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On Bounds for Concave Distortion Risk Measures for Sums of Risks[*]

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Abstract. In this paper we consider the problem of studying the gap between bounds of risk measures of sums of non-independent random variables. Owing to the choice of the context where to set the problem, namely that of distortion risk measures, we first deduce an explicit formula for the risk measure of a discrete risk by referring to its writing as sum of layers. Then, we examine the case of sums of discrete risks with identical distribution. Upper and lower bounds for risk measures of sums of risks are presented in the case of concave distortion functions. Finally, the attention is devoted to the analysis of the gap between risk measures of upper and lower bounds, with the aim of optimizing it.

Keywords: Distortion risk measures; discrete risks; concave risk measures; upper and lower bounds; gap between bounds.

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1 Introduction

Recently in actuarial literature, the study of the impact of dependence among risks has become a major and flourishing topic: even if in traditional risk theory individual risks have been usually assumed to be independent, this assumption is very convenient for tractability but it is not generally realistic. Think for example to the aggregate claim amount in which any random variable represents the individual claim size of an insurer's risk portfolio. When the risk is represented by residential dwellings exposed to danger of an earthquake in a given location or by adjoining buildings in fire insurance, it is unrealistic to state that individual risks are not correlated, because they are subject to the same claim causing mechanism. Several notions of dependence were introduced in literature to model the fact that larger values of one of the component of a multivariate risk tend to be associated with larger values of the others. In financial or actuarial situations one often encounters random variables of the type

$$S = \sum_{i=1}^{n} X_i$$

where the terms X_i are not mutually independent and the multivariate distribution function of the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is not completely specified but one only knows the marginal distribution functions of the risks. To be able to make decisions, in such cases it may be helpful to determine approximations for the distribution of S, namely upper and lower bounds for risk measures of the sum of risks S, in such a way that it is possible to consider a riskiest portfolio and a safest portfolio, where riskiness and safety are both evaluated in terms of risk measures.

With the aim of studying the gap between the riskiest and the safest portfolio, the present contribution addresses the analysis to a particular class of risk measures, namely that of distortion risk measures introduced by Wang [8]. In this class the risk measure of a non-negative real valued random variable X is written in the following way

$$W_g(X) = \int_0^{\infty} g(H_X(x))dx$$

where the distortion function g is defined as a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

Given the choice of this context, it is possible to write an explicit formula for the risk measure of a discrete risk by referring to its writing as sum of layers (Campana and Ferretti [1]). Starting from this result, the attention is therefore devoted to the study of bounds of sums of risks in the case of discrete identically distributed random variables. Now the key role is played by the choice of the framework where to set the study: by referring to concave distortion risk measures, in fact, it is possible to characterize the riskiest portfolio where the multivariate distribution refers to mutually comonotonic risks and the safest portfolio where the multivariate distribution is that of mutually exclusive risks. Again, starting from the representation of risks as sums of layers, it is possible to derive explicit formulas for risk measures of upper and lower bounds of sums of risks. The attention is then devoted to the study of the difference between risk measures of upper and lower bounds, with the aim of obtaining some information on random variables for which the gap is maximum or minimum.

The paper is organized as follows. In Section 2 we first review some basic settings for describing the problem of measuring a risk and we remind some definitions and preliminary results in the field of distortion risk measures; then we propose the study of the case of a discrete risk with finitely many mass points in such a way that it is possible to give an explicit formula for its distortion risk measure. Section 3 is devoted to the problem of detecting upper and lower bounds for sums of not mutually independent risks: we present the study of the case of sums of discrete and identically distributed risks in order to obtain upper and lower bounds for concave distortion measures of aggregate claims of the portfolio. Then in Section 4 the attention is focused on the problem of characterizing risks for which the gap between bounds of risk measures is maximum or minimum. Some concluding remarks in Section 5 end the paper.

2 The class of distortion risk measures

As it is well-known, an insurance risk is defined as a non-negative real-valued random variable X defined on some probability space.

Here we consider a set Γ of risks with bounded support $[0, c]$. For each risk $X \in \Gamma$ we denote by H_X its tail function, i.e. $H_X(x) = Pr[X > x]$, for all $x \geq 0$.

A risk measure is defined as a mapping from the set of random variables, namely losses or payments, to the set of real numbers. In actuarial science common risk measures are premium principles; other risk measures are used for determining provisions and capital requirements of an insurer in order to avoid insolvency (see e.g. Dhaene et al. [5]).

In this paper we consider the distortion risk measure introduced by Wang [8]:

$$W_g(X) = \int_0^\infty g(H_X(x))dx \tag{1}$$

where the distortion function g is defined as a non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$. As it is well-known, the *quantile risk measure* and the *Tail Value-at-Risk*are examples of risk measures belonging to this class. In the particular case of a power g function, i.e. $g(x) = x^{1/\rho}$, $\rho \geq 1$, the corresponding risk measure is the *PH-transform risk measure* proposed by Wang [7].

Distortion risk measures satisfy the following properties (see Wang [8] and Dhaene et al. [5]):

P1. Additivity for comonotonic risks

$$W_g(S^c) = \sum_{i=1}^{n} W_g(X_i) \tag{2}$$

where S^c is the sum of the components of the random vector $\mathbf{X^c}$ with the same marginal distributions of $\mathbf{X}$ and with the comonotonic dependence structure.

P2. Positive homogeneity

$$W_g(a\,X) = a\,W_g(X) \quad \text{for any non-negative costant } a; \tag{3}$$

P3. Translation invariance

$$W_g(X+b) = W_g(X) + b \quad \text{for any costant } b; \tag{4}$$

P4. Monotonicity

$$W_g(X) \leq W_g(Y) \tag{5}$$

for any two random variables X and Y where $X \leq Y$ with probability 1.

2.1 Discrete risks with finitely many mass points

In the particular case of a discrete risk $X \in \Gamma$ with finitely many mass points it is possible to deduce an explicit formula of the distortion risk measure $W_g(X)$ of X. The key-point relies on the fact that each risk $X \in \Gamma$ can be written as sum of layers that are pairwise mutually comonotonic risks.

Let $X \in \Gamma$ be a discrete risk with finitely many mass points: then, there exist a positive integer m, a finite sequence $\{x_j\}$, $(j = 0, \cdots, m)$, $0 \equiv x_0 < x_1 < \ldots < x_m \equiv c$ and a finite sequence $\{p_j\}$, $(j = 0, \cdots, m-1)$, $1 \geq p_0 > p_1 > p_2 > \ldots > p_{m-1} > 0$ such that the tail function H_X of X is so defined

$$H_X(x) = \sum_{j=0}^{m-1} p_j \, I_{(x_j \leq x < x_{j+1})}, \quad x \geq 0, \tag{6}$$

where $I_{(x_j \leq x < x_{j+1})}$ is the indicator function of the set $\{x : x_j \leq x < x_{j+1}\}$. Then

$$X = \sum_{j=0}^{m-1} L(x_j, x_{j+1}) \tag{7}$$

where a layer at (x_j, x_{j+1}) of X is defined as the loss from an excess-of-loss cover, namely

$$L(x_j, x_{j+1}) = \begin{cases} 0 & 0 \leq X \leq x_j \\ X - x_j & x_j < X < x_{j+1} \\ x_{j+1} - x_j & X \geq x_{j+1} \end{cases} \tag{8}$$

and the tail function of the layer $L(x_j, x_{j+1})$ is given by

$$H_{L(x_j,x_{j+1})}(x) = \begin{cases} p_j & 0 \leq x < x_{j+1} - x_j \\ 0 & x \geq x_{j+1} - x_j \end{cases} . \tag{9}$$

If we consider a Bernoulli random variable B_{p_j} such that $Pr[B_{p_j} = 1] = p_j = 1 - Pr[B_{p_j} = 0]$ then $L(x_j, x_{j+1})$ is a two-points distributed random variable which satisfies the equality in distribution $L(x_j, x_{j+1}) \stackrel{\mathrm{d}}{=} (x_j - x_{j+1}) \, B_{p_j}$.

Additivity for comonotonic risks and positive homogeneity of distorted risk measures W_g ensure that

$$W_g(X) = \sum_{j=0}^{m-1} W_g(L(x_j, x_{j+1})) = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(p_j).$$

In this way for any discrete risk $X \in \Gamma$ for which representation (6) holds and any distortion function g we can assert that

$$W_g(X) = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(p_j). \tag{10}$$

3 The class of concave distortion risk measures

In the particular case of a concave distortion measure, the related distortion risk measure satisfying properties *P1-P4* is also sub-additive and it preserves stop-loss order. As it is well-known, examples of concave distortion risk measures are the *Tail Value-at-Risk* and the *PH-transform risk measure*, whereas *quantile risk measure* is not a concave risk measure.

In the previous section we deduced an explicit formula for the distortion risk measure $W_g(X)$ when a discrete risk $X \in \Gamma$ with finitely many mass points is considered. This result may be used to obtain upper and lower bounds for sums of discrete and identically distributed risks with common tail function given by (6) when we consider the following framework where to set the study: the Fréchet space consisting of all n-dimensional random vectors $\mathbf{X}$ possessing $(H_{X_1}, H_{X_2}, \ldots, H_{X_n})$ as marginal tail functions, for which the condition $\sum_{i=1}^{n} H_{X_i}(0) \leq 1$ is fulfilled and the distortion function g is assumed to be concave.

3.1 Upper bound for sums of discrete and identically distributed risks

Let $\mathbf{X}$ be a random vector with discrete and identically distributed risks $X_i \in \Gamma$. The least attractive random vector with given marginal distribution functions has the comonotonic joint distribution (see e.g. Dhaene et. al. [3] and Kaas et al. [6]), namely

$$W_g(S) \leq W_g(S^c).$$

Now we want to give an explicit formula for $W_g(S^c)$. Let the common tail function of X_i be written as

$$H_{X_i}(x) = \sum_{j=0}^{m-1} p_j\, I_{(x_j \leq x < x_{j+1})}, \quad x \geq 0 \tag{11}$$

where m is a positive integer and $1 \geq p_0 > p_1 > p_2 > \ldots > p_{m-1} > 0$, $0 \equiv x_0 < x_1 < \ldots < x_m \equiv c$. Then

$$S^c \stackrel{\mathrm{d}}{=} n\, X_1,$$

and by subadditivity of the concave risk measure W_g it follows that

$$W_g(S) \leq W_g(S^c) = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, n\, g(p_j).$$

Namely, under representation (11) the riskiest portfolio S^c exhibits the following risk measure

$$\sum_{j=0}^{m-1} (x_{j+1} - x_j)\, n\, g(p_j). \tag{12}$$

3.2 Lower bound for sums of discrete and identically distributed risks

As in the previous subsection, let $\mathbf{X}$ be a random vector with discrete and identically distributed risks $X_i \in \Gamma$. In the Fréchet space consisting of all n-dimensional random vectors $\mathbf{X}$ possessing $(H_{X_1}, H_{X_2}, \ldots, H_{X_n})$ as marginal tail functions, for which the condition $\sum_{i=1}^{n} H_{X_i}(0) \leq 1$ is fulfilled, the safest random vector is given by (see Dhaene et Denuit, [2]) the vector $\mathbf{X}^e = (X_1^e, X_2^e, \ldots, X_n^e)$ where the components are said to be mutually exclusive because $Pr[X_i^e > 0, X_j^e > 0] = 0$ for all $i \neq j$. Let S^e denote the sum of mutually exclusive risks $X_1^e, X_2^e, \ldots, X_n^e$. In order to have an explicit formula for $W_g(S^e)$, note that its tail function is given by

$$H_{S^e}(x) = \sum_{i=1}^{n} H_{X_i}(x), \quad \text{for all } x \geq 0. \tag{13}$$

Owing to the fact that the common tail function of X_i is written as (11) where $np_0 \leq 1$, the tail function of the sum S^e of mutually exclusive risks becomes

$$H_{S^e}(x) = n \sum_{j=0}^{m-1} p_j\, I_{(x_j \leq x < x_{j+1})}, \quad \text{for all } x \geq 0. \tag{14}$$

Note that S^e can be written as a sum of layers

$$S^e = \sum_{j=0}^{m-1} \tilde{L}(x_j, x_{j+1})$$

where $\tilde{L}(x_j, x_{j+1})$ is a two-points distribution with $\tilde{L}(x_j, x_{j+1}) \stackrel{\mathrm{d}}{=} B_{n\, p_j}$. By considering a concave distortion risk measure, it is

$$W_g(S) \geq W_g(S^e) = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(n\, p_j). \tag{15}$$

In other words, under hypothesis (11) where $np_0 \leq 1$, the safest portfolio S^e exhibits the following risk measure

$$\sum_{j=0}^{m-1} (x_{j+1} - x_j)\, g(n\, p_j). \tag{16}$$

4 Optimal gap between bounds of risk measures

In the previous section lower and upper bounds for sums of discrete and identically distributed risks $X_i \in \Gamma$ have been obtained: the attention now is devoted to the study of the difference between risk measures of upper and lower bounds, with the aim of obtaining some information on random variables for which the gap is maximum or minimum. Starting from formulations (16) and (12) exhibiting bounds for aggregate claims S of the portfolio $\mathbf{X}$, we face the problem of studying the difference between bounds in order to minimize and maximize it. The problem in p_j $(j = 1, \cdots, m-1)$ and x_j $(j = 1, \cdots, m)$ is then

$$\max / \min \sum_{j=0}^{m-1} (x_{j+1} - x_j)[ng(p_j) - g(np_j)] \tag{17}$$

where

$$\begin{array}{l} x_0 \equiv 0; x_j < x_{j+1}; x_m \equiv c; \\ 0 < p_{m-1} < \cdots < p_0 \leq \frac{1}{n}; n > 1; \\ g : [0,1] \to [0,1];\ g(0) = 0;\ g(1) = 1; g \text{ is non-decreasing and concave.} \end{array}$$

Note that the optimization problem is related to the maximization/minimization of the gap between upper and lower bounds for risk measures, namely to the maximization/minimization of the difference $W_g(S^c) - W_g(S^e)$.

Let $\phi_n(p_j)$ be equal to $ng(p_j) - g(np_j)$: since g is concave and $g(0) = 0$, ϕ_n is non-negative; in particular $\phi_n(0) = 0$. Moreover, non-negativity and concavity of g imply that ϕ_n is non-decreasing. After setting $\Delta_j = x_{j+1} - x_j$ the problem becomes

$$\max / \min \sum_{j=0}^{m-1} \Delta_j\ \phi_n(p_j) \tag{18}$$

where

$$\begin{array}{l} \Delta_j > 0;\ \sum_{j=0}^{m-1} \Delta_j = c; \\ 0 < p_{m-1} < \ldots < p_0 \leq \frac{1}{n};\ n > 1; \\ \phi_n : [0, \frac{1}{n}] \to \mathcal{R};\ \phi_n(0) = 0;\ \phi_n(\frac{1}{n}) = ng(\frac{1}{n}) - 1 \geq 0;\ \phi_n \text{ is non-decreasing.} \end{array}$$

Note that the feasible set is not closed, so at any first step a relaxed problem with closed constraints will be faced.

4.1 The problem of minimizing the gap

The problem of minimizing the difference $W_g(S^c) - W_g(S^e)$ may be faced both in terms of p_j both in terms of Δ_j.

a) *Solution with respect to the p_js.*

At a first step the hypothesis that the constraints for the p_js admit equality is assumed (namely we consider the closure of the feasible set); by monotonicity of ϕ_n it follows that $p_j = 0$; moreover, if there exists $0 < \epsilon < \frac{1}{n}$ such that $\phi_n(\epsilon) = 0$ then all the p_js could be set in the interval $(0, \epsilon]$ and a solution would exist also in the original open set.

b) *Solution with respect to the Δ_js.*

Starting from the case that the constraints for the Δ_js admit equality (the closure of the feasible set is assumed), by non-decreasing monotonicity of ϕ_n it follows that the minimum is when $\Delta_{m-1} = c$ and all the other Δ_j are set equal to 0 (that is $x_0 = x_1 = \ldots = x_{m-1} = 0$ and $x_m = c$). In the particular case of a constant function ϕ_n in the interval $[p_{m-1}, p_0]$ the problem would admit interior minima given by any feasible choice of the Δ_js.

4.2 The problem of maximizing the gap

The problem of maximizing the difference $W_g(S^c) - W_g(S^e)$ exhibits the following solutions in terms of p_j and in terms of Δ_j.

a) *Solution with respect to the p_js.*

By referring to the case of closed feasible set (that is the constraints for the p_js admit equality) the optimal solution is given by $p_j = \frac{1}{n}$; if moreover there exists $0 < \epsilon < \frac{1}{n}$ such that ϕ_n is constant in the interval $[\frac{1}{n} - \epsilon, \frac{1}{n}]$, then all the p_js could be set in that interval and a solution would exist also in the original open set.

b) *Solution with respect to the Δ_js.* Under the relaxed hypothesis of equality constraints on Δ_js, the maximum is when $\Delta_0 = c$ and all the other Δ_js are set equal to 0 (that is $x_0 = 0$ and $x_1 = \ldots = x_m = c$). Note that if ϕ_n were constant in the interval $[p_{m-1}, p_0]$ the problem would admit interior maxima: any feasible choice of the Δ_js is solution.

5 Concluding remarks

In this paper we face the problem of studying the gap between bounds for risk measures of sums of discrete and identically distributed risks. Starting from the representation of risks as sums of layers, explicit formulas for risk measures of upper and lower bounds of sums of risks are obtained in the particular case of concave distortion risk measures. A maximization(minimization) problem related to the maximization (minimization) of the gap between risk measures of upper and lower bounds is solved with respect to information characterizing the random vector **X**.

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