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Learning Cancellation Strategies in a Continuous Double Auction Market

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Abstract. This paper deals with two different issues. On one side, it tries to determine if the equilibrium order placement strategies analytically derived in Foucault et al. (2005) are learnable by no-maximizing agents that update their strategies on the only base of their own past experience (via genetic algorithm). Results state outcome (but not strategic) equivalence. On the other side, it relaxes the assumption in the original model by Foucault for which cancellation is not allowed and evaluate market performance. Results are mixed; the introduction of a cancellation option turns out to be beneficial dependently on the key determinants of the market dynamic (i.e., the arrival rate and the percentage of patient traders) and an additional setup variable: the initial level of order aggressiveness in the market.

Keywords: market evaluation, market design, equilibrium strategies, order cancellation, genetic algorithms.

JEL Classification Numbers: C63, D44, D61, D83.

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1 Introduction

"The individual is foolish; the multitude, for the moment is foolish,
when they act without deliberation;
but the species is wise, and, when time is given to it,
as a species it always acts right"
Edmund Burke (1782)

Recent literature has devoted increasing attention to the study of the dynamics of a limit order market. This attention is primarily due to the fact that the most part of the stock exchange markets are governed using continuous double auction protocol's rules (e.g., the Paris Bourse, the Spanish Stock Exchange, the ASX, the NYSE, the Nasdaq and many others). Results on different issues are mixed and no unique answer is provided to many questions.

In this work, we deal with two different key issues: the search for an equilibrium trading strategies result in a dynamic limit order market and the role played by the order cancellation option. Let us introduce them one at a time.

We cite Edmund Burke: "the individual is foolish [...] but the species is wise". The context in which this citation was made is very different from the one that is treated here but we can adapt this citation and interpret its meaning as follow: an individual agent might be not able to really strategically act as a maximizer; it is limited in his own computational abilities and makes mistakes. However, when behave as a species and takes his time (to learn - the species has to survive) he acts in the right direction. That's why our first point at issue is the following. Assume that agents are not able to maximize their expected profits in a continuous double auction; they are only able to learn from their past experience and from their own interactions among the other agents. Are they able to behave in the right way? Do they learn the (same) optimal strategy? We implement a genetic algorithm that drives the evolution of the species and analyze the relevance and the properties of the achieved (equilibrium) results.

As a second stage, and maintaining this evolutionary approach, we deal with the cancellation issue. It is a typical issue in which analytical modeling loses in tractability. In fact, many authors tried, in the last few decades, to introduce it in their mathematical models. The results is an abuse of assumptions (and too many simplifications helps to perform a math-exercise but put away from reality) or a simplification of the cancellation rule itself treated and modelled as an exogenous random variable. The research question that we try to answer is the following: from the point of view of a market designer, does the introduction of a cancellation rule in a continuous double auction protocol (i.e., allowing traders to cancel a submitted order and place a new offer) have any effect on order aggressiveness? Does it help to increase the performance of the market according to specific criteria such as average spread, allocative efficiency, volume and individual average profits?

We conduct both these studies using a computational approach and running simulations. The advantage to use an agent-based model for this analysis is to overcome computational limitations that make impossible to analytically reach a close form solution for the equilibrium result in a less simplified model.

This report is organized as follow. Section 2 summarizes related literature. Section 3 describes the model and the main findings as in [8]. Section 4 introduces the evolutionary algorithm, illustrates the experimental design and setup and reports main results obtained using a described learning approach. Equilibrium order placement strategies analytically derived in [8] and profiles of strategies as outcome of the learning process are compared in terms of market performance according to different criteria (i.e., distribution of spreads, order aggressiveness and the role of the arrival rate). Section 5 relaxes the no-cancellation assumption in the original model, evaluate the performance of the market under the new institutional structure and compare results from the two different scenarios both at individual and aggregate level. Section 6 concludes and presents open scenarios.

2 Related Literature

The increasing attention devoted to the study of the continuous double auction in the last few decades is primarily due to the fact that many of the most important exchange markets are modeled according to this protocol.

The recent literature analyzes different aspects of the limit order books.

Empirical works have tried to establish a link between order book and order flow and to identify the main determinants of the book dynamics [2]. Theoretical papers have developed static and dynamic models of order-driven markets [16, 7, 8]. They analyze the choice between limit and market orders in different ways; [16] develops a two-tick model in which agents make their choice taking into account the consequences of their trading behavior on future traders' order placement strategies, [7] focuses on the no-execution risk and [8] introduces agents' types that differ by the magnitude of their waiting costs and concentrates the attention on the key determinants of the market dynamic: the percentage of patient traders and the order arrival rate. However, all these models make restrictive assumptions in order to achieve analytical solutions. The attempt to relax these assumptions makes them intractable mostly because of the size of the state space and/or the large number of variables and parameters involved. In order to overcome this problem, some authors presents models solved numerically for equilibrium [9].

The hypothesis for which traders are allowed to revise or cancel their orders is often ignored by the literature since it increases dramatically the complexity of the problem. Many theoretical models rule out this possibility, many others simply do not mention how revision and cancellation are treated. However, given the fact that it is a common feature of real exchange markets, it cannot be ignored.

In the most part of the few papers in which cancellation is explicitly examined, it is modeled as a random event. Sometimes it is an exogenous shock that follows a poisson random process; otherwise, it might be modeled using other more complicated functions that denote the probability that a given order (or a given share of orders) is cancelled at a certain point in time [9]. However, experimental evidence show data that "seem too systematic for such a random event explanation" [4]. Sounds more persuasive the idea to consider cancellation and revision as part of the agents' trading strategies, a consequence of their strategic behavior [20, 17].

Literature on cancellation concentrates its attention on two different main aspects. On one side, it tries to identify and explain which are the main *determinants* of the cancellation choice; on the other side, it explains the *effects and the consequences* of cancellation itself.

[5] conducts an experiment with the main goal to capture behavioral regularities in order placement and cancellation. The authors recognized three different factors that have a significant effect on order cancellation and the cancellation rate: the position of an order relative to the best price, the imbalance between buying and selling orders and the number of orders in the order book. A more aggressive order, placed at the best price or better, denotes impatience and it is expected to be canceled faster than an order placed deep inside the book that suggests a high willingness to wait for execution. Using an ad hoc indicator of order imbalance, they show that it is more (less) likely for an order to be cancelled when it is the dominant (dominated) order type on the book. Finally, plotting the cancellation probability as a function of the total number of orders in the book, they surprisingly display that greater is the number of orders in the book, lower is the probability to be canceled. [3] proposes an empirical analysis of cancellation in the Spanish Stock Exchange. The paper define two main reasons for cancellation: the attempt to acquire information about the state of the market (and this leads to fleeting orders that are canceled very fast) and the change in market conditions. The focus of our paper is on the latter. Their results are consistent with the main findings in [5] described above. The choice of cancellation is affected by the size of the spread, wider the spread higher the probability to be canceled, and the position in the order book. In addition, they observe three supplementary determinants of cancellation: the level of volatility, the number of traders and the interaction of trading activity and relative inside spread. In periods characterized by a high level of uncertainty (i.e., high volatility) traders place less competitive orders waiting for better conditions [7] and the probability of cancellation decreases. A decrease in the number of traders[1] in the market implies a lower frequency with which agents arrive at the market. Then, the risk of no execution increases and it might be preferable to cancel an order; see [13]. Viceversa, an increase in the number of traders (between submission and cancellation) reduces the probability of cancellation. Finally, when the size of the spread increases together with the number of transactions we can assume that orders are executed very quickly; this reduces the risk of no execution and, consequently, the probability to be cancelled overall. [6] conducts an empirical analysis on the Australian Stock Exchange with the aim to both document the regularity of limit order revisions and cancellations and to examine their determinants. Cancellation is positively affected by the order size, the price-order priority, the release of bad public news and the risk associated with limit orders. Conversely, it is negatively related with order aggressiveness and good public news.

Looking at the effects of cancellation the attention of the literature is focused on the consequences of the introduction of the option to cancel on market performance. The main findings from the experiments in [4] show that cancellation affects volume but not price-associated variable. The option to allow cancellation increases the number of orders submitted due to a reduction in the risk of no execution and, consequently, the level of transactions. However, the magnitude is higher for the former variable. Another potentially interesting

[1] The authors use the number of transactions as a proxy for the number of traders.

finding is that the variation in traders' payoffs increases when investors can cancel orders; however, this result is not statistically reliable due to the small size of the sample. One goal of this paper is to deeper investigate in this direction.

3 The Model

With the aim to study the dynamics of a limit order market, we initially base our simulation on the analytical model proposed in [8]. In this section we describe that model; following [18], we distinguish three components of their exchange market: the rules that govern exchange (institutional structure), the agents' tastes and endowments (market environment) and the specified trading strategies (agents' behavior).

Institutional Structure. The model describes a limit order market in which agents can trade at most one unit at a time. There is a specified range of admissible prices (determined by the latent information about the security value) and offers and spreads are placed on a discrete grid governed by the tick size $\triangle > 0$[2]. Traders' arrivals follow a Poisson process with parameter $\lambda > 0$. *(Ass.1) Each trader arrives only once, submits a market or a limit order and exits. Submitted orders cannot be canceled or modified. (Ass.3) Buyers and sellers alternate with certainty. The first is a buyer with probability* 1/2.

Market Environment. There is an economy of traders equally divided between buyers and sellers. Valuations and costs lie outside the range of admissible prices[3]. Both buyers and sellers are allowed to be of two types (patients-P and impatients-I) that differ by the magnitude of their waiting costs ($0 \leqslant \delta_P \leqslant \delta_I$). Proportions of buyers (sellers) of a certain type in the population are given by θ_P and θ_I (for patient and impatient traders respectively), where $\theta_P = 1 - \theta_I$.

Agents' Behavior. Bids and asks are determined as distances (J, expressed by the number of tick size) from the best outstanding offer on the opposite side of the book (a and b), given the actual spread (s):

$$\begin{aligned} bid_i &= a - J \\ ask_j &= b + J \end{aligned} \qquad (1)$$

with $J \in \{0, ..., s-1\}$ *(Ass.2) Limit orders must be price improving, that is, narrow the spread by at least one tick.*[4]

[2]Discretization of prices is standard in the literature and many real stock exchanges implement this rule (e.g., Paris Bourse).

[3]We assume valuations (costs) to be equal across buyers (sellers), without loss of generality, to simplify the model and favor computational tractability and comparison.

[4]The choice to introduce a spread improvement rule in the model may appear too much restrictive; we also conjecture that it remarkably affect the results. However, empirical evidence supports this assumption even when that rule is not imposed by the market: *"A large fraction of the order placements improves upon the best bid or ask quote. Such improvements on one side of the quotes tend to occur in succession (undercutting), which reflects competition in the supply of liquidity" ([2], p.4).*

Profits from trade are given by the following expressions:

$$\pi_i = v - p - \delta_i T$$
$$\pi_j = p - c - \delta_j T \quad (2)$$

where $v(c)$ is the valuation (cost) of the buyer (seller), p is the transaction price, $\delta_{i,j}$ is a measure for the waiting costs and T is the time elapsed between the submission of the offer and its cancellation from the book because of trade. At the end of the trading session, agents that do not trade make a negative profits that equals to $-\delta_i T$.

3.1 Main results of the model in Foucault et al. (2005)

To validate our setup, as a first step we reproduce the model in Foucault et al. (2005) by agent-based simulation. The use of such a technique implies two important adjustments. First of all, as stated in Eq. 2, we play with realized (instead of expected) payoffs and time between arrivals. Secondly, the original model assumes a stream of agents over an infinite horizon; the use of an agent-based model requires the number of traders to be finite (both for computational reasons and to allow agents to learn[5]). This necessitates the introduction of a halting rule: the trading session ends when all traders had their chance to submit an order and no more trades are possible.

Details on the validation exercise are available in appendix A. To favor comparison between results in the analytical model by Foucault et al. (2005) and the learning model that will be presented in the next section, Table 1 summarizes the main findings in [8].

Equilibrium strategies are defined as follow.

Definition 1. *An equilibrium of the trading game is a pair of order placement strategies,* $\sigma^*_P(.)$ *and* $\sigma^*_I(.)$*, such that the orders prescribed by the strategies solve*

$$\max_{j \in \{0,\ldots,s-1\}} \pi_i(j) \cong j\triangle - \delta_i T^*(j) \quad (3)$$

when the expected waiting time $T^*(.)$ *is computed assuming that traders follow strategies* $\sigma^*_P(.)$ *and* $\sigma^*_I(.)$.

[8] claim that the equilibrium order placement strategies analytically derived is unique by construction (p.1179). We conduct a test to check if the equilibrium result is confirmed by simulation. It turns out to be sensible to the number of traders in the market and the number of rounds over which average payoffs are computed, see Appendix B for details.

[5]This choice has an effect on the ability of the system to reach the suggested equilibrium result. Details in appendix B.

Market Resiliency	*The market appears much more resilient when the percentage of patient traders in the market is higher than the percentage of impatient traders.*
Distribution of Spreads	*The distribution of spreads becomes skewed toward low (high) spreads when the percentage of patient (impatient) traders is higher. The average spread is smaller in markets dominated by patient traders (more resilient) since small spreads are more frequent in this case.*
Spread Improvements	*Spread improvements increase (decrease) with the size of the spread when market is dominated by patient (impatient) traders.*
Rate of Arrival (λ)	*(Corollary 2.1) The support of feasible spreads in the fast market is shifted to the left compared to the support of possible spreads in the slow market. (Corollary 2.2) The slow market is more resilient than the fast market. The former effect dominates the latter: Fast markets exhibit lower spreads with respect to slow markets (i.e., higher values of λ decrease the average spread).*
Equilibrium Strategies	*In equilibrium, patient traders tend to submit limit orders, whereas impatient traders submit market orders.*

Table 1: Main results of the model in Foucault et al. (2005)

4 Is the (Unique) Equilibrium Learnable?

The recent behavioral literature strongly suggest that agents are not able to behave as a profit-maximizer due to their own computational limitations. Since we expect that individual agents participate in the market to exploit as much as possible from transactions, a first interesting issue we deal is to try to answer the following question: ***are agents able to learn the maximizing strategies adjusting their trading behavior on the basis of their own past histories?***. We consider this an important point at issue both for theoretical and methodological purposes.

The recent literature has provided many learning models and there is no clear (definitive) reason to choose one of them instead of another. Given the recognized ability of evolutionary model to lead to maximizing results, we decide to implement a genetic algorithm that will be described in detail in the next section.

4.1 The Design of the Evolutionary Algorithm (GA)

Following [1], we describe how trading strategies evolve answering five different questions.

What data structure will you use (*gene*)? The population of N traders is initially divided into two subgroups accordingly with the parameter θ_P; namely, NP (patients traders) and NI (impatient traders). Each of these groups represents a population that evolves autonomously. In fact, we are dealing with a kind of learning that is known in the literature as *type-learning*: "agents of several distinct types interact with each other, but they only learn from successful agents of their own type" [19]. Each trader is endowed with a string that describes a strategy to be implemented: for each possible spread faced (s) it indicates which strategy the trader has to play (J). Not feasible strategies are ruled out. Only pure strategies are considered.

What fitness function will you use? Learning is driven by the average profit of a profile of trading strategies. It is computed over an evaluation window defined by a fixed number of trading sessions (R); strategies are updated using a genetic algorithm every R consecutive days.

What crossover and mutation operators (i.e., *variation operators*) will you use? Children are expected to preserve some part of the parents' structure; mimicking sexual reproduction, the crossover operator maps parents' genes onto children's genes. We choose a *uniform crossover* in order to avoid representational bias[6]: "this crossover operator flips a coin for each locus in the gene to decide which parent contributes its genetic value to which child" (computational expensive). The mutation operator makes small changes in a gene with the aim to (a) be sure that all the possible profiles of strategies have a chance to be exploited and (b) "good" profiles that perform bad by chance and are randomly substituted have a chance to re-enter the game. Inspired by [14] we apply mutation as follow: with probability decreasing in time (as measured by the cumulative number of revision, i.e. with probability 1/n) we substitute one of the siblings' actions with a pure action selected with uniform probability among all the feasible strategies.

How will you select parents from the population and how will you insert children into the population (i.e., *Model of Evolution*)? Parents are selected using the *double tournament selection* method; a subgroup of size m is picked and the single most fit one is taken as a parent. The procedure is repeated in order to choose the second parent; a new different subgroup of m traders is randomly drawn from the same initial pool of N with replacement, i.e. the same parent can be picked twice. Children are placed in the population using the *absolute fitness replacement* method: "we replace the least fit members of the population with the children".

What termination condition will end your algorithm? The simulation ends after a fixed number of trials ($= R\times$ number of revisions by GA).

4.2 Experimental Design and Setup

There is an economy of $N = 1000$ traders; types (*patient* or *impatient*) are assigned according to the parameter θ_P splitting the population into two subgroups, namely NP and NI. Traders in each subgroups are equally divided into buyers and sellers; sellers (buyers) are endowed with one (zero) unit of a good. Agents play a continuous double auction and are

[6] Ashlock (2004, p.39) highlights a problem with single point crossover: "loci near one another in the representation used in the evolutionary algorithm are kept together with a much higher probability than those that are farther apart."

allowed to trade at most one unit in each trading session. The trading session ends when all traders have made their offer and no more trade are possible. Each trading session represents one round; the genetic algorithm applies over an evaluation window of $R = 100$ rounds looking at the average profits earned by each agent. We allow traders to revise their strategies 2000 times; i.e., $R\times$ number of revisions by GA $= 100 \times 2000 = 200000$ trading sessions are played in total. Books are cleared at the end of each round; hence, trading sessions are independent.

Initial conditions turn out to be very relevant in determining results[7]. We initially run four different simulations using the parameters in Table 2, accordingly to the numerical example in [8]. Strategies are chosen as stated by Eq.1; the aggressiveness of the limit or market order is determined by the choice of a strategy randomly drawn using a uniform distribution over the set of feasible J.

	θ_P	λ	δ_P	δ_I	$\triangle$	$[B, A]$
(a) market dominated by patient traders	0.55	1	0.05	0.125	0.0625	[20, 21.25]
(b) market dominated by impatient traders	0.45	1	0.05	0.125	0.0625	[20, 21.25]

Table 2: Parameters of the simulations

At the end of the simulations, strings that describe behavior of traders are stored. Then, we run an additional simulation for each scenario consisting in 200 consecutive independent trading sessions in which agents place offers according to the profiles of strategies obtained by the genetic algorithm implementation. Average values of different variables are computed in order to analyze the performance of the market with respect to results in [8].

We test Foucault's predictions using one of the "equilibrium" profiles we end up with; even if the learning process leads to different profiles of strategies we argue that they all replicate the same features of the model. At the same time, we assume that $\theta_P = 0.55$ (0.45) is representative of a market dominated by patient (impatient) traders.

4.3 Results

Distribution of spreads. Figure 1 (left) is constructed plotting the average frequencies for each spread over the 200 rounds taken into account[8]; Figs.1 (center and right) report on average frequencies over a sample round. As it can be easily seen by direct visual inspection, distribution of spreads is skewed towards the left in both examples and intermediate and

[7]In my PhD Thesis I have analyzed what happens when we initialize the simulation allowing a certain percentage of traders (namely 30% and/or 70%) to place offers accordingly tot he equilibrium strategies in [8]; a sufficiently high number of maximizing agents assure that equilibrium strategies are adopted by almost all the traders in the market. At the same time, different level of order aggressiveness in the initial setup lead to different results; more on that in Sect. **??**

[8]Using the average values, the sum of the frequencies does not sum exactly to one.

high spreads[9] are faced with probabilities zero (or close to zero). An explanation for higher frequencies attached to low spreads is due to the fact that J-limit orders are chosen using a uniform distribution over the support $[0, s-1]$ and low values of J (that determines low created spreads) lie more often in such an interval.

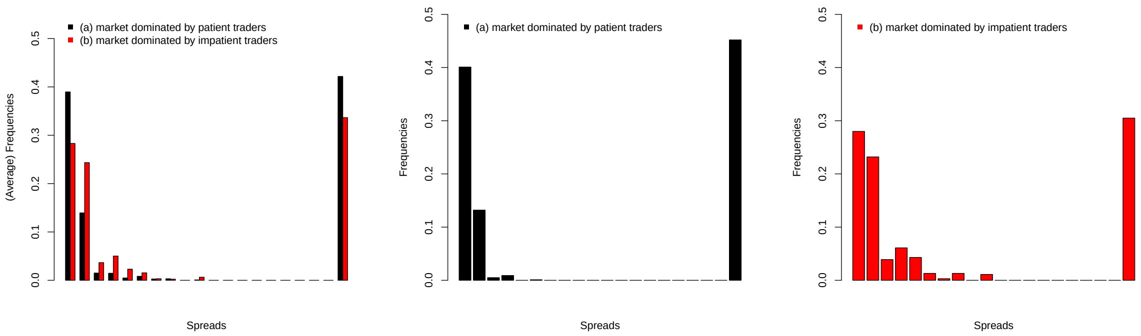


Figure 1: Spread Distribution.

However, even if this conclusion seems to be in contrast with findings in [8], we observe that when $\theta_P < \theta_I$ (i.e., the market is dominated by impatient traders, red bars in Fig. 1) the spread distribution moves toward the right. Table 3 give details of values of frequencies in Figs.1 (center and right); where no values are reported, frequencies are lower than 0.001. Results reflect two different effects. On one side, as stated in [8], a market dominated by patient traders turns out to be more resilient (i.e., the spread goes back faster to the competitive spread) since patient traders play more aggressively (see Fig.2) to speed up execution. This finding is confirmed by simulation. On the other side, differently from [8][10], we simulate the continuous double auction protocol with traders that arrive randomly to place their offers[11]. Hence, the sequence differs in each round and for each scenario and it reflects the composition of the market as described by the parameter θ_P. Given a lower number of impatient traders, that are in general more aggressive of patient traders, liquidity is consumed less rapidly and the inside spread has more time to narrow between market order arrivals.

As to be noticed that traders learn better how to behave in states of the world that are more likely to be experienced. Then, we can point out another important consequence of the shape of the spread distribution; equilibrium order placement strategies cannot be learned and agents behave more aggressively than in equilibrium trying to minimize their waiting

[9] By construction, given the range of admissible prices and the tick size, and according to the numerical examples in [8], a spread of 20 is faced when best offers are placed at the extremes and/or each time in which both books are empty. Hence, probability to be faced for the maximum feasible spread is always positive.

[10] Authors in [8] deal with this issue in a footnote. They claim that "if realizations for traders' types were not held constant, an additional force would make small spreads more frequent when the market is dominated by patient traders. In this case, the liquidity offered by the book is consumed less rapidly, since the likelihood of a market is smaller than in markets dominated by impatient traders. Thus, the inside spreads has more time to narrow between market orders arrivals" ([8], p.1190).

[11] Except by the fact that buyers and sellers alternate with certainty by assumption.

Composition of the market	Spreads 1	2	3	4	5	6	7	8	9	10	11	[...]	19	20
$\theta_P = 0.55$	0.401	0.132	0.005	0.009										0.452
$\theta_P = 0.45$	0.280	0.232	0.039	0.061	0.043	0.013	0.003	0.013		0.011				0.305

Table 3: Frequencies of spreads over a single round.

time and speed up execution. This trading behavior is close to the strategy that maximize the welfare. We will come back on that in the next sections. Even if it seems counterintuitive at a first glance, the generated framework in which high (and intermediate) spreads are faced with zero probability makes the average spread higher when market is dominated by patient traders due to the fact that we have a continuous alternance between the maximum and the lowest feasible spreads, see Table 4[12].

Order placement strategies. Figure 2 represents the expected J-limit orders[13] placed by patient and impatient traders in the two cases of market dominated by patient traders (left) and market dominated by impatient traders (right). Only J chosen given spreads faced with positive probability are shown.

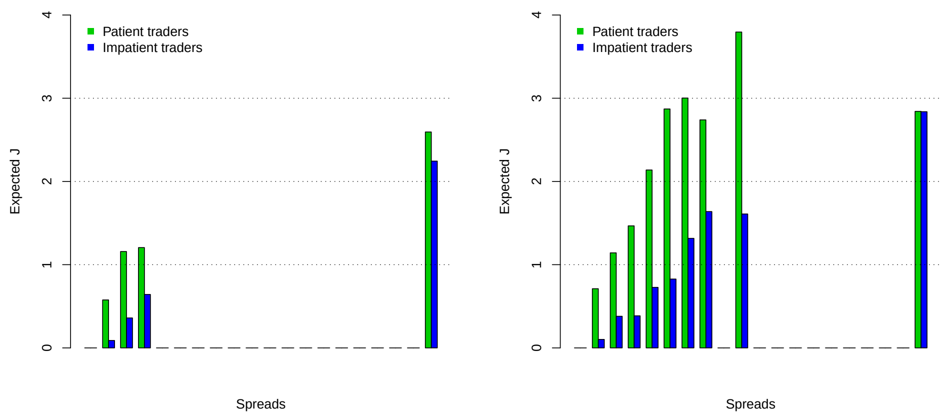


Figure 2: Order placement strategies. Markets dominated by patient (impatient) traders on the left (right).

Impatient traders always play more aggressively than patient traders; this difference is a direct consequence of their lower willingness to wait given the higher discount rate attached to time elapsed between submission and execution of an order.

[12]We can report an average spread of 0.58 (0.51) when $\theta_P >$ (<) θ_I.

[13]Expected Js are computed looking at the share of population who choose a specific J, conditional on the spread. For example, given a current spread of two ticks, by the improvement rule the only feasible choices for an agent who have to submit an offer are to place (1) a J-limit order with J=1 or (2) a market order (i.e., choose J=0). If the 80% of the population make the first choice and the rest adopt the residual strategy, then the expected J will be $0 \times 0.2 + 1 \times 0.8 = 0.8$.

Two additional important considerations concerning impatient and patient traders (respectively) have to be done. Firstly, impatient traders tend to go market when $\theta_P > \theta_I$ (i.e., expected J is almost always less than one). By contrary, when the market composition is reverted, their aggressiveness decreases; they still place market orders when low spreads are faced but they submit limit orders for intermediate spreads and their offers when the spread is maximum is more aggressive than in the opposite scenario(i.e., when market is dominated by patient traders). This seems to be directly connected with the higher percentage of impatient traders; the evolutionary algorithm where fitness is measured by average profits recompenses impatient traders that place a less aggressive offer by chance. In fact, due to the average aggressiveness of impatient traders, they have the chance to conclude a transaction almost immediately at a more favorable price. Secondly, patient traders are more aggressive (to speed up execution) when they prevail in the market. In addition, with the opposite composition of the market the reasoning above also applies to the minority of patient traders.

Simulation results show that in both examples spread improvements increase with the size of the spread. In [8] this is true only for markets dominated by patient traders; however, we can assume that it depends on the fact that in our setup impatient traders do not always submit market orders.

Rate of arrival. In the previous paragraphs, we have analyzed how the percentage of patient traders in the population can affect the order placement strategies and, consequently, the spread distribution. [8] states an additional key determinant of the dynamic of the book: the order arrival rate (represented by the parameter λ).

We conduct a second set of experiments that replicates scenarios in Table 2, replacing parameter $\lambda_1 = 1$ with $\lambda_2 = 0.2$. It leads to a slower market in which the average time between arrivals is given by $\lambda_2 = 1/0.2 = 5 > 1 = 1/1 = 1/\lambda_1$.

We can summarize main results as follow.
On one hand, in slower markets we expect that time between submission and transaction increases and wait for execution becomes more costly; as a consequence, agents are incline to place more aggressive orders. This intuition is confirmed by the lower levels of expected J obtained by simulation. Figure 3 report expected J-limit orders for fast (black) and slow (red) markets for patient (top) and impatient (bottom) traders in the two different scenarios, i.e. $\theta_P > \theta_I$ (left) and $\theta_P < \theta_I$ (right).

	$\theta_P = 0.55$	$\theta_P = 0.45$
Fast Market ($\lambda = 1$)	0.583	0.508
Slow Market ($\lambda = 0.2$)	0.624	0.588

Table 4: Average Spreads.

On the other hand, coherently with our previous analysis, in slow markets the number of

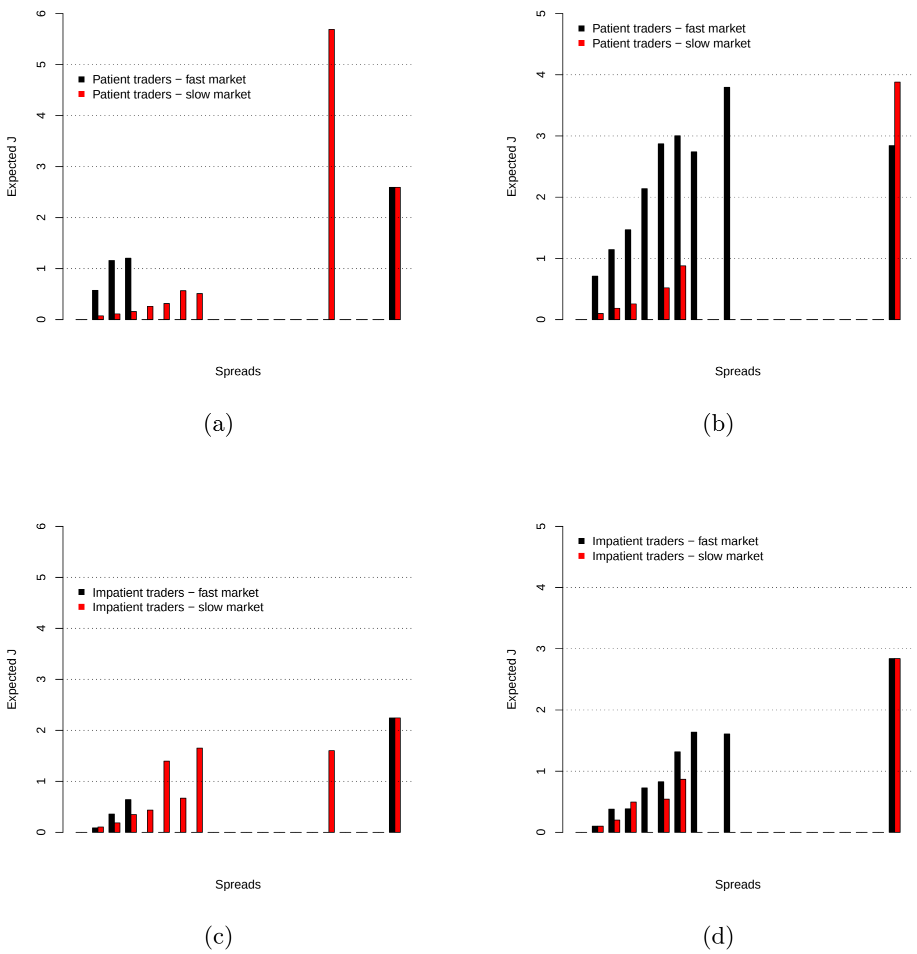


Figure 3: Market dominated by patient (left) and impatient (right) traders.

spread faced increases and the support of feasible spreads shifts to the right. Both these effects are stated in [8]. As pointed out by the authors[14] they lead to different conclusions on which market displays the lower average spread; the former (latter) tends to reduce (increase) the average spread. Given the set of parameters in Table 2, we confirm by agent-based simulation results in [8] for which the latter effect dominates the former and, in most cases, there is a negative correlation between arrival rate (λ) and the average spread, see Table 4.

To sum up, we can conclude that, even if agents are not able to learn equilibrium order placement strategies due to the uniform random initial setup[15] and, then, we cannot state strategic equivalence between equilibrium strategies and learned trading behavior, the two

[14]See Appendix A for details.

[15]An additional motivation that helps to understand why equilibrium results are not reached concerns the fragility of the equilibrium under consideration discussed in appendix B.

different profiles of strategies we end up with can be considered as outcome equivalent with some minor exceptions. A quality control for the strategy outcome of the learning process is offered in Appendix C.

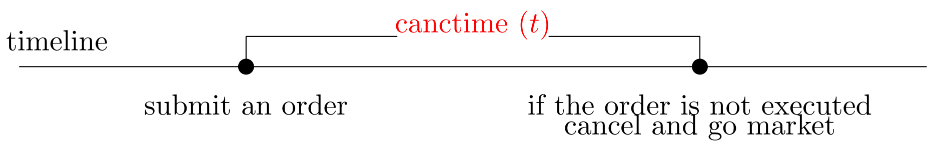
5 Introduce Cancellation

The second contribution of this paper is to study ***what happens when the restrictive assumptions for which orders might not be cancelled is relaxed***. Cancellation is a key issue in the recent literature and it turns out to be very difficult to be analytically analyzed. In fact, the introduction of some cancellation rules causes a significative increment in the level of sophistication of the model and makes much more difficult to figure out a closed form solution for the equilibrium strategies.

We allow traders to cancel their orders under the condition that the "cancellation option" might be exercised only once in each trading session (i.e., if you decide to retract an order, then you cancel your outstanding offer and immediately place a market order, if possible). This seems to be (still) a restrictive assumption but it is the one that keeps the model simple, since you only need to decide your willingness to wait (measured by t), and it may be justified by the fact that in a context in which (a) there are no many traders and (b) the spread-improvement rule applies, if you place an order and it does not pass after a "reasonable" amount of time, it might be convenient to cancel it and there is no room, both in terms of time and profitability, for placing another limit order.
Cancellation costs are not considered.

Let us describe the main changes in the model:

- *Strategies.* Differently from before, each agent choses a pair $(\text{J}, t|s)$ where J is the distance from the best offer on the opposite side of the book that determines his/her own bid/ask and t is the interval of time after which, if the order is not executed, the trader decides to cancel the previous offer and place a (new) market order. t is computed from the point in time in which the offer is placed and is chosen on a discrete grid. Taking into account the value of the grid ($\triangle = 0.0625$), the size of range of admissible prices ($K\triangle = 1.25$) and the number of traders in the market ($N = 1000$), we define the set of admissible $t \in \{2, 4, 8, 16, 32\}$.



- *Halting rule.* A session closes when (a) all traders have placed their offers, (b) no more trades are possible, and (c) no more cancellations are queued waiting to be activated.

Both from a point of view of market evaluation and market design, it is interesting to understand what happens when cancellation is introduced. We apply the same genetic algorithm with the only difference that now two genes for each trader evolve simultaneously: the first one indicates which J the agent will choose, given the current spread s (exactly as before); the second associates a specific t in the feasible set for each admissible J. The process of learning, than, is not more complicated (apart from the increasing computational effort) but it takes more time to converge towards a clear pattern. In Sect. 5.1 main results are summarized. This exercise has (partially) a different meaning from the one performed above. In fact, in the previous section we wanted to see if there was (or not) convergence to the unique and known equilibrium. Here, changing the market structure due to the introduction of an option to cancel an offer, the equilibrium strategies change; given the difficulties in computing them analytically, we conduct a computational analysis to compare situations with the same initial setup and to be able to isolate the effect of learning combined with the cancellation issue.

5.1 Results

As previously done, we summarize the main findings with respect to the two key determinants of the book dynamics stated in [8]: the percentage of patient traders (θ_P) and the rate of arrival (λ).

In this section the aim is twofold. One one side, we want to analyze results due to the introduction of a cancellation option *per sè*; on the other side, we compare outcomes from the two different scenarios, with and without cancellation, in order to figure out if and when the cancellation option might be beneficial (see Sect. 5.2).

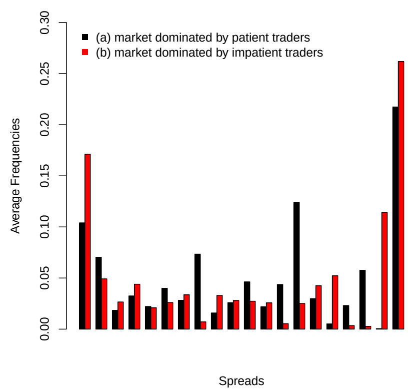


Figure 4: Spread Distribution.

Distribution of spreads. Figure 4 reports the average spread frequencies over 200 rounds in markets dominated by patient (black) or impatient (red) traders simulated using parameters in Table 2. As it can be easily seen, the introduction of the cancellation option enlarges the set of spreads faced with positive probability. This is partially due to the less aggressive behavior of traders with respect to the case of no-cancellation. In fact, the cancellation option might be considered as a sort of insurance/protection from the non execution risk that allow traders to hazard a less aggressive behavior in order to exploit the most from transaction.

Order placement strategies. Figure 5 displays the expected J-limit order placed by patient (green) or impatient (blue) traders in the market dominated by patient (left) and impatient (right) traders. Comparing aggressiveness for spreads that are faced in both cases (i.e., with and without cancellation), the hypothesis for which aggressiveness decreases is confirmed. On average, patient traders seem to place higher J than impatient traders even if the level of aggressiveness reduces compared to the previous scenario without cancellation[16]. In market dominated by impatient traders difference in aggressiveness between patient and impatient traders for low and intermediate spreads are larger; impatient traders reduce aggressiveness more than patients.

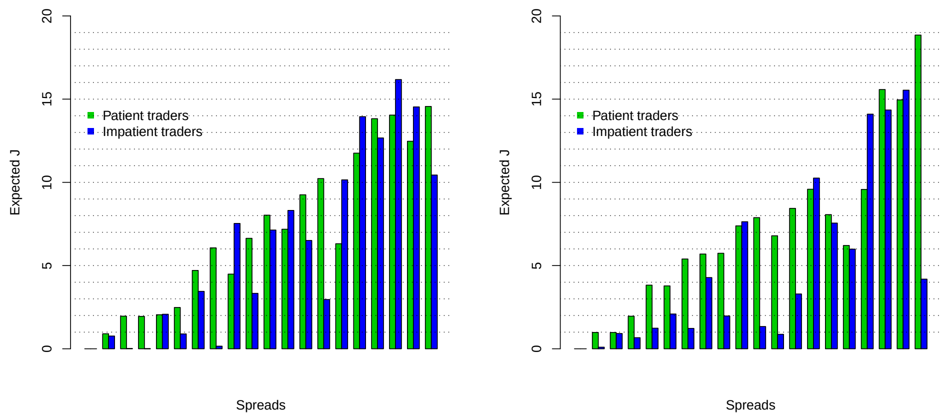


Figure 5: Order Placement Strategies.

As a direct consequence of the positive frequency attached to all the spreads in the support and the less aggressive behavior of traders (that is positively correlated with θ_I, the percentage of impatient traders in the market) the average spread (a) increases with respect to the no-cancellation case and (b) is higher in markets dominated by impatient traders, see Table 5.

[16]In some isolated cases, say high spreads that are visited with very low probability and that represents states of the world in which is difficult to learn the optimizing behavior, impatient traders turn out to be even less aggressive than patient traders.

	$\theta_P = 0.55$	$\theta_P = 0.45$
Fast Market ($\lambda = 1$)	0.710	0.730
Slow Market ($\lambda = 0.2$)	0.663	0.661

Table 5: Average Spreads.

Rate of arrival. We analyze what happens when the rate of arrival is reduced (from $\lambda = 1$ to $\lambda = 0.2$ as in the previous scenario to favor direct comparison).

Contrarily to what we have observed in the no-cancellation scenario, in slow markets intermediate spreads are less likely to be faced; since time is costly, traders who placed a no aggressive order use more often their chance to cancel the offer and complete a transaction to speed up execution and reduce losses. As a consequence, the average spread reduces and, because of the high likelihood of market orders, it is not substantially different with respect to the percentage of patient traders in the market. This behavior will be crucial in increasing the welfare (i.e., the allocative efficiency level reached by the market).

Finally, we propose some considerations concerning order aggressiveness. Since traders use more often their option to cancel in fast markets than in slow markets, the latter appear quite aggressive; however, also the pre-cancellation limit orders placed are usually less aggressive than the expected J chosen in slow markets, both for patient and impatient traders.

5.2 Does Cancellation matter?

In the previous section we have analyzed the effects of the introduction of the cancellation option on market outcome, according to the performance measures used in Sect.4.3. In thi section we concentrate on the evaluation of the market both at individual and aggregate level on the basis of three different additional criteria: the number of transactions completed per trading session (i.e., volume), the individual average profits achieved by agents in the market and the level of efficiency reached by the market itself. Then, comparison between the two scenarios (with and without cancellation) is conducted with the aim to provide some market design implications.

5.2.1 Volume

Volume is defined as the number of transactions per trading session. Since in our simulation we have $N = 1000$ agents equally divided between buyers and sellers, the maximum number of trades in the artificial market is 500.

Table 6 reports the average number of transactions for each case under consideration. As it can be easily seen, the average number of trades is never lower than 494 (477) without (with) cancellation: at most 5% of trades are missing! The reason for that is quite trivial.

	$\theta_P = 0.55$	$\theta_P = 0.45$
Fast Markets	499.415	498.630
$\lambda = 1$	477.68	477.93
Slow Markets	494.74	499.41
$\lambda = 0.2$	495.50	497.05

Table 6: Number of transactions without (black) and with (red) cancellation.

Independently from everything else, one assumption of our model is the spread improvement rule: every trader needs to reduce the actual spread of at least one tick. This assumption not only force the spread to be narrowed but also avoid the possibility to choose a limit order when no more reduction is possible.

With only one (not statistically significant) exception, we assume that volumes are higher when the cancellation option is not available. Slow markets display higher volumes in markets dominated by impatient traders and this effect is amplified in the scenario with cancellation (where the percentage of patient traders in the market does not count) since more traders use their option to cancel their offers and place market orders. Additionally, looking at the total number of patient and impatient traders involved in a trade, in fast market with cancellation more agents (and in particular -not surprisingly- patient traders) fail to trade.

5.2.2 Individual Average Profits

Individual average profits are computed overall and per type over 200 independent rounds.

	$\theta_P = 0.55$	$\theta_P = 0.45$
Fast Markets	1.321	1.294
$\lambda = 1$	1.177	1.184
	(-0.144, P↑, I ↓↓)	(-0.11, P↓↓, I ↓)
Slow Markets	1.066	1.078
$\lambda = 0.2$	1.149	1.135
	(+0.083, P ↑↑, I↑)	(+0.057, P↓, I↑↑)

Table 7: Individual Average Profits without (black) and with (red) cancellation.

Table 7 reports the individual average profits without (black) and with (red) cancellation. In general, we can assume that cancellation is beneficial in slow market; i.e., if the arrival rate (λ) decreases, the cancellation option helps to increase the realized share of the competitive outcomes.

This overall effect comes from a different impact on patient and impatient traders. Patient traders take advantage from cancellation when they dominate the market but their profits are lowered in a market dominated by impatient traders; however, the former advantage (latter disadvantage) increases (decreases) in absolute values when the arrival rate decreases.

From what concern impatient traders, they take advantages from the cancellation option (more when they dominate the market) in slow markets but not in fast markets; this advantage (disadvantage) is higher (lower) when they dominate the market.

Note that in slow (fast) markets dominated by patient (impatient) traders, gains (losses) go in the same direction for both types and players.

More in general, in slow markets average individual profits are lower independently by time and the introduction of a cancellation option; however, the latter reduces this difference in magnitude.

5.2.3 Allocative Efficiency

One of the most common criteria used to evaluate the performance of a market is given by the notion of allocative efficiency. In [8], the authors devote an entire paragraph to make some observations concerning efficiency, defined as the sum of the expected payoffs[17]. Their first claim is that efficiency cannot be fully reached due to waiting costs that constitute a dead-weight loss; then, the aim become to minimize such a cost.

As we will see, it happens when two conditions are fullfilled: (1) waiting time is minimized (no more than one period) and (2) all limit orders generate the competitive spreads (=1 in our examples). Note that [8] assumes that all impatient traders place market orders; then, $EC = \delta_P/\lambda$. In our framework this is not necessarily true and computations about the maximum level of allocative efficiency achievable will be updated.

The benchmark paper states that (in general) efficiency is not achieved in equilibrium due to the fact that the model do not uncover two source of inefficiency: (a) impatient traders sometimes submit limit orders and (b) patient traders post spreads larger than the competitive spreads. Both of these sources of inefficiency will be taken into account in what follows.

The maximum level of efficiency achievable by the market as a whole is usually measured as the competitive outcome (thereon, CO) defined as the difference between values and costs for intramarginal traders[18]:

$$CO = \sum_{i=1}^{q^*}(v_i - c_i) \tag{4}$$

where v_i (c_i) are the ith highest (lowest) value (cost) and q^* is the competitive quantity. Hence, the allocative efficiency measure is defined as follow[19]:

Definition 2. *The allocative efficiency can be defined as the total profit actually earned by all traders divided by the maximum total profit that could have been earned by all traders:*

$$AE = \frac{\sum(v_i - p) + (p - c_j)}{CO}$$

[17]As usual, we deal with ABM then measure of efficiency is based on realized payoffs.

[18]Intramarginal traders are buyers (sellers) with their values (costs) higher (lower) than the equilibrium price. In this implementation we have assumed simmetric reservation values with respect to the range of admissible prices; hence, the equilibrium price coincides with the medium point of this interval.

[19]See [21] for definitions of competitive outcome and allocative efficiency; for the latter, refer also to [18]

where v_i (c_j) is the value (cost) of a buyer i (a seller j) involved in a trade and p is the transaction price for that specific exchange[20].

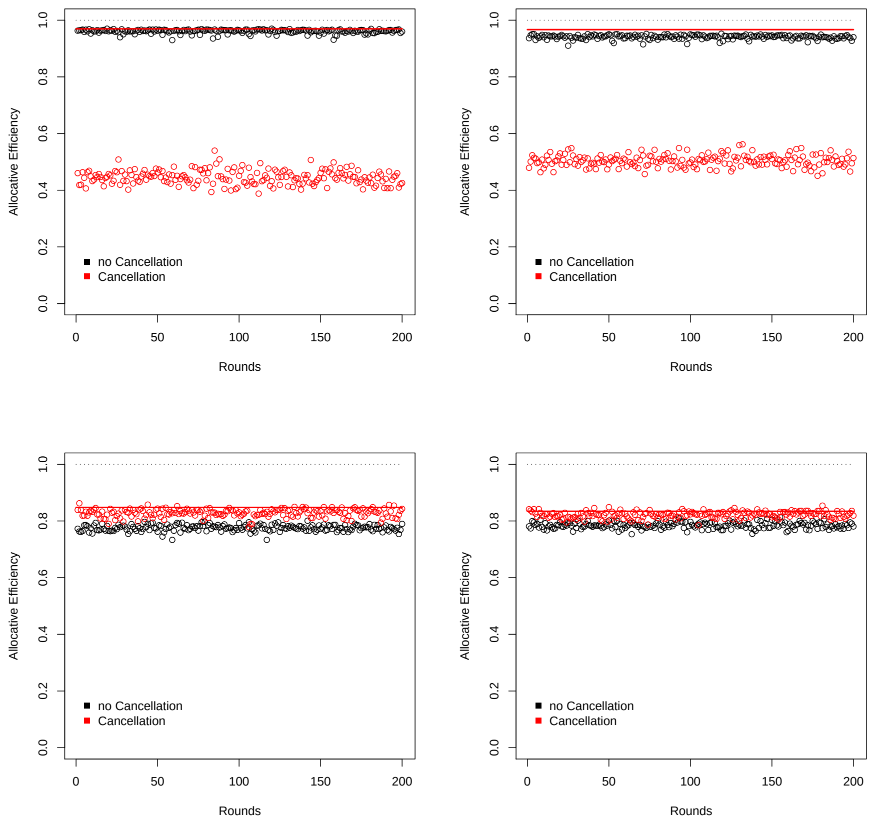


Figure 6: Allocative Efficiency

However, in our model there is something more to be taken into account: the waiting time has a cost that depends on agent's type and this cost has the effect to decrease the maximum payoff achievable.

Definition 3. *The maximum level of allocative efficiency attainable in a market with waiting costs in which buyers (sellers) have the same value, v (costs, c) and alternate with certainty (namely, the first player is a buyer with probability 1/2) can be defined by the following equation:*

$$AE_{\max} = v - c - \frac{1}{\lambda}[\theta_P \delta_P + (1 - \theta_P)\delta_I]$$

[20]Given our set of parameters, according to the Foucault' s numerical examples, we have $v_i = 22, \forall i$ and $c_j = 19.25$, $\forall j$. Hence, $AE_{\max} = N(22 - 19.25)$

where θ_P is the share of patients in the population, δ_P (δ_I) is the waiting cost for patient (impatient) traders and λ is the parameter of the Poisson arrival process.

Proof. Let us assume the case in which the first player is a buyer (wp 1/2). If the following seller will place a market order, then the profit of the buyer will be $\pi_b = v - p - [\theta_P(\delta_P T) + (1-\theta_P)(\delta_I T)]$, where p is the transaction price. Since, on average, the interarrival time of a Poisson process is $1/\lambda$, then we can substitute $T = 1/\lambda$. The seller profit can be written as $\pi_s = p - c$. Then, the total gain from transaction will be $\pi = v - c - [\theta_P(\delta_P T) + (1-\theta_P)(\delta_I T)]$. By symmetry, the same happens if the first player is a seller (with complementary probability 1/2) from which the result follows. □

With the aim to compare the level of efficiency for the two markets under consideration (namely, with and without cancellation), we can use as a reference point (technically, the de- nominator in the AE definition equation) alternatively the CO (dotted line at 1) or the $AE_{\max}$ (red line) as described above. This choice does not affect the direction of our results.

Note that the maximum level of efficiency that can be attained ruling out cancellation is higher by construction. That's why we use it as a benchmark.

Figure 6 describes the allocative efficiency attained by traders in the market over 200 rounds. In the first (second) line results from fast (slow) markets are reported; the two columns represent markets dominated by patient (left) and impatient (right) traders.

As it can be easily seen by direct visual inspection, coherently with previous results on individual average profits, cancellation turns out to be beneficial in slow markets.

	with cancellation	without cancellation
Fast markets	higher	lower
Slow markets	lower	higher

Table 8: Comparison of level of allocative efficiency achieved in markets dominated by impatient traders, when $\theta_I > \theta_P$ (with respect to the opposite case: $\theta_I < \theta_P$).

In slow markets, allocative efficiency is higher (lower) than in fast markets with (without) cancellation. However, the difference in the level of efficiency achieved reduces when the market is dominated by impatient traders. In fact, under this scenario (i.e., $\theta_I > \theta_P$) allocative efficiency is higher (lower) with (without) cancellation in fast markets and viceversa in slow markets.

Finally, note that in the cases in which only extremes spreads are faced the level of efficiency reached by the market is close to the maximum value achievable; this is due to the fact that the two conditions for which welfare is maximized are fulfilled (it is confirmed by simulation plotting the book dynamic, see [15] and Sect.4.3).

6 Concluding Remarks

In this paper we deal with two different research questions.

Initially, we try to understand if the unique order placement strategies analytically derived in [8] are learnable by no-maximizing agents that place their offers (only) on the basis of their own past experience. We model learning via genetic algorithm. As a result, we have shown in Sect. 4 that the outcome of the learning process is different from the predicted (maximizing) one. There are several reasons for which equilibrium strategies are not acquired; we focus our attention on two main determinants of the result: the fragility of the equilibrium in [8] (discussed in Appendix B) and the initialization choice. In fact, we assume that agents initially submit a J-order randomly drawn from a uniform distribution over the feasible set (given spread); hence, the likelihood to place low J is higher than the probability to submit high J and, as a consequence, the initial overall aggressiveness of traders in the market is quite high. This implies that equilibrium strategies for intermediate and high spreads are very difficult to be learned. However, the two profiles of strategies we end up with can be considered as outcome equivalent with some minor exception.

In the second part of the paper we relax the assumption for which cancellation is not allowed in order to evaluate market performance under the new institutional structure. It cannot be stated that cancellation is (or is not) beneficial *per sè*. In fact, its consequences in the market outcome, according to the performance criteria considered in the paper, depend on the joint effects of different parameters of the model. In particular, we claim that the main determinants of the cancellation results are the same key parameters that determine the book dynamics: the **percentage of patient traders in the market** (θ_P) and the **time arrival rate** (λ). At first glance, cancellation turns out to be beneficial (overall) when θ_P or λ decreases (i.e., in market dominated by impatient traders or slow markets). However, effects are different for patient and impatient traders since they have different order placement strategies (driven by waiting costs) and different chances to exploit and take advantages from the cancellation option. In addition, an important role is played by the **initial level of order aggressiveness in the market**. We have conducted some supplementary simulations changing the distribution from whcih J is randomly drawn; as a preliminary result, the main findings of this paper still hold for markets that have an initial very low or very high level of order aggressiveness but not for markets in which intermediate J are initially more likely to be played.

Finally, it has to be pointd out that there are at least three additional rules and/or parameters that seem to have an effect on the consequences of the introduction of the cancellation option on market outcome: (a) the spread improvement rule (that rules out strategic behavior -favored by cancellation- to place offers that do not reduce the spread), (b) the tick size $\triangle$ (that contributes to determine the depth of the book and, for low values, plays a role similar to the spread improvement rule) and (c) the set of feasible t. Deeper investigation is left for future research.

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A Foucault et al. (2005) as an Agent-Based Model - Validation

Determinant of the book dynamic #1:
The percentage of patients traders

Book Dynamics and Resiliency. Figure 7 shows the book dynamics in two markets characterized by different bidding behaviors. It reports (left) two numerical example contained in the original paper and replicates them (right) by running simulations. The two scenarios differ only in the percentage of patient traders in the market (55% in the first line, 45% in the second line). Figures are rescaled to favor direct comparison. The competitive spread is reached quite fast in a market with a majority of patient traders; in a market characterized by the opposite composition the quoted spread remains quite large during all the trading sessions.

Result 1. The market appears much more resilient when the percentage of patient traders in the market is higher than the percentage of impatient traders.

This result is imputed by the authors to the bidding behavior: the equilibrium strategy suggested (and analytically derived) for markets in which patient traders prevail force them to submit more aggressive orders to speed up execution. The authors use the same sequence of traders types in each example to be able to disentangle this effect; in fact, they claim that when realizations of traders types differ, a market dominated by impatient traders shows small spreads less frequently due to an additional reason: the likelihood of a market order is higher. Clearly, in a framework based on a finite horizon (and hence, on a finite number of traders N) the use of the same sequence of traders types does not reflect differences in the

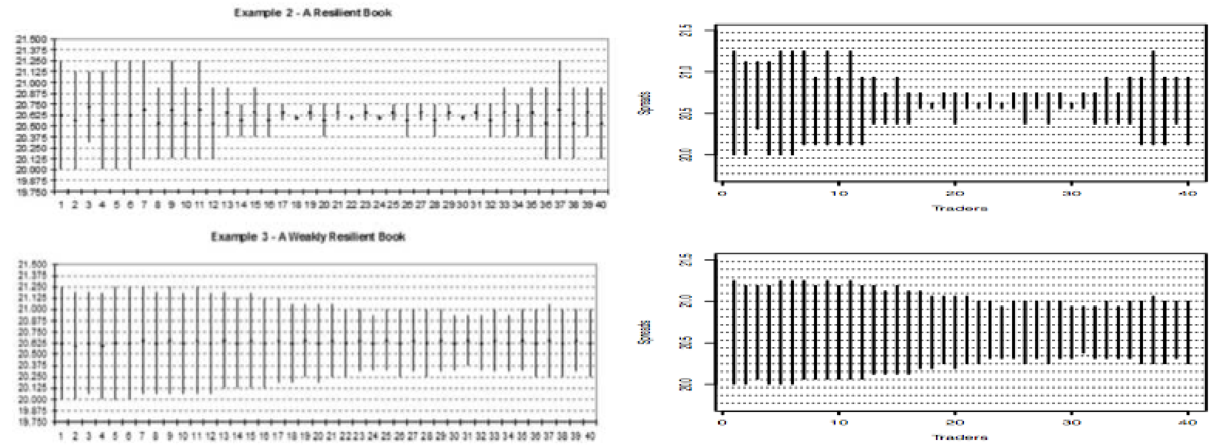

Figure 7: Source: Foucault et al. (2005), Fig. 3 p.1190 (left), Replication by ABM (right).

composition of the market.

The use of two different (randomly chosen) sequences respecting the percentage of patient and impatient traders does not significantly affect the result. Figure 8 reports the book dynamics for markets populated by $N = 10000$ traders, distributed accordingly to the different percentage of patient traders in the two examples. The average spreads in the top-left market is 0.529, while the average spread in the top-right market is 1.005. Results are confirmed. The first line shows the book dynamics of the first five-thousand submissions. The second line shows the book dynamics of the first forty submissions (to favour direct visual inspection). Even if the sequence of traders differs from the one used in Fig. 7, the book dynamics appear very similar and consistent with the finding in the paper.

Distribution of spreads. Figure 9 presents the stationary distribution of spreads in the two markets described above. As usual, we report and compare original and simulated results.

Result 2. The distribution of spreads becomes skewed toward low (high) spreads when the percentage of patient (impatient) traders is higher. The average spread is smaller in markets dominated by patient traders (more resilient) since small spreads are more frequent in this case.

State of the book and order aggressiveness. Spread improvements are a measure of order aggressiveness. They describe the amount (in number of ticks) by which the agent reduce the spread placing his offer. For example, given a spread of twenty ticks an optimal strategy for a patient agent is to place a limit order with $J^* = 18$; since J^* represents the new spread created by the offer itself, this means that the original spread is reduced by two

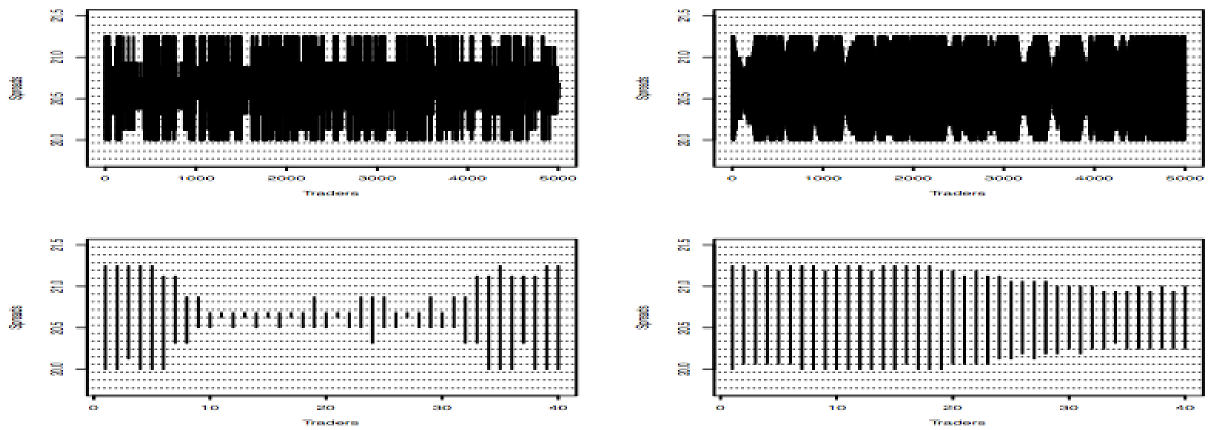

Figure 8: Book Dynamics (random sequence).

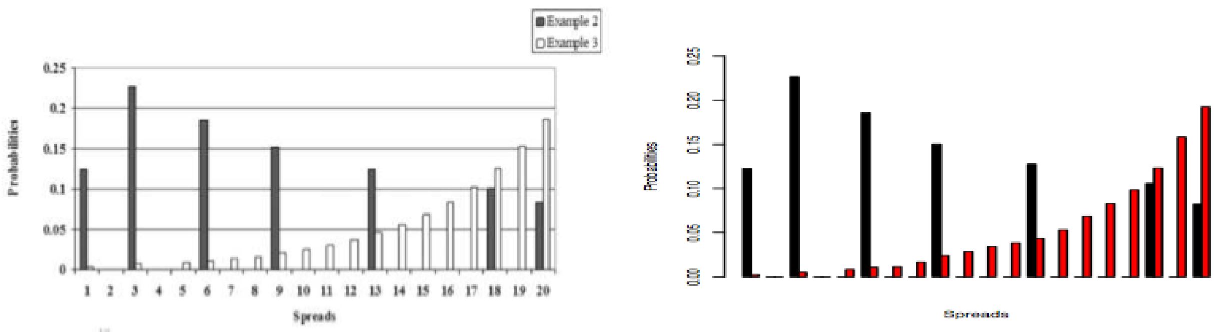

Figure 9: Source: Foucault et al. (2005), Fig. 4 p.1192 (left), Replication by ABM (right).

ticks (namely, 20-18=2). Figure 10 reports the (optimal) spread improvements: as in the previous picture, results from a market dominated by patient (impatient) traders are shown in black (red).

Result 3. Spreads improvements increase (decrease) with the size of the spread when the market is dominated by patient (impatient) traders.

Determinant of the book dynamic #2:
Measures of trading activities

The market composition determines resiliency; then, it is indirectly related to some measures of trading activities, namely (a) the order arrival rate and (b) the time between trades. We are interested in the former, since it is a key determinant of the book dynamics. [8]

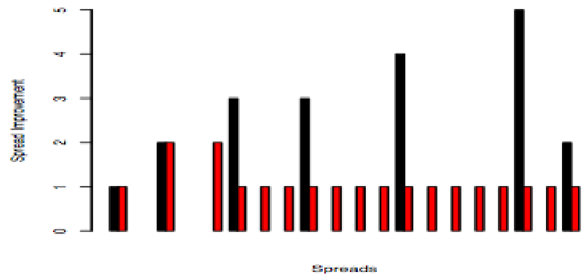

Figure 10: Spread Improvements by ABM.

analyzes this relationship leading to the conclusions described below.

The order arrival rate. The authors identify two different effects with opposite impact on the average spread. On one hand, the average spread tends to be inversely related to the order arrival rate *(Corollary 2.1, p.1193) The support of possible spreads in the fast market is shifted to the left compared to the support of possible spreads in the slow market.* Figure 11 shows the spread distributions for fast (black, $\lambda = 1$) and slow (red, $\lambda = 0.2$) markets dominated by patient (left) and impatient (right) traders. Agents maximize following optimal strategies in [8][21].

As stated in [8], higher spreads appear with higher probability in a slow market. On the other hand, market resiliency is inversely related to the order arrival rate *(Corollary 2.2, p.1193) The slow market is more resilient than the fast market.*

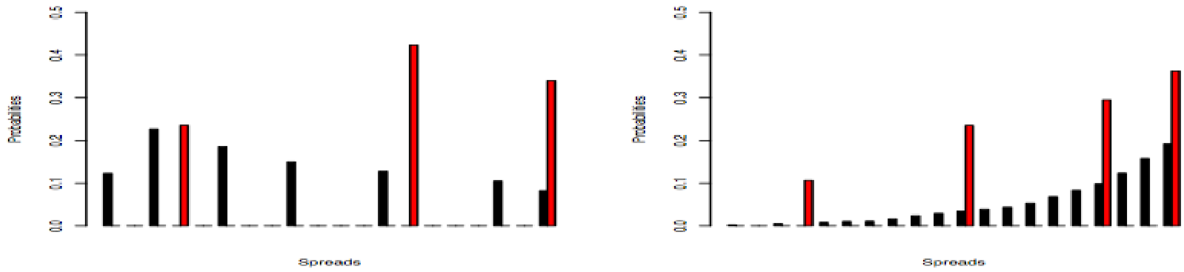

Figure 11: Spreads Distributions by ABM model: Fast market vs Slow market.

Figure 12 shows the spread improvements. Traders are more aggressive in a slow market (red), i.e., the spread is reduced more than in the fast market (black). This holds for both markets dominated by patient traders (left) and dominated by impatient traders (right), and

[21]Computations for equilibrium order placement strategies in slow markets with $\lambda = 0.2$ are reported in appendix A.1.

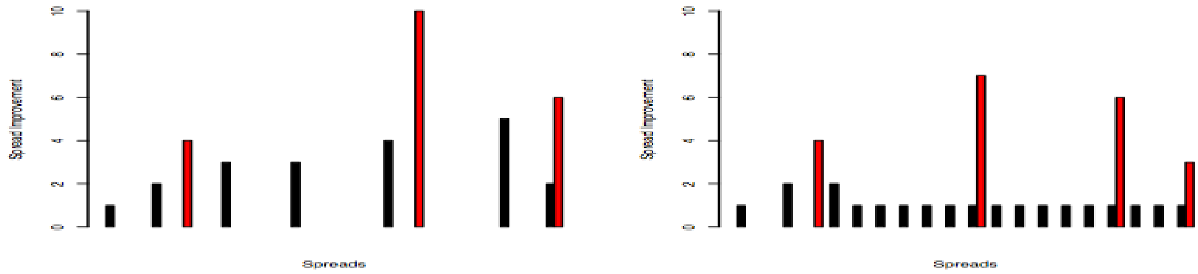

Figure 12: Spreads Improvements by ABM model: Fast market vs Slow market.

leads to more resilient books. Given two different values of λ (that characterize slow and fast markets), which market exhibits the smaller spread on average depends on which effect prevails. [8] concludes by computation that, for a large set of parameters values, the former effect (Corollary 2.1) dominates the latter; hence, in most cases there is a negative correlation between the order arrival rate (λ) and the average spread.

Result 4. Fast market exhibits lower spreads with respect to slow markets (i.e., higher values of λ decrease the average spread).

Figure 13 shows the book dynamics for the first forty traders in fast (black) and slow (red) markets characterized by different percentage of patient traders ($\theta_P = 0.55$ in the first line, $\theta_P = 0.45$ in the second line). Table 9 reports average spreads in fast and slow markets populated by $N = 10.000$ traders. Results 4 in [8] is confirmed by simulation for markets dominated by patient traders but not always for markets that exhibit the opposite composition of traders types. This conclusion is not surprisingly due to the (previously mentioned) fact that the likelihood of a market order is higher in the latter markets and this implies that small spreads are shown less frequently. This effect tends to be amplified by a higher order arrival rate (i.e., in a fast market).

	$\theta_P = 0.55$	$\theta_P = 0.45$
Fast Markets ($\lambda = 1$)	0.5305	1.0000
Slow Markets ($\lambda = 0.2$)	0.8583	0.9537

Table 9: Average Spreads in Fast/Slow Markets populated by different percentage of patient traders (N=10000).

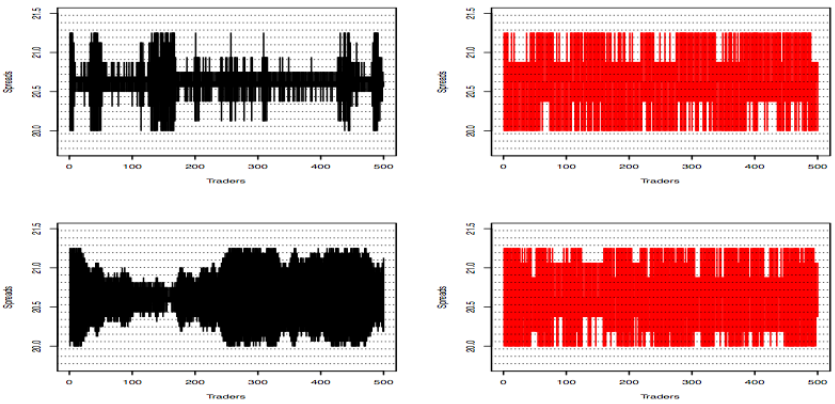

Figure 13: Market Resiliency by ABM model: Fast market vs Slow market.

A.1 Equilibrium Order Placement Strategies in Slow Markets (numerical example, $\lambda = 0.2$)

Since the condition $s_c = K$ (i.e., the cutoff spread for which impatient traders submit market orders is equal to the maximum spread) holds given both the parameter sets in Tables 10, 11, *impatient traders* optimal placement strategy is always to place a (market) order with $J^* = 0$.

Equilibrium strategies for *patient traders* derive from proposition 5, p.1183.

Proposition 1. *The set of equilibrium spreads is given by:*
$n_1 = j_P^*$, $n_q = K$
$n_h = n_1 + \sum_{k=2}^{h} \Psi_k$, $h = 2, ..., q-1$
where $\Psi_K = CF(2\rho^{k-1}\dfrac{\delta_P}{\lambda\triangle})$
and q *is the smallest integer such that* $j_P^* + \sum_{k=2}^{q} \Psi_K \geqslant K$

Note: CF is the ceiling function; it returns the smaller integer greater than or equal to the object.

Case (1)

θ_P	δ_P	δ_I	$\triangle$	λ
0.55	0.05	0.125	0.0625	0.2

Table 10: Parameter set - market dominated by patient traders.

$$\begin{cases} j_P^* = CF(\frac{\delta_P}{\lambda\triangle}) \Rightarrow j_P^* = 4 \\ \rho = \frac{\theta_P}{\theta_I} = 1.22 \\ \\ \Psi_2 = CF(9.76) = 10 \\ \Psi_3 = CF(11.9072) = 12 \\ \\ n_1 = j_P^* = 4 \\ n_2 = n_1 + \Psi_2 = 4 + 10 = 14 \\ n_3 = n_1 + \Psi_2 + \Psi_3 = 4 + 10 + 12 = 26 > 20 = K \Rightarrow q = 3. \end{cases}$$

Case (2)

θ_P	δ_P	δ_I	$\triangle$	λ
0.45	0.05	0.125	0.0625	0.2

Table 11: Parameter set - market dominated by impatient traders.

$$\begin{cases} j_P^* = CF(\frac{\delta_P}{\lambda\triangle}) \Rightarrow j_P^* = 4 \\ \rho = \frac{\theta_P}{\theta_I} = 0.81 \\ \\ \Psi_2 = CF(6.48) = 7 \\ \Psi_3 = CF(5.2488) = 6 \\ \Psi_4 = CF(4.25) = 5 \\ \\ n_1 = j_P^* = 4 \\ n_2 = n_1 + \Psi_2 = 4 + 7 = 11 \\ n_3 = n_1 + \Psi_2 + \Psi_3 = 4 + 7 + 6 = 17 \\ n_4 = n_1 + \Psi_2 + \Psi_3 + \Psi_4 = 4 + 7 + 6 + 5 > 20 = K \Rightarrow q = 4. \end{cases}$$

spread faced	Equilibrium strategies (J^*) - patient traders - Caso (1)	Caso (2)
1	0	0
2	0	0
3	0	0
4	0	0
5	4	4
6	4	4
7	4	4
8	4	4
9	4	4
10	4	4
11	4	4
12	4	11
13	4	11
14	4	11
15	14	11
16	14	11
17	14	11
18	14	17
19	14	17
20	14	17

Table 12: Equilibrium order placement strategies for patient traders. In redstrategies on equilibrium path.

B Fragility of the Equilibrium Order Placement Strategies

We use a test to check if the equilibrium result is confirmed by simulation. The baseline setup replicates the parameters (of the model) used in the numerical example in Foucault et al. (2005) for both market dominated by patient traders and market dominated by impatient traders. The suggested strategy-profile to be tested (according to Eq. 3) are the following:

spread 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	composition of the market
0	1	1	3	3	3	6	6	6	9	9	9	9	13	13	13	13	13	18	18	$\theta_P = 0.55$
0	1	1	3	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\theta_P = 0.45$

Table 13: Equilibrium order placement strategies for patient traders (impatients place market orders), Table 2, p. 1187 (Foucault et al., 2005). Parameter values: $\lambda = 1$, $\triangle = 0.0625$, $\delta_P = 0.05$, $\delta_I = 0.125$.

A randomly selected agent play a certain number of independent rounds (R); at each round he can submit his offer only once, choosing a random strategy in the feasible set given the actual spread. All the other agents play accordingly to the strategy reported in Table 13.

Average payoffs are computed for strategies on and off the equilibrium path (i.e., for every possible deviation), conditional on spread. Then, the maximizing strategy is identified and correspondence with the equilibrium strategy in [8] is checked.

We run two simulations for each market, one for each type of players (patient or impatient). By symmetry, the difference in role (i.e., buyers or sellers) need not to be considered.

We consider three different scenarios. The benchmark is a market populated by $N = 1000$ traders that play for 1.000.000 consecutive and independent trading sessions; in the other two cases, the number of agents in the market and the number of rounds over which average payoffs are computed are (alternatively) reduced to test their role in reaching the expected equilibrium. Table 14 reports results for patient traders in a market dominated by patient traders. Results are qualitatively equivalent for impatient traders.

	Equilibrium Spreads						
	1	3	6	9	13	18	20
(a) N=1000, R=1.000.000	0	1	3	6	9	13	18
(b) N=200, R=1.000.000	0	1	1	6	9	13	18
(c) N=1000, R=100.000	0	1	1	6	12	6	18
Foucault	0	1	3	6	9	13	18

Table 14: Test on equilibrium, $\theta_P = 0.55$, maximizing strategies for patient traders.

As it can be easily seen, the equilibrium result is confirmed: given a sufficiently large number of traders and rounds over which average payoffs are computed, the strategy suggested in Foucault et al. (2005) maximizes the payoffs of patient (who place limit orders) and impatient (who place market orders) traders in both markets. The choice of a high N is crucial. In fact, as mentioned in the paper, [8] construct a model over an infinite horizon, considering a stream of traders who continuously submit an order. If we redo this exercise with a lower N the same equilibrium result is no longer obtained.

At the same time, the choice of R (i.e., number of rounds over which average payoffs are computed) is even more decisive. In fact, the size of the range of admissible prices is not particularly large compared with the number of traders in the market, valuations and costs do not differ across buyers and sellers and are chosen outside the interval in which agents can submit their offers. As a joint effect of these facts, differences in payoffs are minimal and a large number of observations is needed to figure out which strategy is really the optimizing one.

C Quality Control

The genetic algorithm implemented leads to an "equilibrium" profile as it can be learned by agents. This equilibrium differs from the one discussed in Foucault et al. (2005) for many reasons. Firstly, an agent-based simulation, consistently with what happens in real markets, considers a finite number of traders for each trading session; as a consequence, the equilibrium suggested by the authors cannot be reached unless we use a quite high number of traders and reviews of strategies are based on average profits over a large evaluation window. Secondly, the reached result in terms of strategies chosen by the agent is not unique; learning takes time and is strictly related to both the initial conditions (i.e., random initialization of the agents strings that describe the strategy), the kind of mutation implemented, the parameters of the model and the path of learning itself. So, the learning process may leads to different profile of strategies.

We have to deal with an important point at issue; we need to test if we actually obtain an equilibrium strategy profile. To do that, we provide a quality control for the GA strategies.

Definition 4. *Given $\epsilon > 0$, a (possibly randomized) strategy profile σ is an ϵ-equilibrium if*

$$u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(s_i, \sigma_{-i} - \epsilon), \forall i, s_i \in S_i$$

When $\epsilon = 0$, we go back to the standard definition of a Nash Equilibrium.

We can consider ϵ as a measure of "how much" the strategy profile that we are going to test approximates an equilibrium; the lower the ϵ, the lower is the temptation for an agent to deviate (deviation is weakly profitable), and the closer we are to an equilibrium. Two computational problems arise given our setup: (a) our genetic algorithm (as behavior of learning processes in general) does not lead to the same equilibrium strategy profile and (b) agents of the same type (namely, patients and impatients) do not converge to a unique strategy in each GA-implementation. To deal with these issues, we set up a quality control as described below. We consider 10 different profile of strategies, obtained by specific GA simulations. For each of them, we randomly pick up a specific number of agents (namely, $N/10$ - where N is the total number of agents in the market). Then, a simulation is run for all traders under examination (separately); taking as fixed the strategies of all the other players, deviations from the original strategy to each feasible alternative pure strategy are experimented. We list average profits gained by the tested agent over the number of rounds played for each possible alternative strategy.

Finally, we compare the original strategy to be tested with the one that comes out choosing for each actually faced spread the pure strategy that leads to the maximum (average) profit (ceteris paribus). Since spreads might be faced with different frequencies, expected profits are weighted according to these frequencies. The difference between the maximum expected profits and the expected profits given the original strategies are computed; they represents our $\epsilon(s)$. We also compute the median and the mean value of ϵ for each scenario (when scenarios differ by the number of traders in the market). This gives an intuition of how ϵ might be distributed.

N	number of picked up agents	number of simulations	$\epsilon_{\max}$ (type)	ϵ_{median}	ϵ_{mean}
1000	100	1000	0.05 (P)	0.0312	0.0289

Table 15: ϵ-equilibrium test.

Table 15 summarizes the results. Results do not seem to be too far from an equilibrium: ϵ is sufficiently small. ϵ_{median} and ϵ_{mean} do not differ significantly; this suggests that

$\epsilon_{\max}$ cannot be considered an exceptional result but it represents quite well a worst-case benchmark[22].

[22]We also checked markets with different size. As expected, the number of traders in the market seems to be a key-determinant the value of ϵ; it is quite intuitive: higher N, finer is the learning process and, as a consequence, the better is the equilibriums approximation.