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# APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY $\alpha$-STABLE LÉVY MOTION

*Abstract.* In this paper we present a result on convergence of approximate solutions of stochastic differential equations involving integrals with respect to $\alpha$-stable Lévy motion. We prove an appropriate weak limit theorem, which does not follow from known results on stability properties of stochastic differential equations driven by semimartingales. It assures convergence in law in the Skorokhod topology of sequences of approximate solutions and justifies discrete time schemes applied in computer simulations. An example is included in order to demonstrate that stochastic differential equations with jumps are of interest in constructions of models for various problems arising in science and engineering, often providing better description of real life phenomena than their Gaussian counterparts. In order to demonstrate the usefulness of our approach, we present computer simulations of a continuous time $\alpha$-stable model of cumulative gain in the Duffie–Harrison option pricing framework.

**1. Introduction.** Recent practical and theoretical studies of various physical and biological problems (see, e.g., Buldyrev *et al.* (1993) and Wang (1992)), signal processing (Shao and Nikias (1993)), finance models (Embrechts and Schmidli (1994), Rachev and Samorodnitsky (1993)), queueing networks (Kella (1993)), level crossing problems with their applications (Adler, Samorodnitsky and Gadrich (1993) or Michna and Rychlik (1995)), etc., reinforce the need for infinite variance stochastic models, including pro-

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cesses with discontinuous trajectories. Of interest are problems involving $\alpha$-stable processes and, in particular, stochastic models described by stochastic differential equations with jumps, driven by $\alpha$-stable random measures.

In this paper we are interested in approximate methods for a stochastic differential equation (SDE) driven by an $\alpha$-stable Lévy motion process $\{L_{\alpha,\beta}(t) : t \geq 0\}$, i.e., we discuss constructions of stochastic processes $\mathbf{X} = \{X(t) : t \geq 0\}$ with values in $\mathbb{R}$, which solve SDEs with given drift and diffusion coefficients. Such an SDE can be written as

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dL_{\alpha,\beta}(t), \quad t > 0, \quad X(0) = X_0,$$

and has, in fact, strict mathematical meaning in the following integral form:

$$X(t) = X_0 + \int_0^t a(s, X(s-))\,ds + \int_0^t b(s, X(s-))\,dL_{\alpha,\beta}(s), \quad t \geq 0. \tag{1.1}$$

Let us briefly recall that a stochastic process $\{L_{\alpha,\beta}(t) : t \geq 0\}$ is called an *$\alpha$-stable Lévy motion* if

1. $L_{\alpha,\beta}(0) = 0$ a.s.;
2. $L_{\alpha,\beta}(t)$ has independent increments;
3. $L_{\alpha,\beta}(t) - L_{\alpha,\beta}(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$,

where $S_\alpha(\sigma, \beta, \mu)$ stands for an *$\alpha$-stable random variable*, which is uniquely determined by its characteristic function involving four parameters: $\alpha \in (0, 2]$, the index of stability; $\beta \in [-1, 1]$, the skewness parameter; $\sigma \in (0, \infty)$, the scale parameter; $\mu \in (-\infty, \infty)$, the shift, and which has the form

$$\log \phi(\theta) = \begin{cases} -\sigma^\alpha |\theta|^\alpha \{1 - i\beta \operatorname{sgn}(\theta) \tan(\alpha\pi/2)\} + i\mu\theta, & \alpha \neq 1, \\ -\sigma|\theta|\{1 + i\beta(2/\pi) \operatorname{sgn}(\theta) \ln|\theta|\} + i\mu\theta, & \alpha = 1. \end{cases} \tag{1.2}$$

Notice that $S_2(\sigma, 0, \mu)$ gives the Gaussian distribution $\mathcal{N}(\mu, 2\sigma^2)$, so $L_{2,0}(t) = \sqrt{2}B(t)$, where $\{B(t) : t \geq 0\}$ stands for the Brownian motion process. This means that (1.1) includes as a special case the following, very well known SDE:

$$X(t) = X_0 + \int_0^t a(s, X(s))\,ds + \int_0^t b(s, X(s))\,dB(s). \tag{1.3}$$

Problems of existence and regularity of solutions to such SDEs have been studied for a long time (see e.g., Karatzas and Schreve (1988)). The numerical analysis of stochastic differential systems driven by Brownian motion—focusing essentially on such problems as mean-square convergence of various discrete time approximate schemes, pathwise approximation or approximation of expectations of the solution, etc.—has also been developed for many years. There is an extensive literature concerning this subject; see, e.g., Pardoux and Talay (1985) or the monograph of Kloeden and Platen (1992).

Discrete time algorithms and results of computer simulations applied to different stochastic models are presented in Kloeden, Platen and Schurz (1994).

This is in sharp contrast to the case of (1.1). There exists a vast literature concerning various classes of $\alpha$-stable processes (see, e.g., Weron (1984) or Samorodnitsky and Taqqu (1994)), but little has been written about SDEs with respect to $\alpha$-stable random measures. However, basic numerical algorithms and computer simulation methods involving statistical estimation techniques, as well as some convergence results, are presented in Janicki and Weron (1994a), (1994b), together with various examples of application in stochastic modeling.

Fundamental properties of stochastic integrals with $\alpha$-stable random measures as integrators can be derived from the general theory of integrals with respect to semimartingales (see, e.g., Protter (1990)). Also some theorems on existence of solutions to SDEs driven by $\alpha$-stable random measures can be obtained in the same way (Protter (1990), Chapter V), since they belong to the general class of SDEs driven by semimartingales, i.e. equations of the form

$$X(t) = X_0 + \int_0^t f(X(s-))\,dY(s), \qquad t > 0, \tag{1.4}$$

with the solution $\mathbf{X} = \{X(t) : t \geq 0\}$ with values in $\mathbb{R}^d$, where $\{Y(t) : t \geq 0\}$ stands for a given semimartingale process with values in $\mathbb{R}^m$ and $f$ denotes a given function from $\mathbb{R}^d$ into $\mathbb{R}^d \times \mathbb{R}^m$. In fact, it is not difficult to see that an $\alpha$-stable Lévy motion process belongs to the class of semimartingales. It is enough to notice that such a stochastic process is infinitely divisible and can be represented by its characteristic function given by the Lévy–Khinchin formula

$$\mathbb{E}e^{i\theta L_{\alpha,\beta}(t)} = \exp(t\psi(\theta)), \tag{1.5}$$

where

$$\psi(\theta) = ib\theta - \frac{1}{2}c\theta^2 + \int_{-\infty}^{\infty} \left(e^{i\theta u} - 1 - \frac{i\theta u}{1+u^2}\right) d\nu(u), \tag{1.6}$$

and $d\nu(u) = d\nu_{\alpha,\beta}(u)$ is defined by

$$d\nu(u) = \begin{cases} \alpha\{C^+\mathbb{I}_{(0,\infty)}(u) + C^-\mathbb{I}_{(-\infty,0)}(u)\}|u|^{-\alpha-1}du, & 0 < \alpha < 2, \\ 0, & \alpha = 2, \end{cases} \tag{1.7}$$

where $C^+$ and $C^-$ denote nonnegative constants depending on $\beta$ such that $C^+ + C^- > 0$.

The problem of stability of (1.4) consists in investigation of conditions under which the sequence $\{\mathbf{X}_n\}_{n=1}^{\infty}$ of processes $\mathbf{X}_n = \{X_n(t) : t \geq 0\}$

solving the SDE

$$X_n(t) = X_{n,0} + \int_0^t f_n(X_n(s-))\, dY(s) \tag{1.8}$$

converges weakly in law (in the sense of weak convergence of underlying measures on the space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ of cadlag functions, endowed with the Skorokhod topology). There is a vast literature on this subject, concerning general semimartingales (see, e.g., Słomiński (1989), Jakubowski, Mémin and Pages (1989) or Kurtz and Protter (1991)), as well as semimartingales represented by Poisson counting measures (see, e.g., Kasahara and Watanabe (1986), and Kasahara and Yamada (1991)).

The main goal of this work is to prove a similar result, concerning the problem of convergence to the solution $\mathbf{X}$ to (1.1) of a sequence of processes $\mathbf{X}_n$ defined by the SDE

$$X_n(t) = X_{n,0} + \int_0^t a_n(s, X_n(s-))\, dl_n(s) + \int_0^t b_n(s, X_n(s-))\, dL_n(s), \tag{1.9}$$

with appropriately chosen $X_{n,0}$, $a_n$, $b_n$, $dl_n$, and $dL_n$.

A weak functional limit theorem discussed here does not follow from known results on stability properties of stochastic differential equations driven by semimartingales. It assures weak convergence in the Skorokhod topology in the space $\mathbb{D}([0,\infty),\mathbb{R})$ of sequences of approximate solutions to (1.1) defined by (1.9), and justifies discrete time schemes applied in computer simulations.

In Section 2 we present in detail the method of numerical continuous and discrete time approximations for the problem described by (1.1) and prove the main theorem concerning the convergence of approximate solutions. In order to demonstrate the usefulness of our approach in applications, in Section 3 we present a discrete time algorithm of computer construction of solutions to (1.1) and in Section 4 an example of computer simulations concerning one important and rather well known problem from financial mathematics.

**2. Main result.** Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis and $L^0(\Omega, \mathcal{F}, P)$ the space of all random variables (measurable functions) on $\Omega$.

We consider the SDE the (1.1) driven by a stochastic measure determined by increments of an $\alpha$-stable Lévy motion $\{L_{\alpha,\beta}(t) : t \geq 0\}$ with the index of stability $\alpha \in (0, 2]$ and skewness parameter $\beta \in [-1, 1]$, where $X_0$ denotes a given fixed $\mathcal{F}_0$-measurable random variable and a solution $\mathbf{X} = \{X(t) : t \geq 0\}$ is defined as an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted cadlag (càdlàg—*continu à droite, limites à gauche* in French) stochastic process (i.e., a stochastic process whose trajectories are right continuous and have bounded left limits, which implies that $\mathbf{X}$ is represented by an appropriate measure on the space $\mathbb{D}([0,\infty),\mathbb{R})$ of cadlag functions on $[0,\infty)$ endowed with the Skorokhod topology; see, e.g., Parthasarathy (1969), Chapter VII or Kurtz and Protter (1991)).

Our aim is to derive from (1.9) some explicit methods of construction of sequences $\{\mathbf{X}_n\}_{n=1}^{\infty}$ of stochastic processes $\mathbf{X}_n = \{X_n(t)\}$ converging to a solution $\mathbf{X} = \{X(t)\}$ of the problem (1.1).

For constructions of approximations to $\alpha$-stable random measures we will constantly use a sequence $\{Y_j\}_{j=1}^{\infty}$ of i.i.d. random variables with a common distribution $F$, living on $(\Omega', \mathcal{F}', P')$ and belonging to the domain of attraction of $\alpha$-stable laws, i.e., such that

$$\frac{1}{\phi(n)}\sum_{j=1}^{n} Y_j \xrightarrow{\mathcal{L}} L_{\alpha,\beta}(1) \quad \text{as } n\to\infty, \tag{2.1}$$

where $\phi(n) = n^{1/\alpha}\psi(n)$ and $\psi = \psi(u)$ denotes an appropriately chosen slowly varying function. For further convenience let us also introduce

$$c_{nk}^{\varepsilon} := \int\limits_{0<|u|\le\varepsilon} u^k \, dF(\phi(n)u). \tag{2.2}$$

From now on, let $\mathbf{X}_n = \{X_n(t) : t \ge 0\}$, for $n = 1,2,\ldots$, denote a sequence of cadlag stochastic processes $\mathbf{X}_n$ defined on $(\Omega', \mathcal{F}', P')$ endowed with a natural filtration $\{\mathcal{F}^{(n)}\}_{t\ge0}$ of $\mathbf{X}_n$, and satisfying the SDE

$$\begin{aligned} X_n(t) = X_{n,0} &+ \int\limits_0^t a_n(s, X_n(s-))\, dl_n(s) \\ &+ \frac{1}{\phi(n)}\sum_{j=1}^{[nt]} b_n\left(\frac{j}{n}, X_n\left(\frac{j}{n}-\right)\right)Y_j, \end{aligned} \tag{2.3}$$

where $l_n = l_n(s)$ denotes a given deterministic nonnegative and nondecreasing function on $[0,\infty)$, and $a_n = a_n(s,x)$, $b_n = b_n(s,x)$ are measurable functions such that the solution $\mathbf{X}_n$ exists.

Our aim is to prove the following theorem on convergence of approximate solutions of (1.1), which does not follow directly from known results on stability properties of (1.4) driven by a semimartingale. It assures weak convergence in the Skorokhod topology of sequences of approximate solutions and justifies discrete time schemes applied in computer simulations.

THEOREM 2.1. *Suppose that the coefficients $a = a(s,x)$ and $b = b(s,x)$ in* (1.1) *are continuous. Suppose that* (1.1) *has a unique weak solution in* $\mathbb{D}([0,\infty),\mathbb{R})$. *Assume that for any $T > 0$ and $R > 0$, there exists a constant*

*$M > 0$ such that, for all $n \geq 1$,*

$$\sup_{0 \leq s \leq T,\, |x| \leq R} |a_n(s,x)| \leq M, \tag{2.4}$$

$$\sup_{0 \leq s \leq T,\, |x| \leq R} |b_n(s,x)| \leq M. \tag{2.5}$$

*Suppose that for every sequence $\{(s_n, x_n)\}$ converging to $(s,x)$,*

$$a_n(s_n, x_n) \to a(s,x), \qquad b_n(s_n, x_n) \to b(s,x), \tag{2.6}$$

*and also*

$$l_n(t) \to t \quad \text{for all } t \in [0,\infty), \tag{2.7}$$

$$X_{n,0} \to X_0 \quad \text{weakly as } n \to \infty. \tag{2.8}$$

*Then the sequence $\{X_n(t)\}_{n=1}^{\infty}$ of solutions to (2.3) converges weakly to the solution $X(t)$ of (1.1) in the Skorokhod topology.*

Proof. Our proof leans to some extent on a result of Kasahara and Yamada (1991) concerning the functional central limit theorem for solutions to SDEs driven by semimartingales represented by Poisson random measures. We start by introducing such a representation for $\alpha$-stable Lévy motion, apply it to the description of sequences of approximate solutions, check suitable tightness criteria for them, ending up with a conclusion on their weak convergence in the space $\mathbb{D}([0,\infty),\mathbb{R})$. The proof is split into a few steps.

Step 1. Let us rewrite equation (1.1) using a Poisson random measure generated by an $\alpha$-stable Lévy motion, recalling first a formal definition of a Poisson random measure. Let $(S, \mathcal{S}, n)$ be a measure space and $\mathcal{S}_{\mathrm{f}} = \{A \in \mathcal{S} : n(A) < \infty\}$. A *Poisson random measure* $N$ on $(S, \mathcal{S}, n)$ is an independently scattered $\sigma$-additive set function $N : \mathcal{S}_{\mathrm{f}} \to L^0(\Omega, \mathcal{F}, P)$ such that for each $A \in \mathcal{S}_{\mathrm{f}}$, $N(A)$ has a Poisson distribution with mean $n(A)$, that is,

$$P(N(A) = k) = e^{-n(A)} \frac{(n(A))^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots,$$

and (the mean) $n$ is a control measure of $N$. Let us recall briefly the Lévy–Itô theorem characterizing Lévy processes, i.e., right continuous processes with stationary independent increments (see Ikeda and Watanabe (1981), Chapter II, or Protter (1990), Chapter I). Let $\mathbf{L} = \{L(t) : t \geq 0\}$ be a Lévy process. Its trajectories $\lambda = \lambda(t)$ all belong to the space $\mathbb{D}([0,\infty),\mathbb{R})$. For any $A \in \mathcal{S}_{\mathrm{f}}$ such that $0 \notin \overline{A}$ one can define

$$N([t_1,t_2], A, \lambda) = \mathrm{card}\{t_1 \leq s \leq t_2 : \Delta\lambda(s) \in A\},$$

where $\Delta\lambda(s)$ denotes the jump of the trajectory $\lambda$ at the point $s$, and then

$$N([t_1,t_2], A) := N([t_1,t_2], A, \mathbf{L}).$$

So, $N$ generates a random measure on $[0,\infty)\times(\mathbb{R}\setminus\{0\})$ which is a Poisson measure with the mean

$$\mathbb{E}N(ds,du)=ds\,d\nu(u),$$

where $\nu$ is a corresponding (deterministic) Lévy measure. The Lévy–Itô theorem states that any Lévy process $\mathbf{L}=\{L(t)\}$ takes the form

$$L(t)=\zeta B(t)+\gamma t+\int_0^t\int_{|u|\le 1}u\,\widetilde{N}(ds,du)+\int_0^t\int_{|u|>1}u\,N(ds,du), \tag{2.9}$$

where $B=B(t)$ denotes a standard Brownian motion, $\widetilde{N}:=N-\mathbb{E}N$, and $\zeta$, $\gamma$ are some fixed constants. So, thanks to (1.5)–(1.7), for an $\alpha$-stable Lévy motion $\{L_{\alpha,\beta}(t)\}$ we get

$$L_{\alpha,\beta}(t)=\gamma t+\int_0^t\int_{|u|\le 1}u\,\widetilde{N}(ds,du)+\int_0^t\int_{|u|>1}u\,N(ds,du) \tag{2.10}$$

for $\alpha\in(0,2)$ (and any $\beta\in[-1,1]$), and

$$L_{2,\beta}(t)=L_{2,0}(t)=\sqrt{2}B(t).$$

In what follows we have to deal with stochastic processes defined by a stochastic integral with an $\alpha$-stable Lévy motion as integrator. Namely, for a given caglad (*continu à gauche*, *limites à droite*) process $\{H(t):t\in[0,\infty)\}$ one can construct a new process

$$Y(t)=\int_0^t H(s)\,dL_{\alpha,\beta}(s),$$

and, applying (2.10), get the representation

$$\begin{aligned}Y(t)=\gamma\int_0^t H(s)\,ds&+\int_0^t\int_{0<|u|\le 1}H(s)u\,\widetilde{N}(ds,du)\\&+\int_0^t\int_{|u|>1}H(s)u\,N(ds,du),\end{aligned} \tag{2.11}$$

which is correctly defined thanks to the fact that $\int_0^t\int_{|u|\le 1}u\,\widetilde{N}(ds,du)$ and $\int_0^t\int_{|u|>1}u\,N(ds,du)$ are semimartingales.

S t e p 2. It is known that the Poisson measure $N$ can be approximated by discrete measures of the form

$$N_n(ds,du)=\sum_{k=1}^{\infty}\delta(k/n,Y_k/\phi(n))(ds,du),$$

where $\delta = \delta(x,y)$ denotes Dirac's distribution function on $\mathbb{R}^2$, so $\mathbb{E}N_n = \widehat{N}_n(ds,du) = d\varrho_n(s)d\nu_n(u)$, $\varrho_n(s) = [ns]/n$, $\nu_n(u) = nF(\phi(n)u)$.

For the sequence $\{Y_j\}_{j=1}^{\infty}$ satisfying (2.1) we can construct a sequence $\{\mathbf{X}_n\}_{n=1}^{\infty}$ of cadlag stochastic processes $\mathbf{X}_n = \{X_n(t) : t \geq 0\}$ adapted to $\{\mathcal{F}_t^{(n)}\}$ and satisfying the SDE

$$(2.12)\quad X_n(t) = X_{n,0} + \int_0^t a_n(s, X_n(s-))\, dl_n(s) + \gamma \int_0^t b_n(s, X_n(s-))\, dl_n(s) + \int_0^t \int_{0<|u|\leq 1} b_n(s, X_n(s-))u\, \widetilde{N}_n(ds,du) + \int_0^t \int_{|u|>1} b_n(s, X_n(s-))u\, N_n(ds,du).$$

The sequence $\{\mathbf{X}_n\}_{n=1}^{\infty}$ approximates the process $\mathbf{X}$ solving equation (1.1). Let us replace (2.12) by

$$(2.13)\quad X_n(t) = X_{n,0} + \int_0^t a_n(s, X_n(s-))\, dl_n(s) + \gamma \int_0^t b_n(s, X_n(s-))\, dl_n(s) + \frac{1}{\phi(n)} \sum_{j=1}^{[nt]} b_n\left(\frac{j}{n}, X_n\left(\frac{j}{n}-\right)\right) Y_j, - c_{n1}^1 \sum_{j=1}^{[nt]} b_n\left(\frac{j}{n}, X_n\left(\frac{j}{n}-\right)\right).$$

Step 3. Now our aim is to check that $\widehat{N}_n((0,t] \times A) \stackrel{P}{=} t\nu(A)$, for any fixed measurable set $A \subset \mathbb{R}_a := (-\infty,-a] \cup [a,\infty)$ (for some $a > 0$) such that $\nu(\partial A) = 0$. We must prove that for the sequence of measures $\nu_n$ defined above we have the convergence

$$\nu_n(A) \to \nu(A) \quad \text{as } n \to \infty,$$

for any such $A$.

We see that $\nu_n(\mathbb{R}_a) \to \nu(\mathbb{R}_a)$ as $n \to \infty$ (see Feller (1971), Sect. XVII.1). Let $\vartheta \subset \mathbb{R}_a$ and $\vartheta$ be open. By regularity of $\nu$, for every $\varepsilon > 0$ there exists a compact subset $K$ in $\vartheta$ such that

$$\nu(\vartheta) < \nu(K) + \varepsilon.$$

From the Urysohn lemma we obtain a function $f$ such that $K \subset \mathrm{supp}(f) \subset \vartheta$, $|f| \leq 1$ and $f \equiv 1$ on $K$. Since $\nu_n$ converges in the vague topology to $\nu$, it follows that $\lim_n \nu_n g = \nu g$ for all continuous functions $g$ with compact support, where $\nu g = \int_{\mathbb{R}\setminus\{0\}} g\, d\nu$ (for the definition of the vague topology and some details on the corresponding convergence we refer the reader to Jagers (1974) and Resnick (1987)). So,

$$\liminf_n \nu_n(\vartheta) \geq \liminf_n \nu_n f = \nu f \geq \nu(K) > \nu(\vartheta) - \varepsilon,$$

and thus $\lim_n \nu_n(A) = \nu(A)$.

Step 4. Further, we have the estimate

$$\mathbb{E}\Big[\int_0^t \int_{0<|u|\leq 1} u^2\, \widehat{N}_n(ds, du)\Big] = \int_0^t \int_{0<|u|\leq 1} u^2\, \widehat{N}_n(ds, du) = [nt]c^1_{n2} < \infty.$$

Step 5. Next, we have to check that for some $\varrho > 0$ we have

$$\lim_{\varepsilon\downarrow 0} \limsup_n P(|[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n] - \varrho^2 t| > \delta) = 0, \qquad \delta > 0, \tag{2.14}$$

where $Y^\varepsilon_n(t) := \int_0^t \int_{0<|u|<\varepsilon} u\, \widetilde{N}_n(ds, du)$ and $[\mathbf{X}, \mathbf{X}]$ denotes the quadratic variation process for a given martingale $\mathbf{X}$ (see, e.g., Protter (1990), Chapter II). So we have

$$Y^\varepsilon_n(t) = \frac{1}{\phi(n)} \sum_{j=1}^{[nt]} Y_j \mathbb{I}_{(0,\varepsilon)}\Big(\frac{|Y_j|}{\phi(n)}\Big) - [nt]c^\varepsilon_{n1}.$$

The quadratic variation process has the form

$$[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n](t) = \sum_{j=1}^{[nt]} \Big(\frac{Y_j}{\phi(n)} \mathbb{I}_{(0,\varepsilon)}\Big(\frac{|Y_j|}{\phi(n)}\Big) - c^\varepsilon_{n1}\Big)^2. \tag{2.15}$$

The expected value of $[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n]$ takes the form

$$\mathbb{E}[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n](t) = [nt](c^\varepsilon_{n2} - (c^\varepsilon_{n1})^2). \tag{2.16}$$

If $0 < \alpha < 2$ then the left hand side of (2.16) converges to $t\int_{|u|\leq\varepsilon} u^2\, \nu(du)$ (see e.g. Kasahara and Maejima (1986)). So,

$$\lim_{\varepsilon\downarrow 0} \limsup_n \mathbb{E}|[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n](t)| = 0,$$

and hence $\varrho = 0$.

For $\alpha = 2$ we have to show that $\varrho = \zeta$ in (2.14), with $\zeta$ defined by (2.9). So we have

$$\begin{aligned} &P(|[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n](t) - \varrho^2 t| > \delta) \\ &\qquad \leq P(|[\mathbf{Y}^\varepsilon_n, \mathbf{Y}^\varepsilon_n](t) - [nt]b^\varepsilon_n| > \delta/2) + P(|[nt]b^\varepsilon_n - \zeta^2 t| > \delta/2) \end{aligned} \tag{2.17}$$

where

$$b^\varepsilon_n = \mathbb{E}\Big(\frac{Y_j}{\phi(n)} \mathbb{I}_{(0,\varepsilon)}\Big(\frac{|Y_j|}{\phi(n)}\Big) - c^\varepsilon_{n1}\Big)^2$$

and we know that $nb_n^\varepsilon \to \zeta^2$ as $n \to \infty$ (see Kasahara and Maejima (1986), (1988)). We obtain

$$(2.18)\quad P(|[\mathbf{Y}_n^\varepsilon, \mathbf{Y}_n^\varepsilon](t) - [nt]b_n^\varepsilon > \delta/2)$$

$$= P\left(\left|\sum_{j=1}^{[nt]}\left\{\left(\frac{Y_j}{\phi(n)}\mathbb{I}_{(0,\varepsilon)}\left(\frac{|Y_j|}{\phi(n)}\right) - c_{n1}^\varepsilon\right)^2 - b_n^\varepsilon\right\}\right| > \frac{\delta}{2}\right)$$

$$\leq \frac{4}{\delta^2}\sum_{j=1}^{[nt]}\mathbb{E}\left\{\left(\frac{Y_j}{\phi(n)}\mathbb{I}_{(0,\varepsilon)}\left(\frac{|Y_j|}{\phi(n)}\right) - c_{n1}^\varepsilon\right)^2 - b_n^\varepsilon\right\}^2$$

$$= \frac{4}{\delta^2}[nt]\left\{\mathbb{E}\left(\frac{Y_j}{\phi(n)}\mathbb{I}_{(0,\varepsilon)}\left(\frac{|Y_j|}{\phi(n)}\right) - c_{n1}^\varepsilon\right)^4 - (b_n^\varepsilon)^2\right\}$$

$$\leq \frac{4}{\delta^2}[nt](4\varepsilon^2 b_n^\varepsilon - (b_n^\varepsilon)^2).$$

So,

$$\lim_{\varepsilon\downarrow 0}\limsup_n \frac{4}{\delta^2}[nt](4\varepsilon^2 b_n^\varepsilon - (b_n^\varepsilon)^2) = \lim_{\varepsilon\downarrow 0}\frac{4}{\delta^2}\zeta^2 t 4\varepsilon^2 = 0.$$

Step 6. The final question is whether there exists a constant $\gamma > 0$ such that the sequence

$$\left\{\int_0^t \int_{0<|u|\leq\gamma} u^2\,\widehat{N}_n(ds, du)\right\}_{n=1}^\infty$$

is $C$-tight in $\mathbb{D}([0,\infty),\mathbb{R})$. In our case we get, however,

$$(2.19)\quad \lim_n \int_0^t \int_{0<|u|\leq 1} u^2\,\widehat{N}_n(ds, du) = \lim_n\,[nt]\int_{0<|u|\leq 1} u^2\,dF(\phi(n)u) = td_\alpha,$$

for some constant $d_\alpha > 0$ (see Feller (1971), Section XVII.5).

Step 7. Our assumptions on equations (1.1) and (2.3), and the argument presented in steps 3 to 6 allow us to repeat, with some obvious modifications, the argument applied in the proof of Theorem 2 of Kasahara and Yamada (1991). It is enough to make an appropriate use of Theorems I.7.1 and II.6.3 of Ikeda and Watanabe (1981) and Theorem 5.2 of Kasahara and Watanabe (1986) in order to conclude that the sequence $\{\mathbf{X}_n\}$ of solutions to (2.13) converges weakly to the solution $\mathbf{X}$ to (1.1) as $n \to \infty$.

Step 8. Considering equations (1.1) and (2.13) with $a_n(s,x) = a(s,x) \equiv 0$ and $b_n(s,x) = b(s,x) \equiv 1$ and $l_n = \varrho_n$ and using weak convergence of $\mathbf{X}_n$

to $\mathbf{X}$ we see that

$$nc_{n1}^{1} \to \gamma \quad \text{as } n \to \infty. \tag{2.20}$$

By the usual cut-off method (see Kesten and Papanicolaou (1979)) we may and do assume that $a_n$ and $b_n$ satisfy (2.4) and (2.5) with $R = \infty$. Since the solution of equation (1.1) is unique, to prove that $\mathbf{X}_n$ in (2.3) converges weakly to $\mathbf{X}$ it is enough to prove that

$$\int_0^t b(s, X(s-))\, ds - \int_0^t b_n(s, X_n(s-))\, dl_n \to 0 \quad \text{as } n \to \infty \tag{2.21}$$

in probability in the Skorokhod topology.

Since $\mathbf{X}_n$ in (2.13) converges weakly to $\mathbf{X}$, using the Skorokhod representation theorem (see e.g. Ethier and Kurtz (1986)) we may assume that $\mathbf{X}_n$ in (2.13) converges to $\mathbf{X}$ a.s. in the Skorokhod topology. Then, recalling (2.6) for $s_n \to s$ as $n \to \infty$,

$$b_n(s_n, X_n(s_n-)) \to b(s, X(s-)) \tag{2.22}$$

a.e. on $[0,T]$ where $T > 0$.

Let us now consider

$$\begin{aligned} \Big| \int_0^t b(s, X(s-))\, ds - \int_0^t b_n(s, X_n(s-))\, dl_n \Big| &\le \int_0^T |b_n(s, X_n(s-)) - b(s, X(s-))|\, dl_n \\ &\quad + \Big| \int_0^t b(s, X(s-))\, ds - \int_0^t b(s, X(s-))\, dl_n \Big|. \end{aligned} \tag{2.23}$$

The first term on the right hand side tends to 0 because the measure $dl_n(s)$ converges weakly to Lebesgue measure $ds$ on $[0,T]$ and $|b_n(s, X_n(s-)) - b(s, X(s-))| \le 2M$ for $0 \le s \le T$ and by (2.22) and Theorem 5.5 in Billingsley (1968).

Let $\{p_{k,X}\}_{k=1}^{\infty}$ be a sequence of simple functions which converges uniformly to $b(s, X(s-))$ on $[0,T]$ as $k \to \infty$ such that

$$p_{k,X}(s) = \sum_{i=1}^{m(k)} p_i^{(k)} \mathbb{I}_{A_i^{(k)}}(s).$$

Since $b(s, X(s-))$ is a caglad function, we can construct $p_{k,X}$ in such a way that $A_i^{(k)}$ are each a countable sum of intervals so $l_n(A_i^{(k)}) \to l(A_i^{(k)})$ as $n \to \infty$, where $l$ is Lebesgue measure.

Hence

$$\begin{aligned}(2.24)\quad \mathcal{D} &:= \Big|\int_0^t b(s,X(s-))\,ds - \int_0^t b(s,X(s-))\,dl_n\Big| \\ &\le \int_0^T |b(s,X(s-)) - p_{k,X}(s)|\,ds + \int_0^T |b(s,X(s-)) - p_{k,X}(s)|\,dl_n \\ &\quad + \Big|\int_0^t p_{k,X}(s)\,ds - \int_0^t p_{k,X}(s)\,dl_n(s)\Big|,\end{aligned}$$

and thus we get

$$\begin{aligned}\mathcal{D} &\le \varepsilon(k) + \varepsilon(k) + \Big|\sum_{i=1}^{m(k)} p_i^{(k)}(l(A_i^{(k)} \cap [0,t]) - l_n(A_i^{(k)} \cap [0,t]))\Big| \\ &\le 2\varepsilon(k) + \sum_{i=1}^{m(k)} |p_i^{(k)}| \cdot |l(A_i^{(k)} \cap [0,t]) - l_n(A_i^{(k)} \cap [0,t])|,\end{aligned}$$

and finally,

$$\begin{aligned}(2.25)\quad \mathcal{D} \le 2\varepsilon(k) + \sum_{i=1}^{m(k)} |p_i^{(k)}| \{ &|l(A_i^{(k)} \cap [0,t_i^*]) - l_n(A_i^{(k)} \cap [0,t_i^*])| \\ &\vee |l(A_i^{(k)} \cap [0,t_i^*)) - l_n(A_i^{(k)} \cap [0,t_i^*))|\},\end{aligned}$$

where $t_i^*$ is such that

$$\begin{aligned}\{|l(A_i^{(k)} \cap [0,t_i^*]) - l_n(A_i^{(k)} \cap [0,t_i^*])| \vee |l(A_i^{(k)} \cap [0,t_i^*)) - l_n(A_i^{(k)} \cap [0,t_i^*))|\} \\ \ge |l(A_i^{(k)} \cap [0,t]) - l_n(A_i^{(k)} \cap [0,t])|\end{aligned}$$

for all $0 \le t \le T$ such $t_i^*$ exists because $|l(A_i^{(k)} \cap [0,t]) - l_n(A_i^{(k)} \cap [0,t])|$ is a cadlag function of $t$. So

$$\Big|\int_0^t b(s,X(s-))\,ds - \int_0^t b(s,X(s-))\,dl_n\Big| \le 2\varepsilon(k) + m(k)2M\varepsilon(n).$$

Since $\varepsilon(k) \to 0$ and $\varepsilon(n) \to 0$ as $k \to \infty$ and $n \to \infty$, we get (2.21). ■

COROLLARY 2.1. *Consider a stochastic differential equation*

$$(2.26)\quad X(t) = X_0 + \int_0^t a(s)\,ds + \int_0^t b(s,X(s-))\,dL_{\alpha,\beta}(s), \qquad t \ge 0,$$

*and suppose that the assumptions of Theorem* 2.1 *are satisfied. Then the* *sequence of processes defined by*

$$X_n(t) = X_{n,0} + \int_0^t a_n(s)\,dl_n(s), \qquad 0 \le t < \frac{1}{n},$$

$$X_n(t) = X_{n,0} + \int_0^t a_n(s)\,dl_n(s) + b_n\left(\frac{1}{n}, X_n\left(\frac{1}{n}-\right)\right)\frac{Y_1}{\phi(n)}, \qquad \frac{1}{n} \le t < \frac{2}{n},$$

*and for* $k = 1, 2, \ldots,$

$$X_n(t) = X_n\left(\frac{k}{n}-\right) + \int_{k/n}^t a_n(s)\,dl_n(s) + b_n\left(\frac{k}{n}, X_n\left(\frac{k}{n}-\right)\right)\frac{Y_k}{\phi(n)}, \qquad \frac{k}{n} \le t < \frac{k+1}{n},$$

*converges weakly to the solution of* (2.26), *provided that* $X_{n,0}$ *converges weakly to* $X_0$, *and the* $Y_j$ *satisfy* (2.1).

Proof. It is enough to notice that $\mathbf{X}_n$ satisfies equation (2.3). ∎

Considering (2.26) with $a(s) \equiv 0$ and its approximation with $a_n \equiv 0$ we obtain immediately the following statement.

COROLLARY 2.2. *Consider the SDE*

$$X(t) = X(0) + \int_0^t b(s, X(s-))\,dL_{\alpha,\beta}(s), \qquad t \ge 0, \tag{2.27}$$

*and suppose the assumptions of Theorem* 2.1 *are satisfied. Then the sequence*

$$X_n(t) = X_n\left(\frac{[nt]-1}{n}\right) + b_n\left(\frac{[nt]}{n}, X_n\left(\frac{[nt]-1}{n}\right)\right) \cdot \frac{1}{\phi(n)} Y_{[nt]}$$

*converges weakly to the solution of* (2.27), *where the sequence* $\{Y_j\}$ *satisfies* (2.1) *and* $X_n\left(\frac{1}{n}-\right) = X_{n,0}$.

The process $\mathbf{X}_n$ has the following simple description:

$$X_n(t) = X_{n,0}, \qquad 0 \le t < \frac{1}{n},$$

$$X_n(t) = X_n(0) + b_n\left(\frac{1}{n}, X_n(0)\right)\frac{Y_1}{\phi(n)}, \qquad \frac{1}{n} \le t < \frac{2}{n},$$

and for $k = 1, 2, \ldots,$

$$X_n(t) = X_n\left(\frac{k-1}{n}\right) + b_n\left(\frac{k}{n}, X_n\left(\frac{k-1}{n}\right)\right)\frac{Y_k}{\phi(n)}, \qquad \frac{k}{n} \le t < \frac{k+1}{n}.$$

COROLLARY 2.3. *Let $f = f(s)$ be a deterministic caglad function (i.e., it is left continuous and has right limits for all $s$). Then*

$$\frac{1}{\phi(n)} \sum_{j=1}^{[nt]} f\left(\frac{j}{n}\right) Y_j \tag{2.28}$$

*converges in law to the integral*

$$\int_0^t f(s)\, dL_{\alpha,\beta}(s). \tag{2.29}$$

Proof. It is enough to take (2.26) with $a(s) \equiv 0$ and $b(s,x) = f(s)$ and to choose $X_0 = 0$. ∎

COROLLARY 2.4. *Consider the SDE*

$$X(t) = X_0 + \int_0^t X(s-)\, ds + \int_0^t b(s, X(s-))\, dL_{\alpha,\beta}(s), \qquad t \ge 0, \tag{2.30}$$

*and suppose the assumptions of Theorem* 2.1 *are satisfied. Then*

$$\begin{aligned} X_n(t) &= \left[X_n\left(\frac{[nt]}{n} -\right) + b_n\left(\frac{[nt]}{n}, X_n\left(\frac{[nt]}{n} -\right)\right)\frac{Y_{[nt]}}{\phi(n)}\right] \\ &\quad \times \exp\left\{t - \frac{[nt]}{n}\right\}, \\ X_{n,0} &= X_0, \end{aligned} \tag{2.31}$$

*converges weakly to the solution of* (2.30) *with* $\{Y_j\}$ *satisfying* (2.1).

Proof. $X_n$ in (2.31) satisfies the equation (2.3). The formula (2.31) can be written as

$$\begin{aligned} X_n(t) &= X_{n,0} \exp\{t\}, \qquad 0 \le t < \frac{1}{n}, \\ X_n(t) &= \left[X_n\left(\frac{1}{n} -\right) + b_n\left(\frac{1}{n}, X_n\left(\frac{1}{n} -\right)\right)\frac{Y_1}{\phi(n)}\right] \\ &\qquad \times \exp\left\{t - \frac{1}{n}\right\}, \qquad \frac{1}{n} \le t < \frac{2}{n}, \end{aligned}$$

and for $k/n \le t < k + 1/n$,

$$X_n(t) = \left[X_n\left(\frac{k}{n} -\right) + b_n\left(\frac{k}{n}, X_n\left(\frac{k}{n} -\right)\right)\frac{Y_j}{\phi(n)}\right] \exp\left\{t - \frac{k}{n}\right\},$$

which ends the proof. ∎

**3. Numerical algorithm.** Now we present briefly the method of discrete time approximation of equation (1.1) and discuss its convergence.

Looking for an approximation of the process $\{X(t) : t \geq 0\}$ solving equation (1.1) we have to approximate this equation by a time discretized explicit system of the form

$$X_n\left(\frac{k+1}{n}\right) = \Phi\left(X_n\left(\frac{k}{n}\right), L_{\alpha,\beta}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)\right), \tag{3.1}$$

where the stochastic stable measure $L_{\alpha,\beta}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right)$ of the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ is defined by

$$L_{\alpha,\beta}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right) := L_{\alpha,\beta}\left(\frac{k+1}{n}\right) - L_{\alpha,\beta}\left(\frac{k}{n}\right)$$
$$\sim S_\alpha(n^{-1/\alpha}, \beta, 0) \sim \frac{1}{n^{1/\alpha}} S_\alpha(1, \beta, 0),$$

since to assure proper approximation of the $\alpha$-stable random measure $dL_{\alpha,\beta}$ it is enough to take $\phi(n) = n^{1/\alpha}$ and an i.i.d. sequence $Y_j \sim S_\alpha(1, \beta, 0)$ for $j = 1, \ldots, n$, and to notice that (2.1) takes the form

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} Y_j \stackrel{\mathrm{d}}{=} L_{\alpha,\beta}(1).$$

*Numerical algorithm.* In accordance with what was presented above, the discrete time scheme can be described as follows:

$$X_n\left(\frac{k+1}{n}\right) = X_n\left(\frac{k}{n}\right) + \frac{1}{n} a\left(\frac{k}{n}, X_n\left(\frac{k}{n}\right)\right) + b\left(\frac{k}{n}, X_n\left(\frac{k}{n}\right)\right) L_{\alpha,\beta}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right), \tag{3.2}$$

for $k = 0, 1, \ldots$

Convergence of this scheme follows directly from Theorem 2.1.

THEOREM 3.1. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. Then the sequence*

$$X_n(t) = X_n\left(\frac{[nt]}{n}\right) + \frac{1}{n} a\left(\frac{[nt]}{n}, X_n\left(\frac{[nt]}{n}\right)\right) + b\left(\frac{[nt]}{n}, X_n\left(\frac{[nt]}{n}\right)\right) L_{\alpha,\beta}\left(\left[\frac{[nt]}{n}, \frac{[nt]+1}{n}\right)\right) \tag{3.3}$$

*converges in law to the solution* $X(t)$ *of* (1.1).

**4. An example.** In order to demonstrate the usefulness of our approach, we present here the results of computer simulations concerning one exemplary problem from mathematical finance theory. Other possible fields of application are discussed in Janicki and Weron (1994a).

EXAMPLE 4.1 (Continuous time $\alpha$-stable model of cumulative gain in finance theory). Specifying a problem discussed by Duffie and Harrison (1993) we want to show how Corollary 2.1 can be applied in finance theory, and to demonstrate how an application of $\alpha$-stable random measures modifies the chosen model. We approximate the continuous time model with its discrete time counterpart. Consider the stochastic differential equation describing the price of a stock owned:

$$S(t) = S_0 + \int_0^t S(s-)\, dZ(s), \tag{4.1}$$

assuming that $\mathbf{Z} = \{Z(t) : t \geq 0\}$ denotes an $\alpha$-stable Lévy motion $\{L_{\alpha,\beta}(t) : t \geq 0\}$. Keeping in mind that the solution to (4.1) defines the stochastic exponential (see Protter (1990), Sections II.8 and V.4) and recalling Corollary 2.1 we see that the process

$$S_n(t) = S_{n,0} \prod_{k=1}^{[nt]} \left(1 + \frac{Y_k}{\phi(n)}\right) \tag{4.2}$$

converges weakly to $\{S(t)\}$, where the sequence $\{Y_k\}$ satisfies (2.1) and the sequence $\{S_{n,0}\}$ converges in law to $S_0$. We conclude that the periodic rate of return on the stock is $Y_k/\phi(n)$. The cumulative gain up to time $t$ can be written as

$$G_n(t) = \int_0^t \theta_n(s-)\, dS_n(s), \tag{4.3}$$

where $\theta_n(t) = \theta^n_{[nt]}$ and $\theta^n_k$ denotes the number of shares of the stock during the $(k+1)$th period.

Let $Z_n(t) = \sum_{i=1}^{[nt]} Y_k/\phi(n)$, and define $\mathbf{Z}^\delta_n = \mathbf{Z}_n - J_\delta(\mathbf{Z}_n)$, where

$$J_\delta(x) = \sum_{s \leq t} \left(1 - \frac{\delta}{|x(s) - x(s-)|}\right)^+ (x(s) - x(s-)).$$

The process $\mathbf{Z}^\delta_n$ takes the form

$$Z^\delta_n(t) = \sum_{k=1}^{[nt]} \left[\left(\frac{Y_k}{\phi(n)}\right) \mathbb{I}_{(-\delta,\delta)}\left(\frac{Y_k}{\phi}\right) + \delta \operatorname{sgn}\left(\frac{Y_k}{\phi(n)}\right) \mathbb{I}_{(-\infty,-\delta]\cup[\delta,\infty)}\left(\frac{Y_k}{\phi}\right)\right], \tag{4.4}$$

and is of finite variation. Let $T_t(\mathbf{Z}) := \sup \sum_i |Z(t_{i+1}) - Z(t_i)|$ (where the supremum is taken over all partitions of $[0,t]$). Then

$$T_t(\mathbf{Z}_n^\delta) = \sum_{k=1}^{[nt]} \left|\frac{Y_k}{\phi(n)}\right| \mathbb{I}_{(-\delta,\delta)}\left(\frac{Y_k}{\phi(n)}\right) + \delta \mathbb{I}_{(-\infty,-\delta]\cup[\delta,\infty)}\left(\frac{Y_k}{\phi}\right). \tag{4.5}$$

Define the stopping time $\tau_n^d := \inf\{t : T_t(\mathbf{Z}_n^\delta) \geq d\}$ $(\tau_n^d < \infty)$. For $\alpha > 0$ there exists $d_\alpha$ such that $P\{\tau_n^{d_\alpha} \leq \alpha\} = P\{T_\alpha(\mathbf{Z}_n^\delta) \geq d_\alpha\} \leq 1/\alpha$ and we have

$$\begin{aligned} T_{t\wedge\tau_n^{d_\alpha}}(\mathbf{Z}_n^\delta) \leq d_\alpha &+ \left|\frac{Y_{n\tau_n^{d_\alpha}+1}}{\phi(n)}\right| \mathbb{I}_{(-\delta,\delta)}\left(\frac{Y_{n\tau_n^{d_\alpha}+1}}{\phi(n)}\right) \\ &+ \delta \mathbb{I}_{(-\infty,-\delta]\cup[\delta,\infty)}\left(\frac{Y_{n\tau_n^{d_\alpha}+1}}{\phi(n)}\right). \end{aligned}$$

So, we get

$$\sup_n \mathbb{E} T_{t\wedge\tau_n^{d_\alpha}}(\mathbf{Z}_n^\delta) \leq d_\alpha + 2\delta. \tag{4.6}$$

Assuming that the sequence $(\mathbf{Z}_n, \theta_n, S_{n,0})$ converges weakly to $(\mathbf{Z}, \theta, S_0)$ in the topology of $\mathbb{D}([0,\infty), \mathbb{R}^2) \times \mathbb{R}$ and making use of Theorem 2.2 of Kurtz and Protter (1991) we conclude that the sequence $\{\mathbf{G}_n\}_{n=1}^\infty$ converges weakly to a process $\mathbf{G} = \{G(t)\}$ which describes the cumulative gain up to time $t$ and is given by

$$G(t) = \int_0^t \theta(s-)\, dS(s). \tag{4.7}$$

Figures 4.1 and 4.2 give graphical representations of the process $\{G(t)\}$ given by equation (4.7) with $\theta(t) = 1$, and constructed with the use of a process $\{S(t)\}$ solving the following, quite realistic, stochastic problem:

$$S(t) = S_0 + \lambda \int_0^t S(s-)\, dL_{\alpha,\beta}(s),$$

which can be understood as a proposal of a possible generalization of the equation

$$X(t) = X_0 + \lambda \int_0^t X(s)\, dB(s),$$

studied, e.g., in Duffie and Harrison (1993) in a more general context. Here we have chosen $S_0 = 100.0$ a.s., $\lambda = 0.01$, and $\beta = 0$.

Detailed descriptions of statistical methods and computer techniques developed for the algorithms providing the presented results can be found in Janicki and Weron (1994a). Now, let us only mention that Figures 4.1 and 4.2 present 5 realizations (thin lines, with vertical intervals marking large upward or downward jumps of trajectories) of the constructed process

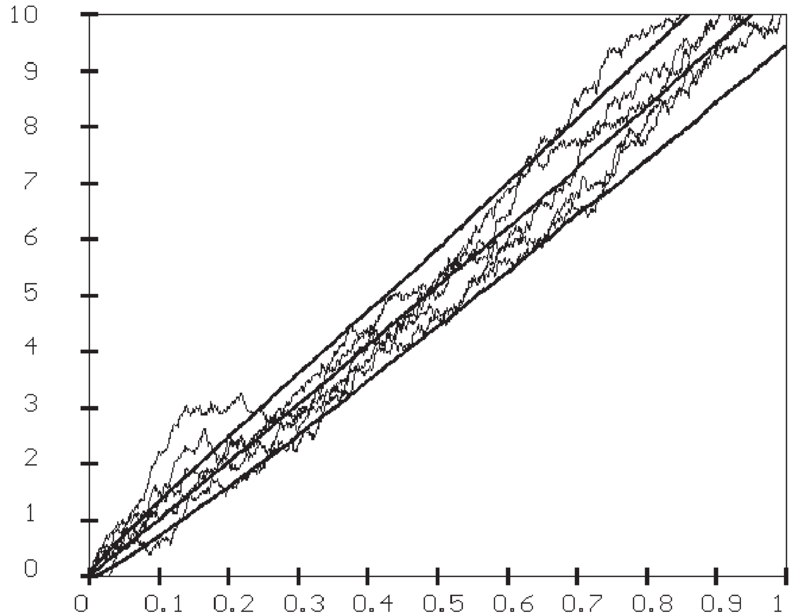


Fig. 4.1. Computer simulation and visualization of the cumulative stochastic gain process $\{G(t) : t \in [0,1]\}$ with respect to Brownian motion ($\alpha = 2$)

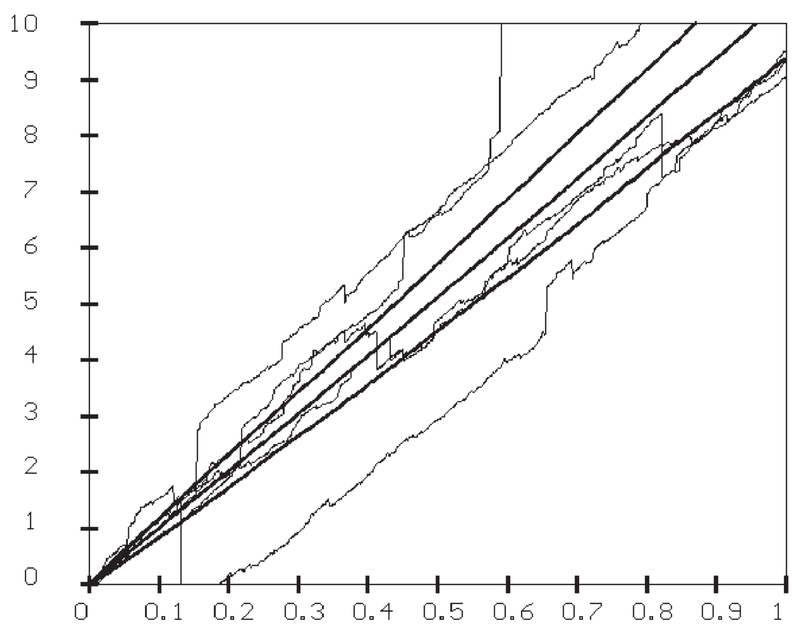


Fig. 4.2. Computer simulation and visualization of the cumulative stochastic gain process $\{G(t) : t \in [0,1]\}$ with respect to $\alpha$-stable Lévy motion with $\alpha = 1.3$, $\beta = 0$

(with $\alpha = 2$ or $\alpha = 1.3$) and estimators of 3 quantile lines (thick lines), i.e. lines $q_i = q_i(t)$ defined by a condition $P\{G(t) \geq q_i(t)\} = p_i$, for $p_i = 0.25, 0.5, 0.75$, constructed on the basis of a statistical sample containing 5000 approximate realizations of appropriate random variables. The time discretizing parameter $n$ was fixed to be equal to 1000.

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