high ability is the one with low marginal cost, and vice versa. We use high (low) ability and low (high) cost interchangeably to describe the strength of a contestant. Each contestant's cost is independently drawn from the common distribution F with a continuous density f,[4] and

[3]Therefore, the contest is equivalent to an all-pay auction in which the bidder with the highest bid wins but all bidders, winning or losing, have to pay their bids.

[4]Since most distribution functions we shall use for deriving results are distributions of

the support is $[m, 1]$ where $m > 0$.[5] The distribution function F is assumed to be common knowledge. We use capital letter C to denote the random variable of contestants' cost of effort. Then, in our notations described in Appendix A, $C_{(k,n)}$, $k = 1, 2, ..., n$, denotes the corresponding order statistics of C, where $C_{(1,n)}$ is the random variable of the lowest cost among n contestants, $C_{(2,n)}$ is the random variable of the second lowest cost, and so on.

The novel feature of our model is that, after the first-round winners are determined, individual abilities are publicly revealed before the second-round contest takes place.[6]

The exact timing of the game is as follows:

1. Contestants are divided into t groups in the first round with $k = n/t$ contestants in each group.
2. Nature draws a cost of effort, c, for each contestant from the distribution F. Each contestant knows his value of c and only that other contestant's costs are independently distributed according to F.
3. In each of the t sub-contests, contestants compete with $k - 1$ rivals for the prize W_1 and the opportunity to enter the second round.
4. The t winners in the first round are determined and awarded the runner-up prizes W_1. They observe the abilities of each other.
5. Given the finalists' abilities, the t contestants decide how much effort to exert in the second round.
6. The prize W_2 is awarded to the final winner.

order-statistics, to ease the description we shall call F the *primitive distribution*. More generally, primitive distribution will be used to denote any distribution of unranked random variables. Refer to Appendix A for the details of their relations.

[5] m is assumed to be strictly positive to avoid infinite effort when cost is zero.

[6] Since only the first-round winners are going to compete in the second round, it makes no difference whether the abilities of all n contestants or just the t first-round winners are revealed.

3 Equilibrium Analysis of the Contest

In this section, we analyze the monotonic symmetric equilibrium of the two-stage contest. To expedite the comparison with the contest without ability revelation, we also reproduce some results from Moldovanu and Sela (2006).

3.1 Ability Revelation

The approach of our equilibrium analysis is a constructive one. We derive the expected value of the second-round participation as a part of the first-round prize based on the conjecture that there exists a monotonic symmetric equilibrium. Then, we derive the first-round equilibrium effort function and verify that the equilibrium we obtain is indeed monotonic. The equilibrium we derive is a unique monotonic symmetric equilibrium.

Second-Round Effort

The second-round contest is a game of complete information and the analysis of such a contest in Hillman and Riley (1989) can be used to derive the equilibrium.

Denote y_i to be the effort exerted by contestant i in the second round. Then, he faces the following maximization problem

$$\max_{y_i} P_i[y_1, .., y_i, .., y_t]W_2 - c_i y_i,$$

where

$$P_i[y_1, .., y_i, .., y_t] = \begin{cases} 1, & \text{if } y_i > y_j \text{ for all } j \neq i, \\ 0, & \text{if } y_i < y_j \text{ for at least one } j \neq i, \\ 1/|\max\{y_1, ..., y_t\}|, & \text{if } y_i \in \max\{y_1, ..., y_t\}, \end{cases}$$

where $|...|$ stands for the cardinality of a set. Hillman and Riley (1989) show that in an equilibrium only the two contestants with highest abilities participate in the competition, and they both adopt mixed strategies.[7] This gives

[7] Baye et al. (1996) consider the same strategic environment in the setting of all-pay

the following characterization of our second-round effort spending: After the ability revelation, only the two highest-ability (lowest-cost) finalists participate while the remaining contestants stay out of the competition. Formally,

Lemma 1. *In the second round finalist 1 with cost c_1 and finalist 2 with cost c_2 such that $c_1 < c_2 < c_j$, for all $j \neq 1, 2$, randomize their effort with distribution functions $B_1(y)$ and $B_2(y)$ respectively, where*

$$B_1(y) = \begin{cases} (c_2/W_2)y, & \text{for } 0 \leq y < W_2/c_2, \\ 1, & \text{for } y \geq W_2/c_2, \end{cases} \tag{1}$$

and

$$B_2(y) = \begin{cases} 1 - c_1/c_2, & \text{for } y = 0, \\ (1 - c_1/c_2) + (c_1/W_2)y, & \text{for } 0 < y < W_2/c_2, \\ 1, & \text{for } y \geq W_2/c_2. \end{cases} \tag{2}$$

All other participants $j \neq 1, 2$, exert zero effort.

We refer to Hillman and Riley (1989) and Baye et al. (1996) for the details of the proof.

Note that the expected equilibrium payoffs of finalists 1 and 2 are respectively $W_2(1 - c_1/c_2)$ and zero. As such, in the second-round only the highest-ability contestant has a positive expected payoff. The expected payoffs of all other finalists are zero.

First-Round Effort

There are effectively two prizes for the first-round winners: each winner is awarded W_1 *and the opportunity to participate in the second round* where he stands some chance to win W_2. How much a contestant values the later depends on his expected payoff from the second round. These two prizes enter into the maximization problem of a contestant as the revenue side of

auctions and find other asymmetric equilibria. In this paper, we focus on the equilibrium described in Hillman and Riley (1989).

his expected payoff when he decides how much effort to exert in the first round. While W_1 is exogenously given, the value of another prize - the "entry ticket" to the second round - depends on the ability of the contestant. We show above that in the second round only the highest-ability contestant has positive (post-revelation) expected payoff. Denote this expected payoff by $\pi(c_i)$. We have

$$\pi(c_i) = \begin{cases} W_2(1 - c_i/c_2), & \text{if } c_i < c_j \text{ for all } j \neq i, \\ 0, & \text{if } c_i > c_j \text{ for at least one } j \neq i, \end{cases} \tag{3}$$

where c_2 is any realized value of the cost of the second highest-ability contestant in the second round.

In the first round when a contestant evaluates the value of the second-round participation, he takes into account the likelihood of winning in the second round as well as the expected value of π (conditioned on winning). Note also that the relevant distribution in the second round with respect to which the expectation is taken is $F_{(1,k)}$, the distribution of the costs of t finalists who beat their rivals in the sub-contests of k contestants. Statistically, $F_{(1,k)}$ is just the distribution of the first-order statistics among k random variables, and it becomes the primitive distribution of the second-round contest. For convenience, we denote $G = F_{(1,k)}$.

Given (3) the expected value of the second-round participation, Π,is given by the following proposition.

Proposition 1. *The expected value of the second-round participation to a contestant with the effort cost c is*

$$\Pi(c) = W_2 \int_c^1 \left(1 - \frac{c}{s}\right) dG_{(1,t-1)}(s), \tag{4}$$

where $G_{(1,t-1)}$ is the distribution of the first order-statistics among $t-1$ random variables with the primitive distribution $G = F_{(1,k)}$.

Expression (4) substantiates formally our preceding discussion that the expected value of the second-round participation depends on individual's ability, and the relation is a negative one:

$$\Pi'(c) = -W_2 \int_c^1 \frac{1}{s} dG_{(1,t-1)}(s) < 0. \tag{5}$$

This means that the expected value of the second-round "entry ticket" is higher for contestants with higher ability (or lower cost).

Note that if there exists a strictly monotonic symmetric equilibrium, a contestant who exerts the highest effort in the first round in the equilibrium is also the one with the highest ability.

To proceed, we assume that contestants $j \neq i$ follow a symmetric, decreasing, and differentiable equilibrium strategy $x_j(c_j)$ in the first round. Suppose that contestant i receives a cost x_i and exerts effort $\tilde{x}_i$. We wish to determine the optimal $\tilde{x}_i$. If contestant i exerts effort $\tilde{x}_i$, his probability of winning the first-round sub-contest is given by

$$\begin{aligned}
P[\tilde{x}_i > x_j(c_j), \forall j \neq i] &= P[\tilde{x}_i > x(c_j), \forall j \neq i] \quad \text{[by symmetry]} \\
&= P[x^{-1}(\tilde{x}_i) < x^{-1}(x(c_j)), \forall j \neq i] \quad [x(c) \text{ is decreasing}] \\
&= P[x^{-1}(\tilde{x}_i) < c_j, \forall j \neq i] \quad \text{[by strict monotonicity]} \\
&= \prod_{j \neq i} P[x^{-1}(\tilde{x}_i) < c_j] \quad \text{[by independence]} \\
&= [1 - F(x^{-1}(\tilde{x}_i))]^{k-1} \quad [\text{by common distribution } F] \\
&= F_1^k(x^{-1}(\tilde{x}_i)) \quad \text{[by formula (A.3)]}.
\end{aligned} \tag{6}$$

Therefore, contestant i faces the following problem in the first round:

$$\max_{\tilde{x}} [W_1 + \Pi(x^{-1}(\tilde{x}_i))] F_1^k(x^{-1}(\tilde{x}_i)) - c\tilde{x}_i. \tag{7}$$

An obvious interpretation of the first-round expected payoff can be seen from expression (7). W_1 is the runner-up prize in the first round and $\Pi(x^{-1}(\tilde{x}_i))$ is the expected value of the opportunity to participate in the second round. The probability of winning them both in the first round is $F_1^k(x^{-1}(\tilde{x}_i))$, and $c\tilde{x}_i$ is the cost that has to be paid with probability one in the all-pay auction environment.

There is, however, another interpretation of (7). By (A.4) and (A.3), it can be shown that $G \equiv 1 - [1 - F]^k$. Then, the probability in (6) can be rewritten as

$$\begin{aligned}
[1 - G(x^{-1}(\tilde{x}_i))]^{t-1} &= [1 - F(x^{-1}(\tilde{x}_i))]^{n-k} \\
&\equiv \frac{[1 - F(x^{-1}(\tilde{x}_i))]^{n-1}}{[1 - F(x^{-1}(\tilde{x}_i))]^{k-1}} \\
&\equiv \frac{F_1^n(x^{-1}(\tilde{x}_i))}{F_1^k(x^{-1}(\tilde{x}_i))},
\end{aligned}$$

where we use (A.3) again and recall that $k = n/t$. Denote the pre-revelation expected payoff in the second round:

$$z = E[W_2(1 - [x^{-1}(\tilde{x}_i)/C_{(1,t-1)}])|C_{(1,t-1)} > (x^{-1}(\tilde{x}_i))].$$

The maximization problem (7) can then be rewritten as

$$\begin{aligned}
&\max_{\tilde{x}_i}[W_1 + \Pi(x^{-1}(\tilde{x}_i))]F_1^k(x^{-1}(\tilde{x}_i)) - c\tilde{x}_i \\
\Leftrightarrow &\max_{\tilde{x}_i} W_1 F_1^k(x^{-1}(\tilde{x}_i)) + z\frac{F_1^n(x^{-1}(\tilde{x}_i))}{F_1^k(x^{-1}(\tilde{x}_i))}F_1^k(x^{-1}(\tilde{x}_i)) - c\tilde{x}_i \qquad (8) \\
\Leftrightarrow &\max_{\tilde{x}_i} W_1 F_1^k(x^{-1}(\tilde{x}_i)) + zF_1^n(x^{-1}(\tilde{x}_i)) - c\tilde{x}_i.
\end{aligned}$$

Expression (8) gives an alternative interpretation of the expected payoff: *In the first round*, a contestant wins W_1, if he exerts the highest effort among k contestants in his sub-contest, and wins the second-round prize z, if he exerts the highest effort among all n contestants.[8]

Of course, all the above depends on the existence of a monotonic symmetric equilibrium, which we describe in the following proposition.

[8]This does not mean that the contestant wins the real prize W_2 for sure because in the second-round a mixed strategy is played and there is a positive probability that he loses even though he exerts the highest effort in the first round.

Proposition 2. *The equilibrium effort function in the first round of the two-stage contest is characterized by the following strictly decreasing function:*

$$x(c) = \int_c^1 \left(W_1 + \Pi(s) - \Pi'(s) \left[\frac{1 - F_{(1,k-1)}(s)}{f_{(1,k-1)}(s)} \right] \right) \frac{1}{s} dF_{(1,k-1)}(s), \quad (9)$$

where $\Pi(s)$ *and* $\Pi'(s)$ *are defined as in* (4) *and* (5) *respectively.*

Let us sum up. First, we derive the expected value of participating in the second-round for any-ability contestant. Then, given the expected value, we find the optimal first-round effort. Therefore, Lemma 1 and Proposition 2 characterize the monotonic symmetric subgame perfect equilibrium of the two-stage contest with ability revelation.

We devote the rest of this section to the two-stage contest without ability revelation. A thorough comparison of contests with and without ability revelation is in the next section.

3.2 No Ability Revelation

When the advancing contestants do not observe the first-round winners' abilities, our set-up reduces to the linear-cost case of Moldovanu and Sela (2006). In their two-stage contest, the advancing contestants update their beliefs only on the basis that their second-round rivals have all won in the first round. These updated beliefs tell the finalists that their costs are all coming from $G = F_{(1,k)}$, but they do not know the exact cost their rivals posses as they can only infer that a first-round winner is the one with the lowest cost among k first-round contestants. Hence, in the second round the incomplete information environment is preserved. Let $u_i : [m, 1] \to \mathbb{R}_+$ and $v_i : [m, 1] \to \mathbb{R}_+$ be contestant i's effort functions in the first and the second round of the contest respectively. The following proposition characterizes the equilibrium of the two-stage contest without ability revelation.

Proposition 3. *Without ability revelation, the equilibrium of the two-stage contest is characterized by a pair of effort functions,* (u, v)*, where*

$$u(c) = [W_1 + \Pi(c)] \int_c^1 \frac{1}{s} dF_{(1,k-1)}(s), \quad (10)$$

is the equilibrium effort function in the first round, and

$$v(c) = W_2 \int_c^1 \frac{1}{s} dG_{(1,t-1)}(s), \tag{11}$$

is the equilibrium effort function in the second round.

Note that the maximization problem associated with the no-revelation first-round equilibrium effort is

$$\max_{\tilde{u}} [W_1 + \Pi(c)] F_1^k(u^{-1}(\tilde{u})) - c\tilde{u}.$$

Comparing this to (7), we can see that they differ by the "pre-equilibrium" version of the expected value $\Pi(x^{-1}(\tilde{x}))$. It turns out that, with ability revelation, contestants have extra incentive to exert higher effort (given their ability types) in the first round.

Proposition 4. *The first-round individual effort is higher in a contest with ability revelation. In particular, $x(c) \geq u(c)$ for all $c \in [m, 1]$ with strict inequality except at $c = 1$. Furthermore, the difference is independent of the first-round prize W_1.*

If a contest designer can decide whether to reveal abilities after the first round, then since $x(c) - u(c)$ is independent of W_1 leads to the following result.

Corollary 1. *Suppose that there is a fixed amount of total prize ω to be allocated. Then, $W_1 = 0$ and $W_2 = \omega$ maximize the first-round effort difference between two-stage contests with and without performance revelation.*

The fact that the ability revelation contributes to a higher expected effort is only a partial result because we have not considered the effect on the second round. It turns out, as to be shown in the next section, a contrasting result is obtained for the second round.

4 Comparison of Contests with and without Ability Revelation

4.1 General Results

In this section, we compare contests with and without ability revelation. Our main finding is that a contest with ability revelation elicits higher total effort in the first round. On the other hand, it generates lower total effort in the second round.

Let us first turn to two lemmas which characterize the stage-wise total effort with and without ability revelation.

Lemma 2. *With ability revelation, the first-round total expected effort, $R_1^{(x,y)}$, and the second-round total expected effort, $R_2^{(x,y)}$, are*

$$R_1^{(x,y)} = n \int_m^1 \left[\int_c^1 \left(W_1 + \Pi(s) - \Pi'(s) \left[\frac{1 - F_{(1,k-1)}(s)}{f_{(1,k-1)}(s)} \right] \right) \frac{1}{s} dF_{(1,k-1)}(s) \right] dF(c), \tag{12}$$

and

$$R_2^{(x,y)} = \frac{W_2}{2} \left[\int_m^1 \frac{1}{c} dG_{(2,t)}(c) + \int_m^1 \int_m^c \frac{s}{c^2} g_{(1,2,t)}(s,c) ds dc \right], \tag{13}$$

where $g_{(1,2,t)}$ is the joint density of the first-order and second-order statistics among t random variables.

Lemma 3. *Without ability revelation, the first-round total expected effort, $R_1^{(u,v)}$, and the second-round total expected effort, $R_2^{(u,v)}$, are*

$$R_1^{(u,v)} = n \int_m^1 \left[(W_1 + \Pi(c)) \int_c^1 \frac{1}{s} dF_{(1,k-1)}(s) \right] dF(c), \tag{14}$$

and

$$R_2^{(u,v)} = W_2 \int_m^1 \frac{1}{c} dG_{(2,t)}(c). \tag{15}$$

The stage-wise comparison of the total expected effort gives the following results.

Proposition 5. *The first-round total expected effort is always higher with ability revelation, i.e.,* $R_1^{(x,y)} > R_2^{(u,v)}$*.*

A contrasting result is, however, obtained for the second round.

Proposition 6. *The second-round total expected effort is always lower with ability revelation, i.e.,* $R_2^{(u,v)} > R_2^{(x,y)}$*.*

Two numerical examples which conclude our comparisons demonstrate that the expected total two-round effort can be either higher or lower with the ability revelation.

4.2 Numerical Examples

We consider two numerical examples in this subsection. In both examples there are $n = 4$ contestants equally divided into $k = 2$ groups in the first round with $t = 2$ in each. Contestants' costs, c, are assumed to be drawn from the interval $[0.5, 1]$. The difference between the examples will be on the primitive distribution function F. We begin with the uniform distribution.

Example 1

Suppose that $F(c) = 2c - 1$, then

$$\begin{aligned} &F(c) = F_{(1,1)}(c) = 2c - 1, \\ &G(c) = G_{(1,1)}(c) = F_{(1,2)} = 2(2c-1) - (2c-1)^2, \\ &G_{(2,2)}(c) = [G(c)]^2 = 4(2c-1)^2 - 4(2c-1)^3 + (2c-1)^4, \\ &g_{(1,2,2)}(s,c) = 32[1-(2s-1)][1-(2c-1)]. \end{aligned}$$

And we have

$$\begin{aligned} R_1^{(x,y)} &= 2.45482W_1 + 0.26408W_2, & R_2^{(x,y)} &= 1.28347W_2, \\ R_1^{(u,v)} &= 2.45482W_1 + 0.22306W_2, & R_2^{(u,v)} &= 1.39560W_2. \end{aligned}$$

The stage-wise comparison is consistent with our theoretical prediction. Note that the total effort in the contest with ability revelation is

$$R^{(x,y)} = R_1^{(x,y)} + R_2^{(x,y)} = 2.45482W_1 + 1.54755W_2,$$

and the total effort in the contest without ability revelation is

$$R^{(u,v)} = R_1^{(u,v)} + R_1^{(u,v)} = 2.45482W_1 + 1.61866W_2.$$

Therefore, the difference is

$$R^{(u,v)} - R^{(x,y)} = 0.07111W_2 > 0.$$

Hence, as long as the total effort is concerned, no ability revelation dominates. The following example gives a contrasting result.

Example 2

Suppose that $F(c) = (2c-1)^8$, then

$$\begin{aligned}
&F(c) = F_{(1,1)}(c) = (2c-1)^8,\\
&G(c) = G_{(1,1)}(c) = F_{(1,2)} = 2(2c-1)^8 - (2c-1)^{16},\\
&G_{(2,2)}(c) = [G(c)]^2 = 4(2c-1)^{16} - 4(2c-1)^{24} + (2c-1)^{32},\\
&g_{(1,2,2)}(s,c) = 2048[(2s-1)^7 - (2s-1)^{15}][(2c-1)^7 - (2c-1)^{15}].
\end{aligned}$$

And we have

$$\begin{aligned}
R_1^{(x,y)} &= 2.06238W_1 + 0.06090W_2, & R_2^{(x,y)} &= 1.03594W_2,\\
R_1^{(u,v)} &= 2.06238W_1 + 0.02078W_2, & R_2^{(u,v)} &= 1.05711W_2.
\end{aligned}$$

The total effort in the contest with ability revelation is

$$R^{(x,y)} = R_1^{(x,y)} + R_2^{(x,y)} = 2.06238W_1 + 1.09684W_2,$$

and the total effort in the contest without ability revelation is

$$R^{(u,v)} = R_1^{(u,v)} + R_1^{(u,v)} = 2.06238W_1 + 1.07789W_2.$$

Therefore, the difference is

$$R^{(u,v)} - R^{(x,y)} = -0.01895W_2 < 0.$$

Thus, the total effort is higher with ability revelation in this example.

5 Concluding Remarks

This paper studies how ability revelation affects the effort spending in a two-stage elimination contest. Our focus has been on a full ability revelation. We derive the monotonic symmetric subgame perfect equilibrium and show that the first-round total expected effort is always higher with ability revelation while the opposite is true for the second-round total expected effort. Two numerical examples demonstrate that the two-round total expected effort can either be higher or lower with ability revelation.

Another open research question that we believe is worth pursuing is to consider other forms of ability revelation which preserve private information in the second round. The set of ability revelations under this category can include one which reveals the rank of the contestants in terms of their abilities. Another less informative feedback will be the announcement of some statistics such as the mean or median ability. This kind of ability revelation has been particularly important in schools where instructors or teachers reveal the mean or median scores after each test. We believe the framework here can be extended to study these other forms of information revelation in contests or tournaments. While the equilibrium derived in this paper is a separating equilibrium, a less informative revelation may give rise to a pooling equilibrium.

Appendix A. Order Statistics

This paper utilizes several results in the theory of order-statistics. Yet, the conventional notation will be different from those in the auction literature (e.g., Krishna, 2002). To avoid confusion, we spell out here our notation. Some useful results in order-statistics will also be included.[9]

Suppose Y_1, Y_2,...,Y_n are a set of identical and independently distributed random variables with an underlying distribution function F. We denote the corresponding order statistics as $Y_{(1,n)}$, $Y_{(2,n)}$,...,$Y_{(n,n)}$ where $Y_{(1,n)}$ is the first (lowest) order-statistics among all n random variables.[10] The corresponding distribution functions of the order statistics are denoted by $F_{(1,n)}$, $F_{(2,n)}$,...,$F_{(n,n)}$. These order-statistical distributions $F_{(k,n)}$, $k = 1, ..., n$ relate to the primitive distribution F in the following way:

$$\begin{aligned} F_{(k,n)}(y) &= \sum_{r=k}^{n} \binom{n}{r} [F(y)]^r [1 - F(y)]^{n-r} \\ &= \sum_{r=k}^{n} \frac{n!}{r!(n-r)!} [F(y)]^r [1 - F(y)]^{n-r} \end{aligned} \tag{A.1}$$

and the corresponding density function is

$$f_{(k,n)}(y) = \frac{n!}{(k-1)!(n-k)!} [F(y)]^{k-1} [1 - F(y)]^{n-k} f(y). \tag{A.2}$$

We denote $F_k^n(y)$, $1 \leq k \leq n$, the *probability* that for a fixed value y, there are $k-1$ random variables that are lower than y and there are $n-k$ random

[9]Since our paper extends Moldovanu and Sela (2006), to the convenience of the readers, we adopt similar notations as they do. Note that the order-statistical notations adopted in contest models of our sort, while different from those in the auction literature, are standard within statistics itself. For textbook reference on the subject, see, for example, David and Nagaraja (2003).

[10]That is, $Y_{(1,n)} \leq Y_{(2,n)} \leq ... \leq Y_{(n,n)}$. Note that because of this inequality relation, the order statistics are necessarily dependent in spite of independence of Y_i.

variables that are higher than y. Then,

$$F_k^n(y) = \binom{n-1}{k-1}[F(y)]^{k-1}[1-F(y)]^{n-k} = \frac{(n-1)!}{(k-1)!(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}. \tag{A.3}$$

The following relation, which follows immediately from the definitions above, is used frequently in our derivation:

$$F_1^n(y) \equiv 1 - F_{(1,n-1)}(y). \tag{A.4}$$

To interpret (A.4), note that according to our definition, $F_1^n(y)$ is the probability that there are $n-1$ random variables among n of them that are higher than y, or equivalently, the probability that y is the lowest among n random variables. $F_{1,n-1}(y)$ is the probability that there is *at least* one of the $n-1$ random variables that are less than y, and the complement of this event is exactly the event that is described by $F_1^n(y)$.

Finally, the joint density of two order statistics, $Y_{(i,n)}$ and $Y_{(j,n)}$, $f_{(i,j,n)}(x,y)$, where $1 \leq i < j \leq n$, relates to the primitives in the following way

$$f_{(i,j,n)}(x,y) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!}f(x)f(y)[F(x)]^{i-1} \times [F(y)-F(x)]^{j-1-i}[1-F(y)]^{n-j}. \tag{A.5}$$

Appendix B. Proofs

Proof of Proposition 1. The probability that a second-round contestant is the highest-ability contestant among all t finalists is given by

$$P\Big[c = \min_i\{c_i\}\Big] = [1-G(c)]^{t-1} \equiv G_1^t(c) \equiv 1 - G_{(1,t-1)}(c),$$

where we use relations (A.3) and (A.4) and $i \in \{1, 2, ..., t\}$ for the t finalists.

Conditioned on being the highest-ability contestant among all t finalists, a contestant with a realized cost c has a pre-revelation expected payoff of $E[W_2(1 - c/C_{(1,t-1)})|C_{(1,t-1)} > c]$, where $C_{(1,t-1)}$ is the lowest (first) order-statistics of the cost of $t-1$ remaining contestants. Since $C_{(1,t-1)}$ is an order statistics after the first-round winners are determined, the primitive distribution (i.e., the distribution of the unranked costs of the finalists) of $C_{(1,t-1)}$ is $G = F_{(1,k)}$. We then have

$$E\left[W_2\left(1-\frac{c}{C_{(1,t-1)}}\right)\middle|C_{(1,t-1)} > c\right] = W_2\left[\frac{1}{1-G_{(1,t-1)}(c)}\int_c^1\left(1-\frac{c}{s}\right)dG_{(1,t-1)}(s)\right].$$

Multiplying the probability with the conditional expectation, we obtain

$$\begin{aligned}\Pi(c) &= [1 - G_{(1,t-1)}(c)] \times W_2\left[\frac{1}{1-G_{(1,t-1)}(c)}\int_c^1\left(1-\frac{c}{s}\right)dG_{(1,t-1)}(s)\right]\\ &= W_2\int_c^1\left(1-\frac{c}{s}\right)dG_{(1,t-1)}(s).\end{aligned}$$

□

Proof of Proposition 2. The first-order condition of the maximization problem (7) is

$$[W_1 + \Pi(x^{-1}(\tilde{x}))]\frac{F_1^{k\prime}(x^{-1}(\tilde{x}))}{x'(x^{-1}(\tilde{x}))} + F_1^k(x^{-1}(\tilde{x}))\frac{\Pi'(x^{-1}(\tilde{x}))}{x'(x^{-1}(\tilde{x}))} = c.$$

At a symmetric equilibrium $x^{-1}(\tilde{x}) = c$, thus

$$[W_1 + \Pi(c)]\frac{F_1^{k\prime}(c)}{x'(c)} + F_1^k(c)\frac{\Pi'(c)}{x'(c)} = c.$$

Re-arranging terms, we obtain the following differential equation

$$x'(c) = \frac{[W_1 + \Pi(c)]F_1^{k\prime}(c) + \Pi'(c)F_1^k(c)}{c}. \tag{B.1}$$

Under the assumption that the equilibrium effort function is monotonic, a contestant with the lowest possible ability (i.e., highest cost) will never win the contest. His best response is therefore to exert zero effort, and this gives us the boundary condition $x(1) = 0$. Solving (B.1) with this boundary condition gives

$$\begin{aligned}x(c) &= -\int_c^1 \frac{[W_1 + \Pi(s)]F_1^{k\prime}(s) + \Pi'(s)F_1^k(s)}{s}ds\\ &= \int_c^1\left(W_1 + \Pi(s) - \Pi'(s)\left[\frac{1-F_{(1,k-1)}(s)}{f_{(1,k-1)}(s)}\right]\right)\frac{1}{s}dF_{(1,k-1)}(s),\end{aligned}$$

where relation (A.4) is used in the second line.

We proceed to prove that $x(c)$ is a strictly decreasing function of c. Note that

$$x'(c) = -\frac{1}{c}\Big(f_{(1,k-1)}(c)[W_1 + \Pi(c)] - \Pi'(c)[1 - F_{(1,k-1)}(c)]\Big).$$

Hence, $x'(c) < 0$ if and only if

$$f_{(1,k-1)}(c)[W_1 + \Pi(c)] > \Pi'(c)[1 - F_{(1,k-1)}(c)].$$

Since $\Pi'(c) < 0$ while all other terms are positive, the above relation always holds and hence the effort function $x(c)$ is strictly decreasing in c.

Finally, we show that the second-order condition is satisfied. Suppose all but contestant i exert first-round effort according to (9). We will demonstrate that, for any cost c_i contestant i maximizes his expected payoff by following the same first-round strategy $x(c)$. Let

$$\pi(\tilde{x}, c) = [W_1 + \Pi(x^{-1}(\tilde{x}))]F_1^k(x^{-1}(\tilde{x})) - c\tilde{x} \tag{B.2}$$

be the expected payoff of contestant i with cost c who exerts effort $\tilde{x}$. We show that the derivative $\pi_{\tilde{x}}(\tilde{x}, c)$ is non-negative for $\tilde{x} < x(c)$ and non-positive for $\tilde{x} > x(c)$ which imply $\pi(\tilde{x}, c)$ is maximized at $\tilde{x} = x(c)$. Differentiating (B.2) with respect to $\tilde{x}$:

$$\begin{aligned}\pi_{\tilde{x}}(\tilde{x}, c) = [W_1 + \Pi(x^{-1}(\tilde{x}))]F_1^{k\prime}(x^{-1}(\tilde{x}))\frac{dx^{-1}(\tilde{x})}{d\tilde{x}} \\ + F_1^k(x^{-1}(\tilde{x}))\Pi'(x^{-1}(\tilde{x}))\frac{dx^{-1}(\tilde{x})}{d\tilde{x}} - c.\end{aligned} \tag{B.3}$$

Further differentiating (B.3) with respect to c gives $\pi_{\tilde{x}c}(\tilde{x}, c) = -1 < 0$ which means that $\pi_{\tilde{x}}(\tilde{x}, \cdot)$ is decreasing in c.

Suppose $\tilde{x} < x(c)$ and denote $\tilde{c}$ to be the cost for which the equilibrium effort is $\tilde{x}$, i.e., $x(\tilde{c}) = \tilde{x}$. Since $x(\cdot)$ is a strictly decreasing, this implies that $\tilde{c} > c$. Then, $\pi_{\tilde{x}}(\tilde{x}, c) \geq \pi_{\tilde{x}}(\tilde{x}, \tilde{c})$. By definition, $x(\tilde{c}) = \tilde{x}$ implies that $\pi_{\tilde{x}}(\tilde{x}, \tilde{c}) = 0$. Hence, it follows that for $\tilde{x} < x(c)$, $\pi_{\tilde{x}}(\tilde{x}, c) \geq 0$. By similar argument, for $\tilde{x} > x(c)$, $\pi_{\tilde{x}}(\tilde{x}, c) \leq 0$. Hence, $\pi(\tilde{x}, c)$ is maximized at $\tilde{x} = x(c)$. □

Proof of Proposition 3. Moldovanu and Sela (2001, 2006) show that

$$u(c) = [W_1 + W_2 G_1^t(c) - cv(c)] \int_c^1 \frac{1}{s} dF_{(1,k-1)}(s). \tag{B.4}$$

Note that using relation (A.4), we have

$$\begin{aligned} \Pi(c) &= W_2 \int_c^1 \left(1 - \frac{c}{s}\right) dG_{(1,t-1)}(s) \\ &\equiv W_2 G_1^t(c) - cW_2 \int_c^1 \left(\frac{1}{s}\right) dG_{(1,t-1)}(s). \end{aligned} \tag{B.5}$$

Substituting (11) into (B.4) and then using (B.5), we have

$$u(c) = [W_1 + \Pi(c)] \int_c^1 \frac{1}{s} dF_{(1,k-1)}(s).$$

□

Proof of Proposition 4. Note that

$$x(c) = -\int_c^1 x'(s) ds = \int_c^1 \left(\frac{1}{s} f_{(1,k-1)}(s)[W_1 + \Pi(s)] - \Pi'(s) \left[\frac{1 - F_{(1,k-1)}(s)}{s} \right] \right) ds, \tag{B.6}$$

and,

$$u(c) = -\int_c^1 u'(s) ds = \int_c^1 \left[\frac{1}{s} f_{(1,k-1)}(s)[W_1 + \Pi(s)] - \Pi'(s) \int_s^1 \frac{1}{\tau} dF_{(1,k-1)}(\tau) \right] ds. \tag{B.7}$$

Thus, the comparison reduces to the one between the two integrands in (B.6) and (B.7). Denote their difference by $\overline{\Delta}$. Then,

$$\overline{\Delta} = \Pi'(s) \left[\int_s^1 \frac{1}{\tau} dF_{(1,k-1)}(\tau) - \int_s^1 \frac{1}{s} dF_{(1,k-1)}(\tau) \right] > 0,$$

since $\Pi'(s) < 0$, and $1/\tau \leq 1/s$ for all $\tau \in [s, 1]$. Hence, $x(c) \geq u(c)$ for all $c \in [m, 1]$. □

Proof of Corollary 1. The result follows immediately from the fact that the difference in first-round total effort with and without ability revelation is independent of W_1. □

Proof of Lemma 2. We denote finalist 1 and finalist 2 as in Lemma 1. Note that the probability distribution associated with finalist 1's mixed strategy, $B_1(y)$, is uniformly distributed on the interval $[0, W_2/c_2]$. Hence, his post-revelation expected effort is $W_2/2c_2$. Similarly argument is applied to finalist 2. His unconditional post-revelation expected effort is $c_1/c_2 \times W_2/2c_2$. Hence, the total post-revelation expected effort in the second round is

$$\frac{W_2}{2}\left[\frac{1}{c_2} + \frac{c_1}{(c_2)^2}\right].$$

Here, c_1 and c_2 are the realized values of the first-order and second-order statistics from the primitive distribution G of the second round. Hence, the (pre-revelation) total expected effort in the second round is

$$R_2^{(x,y)} = E\left[\frac{W_2}{2}\left(\frac{1}{C_{(2,t)}} + \frac{C_{(1,t)}}{(C_{(2,t)})^2}\right)\right].$$

Note that $[1/C_{(2,t)} + C_{(1,t)}/(C_{(2,t)})^2]$ is a function of the first-order and second-order statistics $C_{(1,t)}$ and $C_{(2,t)}$, and they are not independent. Thus, the relevant distribution with which the expectation is formed is the joint density of them, $g_{(1,2,t)}(\cdot,\cdot)$. We then have

$$R_2^{(x,y)} = \frac{W_2}{2}\left[\int_m^1 \int_m^c \left(\frac{1}{c} + \frac{s}{c^2}\right) g_{(1,2,t)}(s,c)dsdc\right]. \tag{B.8}$$

Using relation (A.5), the joint density $g_{(1,2,t)}$ relates to the primitive distribution G and the density function thereof, g, in the following way:

$$g_{(1,2,t)}(s,c) = t(t-1)g(s)g(c)[1-G(c)]^{t-2}. \tag{B.9}$$

With (B.9), we have

$$\int_m^1 \int_m^c \frac{1}{c} g_{(1,2,t)}(s,c)dsdc = \int_m^1 \left[\int_m^c \frac{1}{c} t(t-1)g(s)g(c)[1-G(c)]^{t-2}ds\right]dc$$
$$= \int_t^1 \frac{1}{c} t(t-1)g(c)[1-G(c)]^{t-2}\left[\int_m^c dG(s)\right]dc = \int_t^1 \frac{1}{c} dG_{(2,t)}(c).$$

□

Proof of Lemma 3. It follows from Moldovanu and Sela (2006). □

Proof of Proposition 5. The result follows immediately from Proposition 4. □

Proof of Proposition 6. Let

$$\Delta = \int_m^1 \frac{1}{c} dG_{(2,t)}(c) - \int_m^1 \int_m^c \frac{s}{c^2} g_{(1,2,t)}(s,c) ds dc.$$

Reversing the steps in the proof of Lemma 2, we have

$$\begin{aligned}\Delta &= \int_m^1 \int_m^c \left(\frac{1}{c} - \frac{s}{c^2}\right) g_{(1,2,t)}(s,c) ds dc \\ &= \int_m^1 \left(\frac{1}{c^2}\right) t(t-1) g(c)[1-G(c)]^{t-2} \left[\int_m^c (c-s) dG(s)\right] dc.\end{aligned}$$

Clearly, $(c-s) > 0$ for all $s \in [m, c)$ while $(c-s) = 0$ at $s = c$. Hence,

$$\int_m^c (c-s) dG(s) > 0.$$

Since all other terms in the integral are positive, we have $\Delta > 0$. It immediately follows that

$$\begin{aligned}R_2^{(u,v)} &= W_2 \int_m^1 \frac{1}{c} dG_{(2,t)}(c) = \frac{W_2}{2} \int_m^1 \frac{1}{c} dG_{(2,t)}(c) + \frac{W_2}{2} \int_m^1 \frac{1}{c} dG_{(2,t)}(c) \\ &> \frac{W_2}{2} \int_m^1 \frac{1}{c} dG_{(2,t)}(c) + \frac{W_2}{2} \int_m^1 \int_m^c \frac{s}{c^2} g_{(1,2,t)}(s,c) ds dc = R_2^{(x,y)}.\end{aligned}$$

□

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