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Silvia Faggian, Luca Grosset

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Optimal investment in age-structured goodwill

Silvia Faggian[1], Luca Grosset[2]

[1]Dipartimento di Matematica Applicata, Universita' di Venezia
Ca' Dolfin, Dorsoduro 3825/e - 30123 Venezia
faggian@unive.it

[2]Dipartimento di Matematica Pura ed Applicata
Università degli Studi di Padova
Via Trieste, 63 - 35121 Padova
grosset@math.unipd.it

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Abstract

Segmentation is a core strategy in modern marketing and age-specific segmentation, which is based on the age of the consumers, is very common in practice. A characteristic of age-specific segmentation is the change of the segments composition during time, which may be studied only using dynamic advertising models. Here, we assume that a firm wants to promote and sell a single product in an age segmented market and we model the awareness of this product using an infinite dimensional Nerlove-Arrow goodwill as a state variable. Assuming an infinite time horizon, we use some dynamic programming techniques to solve the problem and to characterize both the optimal advertising effort and the optimal goodwill path in the long run. An interesting feature of the optimal advertising effort is an anticipation effect with respect to the segments considered in the target market due to the time evolution of the segmentation.

Keywords: dynamic programming, advertising, vintage capital.

JEL Classifcation Numbers: E22, M37, C61, C62.

1 Introduction

Segmentation is a core strategy in modern marketing which, coupled with dynamic advertising models, represents an interesting area of research where different disciplines can interact. Market segmentation is the result of partitioning the whole market into distinct consumer groups, each characterized by special sets of attribute values, so that its members will exhibit the same needs and behaviors [1, p.379]. The introduction of segmentation in a dynamic advertising model allows to analyze some phenomena which cannot be studied from a static point of view. For example, segmentation may help the decision maker to compare different advertising media (see [2]), or is useful to analyze advertising strategies distributed over several geographic regions (see [3] and [4]).

In this paper we focus on the demographic segmentation based on consumers' age. Age segmentation is very common in practice, because data may be simply organized according to age of consumers and because some products are spontaneously age-specific. A characteristic of age-specific segmentation, which can be studied only in a dynamic framework, is the change of the segments composition during time. For example, the advertising effort addressed to the "16 years segment" will be found in the "18 years segment" after 2 years. This phenomenon requires models of investment with *vintage capital* [5] and the idea that advertising may fall within this area has already been presented in [6]. In that paper the authors assume that a firm sells a continuum of goods: new goods are continuously launched by the firm onto the market and therefore these goods are differentiated by their vintage. Our approach is different: as in [7] we assume that the firm sells a single product, and the sales of this good in segment a at the time t depend on the goodwill level for this product in segment a at the time t. Even if, from a mathematical point of view, the model is similar to the one described in [6], the economic interpretation in term of marketing variables is definitely different.

The aim of this paper is to study the deep connections between the age-specific segmentation and the equilibrium points in the long run. The paper is organized as follows: in section 2 we described the model and we introduce the characteristic functions of an age-specific segmentation. In section 3 we solve the abstract problem using some Dynamic Programming techniques in infinite dimension and we present an explicit solution for the linear quadratic instance of the model. Finally, in section 4 we show how to apply our results in some real situations and we present some numerical simulations which explicitly show the anticipation effect for the advertising effort with respect to the segments in the target market.

2 The Model

We assume a firm sells a good in a segmented market and wants to organize an advertising campaign to support this good. The market is segmented using age as a demographic variable, which is assumed continuous with values in $[0, \omega]$. As a state variable we use the Nerlove-Arrow goodwill, which summarizes the past investment in advertising. In order to describe the market segmentation we introduce the state variable $G(t, a)$, that represents the goodwill value at time $t \in [0, +\infty)$ for the consumers of age $a \in [0, \omega]$. The goodwill evolution is described (as in [7]) by means of the following partial differential equation

$$\partial_t G(t,a) + \partial_a G(t,a) = -\delta G(t,a) + u(t,a) \tag{1}$$

where $u(t,a) \geq 0$ is the control variable for the decision maker, and represents the advertising effort at time t addressed to the consumer segment of age a. We assume the boundary conditions

$$G(t,0) = 0 \quad \text{for all } t \in [0,+\infty) \tag{2}$$

and the initial condition

$$G(0,a) = g(a) \geq 0 \quad \text{for all} \quad a \in [0,\omega], \tag{3}$$

which mean, respectively, that the goodwill value for the segment of age 0 must be always zero and the initial goodwill value among different segments is given.

Moreover, we assume that the profit flow is linear in the goodwill for each age $a \in [0,\omega]$ and is described by the following function

$$t \mapsto \pi(a) G(t,a)$$

where $\pi(a) > 0$ represents profit rate of the segment a (except for the advertising costs). Hence, the whole profit flow at the time t is described by the function

$$t \mapsto \int_0^\omega \pi(a) G(t,a)\, da.$$

This assumption is quite familiar in dynamic advertising model when closed form solutions are desired (see e.g. [2] and the references therein).

The definition of advertising costs is complicated by the fact that the costs to reach a certain segment need be characteristic of such segment. Hence, we are assuming that the advertising costs are stationary in time, but nonstationary in age. Thus we assume a general advertising cost function such as

$$\begin{array}{rcl} c : [0,\omega] \times [0,+\infty) & \rightarrow & [0,+\infty) \\ (a,u) & \mapsto & c(a,u) \end{array} \tag{4}$$

where $c(a,u)$ is the cost flow to reach the segment a with an advertising effort u. The function c is assumed to be increasing and possibly (but not necessarily) convex in the advertising effort variable u. Under this assumption the total advertising cost flow is

$$t \mapsto \int_0^\omega c(a, u(t,a))\, da$$

We notice that often in literature the advertising cost function is assumed quadratic (see e.g. [8, p. 103]) which corresponds to defining a simple instance of the (4), that is

$$c(a,u) = \frac{\kappa(a)}{2} u^2.$$

Summarizing all our assumptions, we may formulate the problem as follows. A firm wants to organize an advertising campaign (choosing an advertising function $u(t,a) \geq 0$) in order to maximize the functional

$$J[u(t,a), g(a)] = \int_0^{+\infty} \int_0^{\omega} e^{-\rho t} [\pi(a) G(t,a) - c(a, u(t,a))] \, da \, dt \tag{5}$$

under the constraint

$$\begin{cases} \partial_t G(t,a) + \partial_a G(t,a) = -\delta G(t,a) + u(t,a) \\ G(t,0) = 0 \ \text{ for all } t \in [0, +\infty) \\ G(0,a) = g(a) \geq 0 \ \text{ for all } \ a \in [0, \omega] \end{cases}$$

If arrested at the short run, the model falls within the family of age-structured control systems and can be studied using the necessary condition introduced in [9]. In the long run instead, Dynamic Programming techniques in infinite dimension prove to be efficient in computing optimal couples and equilibrium points. Such techniques are also promising in the case, which we leave for future work, in which the profit flow has a nonlinear dependence on the goodwill $G(t,a)$.

In order to clarify the more general case of the next section, we briefly recall the classical linear quadratic instance of this model. If market is not segmented, then we have a single goodwill $G(t)$ and its evolution is described by an ordinary differential equation

$$\dot{G}(t) = -\delta G(t) + u(t) \tag{6}$$

which is the motion equation introduced by Nerlove and Arrow in their seminal paper [14]. The objective functional, assuming a quadratic advertising cost function is

$$J[u(t)] = \int_0^{+\infty} e^{-\rho t} \left[\pi G(t) - \frac{\kappa}{2} [u(t)]^2 \right] dt \tag{7}$$

Using the phase-space analysis it is simple to obtain that the optimal advertising investment u^{opt} is constant

$$u^{opt}(t) = \pi / \kappa (\delta + \rho) \tag{8}$$

and the equilibrium goodwill G_∞ is

$$G_\infty(t) = \pi / \delta\kappa (\delta + \rho) \tag{9}$$

As we want to take an age segmentation into account we have to modify both the motion equation (6) and the objective functional (7). In the motion equation we need to consider time evolution of segments and the possibility of reaching reach different segments with different advertising efforts, we leads to a partial differential equation (1). In the objective functional we need to consider that both revenues and advertising costs are segment-specific, hence the profit is described by (5). Finally, we remark that an equilibrium point as (9) becomes, in an age-structured setting, an equilibrium function of the variable a, which describes segmentation in the long run. One of the aims of this paper is to provide existence of such equilibrium function and study its dependence on the functions $\pi(a)$ and $c(a,u)$ that describe the segmentation.

3 Dynamic Programming in infinite dimension

One of the aim of this paper is to show how infinite dimensional techniques may be powerful and efficient to compute optimal couples and equilibrium points. The technique consists in rephrasing the original control problem for the partial differential equation (1) as a problem for a suitable ordinary differential equation, but in an infinite dimensional setting. The reader is adviced that the unconstrained problem is studied first, while later in subsection **??** we show how the problem with positivity constraints on the control may be obtained as an equivalent unconstrained problem with modified cost functions.

First of all we consider the space of square integrable functions of variable a as the *state space* of the control problem, that is

$$L^2(0,\omega) = \{f : [0,\omega] \to \mathbb{R} : \|f\|_2 < +\infty\}$$

where $\|f\|_2 = \left(\int_0^\omega |f(a)|^2\, da\right)^{\frac{1}{2}}$, and $\langle f, g\rangle$ denotes the scalar product of f and g in $L^2(0,\omega)$. Then the state variable $G(t)$ is that function of $L^2(0,\omega)$ such that $G(t)(a) = G(t,a)$. Similarly one chooses $L^2(0,\omega)$ as the *control space,* and $u(t)$ as the control, where $u(t)$ is that function of $L^2(0,\omega)$ such that $u(t)(a) = u(t,a)$. Hence, if f' denotes the distributional derivative of f, we set

$$H^1(0,\omega) = \left\{f \in L^2(0,\omega) : f' \in L^2(0,\omega)\right\}$$

and introduce the differential operator A with domain $D(A) = \{f \in H^1(0,\omega) : f(0) = 0\}$ such that

$$A : D(A) \subset L^2(0,\omega) \to L^2(0,\omega), \qquad Af(a) = -f'(a) - \delta f(a),$$

so that the original evolution system given by (1), (2), (3) becomes a control system for an ordinary differential equation in the Hilbert space $L^2(0,\omega)$:

$$\begin{cases} \dot{G}(t) = AG(t) + u(t) & t > 0 \\ G(0) = g \in L^2(0,\omega) \end{cases} \tag{10}$$

The boundary condition is enclosed into the definition of the domain $D(A)$ of the operator A. The control operator is the identity function on $L^2(0,\omega)$.

The technique is very well known and may be found in classical books on evolution equations such as [10] or the more recent [11]. In [10], [11] one finds also that the operator A is the generator of a strongly continuous semigroup of operators $\{e^{tA}\}_{t\geq 0}$ with

$$[e^{tA}f](a) = e^{-\delta t}f(a-t)\chi_{[t,\omega]}(a), \tag{11}$$

for all $f \in L^2(0,\omega)$, $a \in [0,\omega]$. Then by means of variation of constants formula the trajectory is given by the following function in $L^2(0,\omega)$

$$G(t) = e^{tA}g + \int_0^t e^{(t-s)A}u(s)ds. \tag{12}$$

By means of (11), the preceding formula can be explicited as

$$G(t,a) = e^{-\delta t}g(a-t)\chi_{[t,\omega]}(a) + \int_0^{\min\{t,a\}} e^{-\delta s}u(t-s,a-s)ds. \tag{13}$$

Regarding the objective functional, we assume that the marginal profit function π is in the space $L^2(0,\omega)$, so that the running revenue R can be described using the scalar product:

$$R : L^2(0,\omega) \to \mathbb{R} \qquad G \mapsto R(G) = \int_0^\omega \pi\,(a)\,G(a)da = \langle \pi, G\rangle \tag{14}$$

while the running advertising investment cost is

$$C : L^2(0,\omega) \to \mathbb{R} \qquad u \mapsto C(u) = \int_0^\omega c\,(a, u(t,a))\,da$$

so that the objective functional is

$$J\,[g,u] = \int_0^{+\infty} e^{-\rho t}\,[\langle \pi, G(t)\rangle - C(u(t))]\;\,dt \tag{15}$$

We notice that this functional is concave in the control-state variables. Moreover, it is structurally similar to (7) but the functions $R(G(t))$ and $C(u(t))$ hide data aggregated with respect to the a variable. The control problem is that of maximizing J over the set of admissible controls

$$\mathcal{U} = L^p_\rho(0,+\infty; L^2(0,\omega)) \tag{16}$$

$$= \left\{ u : [0,+\infty) \to L^2(0,\omega) : \int_0^{+\infty} e^{-\rho t}\,\|u(t)\|_2^p\,dt < +\infty \right\}. \tag{17}$$

Using this notation we define the value function of the optimal control problem in the usual way:

$$V(g) = \sup_{u\in\mathcal{U}} J[g,u].$$

3.1 Discounted demand in abstract terms

Next we list some properties of A^*, the adjoint operator of A, that prove useful in the sequel.

Lemma 1 *If A^* is the adjoint of A, namely*

$$A^* : D(A^*) \subset L^2(0,\omega) \to L^2(0,\omega), \quad A^* f(a) = f'(a) - \delta f(a),$$

with $D(A^*) = \{f \in H^1(0,\omega) : f(\omega) = 0\}$, then e^{tA^*} is the adjoint of e^{tA}, with

$$[e^{tA^*} f](a) = e^{-\delta t} f(a+t)\chi_{[0,\omega-t]}(a).$$

Moreover for all $\rho > -\delta$ the operator $\rho I - A^*$ is invertible with continuous inverse

$$[(\rho I - A^*)^{-1} f](a) = \int_0^{+\infty} e^{-\rho t}[e^{tA^*} f](a)dt = \int_a^{\omega} e^{-(\rho+\delta)(\sigma-a)} f(\sigma)d\sigma.$$

For the proof of this Lemma and for all details on the semigroup and its adjoint the reader is referred to [12].

We now set

$$\bar{\pi}(a) \equiv \int_a^{\omega} e^{-(\rho+\delta)(\sigma-a)}\pi(\sigma)d\sigma \tag{18}$$

and note that this function represents the *discounted demand associated to one unit of goodwill of the segment a*. Indeed the segment of age a becomes of age σ after a time $\sigma - a$, hence the discounted demand of the segment a, which becomes of age σ after a time $\sigma - a$, is

$$e^{-\rho(\sigma-a)}\pi(\sigma).$$

In the meantime the unit of goodwill is exponentially decreased and amounts to $e^{-\delta(\sigma-a)}$. Therefore the discounted demand of a unit of goodwill of the segment a for being of age σ after a time $\sigma - a$ is the integrand in (18)

$$e^{-(\rho+\delta)(\sigma-a)}\pi(\sigma)$$

It is easy to show that if π is in $L^2(0,\omega)$, then $\bar{\pi}$ has a derivative in the sense of distributions and that $\bar{\pi}'$ is also in $L^2(0,\omega)$.

Moreover, the previous lemma yields

$$\bar{\pi}(a) = [(\rho I - A^*)^{-1}\pi](a).$$

3.2 Optimal strategies and trajectories

We assume the following hypotheses:

(H1) $\pi \in L^2(0,\omega)$,

(H2) $C : L^2(0,\omega) \to \mathbb{R} \cup \{+\infty\}$, C measurable and such that $\mathcal{A} = \arg\max \left\{ \langle \overline{\pi}, u \rangle - C(u) : u \in L^2(0,\omega) \right\} \neq \emptyset$.

Remark 2 *Assumption (H2) may be explicited in a number on interesting cases. For instance, assume*

(H3) $C : L^2(0,\omega) \to \mathbb{R} \cup \{+\infty\}$ *convex,* $C \not\equiv +\infty$, l.s.c. *and with superlinear growth, that is*

$$\lim_{||v||_2 \to +\infty} \frac{C(v)}{||v||_2} = +\infty.$$

Moreover, if we denote with

$$C^*(u) = \sup_{v \in L^2(0,\omega)} \{\langle u, v \rangle - C(v)\}$$

the Legendre transform of C*, and with* ∂C^* *the subgradient of the convex function* C^**, then (H3) implies (H2), as*

$$\mathcal{A} = \partial C^*(\overline{\pi})$$

Indeed, the function $u \mapsto \langle \overline{\pi}, u \rangle - C(u)$ *has a maximum at* $\overline{u}$ *if and only if* $\overline{\pi} - \partial C(\overline{u}) \ni 0$ *and, by the well known property of convex conjugate functions,*

$$\overline{\pi} - \partial C(\overline{u}) \ni 0 \Leftrightarrow \overline{\pi} \in \partial C(\overline{u}) \Leftrightarrow \overline{u} \in \partial C^*(\overline{\pi}).$$

Then we come to the main result of this subsection, describing optimal controls and the value function of the control problem. Due to the linearity of the revenue with respect to the goodwill (see (14)), the objective functional may be written separating the dependence on u and g. As a consequence, the optimal control and the value function may be computed *explicitly.*

Theorem 3 *Assume that (H1) and (H2) hold, then the profit functional may be written as*

$$J[g,u] = \langle \overline{\pi}, g \rangle + \int_0^{+\infty} e^{-\rho t} \left[\langle \overline{\pi}, u(t) \rangle - C(u(t)) \right] dt.$$

Moreover, any optimal control u^{opt} *does not depend on* t *and satisfies*

$$u^{opt}(t) \equiv u^{opt} \in \mathcal{A}$$

consequently the optimal value function can be written as,

$$V(g) = \langle \overline{\pi}, g \rangle + \frac{\langle \overline{\pi}, u^{opt} \rangle - C(u^{opt})}{\rho}$$

In particular, it is an affine function of g, so that V is Fréchet-differentiable, with gradient

$$\nabla V(g) = \overline{\pi} \in L^2(0,\omega).$$

Proof. Even if the proof is similar to that given by Barucci and Gozzi in [12], for the reader's convenience we sketch it under our assumptions. The objective functional can be rewritten as

$$\begin{aligned}
& \int_0^{+\infty} e^{-\rho t} \left[\langle \pi, G(t) \rangle - C(u(t)) \right] \, dt \\
= & \int_0^{+\infty} e^{-\rho t} \left\langle \pi, e^{tA} g + \int_0^t e^{(t-s)A} u(s) ds \right\rangle dt - \int_0^{+\infty} e^{-\rho t} C(u(t)) dt \\
= & \int_0^{+\infty} e^{-\rho t} \left\langle e^{tA^*} \pi, g \right\rangle dt + \int_0^{+\infty} e^{-\rho t} \int_0^t \left\langle e^{(t-s)A^*} \pi, u(s) \right\rangle ds dt \\
& - \int_0^{+\infty} e^{-\rho t} C(u(t)) dt
\end{aligned}$$

Using Lemma 1, the first term becomes

$$\int_0^{+\infty} e^{-\rho t} \left\langle e^{tA^*} \pi, g \right\rangle dt = \left\langle \int_0^{+\infty} e^{-\rho t} e^{tA^*} \pi dt, g \right\rangle = \langle \bar{\pi}, g \rangle ,$$

while the second term is

$$\begin{aligned}
& \int_0^{+\infty} e^{-\rho t} \int_0^t \left\langle e^{(t-s)A^*} \pi, u(s) \right\rangle ds dt \\
= & \int_0^{+\infty} e^{-\rho s} \left\langle \int_s^{+\infty} e^{-\rho(t-s)} e^{(t-s)A^*} \pi dt, u(s) \right\rangle ds \\
= & \int_0^{+\infty} e^{-\rho s} \left\langle \int_0^{+\infty} e^{-\rho \sigma} e^{\sigma A^*} \pi d\sigma, u(s) \right\rangle ds = \int_0^{+\infty} e^{-\rho s} \langle \overline{\pi}, u(s) \rangle ds
\end{aligned}$$

and the first formula is proven. Now, we observe that

$$\begin{aligned}
& \sup_{u \in \mathcal{U}} \int_0^{+\infty} e^{-\rho t} \left[\langle \overline{\pi}, u(t) \rangle - C(u(t)) \right] dt \leq \\
& \leq \int_0^{+\infty} e^{-\rho t} \sup_{u \in L^2(0,\omega)} \left[\langle \overline{\pi}, u \rangle - C(u) \right] dt = \frac{\langle \overline{\pi}, \overline{u} \rangle - C(\overline{u})}{\rho}.
\end{aligned} \tag{19}$$

where the supremum in the last formula is attained at any $\overline{u} \in \mathcal{A}$. As all time-constant controls are admissible, the inequality in (19) is an equality. ■

Corollary 4 *Assume that (H1) and (H3) hold, then:*

$$u^{opt}(t) \equiv u^{opt} \in \partial C^*(\overline{\pi})$$

consequently the optimal value function can be written as,

$$V(g) = \langle \overline{\pi}, g \rangle + \frac{C^*(\overline{\pi})}{\rho}.$$

Proof. It is contained in Remark 2.

Remark 5 *If (H3) holds and moreover C^* is differentiable, than $\partial C^*(\overline{\pi})$ is singleton and contains only the Fréchet-differential $\nabla C^*(\overline{\pi})$, hence the optimal control is unique, it is independent of time t, and it is given by the following formula*

$$u^{opt}(t) \equiv u^{opt} = \nabla C^*(\overline{\pi})$$

3.3 Equilibrium points

For the sake of simplicity we now assume that (H3) holds and that $C^* : L^2(0, \omega) \to \mathbb{R}$ is finite and differentiable. Then the optimal feedback map is a constant map $\varphi : L^2(0, \omega) \to L^2(0, \omega)$, $\xi \mapsto \nabla C^*(\overline{\pi})$ so that the closed loop equation is

$$\dot{G}(t) = AG(t) + \nabla C^*(\overline{\pi}) \tag{20}$$

We define an *equilibrium point* for the system as any stationary solution of the closed loop equation (20), that is, a solution G_∞ in $L^2(0, \omega)$ to the equation

$$AG_\infty + \nabla C^*(\overline{\pi}) = 0. \tag{21}$$

Theorem 6 *Assume (H1) (H3) hold, and moreover that $C^* : L^2(0, \omega) \to \mathbb{R}$ is finite and differentiable. Then the unique equilibrium point for the problem (12)(15) is given by*

$$G_\infty(a) = \int_0^a e^{-\delta s} \left[\nabla C^*(\overline{\pi})\right](a - s) ds$$

Moreover, such equilibrium is asymptotically stable, that is any optimal trajectory $G^{opt}(t, a)$ satisfies

$$\lim_{t \to +\infty} G^{opt}(t, a) = G_\infty(a)$$

Proof of Theorem 6 Since the operator A has a continuous inverse $A^{-1} : L^2(0, \omega) \to L^2(0, \omega)$, defined by $A^{-1}f(a) = -\int_0^a e^{-\delta s} f(a - s) ds$, the equation (21) may be written also as

$$G_\infty = -A^{-1} \nabla C^*(\overline{\pi})$$

yielding the desired formula for the unique equilibrium point. The stability of equilibrium is a classical result and depends on the dissipativity of the differential operator A. ∎

3.4 Problems with constraints

We now add a constraint on the control

$$u(t,a) \in [M_1, M_2], \text{ for almost all } t > 0 \text{ and for almost all } a \text{ in } [0,\omega]. \tag{22}$$

and show that we may reformulate the constrained problem as an unconstrained problem with modified running cost. Consider

$$\begin{aligned} \widehat{C}(u) &= \int_0^\omega \widehat{c}(a,u(a))da \\ \widehat{c}(a,w) &= \begin{cases} c(a,w) & w \in [M_1, M_2] \\ +\infty & else \end{cases} \end{aligned}$$

$$\widehat{J}[g,u] = \langle \overline{\pi}, g \rangle + \int_0^{+\infty} e^{-\rho t} \left[\langle \overline{\pi}, u(t) \rangle - \widehat{C}(u(t)) \right] dt. \tag{23}$$

It may be easily shown that the problem of maximizing $J[g,u]$ given by (15) over

$$\mathcal{U}_c = \{u \in \mathcal{U} : \text{ (22) is satisfied}\},$$

subject to (12) is equivalent to the unconstrained problem of maximizing $\widehat{J}[g,u]$ over $\mathcal{U}$ subject to (12). That is, the value function V_c of the constrained problem satisfies

$$V_c(g) = \sup_{u \in \mathcal{U}_c} J[g,u] = \sup_{u \in \mathcal{U}} \widehat{J}[g,u]$$

Indeed, although the supremum of $\widehat{J}$ is computed on the whole class $\mathcal{U}$, a control violating the constraint cannot be optimal, and is not taken into account in the maximization process.

Theorem 7 *Assume (H1) and that (H2) is satisfied with $\widehat{C}$ in place of C. Set also*

$$\mathcal{A}(a) := Argmax\{\overline{\pi}(a)w - c(a,w) : w \in [M_1, M_2]\}$$

and assume that such set is nonempty for a.a. $a \in [0,\omega]$. A control u^{opt} is optimal if and only $u^{opt}(t,a) \equiv u^{opt}(a)$, for a.a. $a \in [0,\omega]$, where $u^{opt}(a) \in \mathcal{A}(a)$ defines a function in $L^2(0,\omega)$. Consequently,

$$V_c(g) = \langle \overline{\pi}, g \rangle + \frac{1}{\rho} \int_0^\omega \left[\overline{\pi}(a) u^{opt}(a) - c(a, u^{opt}(a)) \right] da$$

Proof. The proof is a straightforward application of Theorem 3.■

Remark. If c is a *l.s.c.* function, convex in the variable u, then

$$\overline{\pi}(a) u^{opt}(a) - c(a, u^{opt}(a)) = [\widehat{c}(a,\cdot)]^* (\overline{\pi}(a))$$

where c^* is the Legendre transform of c

$$\left[\widehat{c}(a,\cdot)\right]^*(v) = \sup_{w\in\mathbb{R}} \left[vw - c(a,w)\right]$$

Indeed

$$\begin{aligned}\sup_{w\in\mathbb{R}} \left[vw - \widehat{c}(a,w)\right] &= \max\left\{\sup_{w\in[M_1,M_2]} \left[vw - c(a,w)\right], \sup_{w\notin[M_1,M_2]} \left[vw - c(a,w)\right]\right\} \\ &= \sup_{w\in[M_1,M_2]} \left[vw - c(a,w)\right]\end{aligned}$$

since the latter sup is constantly $-\infty$. Consequently

$$V_c(g) = \langle \overline{\pi}, g\rangle + \frac{1}{\rho}\int_0^\omega \left[\widehat{c}(a,\cdot)\right]^*(\overline{\pi}(a))da$$

3.5 Some useful examples

3.5.1 Pure quadratic costs

In order to make the analysis presented in the previous Section more effective, we assume here that the advertising costs are quadratic:

(H4) $C : L^2(0,\omega) \to \mathbb{R}$,

$$C(u) = \int_0^\omega \frac{\kappa(a)}{2}\left[u(t,a)\right]^2 da = \frac{1}{2}\langle B_\kappa u, u\rangle$$

where $\kappa \in L^\infty(0,\omega)$, $0 < \varepsilon \leq \kappa(a) \leq \overline{\kappa}$, and where B_κ is the moltiplication operator $B_\kappa : L^2(0,\omega) \to L^2(0,\omega)$, $u(\cdot) \mapsto \kappa(\cdot)\, u(\cdot)$.

First of all we notice that (H4) implies (H2) and (H3), hence we can apply here the results obtained in Theorem 3 and derive

$$C^*(v) = \sup_{u\in L^2(0,\omega)} \left\{\langle v,u\rangle - \frac{1}{2}\langle B_\kappa u, u\rangle\right\} = \frac{1}{2}\left\langle B_{\frac{1}{\kappa}} v, v\right\rangle$$

so that C^* is differentiable with $\nabla C^*(\bar{\pi}) = B_{\frac{1}{\kappa}}\bar{\pi}$.. Hence, there exists a unique optimal strategy u^{opt}, not depending on t, and given by

$$u^{opt}(t,a) \equiv \left[B_{\frac{1}{\kappa}}\bar{\pi}\right](a) = \frac{\overline{\pi}(a)}{\kappa(a)}, \tag{24}$$

with associated optimal trajectory is given by

$$G^{opt}(t,a) = e^{-\delta t} g(a-t)\chi_{[t,\omega]}(a) + \int_0^{\min\{t,a\}} e^{-\delta s}\frac{\overline{\pi}(a-s)}{k(a-s)}ds, \tag{25}$$

The value function of the problem is then

$$V(g) = \langle \overline{\pi}, g\rangle + \frac{1}{2\rho}\left\langle B_{\frac{1}{\kappa}}(\bar{\pi}), \bar{\pi}\right\rangle.$$

Moreover, using the results contained in Theorem **6** we can show that there exists a unique equilibrium point for the control system, and such solution is given by

$$G_\infty(a) = \int_0^a e^{-\delta s} \frac{\overline{\pi}(a-s)}{k(a-s)} ds, \tag{26}$$

and such equilibrium is asymptotically stable, that is

$$\lim_{t \to +\infty} G^{opt}(t,a) = G_\infty(a), \quad \forall a \in [0,\omega].$$

We notice that in (24) the positivity constraint is satisfied, hence the solution of the constrained problem coincides with that of the unconstrained problem.

3.5.2 Linear quadratic with positivity constraints

We now assume the data satisfy (H1) and

(H5) $C : L^2(0,\omega) \to \mathbb{R}$,

$$C(u) = \int_0^\omega \eta(a)u(t,a) + \frac{\kappa(a)}{2} \left[u\left(t,a\right)\right]^2 da = \langle \eta, u \rangle + \frac{1}{2} \langle B_\kappa u, u \rangle$$

where

- $\eta \in L^2(0,\omega)$,
- $k \in L^\infty(0,\omega)$, $\varepsilon \leq k(a) \leq \overline{k}$,
- B_k the moltiplication operator $B_k : L^2(0,\omega) \to L^2(0,\omega)$, $u(\cdot) \mapsto \kappa\left(\cdot\right) u(\cdot)$.

(H6) $u(t,a) \geq 0$, for a.a. $(t,a) \in [0,+\infty) \times [0,\omega]$

Again, (H5) implies (H3) and (H2). Moreover throough (H6) we require a positivity constraint on the control.

Theorem 8 *Assume (H1)(H5)(H6). The unique optimal control for the constrained problem is*

$$u_c^{opt}(a) \equiv \left[\frac{\overline{\pi}(a) - \eta(a)}{\kappa(a)}\right]^+ \tag{27}$$

The associated optimal trajectory is given by

$$G_c^{opt}(t,a) = e^{-\delta t} g(a-t)\chi_{[t,\omega]}(a) + \int_0^{\min\{t,a\}} e^{-\delta s} \left[\frac{\overline{\pi}(a-s) - \eta(a-s)}{k(a-s)}\right]^+ ds \tag{28}$$

while the trajectory in the long run is

$$G_c^\infty(a) = \int_0^a e^{-\delta s} \left[\frac{\overline{\pi}(a-s) - \eta(a-s)}{k(a-s)}\right]^+ ds \tag{29}$$

Proof. According to Theorem 7 we consider the modified cost

$$\widehat{c}(a,w)=\begin{cases}\eta(a)w+\frac{\kappa(a)}{2}w^2 & w\geq 0\\ +\infty & w<0\end{cases}$$

so that

$$\begin{aligned}\left[\widehat{c}(a,\cdot)\right]^*(\overline{\pi}(a)) &= \sup_{w\geq 0}\left[\left(\overline{\pi}(a)-\eta(a)\right)w-\frac{\kappa(a)}{2}w^2\right]\\ &= \begin{cases} 0 & \frac{\overline{\pi}(a)-\eta(a)}{\kappa(a)}<0\\ \frac{(\overline{\pi}(a)-\eta(a))^2}{2\kappa(a)} & \frac{\overline{\pi}(a)-\eta(a)}{\kappa(a)}\geq 0\end{cases}\end{aligned}$$

and the maximum is attained in $u^{opt}(a)$ described by (27). The remaining assertions are straightforward. ■

4 Applications

4.1 Pure quadratic costs

First of all we consider only pure quadratic costs so that the simpler setting makes the economic analysis clearer. If (H4) holds, then the optimal advertising effort is

$$u^*(t,a)\equiv\frac{\overline{\pi}(a)}{\kappa(a)} \tag{30}$$

and we do not have to force the positive constraint because the advertising effort defined by (30) is already positive. The optimal advertising effort described in (30) is well known and it represents an application of the golden rule: "marginal revenues equal to marginal costs" (see [2], formula (2.4)). However, it is interesting to notice how marginal revenues are computed in (30). The function $\overline{\pi}$ is defined in (18) and represents the marginal revenue given by each segment. It is interesting to notice that $\overline{\pi}(a)$ may be strictly positive even if $\pi(a)=0$. This phenomenon is not present in [2, formula 2.4] where the marginal revenue given by the segments out of the support of the function π is always zero. This difference is connected with the time evolution of the segmentation: it is profitable to invest in a segment a even if $\pi(a)=0$ when this segment will enter in the support of the function π in the future. In this model advertising efforts anticipate the time evolution and invest also in the segments which will be later on the support of the function π. In order to clarify this situation we present two simple examples. As the novelty in this model is represented by the function (18), we focus on this function and the associated function π and we assume that $\kappa(a)\equiv\bar{\kappa}$.

4.1.1 Target market $[\alpha, \omega]$

Let the target market be $[\alpha, \omega]$ with $\alpha \in (0, \omega)$, so that the firm sells its product only to people of age grater than α. For instance, a car is a product of this type, as it can be sold only to people who are aged above the minimum to posses a driver's licence. If we assume that the marginal revenues for all the segments in the target market are the same, we can write, after rescaling the objecting functional,

$$\pi(a) = \chi_{[\alpha,\omega]}(a)$$

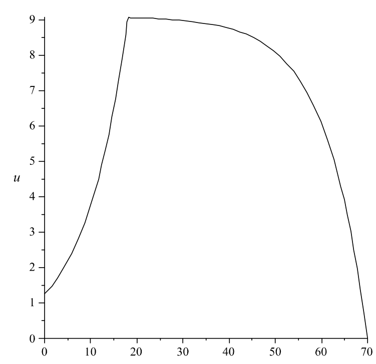
Hence,

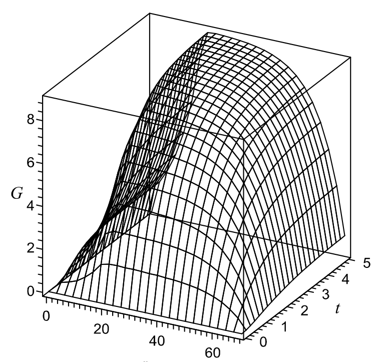
$$\overline{\pi}(a) = \begin{cases} \frac{e^{(\rho+\delta)a}}{(\rho+\delta)} \left[e^{-(\rho+\delta)\alpha} - e^{-(\rho+\delta)\omega} \right] & a \leq \alpha \\ \frac{1}{(\rho+\delta)} \left[1 - e^{-(\rho+\delta)\omega} e^{(\rho+\delta)a} \right] & a > \alpha \end{cases}$$

and the optimal advertising effort is

$$u^{opt}(t, a) = \begin{cases} \frac{1}{\bar{\kappa}(\rho+\delta)} \left[e^{(\rho+\delta)(a-\alpha)} - e^{-(\rho+\delta)(\omega-a)} \right] & a \leq \alpha \\ \frac{1}{\bar{\kappa}(\rho+\delta)} \left[1 - e^{-(\rho+\delta)(\omega-a)} \right] & a > \alpha \end{cases}$$

We observe that this function can be seen as a product between the constant $1/\bar{\kappa}(\rho+\delta)$ (which is the same constant we have found in (8)) and a function depending on a which describes the revenue associated to each segment. Moreover, we notice that even if $[0, \alpha)$ is not in the target market, the advertising flow directed to these segments is not equal to zero. Then it is optimal to anticipate the time evolution of the segmentation. In the following simulation the parameters are chosen as follows: $\alpha = 18, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0$.





As well illustrated in these pictures, we can prove that $u^{opt}(t, a)$ has a maximum at $a = \alpha$ (for all choice of the parameters). Moreover, using (26) we may compute explicitly $G^*_\infty(a)$ and prove that this function is increasing and convex in $[0, \alpha]$, and decreasing in $[\alpha, \omega]$ (for any choice of the parameters).

4.1.2 Target market $[\alpha, \beta]$

In this second example we consider a different target market. To fix ideas we think of a firm which wants to sell a sports car such as a coupé. As in the previous example, the target market for this product is bounded from below by the driver's licence age, and also from above: generally a family man prefers a width car instead of a coupé. Hence, assuming again that the marginal revenues for all the segments in the target market are the same, we can write, after rescaling the objective functional,

$$\pi(a) = \chi_{[\alpha,\beta]}(a)$$

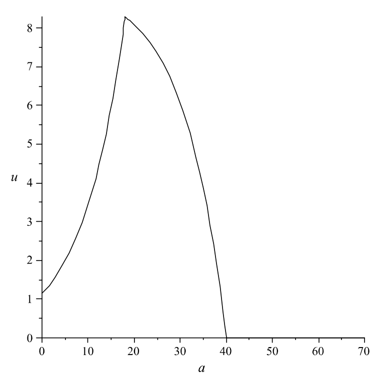
with $\alpha, \beta \in (0, \omega)$ and $\alpha < \beta$. Then

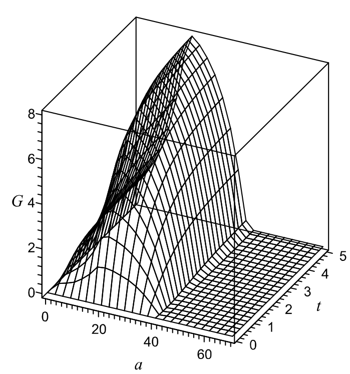
$$\overline{\pi}(a) = \begin{cases} \frac{e^{(\rho+\delta)a}}{(\rho+\delta)} \left[e^{-(\rho+\delta)\alpha} - e^{-(\rho+\delta)\beta}\right] \frac{1}{(\rho+\delta)} & a < \alpha \\ \frac{e^{(\rho+\delta)a}}{(\rho+\delta)} \left[e^{-(\rho+\delta)a} - e^{-(\rho+\delta)\beta}\right] \frac{1}{(\rho+\delta)} & \alpha \le a \le \beta \\ 0 & a > \beta \end{cases}$$

and the optimal advertising effort is

$$u^{opt}(t,a) = \begin{cases} \frac{1}{\bar{\kappa}(\rho+\delta)} \left[e^{(\rho+\delta)(a-\alpha)} - e^{-(\rho+\delta)(\beta-a)}\right] & a < \alpha \\ \frac{1}{\bar{\kappa}(\rho+\delta)} \left[1 - e^{-(\rho+\delta)(\beta-a)}\right] & \alpha \le a \le \beta \\ 0 & a > \beta \end{cases}$$

Here the anticipation effect of the advertising effort is less relevant (i.e. for $a < \alpha$ the advertising effort in the first example is always greater that the advertising effort obtained here). In this example an investment in the segments $[0, \alpha)$ is less profitable because when these segments overrun the threshold β their demand vanishes. For the same reason the advertising effort direct to the segment $(\beta, \omega]$ is always zero. In the following simulation the parameters are chosen as follows: $\alpha = 18, \beta = 40, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0$.





4.2 Linear quadratic costs

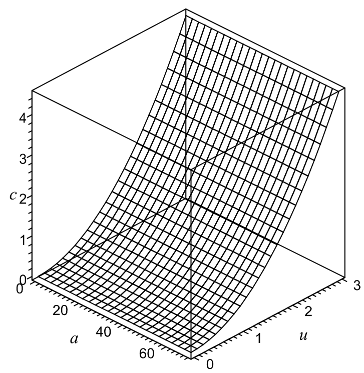
If (H5) holds, then the optimal advertising effort is

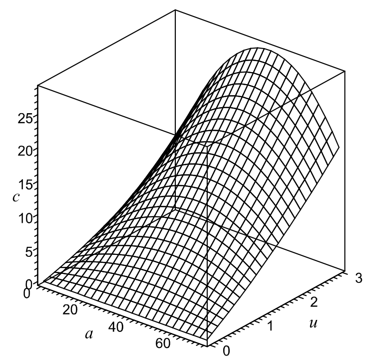
$$u^*(t,a) \equiv \left[\frac{\overline{\pi}(a) - \eta(a)}{\kappa(a)}\right]^+ \tag{31}$$

Now we assume that $\kappa(a) = \bar{\kappa}$, while we defined the function $\eta(a)$ as follows

$$\eta(a) = \bar{\eta}(a + a_1)(a_2 - a) \tag{32}$$

where $a_1 \leq 0$, $a_2 \geq \omega$, while $\bar{\eta} > 0$. It is easy to understand the effect of this new term if we consider these two pictures:





In the picture on the left we have the pure quadratic advertising cost function $c(a,u) = 0.5u^2$, while in the picture on the right we have the linear quadratic advertising cost function $c(a,u) = 0.5u^2 + 0.004(a+5)(85-a)u$. The linear term produces an extra cost which decreases the optimal advertising effort. Using the parameters a_1, a_2 we can model this extra term so that it becomes age dependent. In the formulation (32) the linear term has a maximum for $a = (a_2 - a_1)/2$ and by the definition of a_1 and a_2 we can model the position of this maximum.

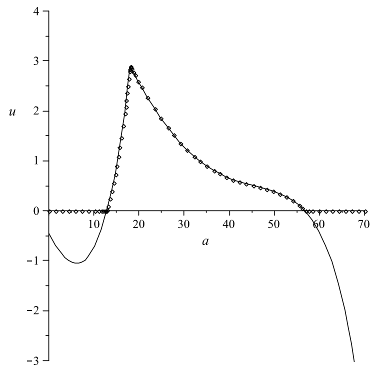
Now it is interesting to consider how this linear term modifies the optimal advertising strategies we have found in the previous examples.

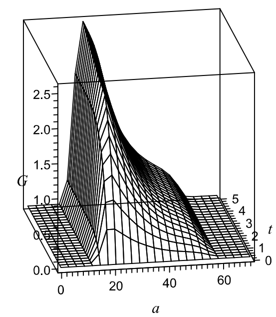
4.2.1 Target market $[\alpha, \omega]$

If (H5) holds with $\kappa(a) \equiv \bar{\kappa}$, $\eta(a) = \bar{\eta}(a + a_1)(a_2 - a)$, and $\pi(a) = \chi_{[\alpha,\omega]}(a)$ then the optimal advertising effort is

$$u^{opt}(t,a) = \begin{cases} \frac{1}{\bar{\kappa}(\rho+\delta)} \left[e^{(\rho+\delta)(a-\alpha)} - e^{-(\rho+\delta)(\omega-a)} - \bar{\eta}(a+a_1)(a_2-a) \right]^+ & a \leq \alpha \\ \frac{1}{\bar{\kappa}(\rho+\delta)} \left[1 - e^{-(\rho+\delta)(\omega-a)} - \bar{\eta}(a+a_1)(a_2-a) \right]^+ & a > \alpha \end{cases}$$

We repeat the previous simulation with $\alpha = 18, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0$, and with the new parameters set as $\bar{\eta} = 0.004, a_1 = 5, a_2 = 85$.





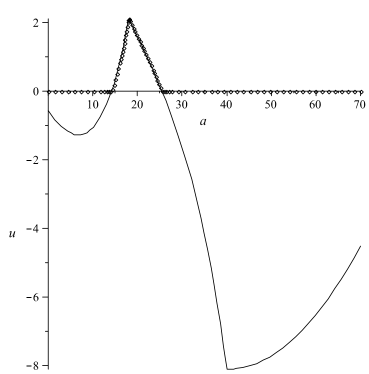
In the picture on the left the dashed line represents the optimal advertising effort, while the other line represents the same function without applying the positive part function. It is interesting to note that the anticipation effect in this simulation becomes less evident.

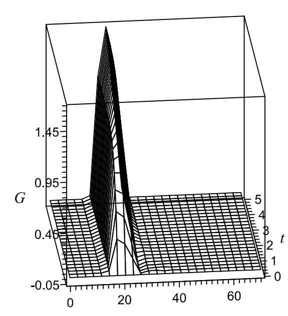
4.2.2 Target market $[\alpha, \beta]$

If (H5) holds with $\kappa(a) \equiv \bar{\kappa}$, $\eta(a) = \bar{\eta}(a + a_1)(a_2 - a)$, and $\pi(a) = \chi_{[\alpha,\beta]}(a)$ then the optimal advertising effort is

$$u^{opt}(t,a) = \begin{cases} \frac{1}{\bar{\kappa}(\rho+\delta)}\left[e^{(\rho+\delta)(a-\alpha)} - e^{-(\rho+\delta)(\beta-a)} - \bar{\eta}(a+a_1)(a_2-a)\right]^+ & a < \alpha \\ \frac{1}{\bar{\kappa}(\rho+\delta)}\left[1 - e^{-(\rho+\delta)(\beta-a)} - \bar{\eta}(a+a_1)(a_2-a)\right]^+ & \alpha \le a \le \beta \\ 0 & a > \beta \end{cases}$$

We repeat the previous simulation with $\alpha = 18, \omega = 70, \delta = 0.1, \rho = 0.01, \bar{\kappa} = 1, g(a) \equiv 0$, and with the new parameters set as $\bar{\eta} = 0.004, a_1 = 5, a_2 = 85$.





5 Conclusion

In this paper we show how Dynamic Programming techniques in infinite dimensions may be exploited to study optimal value of goodwill in the long run for an age-structured problem of optimal advertising. We can characterize the optimal goodwill in the long run and we explicitly find it as an equilibrium point of the closed loop equation of an abstract control problem.

A topic for further research consists in the optimal activation of an advertising channel. Actually, in the real world, a firm cannot decide the advertising effort to direct to each segments. A firm can only activate an advertising channel which hits with different intensities different segments. The solution of the optimal activation problem of an advertising channel in a segmented market in presented in [2], but there the time evolution of an age segmentation is not considered. The extension of the results obtained in this paper to the optimal activation of an advertising channel in an age segmented market seems to be a promising area for further research.

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