Elimination Tournaments where Players Have Fixed Resources

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Abstract. We consider T-round elimination tournaments where players have fixed equal resources instead of cost functions. We show that players always spend more resources in the initial than in the following rounds. The winner-take-all prize scheme and the same number of competitors in each group in each round ensures equal resources allocation across all rounds. Applications for elections and sports are discussed.

Key words: Elimination tournaments; Fixed resources; Elections.
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1. Introduction

We observe tournaments every day. Euro and World Soccer Cups, Super Bowl, American president elections, Summer Olympic Games are just a few examples of tournaments. Over the years, research tournaments have been playing the leading role in the procurement of many innovations. See Taylor (1995), Fullerton and McAfee (1999), Fullerton, Linster, McKee and Slate (2002), Che and Gale (2003) for examples of various research tournaments.

The tournament literature has focused theoretically, see, for example, Lazear and Rosen (1981), Rosen (1981, 1986), Moldovanu and Sela (2001, 2006), Lai and Matros (2006) and empirically, Ehrenberg and Bognanno (1990), Knoeber and Thurman (1994), on players' incentives in tournaments when players have some costs for exerting effort.

This paper analyzes T-round elimination tournaments where players have fixed *equal* resources (budgets for election campaigns, energy and ideas in career games, energy in football tournaments, novelties in chess tournaments, etc.) instead of cost functions and the success function which determines the winner of each round is stochastic. We assume that each player can use her resources only in the tournament and cannot cash it outside. Each player has to allocate her resources optimally across T rounds, given that if she loses one round, she is out. There is a trade off here. On the one hand, the more resources a player spends in a particular round the higher her chance is to win in this round. On the other hand, the player has less chances to win in the following rounds.

We assume that the design - a T-round elimination tournament - is given and focus on the optimal allocation of players into groups and the optimal prize structure, if the designer wants to obtain the best performance in the final, "quality of play," or the highest total spending. In order to solve the designer problem, we first find how players allocate their resources in the symmetric equilibrium. It turns out that players *always* spend (weakly) more resources in the initial than in the following rounds. However, the designer has a tournament design which ensures equal resources allocation across all rounds and provides incentives for players not to lay down on the top. The designer should implement the winner-take-all prize scheme and have the same number of competitors in each group in each round. This tournament design gives the best performance (the highest spending) in the final.

We also show that the tournament designer can influence individual spending by varying the size of the groups in rounds: increasing the size of the group intensifies the competition and increases the individual spending in this round.

Several real-world phenomena in politics, science and sports have the structure of the tournaments with fixed resources. Consider the problem of Major League baseball teams in the playoffs. The playoffs are a multi-stage competition, and only winners in the first round of the playoffs advance to the World Series. The teams must determine which pitchers to start in the first round of the playoffs. A starting pitcher needs several days of rest, so that if a pitcher pitches in the first round, he might not be available for the second round. The problem that the team faces is how to allocate its pitching resources across different rounds. A coach of a soccer team faces an analogous problem in the elimination part of the World Soccer Cup where teams have 22 players for all matches. Alternatively, consider multi-stage elections where candidates have fixed total budgets for their campaign expenditure on advertising.

The main assumption of this paper is that the players have equal (symmetric) resources. Even though players in general are asymmetric, there are many examples when a designer of the tournament wants to create a symmetric environment. This situation arrises if each candidate has the same budget in elections, or if there is a handicap in a sport tournament. For example, Che and Gale (2003) show that if contestants are asymmetric it is optimal to handicap the most efficient one. Gavious, Moldovanu, and Sela (2002) analyze different bidding caps in detail.

President elections in France and Russia have typically two rounds. A candidate may not exceed a campaign spending ceiling of 90 million francs for the first round, and 120 million francs for the second round in France.[1] There is an overall ceiling on the campaign expenditure of each candidate of 250 million roubles, or 300 million roubles if the candidate makes it into the second round in Russia.[2] A typical candidates does not have enough resources to reach the campaign expenditure even for the first round. Therefore such a candidate has to make his allocation decision for the two rounds in advance. Similar situation is in Hong Kong, where the government specifies that

> 15.6. The maximum amount of election expenses for the Election will be prescribed by the Maximum Amount of Election Expenses (Chief Executive) Regulation made by the Chief Executive in Council pursuant to s 45 of the ECICO to limit the maximum amount of expenses a candidate may incur on account of the Election. This limit controls the extent of election campaigns and serves to prevent candidates with ample financial resources from having an unfair advantage.[3]

The salary cap in NBA is supposed to play the same role: to handicap the richest teams. More discussion about salary caps can be found in Fort (2003). Another

[1] See http://www.elysee.fr/elysee/anglais/the_president/his_function/6_questions_about_the_president/6_questions_about_the_president.20030.html

[2] See http://www.russiavotes.org/electorallawchange3.htm

[3] See http://www.info.gov.hk/archive/consult/2001/reo/ce_chap15.doc

example of this kind is some chess tournaments in Sweden when stronger players are handicapped and given less time to play than their weaker opponents.[4] France uses handicaps in horse races.[5,6]

The strategic problem in elimination tournaments with fixed resources is different from the one analyzed in multi-object auctions with budget constraint (see, for example, Benoit and Krishna, 2001) because players in elimination tournaments do not have incentives to deplete the budgets of their rivals in the first round since all losers of this round are eliminated before the second round. It is also different from the problem examined in the contest literature where players have to decide how much effort is needed to win the prize(s) in one contest; see, for example, Dixit (1987, 1999), Baik and Shogren (1992), Baye and Shin (1999), and Moldovanu and Sela (2001).

Krishna and Morgan (1998) and Moldovanu and Sela (2001) show that the winner-take-all prize scheme is often the optimal one for the designer who wants to maximize the total effort of the players in the one-stage tournament when players have cost functions and deterministic success functions. Classical papers Lazear and Rosen (1981) and Rosen (1986) analyze stochastic success functions and show that high differences in prizes in the last round(s) (we obtain the extreme variation of this result: winner-take-all prize scheme) have to provide enough incentives for players to insert the same effort in all rounds.

Moldovanu and Sela (2006) analyze a deterministic model with cost functions. They show that it is optimal to split players in two groups in the first round and to have a final between two winners, if the designer maximizes the expected highest effort.

The rest of the paper is organized as follows. We consider the model and results in Section 2. Section 3 provides a discussion. All proofs can be found in the Appendix.

2. The Model

Consider a T-round elimination tournament where all players have *equal resources.* We assume that there are $n_1 \times ... \times n_T$ risk-neutral players. In round 1, all players are divided into $n_2 \times ... \times n_T \geq 2$ groups with $n_1 \geq 2$ players in each group. Each group determines the winner according to the players' spending in this round. $n_2 \times$

[4] One of such tournaments can be found at http://lass.no-ip.com/lass/ss_index.htm.

[5] Handicapping is simply the additional weight handicap (penalty), measured in Pounds (lbs), that is awarded to a horse that has had more wins relative to other horses in a particular race. The handicap will slow a horse down slightly and an allowance will benefit a horse relative to the rest of the field. This makes a race more competitive. Handicapping in this manner, is an age-old racing tradition that works in real racing and now in virtual racing, where it is part of an owners racing strategy. The base weight in each race on which penalties and allowances are applied is 120lbs. This is the combined base weight of the jockey, saddle and equipment on the horse.

[6] See http://www.digiturf.com/Betting/Handicaps.asp

$... \times n_T$ winners of the first round compete in round 2. All losers of the first round - $(n_1 - 1)\, n_2 \times ... \times n_T$ players - get prize W_0 (they have 0 victories) and are eliminated from the tournament at the end of the first round. In round 2, the winners of the first round are divided into $n_3 \times ... \times n_T \geq 2$ groups with $n_2 \geq 2$ players in each group. Each group determines the winner according to the players' spending in this round. Only $n_3 \times ... \times n_T$ winners of the second round compete in round 3. All losers of the second round - $(n_2 - 1)\, n_3 \times ... \times n_T$ players - get prize W_1 (they have 1 victory) and are eliminated from the tournament at the end of the second round, and so on. Finally, in round T, only n_T players remain. The winner of the final (she has all T victories) gets prize W_T and the losers (they have $T-1$ victories) receive prize W_{T-1}.

We make the standard assumption that prizes increase from round to round:

$$\text{Assumption 1: } W_T \geq W_{T-1} \geq ... \geq W_0 \geq 0. \tag{1}$$

Each player has an initial fixed resource E and must decide how to allocate this resource across all T rounds. We assume that players *cannot* use their resources outside the tournament. Denote the spent part of player i's resource in round k by x_k^i. If player i chooses to use the part $x_k^i \in [0, E]$ of her resource in round k, when her $(n_k - 1)$ opponents in round k choose $\left(x_k^1, ..., x_k^{i-1}, x_k^{i+1}, ..., x_k^{n_k}\right) \in [0, E]^{n_k - 1}$, then player i wins in round k with probability

$$p_k^i \left(x_k^1, ..., x_k^{n_k}\right) = \frac{f\left(x_k^i\right)}{f\left(x_k^i\right) + \sum_{j=1, j\neq i}^{n_k} f\left(x_k^j\right)}, \tag{2}$$

where $f(x)$ is a positive and increasing function:

$$\text{Assumption 2: } f(x) > 0,\ f'(x) > 0 \text{ on the interval } [0, E]. \tag{3}$$

A pure strategy for player i is a rule $(x_1^i, ..., x_T^i)$, which assigns a part of her resource for every round in the tournament such that $\sum_{k=1}^{T} x_k^i = E$, $x_k^i \geq 0$, for any $i \in \{1, ..., n_1 \times ... \times n_T\}$ and $k \in \{1, ..., T\}$. Note that we assume that all players have to specify their resource allocation for all rounds *before* the first round. This assumption allows to concentrate on the resource allocation when players *cannot* signal anything to their opponents. This situation can arise in sports events, if a team have directions for the whole tournament before the first round and cannot communicate with its coach during the tournament, or/and the tournament design might be such that players cannot receive any feedback about their potential opponents.[7] Moreover, a

[7] Lai and Matros (2006) analyze 2-round elimination tournaments where players can signal their abilities. Players have cost functions instead of fixed resources in their model.

coach has to make his pitching resources allocation across different rounds *before* the first round in playoffs in baseball.

Our framework is similar to the famous Blotto games. See, for example, Borel (1953), Blackett (1958), Kvasov (2006) among other. There are two players in a Blotto game. Each player has a fixed resource which she has to allocate simultaneously for all (typically three) battles. A winner in a battle is the player who allocates more resources (deterministic success function) for this battle. For stochastic Blotto games see Matros (2006).

Given the opponents' resource allocation $(x^1, ..., x^{i-1}, x^{i+1}, ..., x^{n_1 ... n_T})$, player i's allocation decision $x^i = (x_1, ..., x_T)$ is determined by the solution of:

$$\max_{x_1,...,x_T} \left\{ \left[1 - P_1\left(x_1^1, ..., x_1^{n_1}\right)\right] W_0 + P_1\left(x_1^1, ..., x_1^{n_1}\right) \left(\left[1 - P_2\left(x_2^1, ..., x_2^{n_2}\right)\right] W_1 + ... + \right.\right.$$

$$\left.\left. ... + P_{T-1}\left(x_{T-1}^1, ..., x_{T-1}^{n_{T-1}}\right) \left(\left[1 - P_T\left(x_T^1, ..., x_T^{n_T}\right)\right] W_{T-1} + P_T\left(x_T^1, ..., x_T^{n_1}\right) W_T\right)\right\}, \tag{4}$$

$$s.t. \sum_{k=1}^{T} x_k = E, \; x_k \geq 0, \tag{5}$$

where P_k is the expected probability of success in round k. This is a weighted average over all possible opponents in round k.

2.1. Existence of the symmetric equilibrium. First we show that there exists a symmetric Nash equilibrium in pure strategies. The properties of the symmetric equilibrium are analyzed after that. The T-round elimination tournament is a symmetric game with $n_1 \times ... \times n_T$ players and at least one symmetric equilibrium, which follows from an application of the Kakutani's Fixed-Point Theorem.

Proposition 1. *Suppose that assumptions* (1), (3) *hold, then the T-round elimination tournament has at least one symmetric equilibrium in pure strategies.*

2.2. Properties of the symmetric equilibrium. Since Proposition 1 establishes the existence of the symmetric equilibrium in pure strategies, properties of this equilibrium can be analyzed. Let $(x_1^*, ..., x_T^*)$ be a symmetric equilibrium, where x_k is a part of the endowment every player spends in round k. It will be shown that a symmetric equilibrium in pure strategies is unique if function $f(x)$ is strictly logconcave (not very convex):

$$\text{Assumption 3: } [\ln f(x)]' > 0 \text{ and } [\ln f(x)]'' < 0 \text{ on the interval } [0, E]. \tag{6}$$

Note that concave and linear functions belong to this class.

We will also assume that the competition is weakly decreasing from round to round:

$$\text{Assumption 4: } n_1 \geq ... \geq n_T. \tag{7}$$

This assumption holds usually in elections, different career games, and most sports events.

Theorem 1. *Suppose that assumptions* (1), (3), (6), *and* (7) *hold. Then, in the symmetric equilibrium* $(x_1^*, ..., x_T^*)$,

$$x_1^* \geq ... \geq x_T^*,$$

for any prize structure $(W_0, W_1, ..., W_T)$.

Theorem 1 shows that players *always* spend more resources in the initial than in the following rounds in the symmetric equilibrium.

We will call the following prize structure

$$W_T > W_{T-1} = W_{T-2} = ... = W_0 \geq 0,$$

a winner-take-all one, and

$$W_T = W_{T-1} = ... = W_1 > W_0 \geq 0$$

a "grand-contest" one. It is obvious that players spend all their resources in the first round in the grand-contest case, because all prizes are the same after the first round and there are no incentives to keep resources more than one round. This tournament design is equivalent to the one Grand contest where $n_2 \times ... \times n_T$ best players all receive the same prize, W_1.

Proposition 2. *Suppose that the prize scheme is a "grand-contest" one. Then, in the symmetric equilibrium* $(x_1^*, ..., x_T^*)$,

$$x_1^* = E,\ x_2^* = ... = x_T^* = 0.$$

Corollary 1. *The highest total spending is achieved, if the prize scheme is a "grand-contest" one.*

Even though a tournament designer often wants to maximize the total spending, in most career games the designer cares about "quality of play" as the game proceeds through its rounds. The quality of the competition on the very top might be especially important for the performance of organizations. Theorem 1 shows that we can *never* have more resources on the top than on the bottom in the symmetric equilibrium. However, it is important to find out what the highest performance can be on the top. The following proposition answers this question.

Proposition 3. *Suppose that assumptions* (1), (3), (6), *and* (7) *hold. Then, in the symmetric equilibrium* $(x_1^*, ..., x_T^*)$, *equal resource allocation across all rounds,* $x_1^* = ... = x_T^*$, *takes place if and only if (i) the prize scheme is the winner-take-all and (ii)* $n_1 = ... = n_T$.

Proposition 3 shows that the extreme reward concentration on the very top has to be in elimination tournaments in order to obtain "the quality of play" across all rounds. This result is consistent with Rosen (1986). Another important result is that the prize scheme is just one of the two requirements. Organizations has to have "proportional" structure - equal number of competitors on each layer, for "the quality of play."

Corollary 2. *The highest performance in the final is achieved if (i) the prize scheme is the winner-take-all and (ii)* $n_1 = ... = n_T$.

Corollary 2 is in contrast with Krishna and Morgan (1998) and Moldovanu and Sela (2001). They show that the winner-take-all prize scheme is the optimal one for the designer who wants to maximize the total effort.

The following proposition demonstrates that the tournament designer can influence the individual equilibrium resource allocation by varying the size of the groups in rounds. The result is very intuitive: higher competition (more players in each group) leads to higher individual spending in this round.

Proposition 4. *Suppose that assumptions* (1), (3), (6), *and* (7) *hold. Then, in the symmetric equilibrium* $(x_1^*, ..., x_T^*)$, x_k^* *increases if* n_k *increases.*

Assumptions (1) and (3) are common and we will illustrate the role of assumption (6) by the following example, where function $f(x)$ is "very convex".

Example. Suppose that there are two rounds, $T = 2$; $n_1 = n_2 = 2$; the resource is equal to one, $E = 1$; $f(x) = e^{(x+1)^2}$; and the following prize structure, $W_0 = 0$, $W_1 = 1$ and $W_2 = 3$. Note that $f'(x) = 2(x+1)e^{(x+1)^2}$. From Proposition 1, there exists a symmetric equilibrium (x_1, x_2). Condition (12) (see the Appendix) becomes

$$2(x_1 + 1) = (2 - x_1)$$

or

$$x_1 = 0.$$

Hence, in the symmetric equilibrium $(x_1^*, x_2^*) = (0, 1)$, every player spends *all resource* in the final round.

3. Discussion

We consider T-round elimination tournaments where risk-neutral players have fixed resources. The symmetric equilibrium in pure strategies is shown to exist and be unique. Moreover, in this equilibrium, all players spend most of their resources in round 1 and least in the last round, T. The intuition is straightforward; the expected payoffs are much higher in round 1 than in all other rounds.

We show that the winner-take-all prize structure and the same number of competitors in each group in each round guarantees equal resource allocation across all rounds. Our result is another explanation for the reward concentration on the very top in organizations. This is consistent with Rosen (1986), who shows that prizes must increase over rounds to provide enough incentives for players to exert the same effort in every round, if players have a trade off between costs and expected high future payoffs.

We show that the designer can also influence individual spending by varying the size of the groups in rounds: increasing the size of the group intensifies the competition and increases the individual spending in this round.

There are several interesting topics for the future research. It deserves a lot of attention to look at elimination tournaments where players can get feedback from previous rounds *before* making allocation decision for the current round. It will be interesting to analyze elimination tournaments where players can have asymmetric resources. Lai and Matros (2006) analyze 2-round elimination tournaments where players have cost functions and can have different abilities. In their model, players signal their abilities in the first round.

Although elimination tournaments are usually associated with sports: tennis, soccer, chess and so on, there are many applications for a hierarchy in a firm, career games, and election campaigns. Some work has been done to test prediction of Lazear and Rosen (1981) theory, see for example Ehrenberg and Bognanno (1990) and Knoeber and Thurman (1994). It will be interesting to test the relationship between prizes/relative prizes and allocation of players' resources in experimental and real-life frameworks.

4. Appendix

Proof of Proposition 1. The proof is a generalization of the result for the two-player symmetric games in Weibull (Proposition 1.5, 1995). The set of all pure strategies for player i is a T dimensional simplex Δ, where vertex k of the simplex is a strategy where the whole endowment E is spent in round k. Simplex Δ is non-empty, convex, and compact. Fix all players but player i, and denote these players as $-i$. Suppose that players $-i$ can only choose the same strategy $x \in \Delta$, which is the diagonal in the simplex $\Delta^{n_1 \times \ldots \times n_T - 1}$. This diagonal is the simplex Δ itself. The best reply correspondence $\beta^i(x,...,x) = \beta^i(x)$ of player i to the same strategies for players $-i$ is upper hemi-continuous. Moreover, $\beta^i(x) \subset \Delta$ is convex and closed. By the Kakutani's theorem, there exists at least one fixed point: $x^* \in \beta(x^*)$, $x^* \in \Delta$. This is true for any player i and leads to the statement of the proposition. **End of proof.**

Proof of Theorem 1. Note that $g(l) = \frac{l-1}{l}$ is an increasing function of l. So,

$$\frac{k-1}{k} > \frac{j-1}{j}, \text{ for any } k > j,$$

or

$$\frac{k-1}{k}\frac{j}{j-1} > 1, \text{ for any } k > l.$$

From assumption (7)

$$\frac{n_j-1}{n_j}\frac{n_k}{n_k-1} > 1, \text{ if } k > j. \tag{8}$$

Given the opponents' symmetric resource allocation $(y_1,...,y_T)$, the player's allocation decision is determined by the solution of

$$\max_{x_1,...,x_{T-1}} \left\{ \frac{(n_1-1)f(y_1)}{f(x_1)+(n_1-1)f(y_1)}W_0 + \right.$$

$$\frac{f(x_1)}{f(x_1)+(n_1-1)f(y_1)}\left[\frac{(n_2-1)f(y_2)}{f(x_2)+(n_2-1)f(y_2)}W_1+...+\right.$$

$$...+\frac{f(x_{T-1})}{f(x_{T-1})+(n_{T-1}-1)f(y_{T-1})}\times$$

$$\left[\frac{(n_T-1)f(E-y_{T-1}-...-y_1)}{f(E-x_{T-1}-...-x_1)+(n_T-1)f(E-y_{T-1}-...-y_1)}W_{T-1}+\right.$$

$$\left.\left.\left.\frac{f(E-x_{T-1}-...-x_1)}{f(E-x_{T-1}-...-x_1)+(n_T-1)f(E-y_{T-1}-...-y_1)}W_T\right]\right]\right\}, \tag{9}$$

$$s.t.\ x_k \geq 0, \text{ for any } k = 1,...,T-1. \tag{10}$$

The first order condition for the problem (9) − (10) is

$$\frac{f(x_1)}{f(x_1)+(n_1-1)f(y_1)}\times...\frac{f(x_{k-1})}{f(x_{k-1})+(n_{k-1}-1)f(y_{k-1})}\times$$

$$\frac{(n_k-1)\,f'(x_k)\,f(y_k)}{[f(x_k)+(n_k-1)\,f(y_k)]^2}\left[-W_{k-1}+\left(\frac{(n_{k+1}-1)\,f(y_{k+1})}{f(x_{k+1})+(n_{k+1}-1)\,f(y_{k+1})}W_{k+1}+...\right)\right]+$$
$$\frac{f(x_1)}{f(x_1)+(n_1-1)\,f(y_1)}\times...\frac{f(x_{k-1})}{f(x_{k-1})+(n_{k-1}-1)\,f(y_{k-1})}\times$$
$$\frac{f(x_k)}{f(x_k)+(n_k-1)\,f(y_k)}\times...\times\frac{f(x_{T-1})}{f(x_{T-1})+(n_{T-1}-1)\,f(y_{T-1})}\times$$
$$\left[\frac{-(n_T-1)\,f'(E-x_{T-1}-...-x_1)\,f(E-y_{T-1}-...-y_1)}{[f(E-x_{T-1}-...-x_1)+(n_T-1)\,f(E-y_{T-1}-...-y_1)]^2}(W_T-W_{T-1})\right]=0, \tag{11}$$

for any $k=1,...,T-1$. In a symmetric equilibrium, $x_{T-1}=y_{T-1},...,x_1=y_1$, and the first order condition (11) becomes

$$\frac{n_k-1}{n_k^2}\frac{f'(x_k)}{f(x_k)}\left[-W_{k-1}+\frac{n_{k+1}-1}{n_{k+1}}W_k+\frac{1}{n_{k+1}}\frac{n_{k+2}-1}{n_{k+2}}W_{k+1}+...+\frac{1}{n_{k+1}}\times...\times\frac{1}{n_T}W_T\right]=$$
$$\left(\frac{1}{n_k}\times...\times\frac{1}{n_{T-1}}\right)\frac{n_T-1}{n_T^2}\frac{f'(E-x_{T-1}-...-x_1)}{f(E-x_{T-1}-...-x_1)}(W_T-W_{T-1}).$$

Finally, we get

$$\frac{n_k-1}{n_k}\frac{f'(x_k)}{f(x_k)}\times$$
$$[(n_{k+1}-1)\,n_{k+2}\times...\times n_T\,(W_k-W_{k-1})+...+(n_T-1)\,(W_{T-1}-W_{k-1})+(W_T-W_{k-1})]=$$
$$=\frac{n_T-1}{n_T}\frac{f'(E-x_{T-1}-...-x_1)}{f(E-x_{T-1}-...-x_1)}(W_T-W_{T-1})\,,\text{ for any }k=1,...,T-1. \tag{12}$$

Assumption (6) guarantees that the left-hand side (LHS) in equation (12) is a strictly decreasing function of x_k on the interval $[0,E]$, and the right-hand side (RHS) in the same equation is a strictly increasing function of x_k on the interval $[0,E]$. It follows from the fact that f'/f is a strictly decreasing function since $f''f-[f']^2<0$, which is a corollary of the assumption (6).

The existence of a symmetric equilibrium in pure strategies follows from Proposition 1. Hence, equation (12) either has no solution and $x_T^*=0$ in a unique pure strategy symmetric equilibrium or has a unique solution x_k^* inside of the interval $(0,E)$, since it defines the intersection of a decreasing and an increasing continuous functions.

Denote $x=x_{T-1}+...+x_{k+1}+x_{k-1}+...+x_1$. Player i has to allocate her resource part $(E-x)$ between round k and the last round T. Note that $x_k\geq x_T$ if and only if $LHS\left(\frac{E-x}{2}\right) \geq RHS\left(\frac{E-x}{2}\right)$. If x_k is equal to $\frac{E-x}{2}$, or the resource parts in period k and the last period are equal, then

$$LHS\left(\frac{E-x}{2}\right) = \frac{n_k-1}{n_k}\frac{f'\left(\frac{E-x}{2}\right)}{f\left(\frac{E-x}{2}\right)}\times$$

$$\left[\left(n_{k+1}-1\right)n_{k+2}\times...\times n_T\left(W_k-W_{k-1}\right)+...+\left(n_T-1\right)\left(W_{T-1}-W_{k-1}\right)+\left(W_T-W_{k-1}\right)\right]$$

and

$$RHS\left(\frac{E-x}{2}\right) = \frac{n_T-1}{n_T}\frac{f'\left(\frac{E-x}{2}\right)}{f\left(\frac{E-x}{2}\right)}\left(W_T-W_{T-1}\right).$$

Note that from (8) and the assumption (1)

$$\frac{n_k-1}{n_k}\times$$

$$\left[\left(n_{k+1}-1\right)n_{k+2}\times...\times n_T\left(W_k-W_{k-1}\right)+...+\left(n_T-1\right)\left(W_{T-1}-W_{k-1}\right)+\left(W_T-W_{k-1}\right)\right]=$$

$$\frac{n_k-1}{n_k}\left(W_T-W_{T-1}\right)+$$

$$\frac{n_k-1}{n_k}\left[\left(n_{k+1}-1\right)n_{k+2}\times...\times n_T\left(W_k-W_{k-1}\right)+...+n_T\left(W_{T-1}-W_{k-1}\right)\right]\geq$$

$$\frac{n_T-1}{n_T}\left(W_T-W_{T-1}\right).$$

Hence, in the symmetric equilibrium $x_k \geq x_T$ for any $k = 1, ..., T-1$ and *any prize scheme* $(W_0, W_1, ..., W_T)$.

Since the right-hand side in the equation (12) is the same for any $k = 1, ..., T-1$, then

$$\frac{n_k-1}{n_k}\frac{f'\left(x_k\right)}{f\left(x_k\right)}\times$$

$$\left[\left(n_{k+1}-1\right)n_{k+2}\times...\times n_T\left(W_k-W_{k-1}\right)+...+\left(n_T-1\right)\left(W_{T-1}-W_{k-1}\right)+\left(W_T-W_{k-1}\right)\right]=$$

$$\frac{n_j-1}{n_j}\frac{f'\left(x_j\right)}{f\left(x_j\right)}\times$$

$$\left[\left(n_{j+1}-1\right)n_{j+2}\times...\times n_T\left(W_j-W_{j-1}\right)+...+\left(n_T-1\right)\left(W_{T-1}-W_{j-1}\right)+\left(W_T-W_{j-1}\right)\right].$$

If $k > j$, then

$$\frac{n_k-1}{n_k}\frac{f'\left(x_k\right)}{f\left(x_k\right)}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] =$$
$$\frac{n_j - 1}{n_j}\frac{f'(x_j)}{f(x_j)}\times$$
$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] +$$
$$\frac{n_j - 1}{n_j}\frac{f'(x_j)}{f(x_j)}\,[(n_{j+1}-1)\,n_{j+2}\times ... \times n_T\,(W_j - W_{j-1}) + ...$$
$$... + (n_k - 1)\,n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1}) + n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1})]\,. \tag{13}$$

Hence from (8) and the assumption (1), in the symmetric equilibrium $x_j \geq x_k$, for any $k > j$. Therefore, the resource allocation in the symmetric equilibrium must be $x_1 \geq x_2 \geq ... \geq x_T$, for *any prize scheme* $(W_0, W_1, ..., W_T)$. **End of proof.**

Proof of Proposition 2. The maximization problem (4)-(5) becomes

$$\max_{x_1,...,x_{T-1}} \frac{\sum_{j=1,j\neq i}^{n_k} f\left(x_1^j\right)}{f(x_1) + \sum_{j=1,j\neq i}^{n_k} f\left(x_1^j\right)} W_0 + \frac{f(x_1)}{f(x_1) + \sum_{j=1,j\neq 1}^{n_1} f\left(x_1^j\right)} W_1,$$
$$s.t.\ x_k \geq 0,\ \text{for any } k = 1, ..., T-1.$$

It is straightforward to see that the optimal strategy is to spend all resources in the first round. **End of proof.**

Proof of Corollary 1. Since all resources are spent in the first round in the symmetric equilibrium, all eliminated players do not have any resources. **End of proof.**

Proof of Proposition 3. If (i) the prize scheme is the winner-take-all and (ii) $n_1 = ... = n_T$, then equation (13) becomes

$$\frac{f'(x_k^*)}{f(x_k^*)}(W_T - W_0) = \frac{f'\left(x_j^*\right)}{f\left(x_j^*\right)}(W_T - W_0)\,,$$

or

$$x_k = x_j, \text{ for any } k, j = 1, ..., T.$$

Suppose that in the symmetric equilibrium $(x_1^*, ..., x_T^*)$, there is equal resource allocation across all rounds, $x_1^* = ... = x_T^* = E/T$. It means that $x_k^* = x_j^* = E/T$ is the solution of the equation (13) :

$$\frac{n_k - 1}{n_k}\frac{f'(x_k)}{f(x_k)}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] =$$

$$\frac{n_j - 1}{n_j}\frac{f'(x_j)}{f(x_j)}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] +$$

$$\frac{n_j - 1}{n_j}\frac{f'(x_j)}{f(x_j)}\left[(n_{j+1}-1)\,n_{j+2}\times ... \times n_T\,(W_j - W_{j-1}) + ...\right.$$

$$\left.... + (n_k - 1)\,n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1}) + n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1})\right],$$

for any $k > j$. Hence, it must be

$$\frac{n_k - 1}{n_k}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] =$$

$$\frac{n_j - 1}{n_j}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] +$$

$$\frac{n_j - 1}{n_j}\left[(n_{j+1}-1)\,n_{j+2}\times ... \times n_T\,(W_j - W_{j-1}) + ...\right.$$

$$\left.... + (n_k - 1)\,n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1}) + n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1})\right],$$

or

$$\left(\frac{n_j - 1}{n_j} - \frac{n_k - 1}{n_k}\right)\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] +$$

$$\frac{n_j - 1}{n_j}\left[(n_{j+1}-1)\,n_{j+2}\times ... \times n_T\,(W_j - W_{j-1}) + ...\right.$$

$$\left.... + (n_k - 1)\,n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1}) + n_{k+1}\times ... \times n_T\,(W_{k-1} - W_{j-1})\right] = 0.$$

From (8) and the assumption (1) it follows that the last equality holds only if (i) the prize scheme is the winner-take-all and (ii) $n_1 = ... = n_T$. **End of proof.**

Proof of Proposition 4. Consider equation (12):

$$\frac{n_k - 1}{n_k}\frac{f'(x_k)}{f(x_k)}\times$$

$$[(n_{k+1}-1)\,n_{k+2}\times ... \times n_T\,(W_k - W_{k-1}) + ... + (n_T - 1)\,(W_{T-1} - W_{k-1}) + (W_T - W_{k-1})] =$$

$$= \frac{n_T - 1}{n_T} \frac{f'(E - x_{T-1} - ... - x_1)}{f(E - x_{T-1} - ... - x_1)} (W_T - W_{T-1}), \text{ for any } k = 1, ..., T-1.$$

Assumption (6) guarantees that the left-hand side (LHS) in equation (12) is a strictly decreasing function of x_k on the interval $[0, E]$, and the right-hand side (RHS) in the same equation is a strictly increasing function of x_k on the interval $[0, E]$. Therefore, if n_k increases, the solution of equation (12), x_k, has to increase too. **End of proof.**

References

[1] Baik, Kyung H., and Shogren, Jason F. "Strategic Behavior in Contests: Comment." *American Economic Review*, 82 (March 1992): 359-62.

[2] Baye, Michael R., and Shin, Onsong. "Strategic Behavior in Contests: Comment." *American Economic Review*, 89 (June 1999): 691-93.

[3] Benoit, Jean-Pierre, and Krishna, Vijay. "Multi-Object Auctions with Budget Constrained Bidders." *Review of Economic Studies*, 68 (2001): 155-79.

[4] Blackett, D. "Pure strategy solutions of Blotto games." *Naval Res. Logistics Quart.*, 5 (1958): 107-110.

[5] Borel, E. "The theory of play and integral equations with skew symmetric kernels." *Econometrica*, 21 (1953): 97-100.

[6] Che, Yeon-Koo, and Gale, Ian. "Optimal Design of Research Contests." *American Economic Review*, 93 (2003): 646-71.

[7] Dixit, Avinash. "Strategic Behavior in Contests." *American Economic Review*, 77 (December 1987): 891-98.

[8] Dixit, Avinash. "Strategic Behavior in Contests: Reply." *American Economic Review*, 89 (June 1999): 694.

[9] Ehrenberg, Ronald G., and Bognanno, Michael L. "Do Tournaments Have Incentives Effects?" *Journal of Political Economy*, 98 (December 1990): 1307-24.

[10] Fort, Rodney. *Sports Economics.* Upper Saddle River, New Jersey, 2003.

[11] Fullerton, Richard; Linster, Bruce; McKee, Michael and Slate, Stephen. "Using Auctions to Reward Tournament Winners: Theory and Experimental Investigations." *RAND Journal of Economics*, 33 (2002): 62-84.

[12] Fullerton, Richard and McAfee, Preston. "Auctioning Entry into Tournaments." *Journal of Political Economy*, 107 (June 1999): 573-605.

[13] Gavious Arieh, Moldovanu, Benny, and Sela, Aner. "Bid costs and endogenous bid caps." *RAND Journal of Economics*, 33 (Winter 2002): 709-722.

[14] Knoeber, Charles R., and Thurman, Walter N. "Testing the Theory of Tournaments: An Empirical Analysis of Broiler Production." *J. Labor Econ.* 12 (April 1994): 155-79.

[15] Krishna, Vijay, and Morgan, John. "The winner-take-all principle in small tournaments." *Advances in Applied Microeconomics,* 7 (1998): 61-74.

[16] Kvasov, D. "Contests with Limited Resources." *Journal of Economic Theory,* forthcoming.

[17] Ernest K. Lai, and Matros, Alexander. "Contest Architecture with Performance Revelation." *Mimeo*, University of Pittsburgh, 2006.

[18] Lazear, Edward P., and Rosen, Sherwin. "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy*, 89 (October 1981): 841-64.

[19] Matros, A. "Stochastic Blotto Games." *Mimeo*, University of Pittsburgh, 2006.

[20] Moldovanu, Benny, and Sela, Aner. "The Optimal Allocation of Prizes in Contests." *American Economic Review*, 91 (June 2001): 542-58.

[21] Moldovanu, Benny, and Sela, Aner. "Contest architecture." *Journal of Economic Theory*, 126 (2006): 70-96.

[22] Rosen, Sherwin. "The Economics of Superstars." *American Economic Review*, 71 (December 1981): 845-58.

[23] Rosen, Sherwin. "Prizes and Incentives in Elimination Tournaments." *American Economic Review*, 76 (September 1986): 701-15.

[24] Taylor, Curtis. "Digging for Golden Carrots: An Analysis of Research Tournaments." *American Economic Review*, 85 (September 1995): 872-90.

[25] Weibull, Jörgen W. *Evolutionary Game Theory.* Massachusetts, MIT Press, 1995.