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A new De Vylder type approximation of the ruin probability in infinite time

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Abstract

In this paper we introduce a generalization of the De Vylder approximation. Our idea is to approximate the ruin probability with the one for a different process with gamma claims, matching first four moments. We compare the two approximations studying mixture of exponentials and lognormal claims. In order to obtain exact values of the ruin probability for the lognormal case we use Pollaczeck-Khinchine formula. We show that the proposed 4-moment gamma De Vylder approximation works even better than the original one.

Keywords: Risk process; Ruin probability; De Vylder approximation; Pollaczeck-Khinchine formula

1. Introduction

In a very interesting paper Grandell (2000) demonstrates that between possible simple approximations of ruin probabilities in infinite time the most successful is the De Vylder approximation, which is based on the idea to replace the risk process with the one with exponentially distributed claims and ensuring that the first three moments coincide.

We introduce a modification to the De Vylder approximation by changing the exponential distribution to the gamma and making the first four moments match. This modification is promising and works in many cases even better than the original approximation. In order to compare De Vylder and 4-moment gamma De Vylder (4MGDV) approximations we consider mixture of two exponentials and lognormal claims. We compute relative errors of the methods with respect to the exact values of the ruin probability. The ruin probability in the lognormal case is calculated via Pollaczeck-Khinchine formula.

In our paper we consider a classical risk model (see e.g. Grandell, 1991). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying Poisson process $\{N_t\}_{t\geq 0}$ with intensity λ, and sequence $\{X_k\}_{k=1}^{\infty}$ of independent, positive, identically distributed random variables, with mean μ and (if existing) μ^2, μ^3, ... being the raw moments. Furthermore, we assume that $\{X_k\}$ and $\{N_t\}$ are independent. The classical risk process $\{R_t\}_{t\geq 0}$ is given by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i, \tag{1}$$

where c is some positive constant and u is nonnegative.

This is the standard mathematical model for an insurance surplus process. The initial capital is u, the Poisson process N_t describes the number of claims in $(0, t]$ interval and claim severities are random, given by sequence $\{X_k\}_{k=1}^{\infty}$. To cover its liability, the insurance company receives premium at a constant rate c, per unit time, where $c = (1+\theta)\lambda\mu$ and $\theta > 0$ is often called the relative safety loading. The loading has to be positive, otherwise c would be less than $\lambda\mu$ and thus with probability 1 the risk business would become negative in infinite time.

For mathematical purposes, it is sometimes more convenient to work with an aggregate loss process $\{S_t\}_{t\geq 0}$, namely $S_t = u - R_t = \sum_{i=1}^{N_t} X_i - ct$. Now, we are going to recall the definition

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of ruin probability, i.e. the probability that the surplus drops below zero. Time to ruin is defined as $\tau(u) = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : S_t > u\}$. Let $M = \sup_{0 \leq t < \infty}\{S_t\}$ and $M_T = \sup_{0 \leq t \leq T}\{S_t\}$. Ruin probability in finite time is given by $\psi(u,T) = \mathbb{P}(\tau(u) \leq T) = \mathbb{P}(M_T > u)$ and ruin probability in infinite time is defined as

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(M > u). \tag{2}$$

In the sequel we assume $c = 1$, but it is not a restrictive assumption. Following Asmussen (2000), let $c \neq 1$ and define $\tilde{R}_t = R_{\frac{t}{c}}$. Then relations between ruin probabilities $\psi(u)$, $\psi(u,T)$ for the process R_t and $\tilde{\psi}(u)$, $\tilde{\psi}(u,T)$ for the process $\tilde{R}_t$ are given by equations: $\psi(u) = \tilde{\psi}(u)$, $\psi(u,T) = \tilde{\psi}(u,Tc)$.

In the classical risk model framework the infinite time ruin probabilities have been studied recently in Grübel and Hermesmeier (1999, 2000) with emphasis on Panjer recursion and transform methods, and Usábel (2001) who presented a method of inverting the Laplace transform. For a detailed discussion how to create accurate bounds and approximations of ψ see Dufresne and Gerber (1989).

2. De Vylder and 4MGDV approximations

The idea of the De Vylder approximation is to replace the risk process with the one with $\theta = \bar{\theta}$, $\lambda = \bar{\lambda}$ and exponential claims with parameter $\bar{\beta}$, fitting first three moments, see De Vylder (1978).

Let

$$\bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}, \qquad \bar{\lambda} = \frac{9\lambda {\mu^{(2)}}^3}{2{\mu^{(3)}}^2}, \qquad \textit{and} \qquad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3{\mu^{(2)}}^2}\theta.$$

Then De Vylder's approximation is given by (see e.g. Grandell, 1991)

$$\Psi_{DV}(u) = \frac{1}{1+\bar{\theta}} e^{-\frac{\bar{\theta}\bar{\beta}u}{1+\bar{\theta}}}. \tag{3}$$

Obviously, in the exponential case the method gives the exact result. For other claim distributions, in order to apply the approximation, the first three moments have to exist.

We now introduce a new 4-moment gamma De Vylder approximation based on the De Vylder's idea to replace the risk process with another one for which the expression for $\psi(u)$ is explicit. We fit the four moments in order to calculate the parameters of the new process with gamma distributed claims and apply the exact formula for the ruin probability in this case which is given e.g. in Grandell and Segerdahl (1971). The risk process with gamma claims is determined by the four parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$. Since

$$\begin{aligned}
\mathbb{E}S_t &= -\theta\lambda\mu t, \\
\mathbb{E}S_t^2 &= \lambda\mu^{(2)}t + (\theta\lambda\mu t)^2, \\
\mathbb{E}S_t^3 &= \lambda\mu^{(3)}t - 3(\lambda\mu^{(2)}t)(\theta\lambda\mu t) - (\theta\lambda\mu t)^2, \\
\mathbb{E}S_t^4 &= \lambda\mu^{(4)}t - 4(\lambda\mu^{(3)}t)(\theta\lambda\mu t) + 3(\lambda\mu^{(2)}t)^2 + 6(\lambda\mu^{(2)}t)(\theta\lambda\mu t)^2 + (\theta\lambda\mu t)^4
\end{aligned}$$

and for the gamma distribution $\bar{\mu}^{(3)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}}(2\bar{\mu}^{(2)} - \bar{\mu}^2)$, $\bar{\mu}^{(4)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2)$, the parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$ must satisfy the equations

$$\theta\lambda\mu = \bar{\theta}\bar{\lambda}\bar{\mu}, \quad \lambda\mu^{(2)} = \bar{\lambda}\bar{\mu}^{(2)}, \quad \lambda\mu^{(3)} = \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2), \quad \lambda\mu^{(4)} = \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2).$$

Hence

$$\bar{\lambda} = \frac{\lambda(\mu^{(3)})^2(\mu^{(2)})^3}{(\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2)}, \qquad \bar{\theta} = \frac{\theta\mu(2(\mu^{(3)})^2 - \mu^{(2)}\mu^{(4)})}{(\mu^{(2)})^2\mu^{(3)}},$$

$$\bar{\mu} = \frac{3(\mu^{(3)})^2 - 2\mu^{(2)}\mu^{(4)}}{\mu^{(2)}\mu^{(3)}}, \qquad \bar{\mu}^{(2)} = \frac{(\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2)}{(\mu^{(2)}\mu^{(3)})^2}.$$

We also need to assume that $\mu^{(2)}\mu^{(4)} < \frac{3}{2}(\mu^3)^2$ and $\mu^{(2)}\mu^{(4)} > \frac{1}{2}(\mu^3)^2$ to ensure that $\bar{\mu}, \bar{\mu}^{(2)} > 0$ and $\bar{\mu}^{(2)} > \bar{\mu}^2$. In case this assumption can not be fulfilled, we simply set $\bar{\mu} = \mu$ and do not calculate the fourth moment. This case leads to

$$\bar{\lambda}=\frac{2\lambda(\mu^{(2)})^2}{\mu(\mu^{(3)}+\mu^{(2)}\mu)},\qquad \bar{\theta}=\frac{\theta\mu(\mu^{(3)}+\mu^{(2)}\mu)}{2(\mu^{(2)})^2},\qquad \bar{\mu}=\mu,\qquad \bar{\mu}^{(2)}=\frac{\mu(\mu^{(3)}+\mu^{(2)}\mu)}{2\mu^{(2)}}. \tag{4}$$

All in all, we get the approximation

$$\psi_{4MGDV}(u)=\frac{\bar{\theta}(1-\frac{R}{\bar{\alpha}})e^{-\frac{\bar{\beta}R}{\bar{\alpha}}u}}{1+(1+\bar{\theta})R-(1+\bar{\theta})(1-\frac{R}{\bar{\alpha}})}+\frac{\bar{\alpha}\bar{\theta}sin(\bar{\alpha}\pi)}{\pi}\cdot I, \tag{5}$$

where

$$I=\int_0^\infty\frac{x^{\bar{\alpha}}e^{-(x+1)\bar{\beta}u}\,dx}{\left[x^{\bar{\alpha}}\left(1+\bar{\alpha}(1+\bar{\theta})(x+1)\right)-cos(\bar{\alpha}\pi)\right]^2+sin^2(\bar{\alpha}\pi)},$$

and $(\bar{\alpha},\bar{\beta})$ are given by $\bar{\alpha}=\frac{\bar{\mu}^2}{\bar{\mu}^{(2)}-\bar{\mu}^2}$, $\bar{\beta}=\frac{\bar{\mu}}{\bar{\mu}^{(2)}-\bar{\mu}^2}$.

In the exponential and gamma case this method gives the exact results. For other claim distributions in order to apply the approximation, the first four (or three) moments have to exist. In Section 3 will show that it gives a slight correction to the De Vylder approximation, which is said in Grandell (2000) to be the best among "simple" approximations.

3. Pollaczeck–Khinchine formula

This time we use the representation (2) of the ruin probability and the decomposition of the maximum M as a sum of ladder heights. Let L_1 be the value that process $\{S_t\}$ reaches for the first time above the zero level. Next, let L_2 be the value which is obtained for the first time above the level L_1; $L_3, L_4, \ldots$ are defined in the same way. The values L_k are called ladder heights. Since the process $\{S_t\}$ has stationary and independent increments, $\{L_k\}_{k=1}^{\infty}$ is the sequence of independent and identically distributed variables. One may show that the number of ladder heights K to the moment of ruin is given by a geometric distribution with parameters $p=\frac{1}{1+\theta}$ and $q=\frac{\theta}{1+\theta}$. Thus, random variable M may be expressed by

$$M=\sum_{i=1}^{K}L_i, \tag{6}$$

This implies that random variable M has a compound geometric distribution given by the distribution function

$$F_M(x)=\frac{\theta}{1+\theta}\sum_{n=0}^{\infty}G^{*n}(x), \tag{7}$$

where G is the defective density

$$g(x)=\frac{1}{\mu(1+\theta)}\bar{F}_X(x)=\frac{1}{1+\theta}b_0(x) \tag{8}$$

and the density

$$b_0(x)=\frac{\bar{F}_X(x)}{\mu}. \tag{9}$$

The above fact together with the representation (2) leads to the Pollaczeck–Khinchine formula for the ruin probability:

$$\psi(u)=\mathbb{P}(M>u)=\frac{\theta}{1+\theta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\theta}\right)^n B_0^{*n}(u), \tag{10}$$

where $\bar{B}_0$ is the tail of the distribution function corresponding to the density b_0.

One can use it to derive explicit solutions for a variety of claim amount distributions, particularly those whose Laplace transform is a rational function, cf. Panjer and Willmot (1992). Unfortunately, the lognormal case in not included. However, in order to calculate the ruin probability the formula can be also applied directly. It incorporates an infinite sum, hence we use the Monte Carlo method. Using (10), the ruin probability $\psi(u)=EZ$, where $Z=1(M>u)$, may be generated as follows, cf. Asmussen (2000).

(1) Generate a random variable K from the geometric distribution with the parameters $p=\frac{1}{1+\theta}$ and $q=\frac{\theta}{1+\theta}$.

(2) Generate random variables $X_1, X_2, \cdots, X_K$ from the density $b_0(x)$.
(3) Calculate $M = X_1 + X_2 + \cdots + X_K$.
(4) If $M > u$, let $Z = 1$, otherwise let $Z = 0$.

The main problem seems to be simulating random variables from the density $b_0(x)$. In the lognormal case the density does not have a closed form. Consequently, in order to generate random variables X_k we use formula (9) and controlled numerical integration.

4. De Vylder vs 4MGDV approximation

We now aim to compare De Vylder and 4-moment gamma De Vylder approximations. To this end we consider the ruin probability as a function of the initial capital u, with two claim amount distributions, namely mixture of two exponentials representing the light-tailed case and lognormal being a prominent example of the heavy-tailed case, cf. Embrechts et al. (1997). In order to show the relative errors of the methods we compare results of the approximations with the exact values.

In the case of mixture of two exponentials distribution, exact values of the ruin probability can be computed using inversion of Laplace transform technics, see e.g. Panjer and Willmot (1992). Figure 1a depicts the exact ruin probability values and results of the De Vylder and 4-moment gamma De Vylder approximations. Figure 1b demonstrates that the relative error of the latter is less than 8% and proves that it gives much better results than the original method which reaches the 50% error.

When the claim amount distribution is lognormal, the formula for the ruin probability does not have a closed form, cf. Thorin and Wikstad (1977). Therefore we employ the Pollaczeck–Khinchine formula to obtain exact results. For the Monte Carlo method purposes we generate 100 blocks of 100000 simulations. We also note that the variance within the results derived from the blocks was always below $3 \cdot 10^{-6}$. Figure 2a illustrates the exact ruin probability values and results of the De Vylder and 4-moment gamma De Vylder approximations. Figure 2b shows that the relative error of the 4-moment gamma De Vylder approximation is always significantly less than the error of the original one.

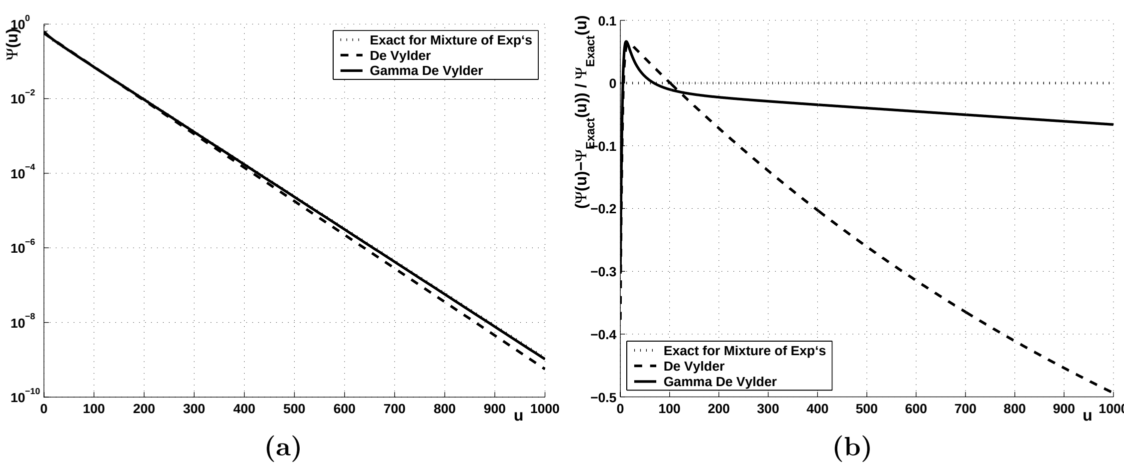

Figure 1. Illustration of the ruin probability (a) and the relative error (b) of the approximations (with respect to exact values). Mixture of exponentials case with $\beta_1 = 0.04$, $\beta_2 = 2$, weights $a_1 = 0.002$, $a_2 = 0.998$, $\theta = 0.1$ and $u \leq 1000$.

Finally, let us note that we have conducted similar studies for other light- and heavy-tailed claim size distributions, e.g. Weibull, Pareto, Burr and loggamma, with different parameters. They justify the thesis the 4-moment gamma De Vylder approximation often works better than the De Vylder approximation.

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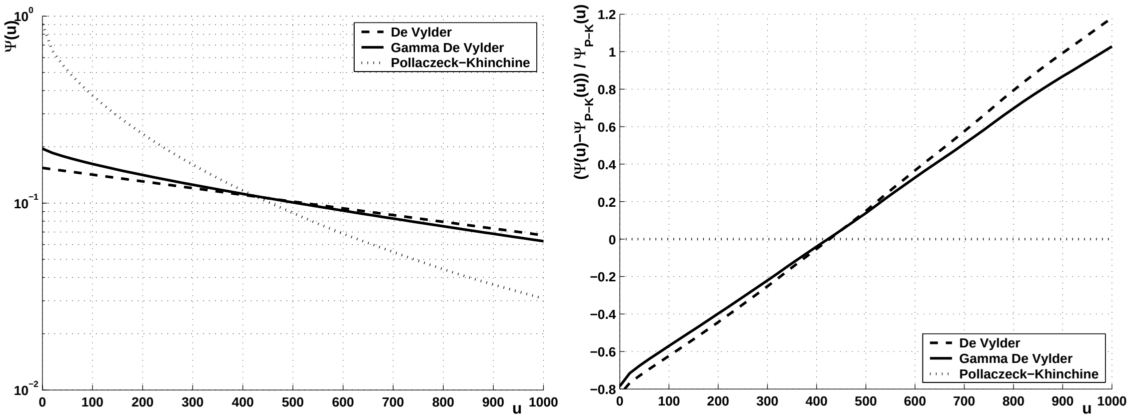

(a) **(b)**

FIGURE 2. Illustration of the ruin probability (a) and the relative error (b) of the approximations (with respect to the values obtained via the Pollaczeck–Khinchine formula). Lognormal case with $\mu = -3$ i $\sigma = 2.1$, $\theta = 0.1$ and $u \leq 1000$.

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