

WORKING PAPER SERIES



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Working Paper n. 184/2008
November 2008

ISSN: 1828-6887

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LIMIT THEOREMS FOR REINFORCED JUMP PROCESSES ON REGULAR TREES

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(November 2008)

Abstract. Consider a vertex-reinforced jump process defined on a regular tree, where each vertex has exactly b children, with $b \geq 3$. We prove the strong law of large numbers and the central limit theorem for the distance of the process from the root. Notice that it is still unknown if vertex-reinforced jump process is transient on the binary tree.

Keywords: Reinforced random walks, stochastic processes, strong law of large numbers, central limit theorem.

JEL Classification Numbers: C02, C00.

AMS Classification Numbers: 60F15, 60F17, 60G50.

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1 Introduction

Let \mathcal{D} be any graph with the property that each vertex is the end point of only a finite number of edges. Denote by $\text{Vert}(\mathcal{D})$ the set of vertices of \mathcal{D} . The following, together with the vertex occupied at time 0 and the set of positive numbers $\{a_\nu: \nu \in \text{Vert}(\mathcal{D})\}$, defines a right-continuous process $\mathbf{X} = \{X_s, s \geq 0\}$. This process takes as values the vertices of \mathcal{D} and jumps only to nearest neighbors, i.e. vertices one edge away from the occupied one. Given $X_s, 0 \leq s \leq t$, and $\{X_t = x\}$, the conditional probability that, in the interval $(t, t + dt)$, the process jumps to the nearest neighbor y of x is $L(y, t)dt$, with

$$L(y, t) := a_y + \int_0^t \mathbb{1}_{\{X_s=y\}} ds, \quad a_y > 0,$$

where $\mathbb{1}_A$ stands for the indicator function of the set A . The positive numbers $\{a_\nu: \nu \in \text{Vert}(\mathcal{D})\}$ are called initial weights, and we suppose $a_\nu \equiv 1$, unless specified otherwise. Such a process is said to be a Vertex Reinforced Jump Process (VRJP) on \mathcal{D} . In this paper we define a process to be recurrent if it visits each vertex infinitely many times a.s., and to be transient otherwise. VRJP was introduced by Wendelin Werner, and its properties were first studied by Davis and Volkov (see [8] and [9]). This reinforced walk defined on the integer lattice is studied in [8] where recurrence is proved. For fixed $b \in \mathbb{N} := \{1, 2, \dots\}$, the b -ary tree, which we denote by \mathcal{G}_b , is the infinite tree where each vertex has $b + 1$ neighbors with the exception of a single vertex, called the root and designated by ρ , that is connected to b vertices. In [9] is shown that VRJP on the b -ary tree is transient if $b \geq 4$. The case $b = 3$ was dealt in [4], where it was proved that the process is still transient. The case $b = 2$ is still open.

We define the distance between two vertices as the number of edges in the unique self-avoiding path connecting them. For any vertex ν , denote by $|\nu|$ its distance from the root. Level i is the set of vertices ν such that $|\nu| = i$. The main result of this paper is the following.

Theorem 1.1 *Let \mathbf{X} be VRJP on \mathcal{G}_b , with $b \geq 3$. There exist constants $K_b^{(1)}, K_b^{(2)} \in (0, \infty)$ such that*

$$\lim_{t \rightarrow \infty} \frac{|X_t|}{t} = K_b^{(1)} \quad a.s., \tag{1.1}$$

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{|X_t| - K_b^{(1)}t}{\sqrt{t}K_b^{(2)}} \leq x \right) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz. \tag{1.2}$$

Durrett, Kesten and Limic have proved in [11] an analogous result for a bond-reinforced random walk, called one-time bond-reinforced random walk, on $\mathcal{G}_b, b \geq 2$. To prove this, they break the path into independent identically distributed blocks, using the classical method of cut points. We also use this approach. Our implementation of the cut point method is a strong improvement of the one used in [3] to prove the strong law of large numbers for the original reinforced random walk, the so-called linearly bond-reinforced random walk, on \mathcal{G}_b , with $b \geq 70$. Aidékon, in [1] gives a sharp criteria for random walk in

a random environment, defined on Galton-Watson tree, to have positive speed. He proves the strong law of large numbers for linearly bond-reinforced random walk on \mathcal{G}_b , with $b \geq 2$.

The reader can find in [16] a survey on reinforced processes. Merkl and Rolles (see [13]) studied the recurrence of linearly-bond reinforced random walk on two-dimensional graphs.

2 Preliminary definitions and properties

From now on, we consider VRJP \mathbf{X} defined on the regular tree \mathcal{G}_b , with $b \geq 3$. For $\nu \neq \rho$, define $\text{par}(\nu)$, called the parent of ν , to be the unique vertex at level $|\nu| - 1$ connected to ν . A vertex ν_0 is a child of ν if $\nu = \text{par}(\nu_0)$. We say that a vertex ν_0 is a descendant of the vertex ν if the latter lays on the unique self-avoiding path connecting ν_0 to ρ , and $\nu_0 \neq \nu$. In this case, ν is said to be an ancestor of ν_0 . For any vertex μ , let Λ_μ be the subtree consisting of μ , its descendants and the edges connecting them, i.e. the subtree rooted at μ . Define

$$T_i := \inf\{t \geq 0: |X_t| = i\}.$$

We give the so-called Poisson construction of VRJP on a graph \mathcal{D} (see [17]). For each ordered pair of neighbors (u, v) assign a Poisson process $P(u, v)$ of rate 1, the processes being independent. Call $h_i(u, v)$, with $i \geq 1$, the inter-arrival times of $P(u, v)$ and let $\xi_1 := \inf\{t \geq 0: X_t = u\}$. The first jump after ξ_1 is at time $c_1 := \xi_1 + \min_v h_1(u, v)(L(v, \xi_1))^{-1}$, where the minimum is taken over the set of neighbors of u . The jump is towards the neighbor v for which that minimum is attained. Suppose we defined $\{(\xi_j, c_j), 1 \leq j \leq i - 1\}$, and let

$$\begin{aligned} \xi_i &:= \inf\{t > c_{i-1}: X_t = u\}, \text{ and} \\ j_v - 1 = j_{u,v} - 1 &:= \text{number of times } \mathbf{X} \text{ jumped from } u \text{ to } v \text{ by time } \xi_i. \end{aligned}$$

The first jump after ξ_i happens at time $c_i := \xi_i + \min_v h_{j_v}(u, v)(L(v, \xi_i))^{-1}$, and the jump is towards the neighbor v which attains that minimum.

Definition 2.1 *A vertex μ , with $|\mu| \geq 2$, is **good** if it satisfies the following*

$$h_1(\mu_0, \mu) < \frac{h_1(\mu_0, \text{par}(\mu_0))}{1 + h_1(\text{par}(\mu_0), \mu_0)} \quad \text{where } \mu_0 = \text{par}(\mu). \quad (2.3)$$

In virtue of our construction of VRJP, (2.3) can be interpreted as follows. When the process \mathbf{X} visits the vertex μ_0 for the first time, if this ever happens, the weight at its parent is exactly $1 + h_1(\text{par}(\mu_0), \mu_0)$ while the weight at μ is 1. Hence condition (2.3) implies that when the process visits μ_0 (if this ever happens) then it will visit μ before it returns to $\text{par}(\mu_0)$, if this ever happens.

Lemma 2.2 *Let*

$$\alpha_b := \mathbb{P}(X_t = \rho \text{ for some } t \geq T_1),$$

and let β_b be the smallest among the positive solutions of the equation

$$x = \sum_{k=0}^b x^k p_k, \quad (2.4)$$

where, for $k \in \{0, 1, \dots, b\}$,

$$p_k := \sum_{j=0}^k \binom{b}{k} \binom{k}{j} (-1)^j \int_0^\infty \frac{1+z}{j+b-k+1+z} e^{-z} dz. \quad (2.5)$$

We have

$$\int_0^\infty \frac{1+z}{b+1+z} b e^{-bz} dz \leq \alpha_b \leq \beta_b. \quad (2.6)$$

Proof. First we prove the lower bound in (2.6). The left-hand side of this inequality is the probability that the first jump after time T_1 is towards the root. To see this, notice that $L(\rho, T_1)$ is equal to $1 + Y$, where

$$Y := \min_{\nu: |\nu|=1} h_1(\rho, \nu)$$

is distributed like an exponential with mean $1/b$. Define $S_2 := \inf\{t > T_1 : X_t \neq X_{T_1}\}$. Then

$$\begin{aligned} \mathbb{P}(X_{S_2} = \rho) &= \int_0^\infty \int_0^\infty e^{-by} (1+z) e^{-(1+z)y} b e^{-bz} dy dz \\ &= \int_0^\infty \frac{1+z}{b+1+z} b e^{-bz} dz. \end{aligned}$$

As for the upper bound in (2.6) we reason as follows. We give an upper bound for the probability that there exists an infinite random tree composed only by good vertices and which has root at one of the children of X_{T_1} . If this event holds, then the process does not return to the root after time T_1 (see the proof of Theorem 3 in [4]). We prove that a particular cluster of good vertices is stochastically larger than a branching process which is supercritical. We introduce the following color scheme. The only vertex at level 1 to be green is X_{T_1} . A vertex ν , with $|\nu| \geq 2$, is green if and only if it is good and its parent is green. All the other vertices are uncolored. Fix a vertex μ . Let C be any event in

$$\mathcal{H}_\mu := \sigma(h_i(\eta_0, \eta_1) : i \geq 1, \text{ with } \eta_0 \sim \eta_1 \text{ and both } \eta_0 \text{ and } \eta_1 \notin \Lambda_\mu), \quad (2.7)$$

that is the σ -algebra that contains the information about X_t observed outside Λ_μ . Given $C \cap \{\mu \text{ is green}\}$, the distribution of $h_1(\text{par}(\mu), \mu)$ is stochastically dominated by an exponential(1). To see this, first notice that $h_1(\text{par}(\mu), \mu)$ is independent of C . Let $D := \{\text{par}(\mu) \text{ is green}\} \in \mathcal{H}_\mu$. Reasoning as in Theorem 3 of [4], there exists a random variable W independent of $h_1(\text{par}(\mu), \mu)$ (see the definition of good vertices), such that

$$\{\mu \text{ is green}\} = \{h_1(\text{par}(\mu), \mu) < W\} \cap D.$$

We have

$$\begin{aligned}
& \mathbb{P}\left(h_1(\text{par}(\mu), \mu) \geq x \mid \{\mu \text{ is green}\} \cap C\right) \\
&= \mathbb{P}\left(h_1(\text{par}(\mu), \mu) \geq x \mid \{h_1(\text{par}(\mu), \mu) < W\} \cap C \cap D\right) \\
&\leq \mathbb{P}\left(h_1(\text{par}(\mu), \mu) \geq x\right).
\end{aligned} \tag{2.8}$$

In the last inequality, we used the fact that $h_1(\text{par}(\mu), \mu)$ is independent of W, C and D . The inequality (2.8) implies that if μ_1 is a child of μ and $C \in \mathcal{H}_\mu$ we have

$$\mathbb{P}\left(\mu_1 \text{ is green} \mid \{\mu \text{ is green}\} \cap C\right) \geq \mathbb{P}\left(\mu_1 \text{ is green}\right). \tag{2.9}$$

Hence the cluster of green vertices is stochastically larger than a Galton–Watson tree where each vertex has k offspring, $k \in \{0, 1, \dots, b\}$, with probability p_k defined in (2.5). To see this, fix a vertex μ and let μ_i , with $i \in \{0, 1, \dots, b\}$ be its children. It is enough to realize that p_k is the probability that exactly k of the $h_1(\mu, \mu_i)$, with $i \in \{0, 1, \dots, b\}$, are smaller than $(1 + h_1(\text{par}(\mu), \mu))^{-1} h_1(\mu, \text{par}(\mu))$. As the random variables $h_1(\mu, \mu_i)$, $h_1(\mu, \text{par}(\mu))$ and $h_1(\text{par}(\mu), \mu)$ are independent exponentials with parameter one, we have

$$\begin{aligned}
p_k &= \binom{b}{k} \int_0^\infty \int_0^\infty \mathbb{P}\left(h_1(\mu_0, \mu) < \frac{y}{1+z}\right)^k \mathbb{P}\left(h_1(\mu_0, \mu) \geq \frac{y}{1+z}\right)^{b-k} e^{-y} e^{-z} dy dz \\
&= \binom{b}{k} \int_0^\infty \int_0^\infty (1 - e^{-\frac{y}{1+z}})^k e^{-\frac{y}{1+z}(b-k)} e^{-y} e^{-z} dy dz \\
&= \sum_{j=0}^k \int_0^\infty \int_0^\infty \binom{b}{k} \binom{k}{j} (-1)^j e^{-y(j+b-k+1+z)/(1+z)} e^{-z} dy dz \\
&= \sum_{j=0}^k \binom{b}{k} \binom{k}{j} (-1)^j \int_0^\infty \frac{1+z}{j+b-k+1+z} e^{-z} dz.
\end{aligned} \tag{2.10}$$

From the basic theory of branching processes we know that the probability that this Galton–Watson tree is finite (i.e. extinction) equals the smallest positive solution of the equation

$$x - \sum_{k=0}^b x^k p_k = 0. \tag{2.11}$$

The proof of (2.6) follows from the fact that $1 - \beta_b \leq 1 - \alpha_b$. This latter inequality is a consequence of the fact that the cluster of green vertices is stochastically larger than the Galton–Watson tree, hence its probability of non-extinction is not smaller. As the Galton–Watson tree is supercritical, we have $\beta_b < 1$. \square

For example, if we consider VRJP on \mathcal{G}_3 , Lemma 2.2 yields

$$0.3809 \leq \alpha_3 \leq 0.8545.$$

Definition 2.3 *Level $j \geq 1$ is a cut level if the first jump after T_j is towards level $j + 1$, and after time T_{j+1} the process never goes back to X_{T_j} , and*

$$L(X_{T_j}, \infty) < 2 \quad \text{and} \quad L(\text{par}(X_{T_j}), \infty) < 2.$$

Define l_1 to be the cut level with minimum distance from the root, and for $i > 1$,

$$l_i := \min\{j > l_{i-1} : j \text{ is a cut level}\}.$$

Define the i -th cut time to be $\tau_i := T_{l_i}$. Notice that $l_i = |X_{\tau_i}|$.

3 l_1 has an exponential tail

For any vertex $\nu \in \text{Vert}(\mathcal{G}_b)$, we define $\text{fc}(\nu)$, which stands for **first child** of ν , to be the (a.s.) unique vertex connected to ν satisfying

$$h_1(\nu, \text{fc}(\nu)) = \min\{h_1(\nu, \mu) : \text{par}(\mu) = \nu\}. \quad (3.12)$$

The root ρ is not a first child. Notice that condition (3.12) does not imply that the vertex $\text{fc}(\nu)$ is visited by the process. If \mathbf{X} visits it, then it is the first among the children of ν to be visited.

For any pair of distributions f and g , denote by $f \bar{*} g$ the distribution of $\sum_{k=1}^V M_k$, where

- V has distribution f , and
- $\{M_k, k \in \mathbb{N}\}$ is a sequence of i.i.d random variables, independent of V , each with distribution g .

Recall the definition of $p_i, i \in \{0, \dots, b\}$, given in (2.5). Denote by $\mathbf{p}^{(1)}$ the distribution which assigns to $i \in \{0, \dots, b\}$ probability p_i . Define, by recursion, $\mathbf{p}^{(j)} := \mathbf{p}^{(j-1)} \bar{*} \mathbf{p}^{(1)}$, with $j \geq 2$. The distribution $\mathbf{p}^{(j)}$ describes the number of elements, at time j , in a population which evolves like a branching process generated by one ancestor and with offspring distribution $\mathbf{p}^{(1)}$. If we let

$$m := \sum_{j=1}^b j p_j,$$

then the mean of $\mathbf{p}^{(j)}$ is m^j . The probability that a given vertex is good is

$$\int_0^\infty \frac{1}{2+z} e^{-z} dz = 0.36133 \dots$$

Hence

$$m = b \cdot 0.36133 > 1,$$

because we assumed $b \geq 3$.

Let $q_0 = p_0 + p_1$, and for $k \in \{1, 2, \dots, b-1\}$ set $q_k = p_{k+1}$. Set \mathbf{q} to be the distribution which assigns to $i \in \{0, \dots, b-1\}$ probability q_i . For $j \geq 2$, let $\mathbf{q}^{(j)} := \mathbf{p}^{(j-1)} \bar{*} \mathbf{q}$. Denote by $q_i^{(j)}$ the weight that the distribution $\mathbf{q}^{(j)}$ assigns to $i \in \{0, \dots, (b-1)b^{j-1}\}$. The mean of $\mathbf{q}^{(j)}$ is $m^{j-1}(m-1)$. From now on, ζ denotes the smallest positive integer in $\{2, 3, \dots\}$ such that

$$m^{\zeta-1}(m-1) > 1. \quad (3.13)$$

For any vertex ν of \mathcal{G}_b let Θ_ν be the set of vertices μ such that

- μ is a descendant of ν ,
- the difference $|\mu| - |\nu|$ is a multiple of ζ ,
- μ is a first child.

Define $\tilde{\Theta}_\nu$ to be set of vertices in Θ_ν and their descendants. Denote by \mathcal{C}_ν the connected cluster of good vertices containing ν . If ν is not good then \mathcal{C}_ν is empty. Let Σ_ν be the subtree of \mathcal{G}_b consisting of ν , its descendants which are not contained in $\tilde{\Theta}_\nu$, and the edges connecting them. Set $\tilde{\nu} = \text{fc}(\nu)$ and let

$$A(\nu) := \left\{ \text{there exists a child } \mu \text{ of } \tilde{\nu} \text{ such that } \mathcal{C}_\mu \cap \text{Vert}(\Sigma_\nu) \cap \text{Vert}(\Lambda_{\tilde{\nu}}) \text{ is infinite} \right\}. \quad (3.14)$$

The event $A(\nu)$ holds if and only if there exists a child of $\tilde{\nu}$ which is the root of an infinite subtree of Σ_ν composed only by good vertices. For $i \in \mathbb{N}$, let $A_i := A(X_{T_i})$.

Proposition 3.1 *The events $A_{i\zeta}$, with $i \in \mathbb{N}$, are independent.*

Proof. We remark the fact that $\zeta \geq 2$. Choose integers $0 < i_1 < i_2 < \dots < i_k$, with $i_j \in \zeta\mathbb{N} := \{\zeta, 2\zeta, 3\zeta, \dots\}$ for all $j \in \{1, 2, \dots, k\}$. It is enough to prove that

$$\mathbb{P}\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k \mathbb{P}(A_{i_j}). \quad (3.15)$$

We proceed by backward recursion. Fix a vertex ν at level i_k . The set $A(\nu)$ belongs to the sigma-algebra generated by $\{P(u, w) : u, w \in \text{Vert}(\Lambda_\nu)\}$. On the other hand, the set $\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = \nu\}$ belongs to $\{P(u, w) : u \notin \text{Vert}(\Lambda_\nu)\}$. As the two events belong to disjoint collections of independent Poisson processes, they are independent. As $\mathbb{P}(A(\nu)) = \mathbb{P}(A(\rho))$, we have

$$\begin{aligned} \mathbb{P}\left(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j}\right) &= \sum_{\nu: |\nu|=i_k} \mathbb{P}\left(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = \nu\}\right) \\ &= \sum_{\nu: |\nu|=i_k} \mathbb{P}\left(A(\nu) \cap \bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = \nu\}\right) = \sum_{\nu: |\nu|=i_k} \mathbb{P}(A(\nu)) \mathbb{P}\left(\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = \nu\}\right) \\ &= \mathbb{P}(A(\rho)) \sum_{\nu: |\nu|=i_k} \mathbb{P}\left(\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = \nu\}\right) = \mathbb{P}(A(\rho)) \mathbb{P}\left(\bigcap_{j=1}^{k-1} A_{i_j}\right). \end{aligned} \quad (3.16)$$

The events $A(\nu)$ and $\{X_{T_{i_k}} = \nu\}$ are independent, and in virtue of the self-similarity property of the regular tree we get $\mathbb{P}(A(\rho)) = \mathbb{P}(A_{i_k})$. Hence

$$\mathbb{P}\left(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j}\right) = \mathbb{P}(A_{i_k}) \mathbb{P}\left(\bigcap_{j=1}^{k-1} A_{i_j}\right). \quad (3.17)$$

Reiterating (3.17) we get (3.15). □

Lemma 3.2 Define γ_b to be the smallest positive solution of the equation

$$x = \sum_{k=0}^{b-1} x^k q_k^{(\zeta)}, \quad (3.18)$$

where ζ and $(q_k^{(n)})$ have been defined at the beginning of this section. We have

$$\mathbb{P}(A_i) \geq 1 - \gamma_b > 0, \quad \forall i \in \mathbb{N}. \quad (3.19)$$

Proof. Fix $i \in \mathbb{N}$ and let $\nu^* = X_{T_i}$. We adopt the following color scheme. The vertex $\text{fc}(X_{T_i})$ is colored blue. A descendant μ of ν^* is colored blue if it is good, its parent is blue, and either

- $|\mu| - |\nu^*|$ is not a multiple of ζ , or
- $\frac{1}{\zeta}(|\mu| - |\nu^*|) \in \mathbb{N}$ and μ is not a first child.

Vertices which are not descendants of ν^* are not colored. Following the reasoning given in the proof of Lemma 2.2, we can conclude that the number of blue vertices at levels $|\nu^*| + j\zeta$, with $j \geq 1$, is stochastically larger than the number of individuals in a population which evolves like a branching process with offspring distribution $\mathbf{q}^{(\zeta)}$, introduced at the beginning of this section. Again, from the basic theory of branching processes we know that the probability that this tree is finite equals the smallest positive solution of the equation (3.18). In virtue of (3.13) we have that $\gamma_b < 1$. \square

The proof of the following Lemma can be found in [10] pages 26-27 and 35.

Lemma 3.3 Suppose U_n is Binomial(n, p). For $x \in (0, 1)$ consider the entropy

$$H(x|p) := x \ln \frac{x}{p} + (1-x) \ln \frac{1-x}{1-p}.$$

We have the following large deviations estimate, for $s \in [0, 1]$,

$$\mathbb{P}(U_n \leq sn) \leq 2 \exp\left\{-n \inf_{x \in [0, s]} H(x|p)\right\}.$$

Denote by $[x]$ the largest integer smaller than x .

Theorem 3.4 For VRJP defined on \mathcal{G}_b , with $b \geq 3$, and $s \in (0, 1)$, we have

$$\mathbb{P}(l_{[sn]} \geq n) \leq 2 \exp\left\{-[n/\zeta] \inf_{x \in [0, s]} H\left(x | (1 - \gamma_b)\varphi_b\right)\right\}, \quad (3.20)$$

where γ_b was defined in Lemma 3.2, and

$$\varphi_b := \left(1 - e^{-b}\right) \left(1 - e^{-(b+1)}\right) \frac{b}{b+2}. \quad (3.21)$$

Proof. We say that level j is of type **A** if A_j holds. In virtue of Proposition 3.1 the sequence $\mathbb{1}_{A_{k\zeta}}$, with $k \in \mathbb{N}$, is composed by i.i.d. random variables. The random variable $\sum_{j=1}^{\lfloor n/\zeta \rfloor} \mathbb{1}_{A_{j\zeta}}$ has binomial distribution with parameters $(\mathbb{P}(A(\rho)), \lfloor n/\zeta \rfloor)$. We say that level j is of type **B** if the first jump after T_j is towards level $j+1$ and $L(X_{T_j}, T_{j+1}) < 2$, and $L(\text{par}(X_{T_j}), T_{j+1}) < 2$.

Let \mathcal{F}_t be the smallest sigma-algebra defined by the collection $\{X_s, 0 \leq s \leq t\}$. For any stopping time S define $\mathcal{F}_S := \{A: A \cap \{S \leq t\} \in \mathcal{F}_t\}$. Now we show

$$\mathbb{P}(i \text{ is of type } \mathbf{B} \mid \mathcal{F}_{T_{i-1}}) \geq \left(1 - e^{-b}\right) \left(1 - e^{-(b+1)}\right) \frac{b}{b+2} = \varphi_b, \quad (3.22)$$

where the inequality holds a.s.. In fact, by time T_i the total weight of the parent of X_{T_i} is stochastically smaller than $1 +$ an exponential of parameter b , independent of $\mathcal{F}_{T_{i-1}}$. Hence the probability that this total weight is less than 2 is larger than $1 - e^{-b}$. Given this, the probability that the first jump after T_i is towards level $i+1$ is larger than $b/(b+2)$. Finally, the conditional probability that $T_{i+1} - T_i < 1$ is larger than $1 - e^{-(b+1)}$. This implies, together with $\zeta \geq 2$, that the random variable $\sum_{j=1}^{\lfloor n/\zeta \rfloor} \mathbb{1}_{\{j\zeta \text{ is of type } \mathbf{B}\}}$ is stochastically larger than a binomial with parameters φ_b and n .

We prove that for any $x > 0$

$$\mathbb{P}(A_\nu \mid h_1(\nu, \text{fc}(\nu)) \leq x) \geq \mathbb{P}(A_\nu). \quad (3.23)$$

To see this, in virtue of (2.3) we have the indicator function of the event that ν is good is a decreasing function of $h_1(\nu, \text{fc}(\nu))$, and for any vertex μ

$$\mathbb{P}(\mu \text{ is good} \mid \text{par}(\mu) \text{ is good}) \geq \mathbb{P}(\mu \text{ is good}),$$

as proved in [4] in the proof of Theorem 3.

For any $i \in \mathbb{N}$, and any vertex ν with $|\nu| = i\zeta$, set

$$\begin{aligned} Z &:= \min\left(1, \frac{h_1(\nu, \text{par}(\nu))}{1 + h_1(\text{par}(\nu), \nu)}\right) \\ E &:= \{X_{T_{i\zeta}} = \nu\} \cap \{L(\text{par}(\nu), T_{i\zeta}) < 2\}. \end{aligned}$$

The random variable Z and the event E are both measurable with respect the sigma-algebra

$$\tilde{\mathcal{H}}_\nu := \sigma\left\{P(\text{par}(\nu), \nu), \{P(u, w): u, w \notin \text{Vert}(\Lambda_\nu)\}\right\},$$

and such that

$$\begin{aligned} \mathbb{P}(i\zeta \text{ is of type } \mathbf{A} \mid \{i\zeta \text{ is of type } \mathbf{B}\} \cap \{X_{T_{i\zeta}} = \nu\}) &= \mathbb{P}(A(\nu) \mid \{h_1(\nu, \text{fc}(\nu)) < Z\} \cap E) \\ &\geq \mathbb{P}(A(\rho)) = \sum_{\nu: |\nu|=i\zeta} \mathbb{P}(A(\nu) \cap \{X_{T_{i\zeta}} = \nu\}) = \mathbb{P}(i\zeta \text{ is of type } \mathbf{A}), \end{aligned}$$

where the first inequality comes from (3.23), and we used the independence between $h_1(\nu, \text{fc}(\nu))$ and $\tilde{\mathcal{H}}_\nu$. The next equality comes from symmetry. Hence

$$\mathbb{P}(i\zeta \text{ is of type } \mathbf{A} \mid i\zeta \text{ is of type } \mathbf{B}) \geq \mathbb{P}(i\zeta \text{ is of type } \mathbf{A}). \quad (3.24)$$

A level is of type **AB** if it is both of type **A** and **B**. A level of type **AB** is a cut level. Define

$$e_n := \sum_{i=1}^{\lfloor n/\zeta \rfloor} \mathbb{1}_{\{\text{level } i\zeta \text{ is of type } \mathbf{AB}\}}.$$

In virtue of (3.22), (3.24) and Proposition 3.1 we have that e_n is stochastically larger than a binomial($(1 - \gamma_b)\varphi_b, \lfloor n/\zeta \rfloor$). Applying Lemma 3.3, we have

$$\mathbb{P}(l_{\lfloor sn \rfloor} \geq n) \leq \mathbb{P}(e_n \leq \lfloor sn \rfloor) \leq 2 \exp \left\{ - \lfloor n/\zeta \rfloor \inf_{x \in [0, s]} H(x \mid (1 - \gamma_b)\varphi_b) \right\}.$$

□

Corollary 3.5 *For $n > 1/((1 - \gamma_b)\phi_b)$, by choosing $s = 1/n$ in Theorem 3.4, we have*

$$\begin{aligned} \mathbb{P}(l_1 \geq n) &\leq 2 \exp \left\{ - \lfloor n/\zeta \rfloor \inf_{x \in [0, 1/n]} H(x \mid (1 - \gamma_b)\varphi_b) \right\} \\ &= 2 \exp \left\{ - \lfloor n/\zeta \rfloor H\left(\frac{1}{n} \mid (1 - \gamma_b)\varphi_b\right) \right\}. \end{aligned} \tag{3.25}$$

4 τ_1 has finite $11/5$ -moment

The goal of this section is to prove the finiteness of the $11/5$ moment of the first cut time. We adopt the following strategy

- first we prove the finiteness of all moments for the number of vertices visited by time τ_1 , then
- we prove that the total time spent at each of these sites has finite $12/5$ -moment.

Fix $n \in \mathbb{N}$ and let

$$\begin{aligned} \Pi_n &:= \text{number of distinct vertices that } \mathbf{X} \text{ visits by time } T_n, \\ g_n(k) &:= \text{number of distinct vertices that } \mathbf{X} \text{ visits at level } k \text{ by time } T_n. \end{aligned}$$

Let $t(\nu) := \inf\{t \geq 0: X_t = \nu\}$. We construct an upper bound for the number of vertices visited at level k , with $1 \leq k \leq n$. Let η_1 be the first vertex at level $k - 1$ to be visited by \mathbf{X} , and let $\varsigma_1 := \text{par}(\eta_1)$. Suppose we have defined $\eta_1, \varsigma_1, \dots, \eta_{m-1}, \varsigma_{m-1}$. Let η_m be the first vertex at level $k - 1$, that is not a child of any of $\varsigma_1, \varsigma_2, \dots, \varsigma_{m-1}$, to be visited (it might not exist). On the set $\{t(\eta_m) < \infty\}$, let $\varsigma_m = \text{par}(\eta_m)$. Define

$$f_n(k) := b^2 + b^2 \sum_{m=1}^{b^{k-2}} \mathbb{1}_{A^c(\varsigma_m)} \mathbb{1}_{\{t(\eta_m) < \infty\}}.$$

The event $A(\varsigma_m)$ is subset of $\{t(\eta_j) = \infty, \forall j \geq m + 1\}$. Hence $f_n(k)$ overcounts the number of vertices at level k which are visited, i.e. $g_n(k) \leq f_n(k)$.

Lemma 4.1 For any $m \in \mathbb{N}$, we have

$$\mathbb{P}(f_n(k) \geq mb^2) \leq (1 - \gamma_b)^{m-1}.$$

Proof. Recall the definition of \mathcal{H}_ν from (2.7). Suppose $\{\eta_i = \nu\}$ and let $C \in \mathcal{H}_\nu$. The event

$$\{t(\eta_i) < \infty\} \cap \{\eta_i = \nu\}$$

depends on the Poisson processes $\{P(\text{par}(\nu), \nu), \{P(u, w): u, w \notin \text{Vert}(\Lambda_\nu)\}\}$. In particular there exists a random variable W independent of $h_1(\text{par}(\nu), \nu)$ and an event D in \mathcal{H}_ν , such that

$$\{t(\eta_i) < \infty\} \cap \{\eta_i = \nu\} = \{h_1(\text{par}(\nu), \nu) < W\} \cap D.$$

Hence

$$\begin{aligned} \mathbb{P}(A(\zeta_i) \mid \{t(\eta_i) < \infty\} \cap \{\eta_i = \nu\} \cap C) &= \mathbb{P}(A(\text{par}(\nu)) \mid \{h_1(\text{par}(\nu), \nu) < W\} \cap D \cap C) \\ &\geq \mathbb{P}(A(\text{par}(\nu))) = \mathbb{P}(A(\zeta_i)) \geq (1 - \gamma_b). \end{aligned} \tag{4.26}$$

The first inequality comes from the fact that the cardinality of the intersection between the descendants of $\text{fc}(\nu)$ and the connected cluster of good vertices containing $\text{fc}(\nu)$ is a decreasing function of $h_1(\text{par}(\nu), \nu)$ (see the definition of good vertices). We also used the independence between $h_1(\text{par}(\nu), \nu)$ and \mathcal{H}_ν . The next equality comes from a symmetry argument. □

Let a_n, c_n be numerical sequences. We say that $c_n = O(a_n)$ if c_n/a_n is bounded.

Lemma 4.2 For $p \geq 1$, we have $\mathbb{E}[\Pi_n^p] = O(n^p)$.

Proof. Consider first the case $p > 1$. Notice that $g_n(0) = g_n(n) = 1$. By Jensen's inequality

$$\mathbb{E}[\Pi_n^p] = \mathbb{E} \left[\left(2 + \sum_{k=1}^{n-1} g_n(k) \right)^p \right] \leq n^p \mathbb{E} \left[\sum_{k=1}^{n-1} \frac{g_n^p(k)}{n} + \frac{2^p}{n} \right] \leq n^p \mathbb{E} \left[\sum_{k=1}^{n-1} \frac{f_n^p(k)}{n} + \frac{2^p}{n} \right] = O(n^p). \tag{4.27}$$

As for the case $p = 1$,

$$\mathbb{E}[\Pi_n] \leq 2 + \sum_{k=1}^{n-1} \mathbb{E}[f_n(k)] = O(n).$$

Let □

$$\Pi := \sum_{\nu \in \text{Vert}(\mathcal{G}_b)} \mathbb{1}_{\{\nu \text{ is visited before time } \tau_1\}}.$$

Lemma 4.3 For any $p > 0$ we have $\mathbb{E}[\Pi^p] < \infty$.

Proof. In virtue of Lemma 4.2, $\mathbb{E} \left[\Pi_n^{2p} \right]^{\frac{1}{2}} \leq C_{b,p}^{(1)} n^p$, for some positive constant $C_{b,p}^{(1)}$. Hence using Cauchy-Schwartz,

$$\begin{aligned} \mathbb{E} [\Pi^p] &= \sum_{n=1}^{\infty} \mathbb{E} \left[\Pi_n^p \mathbb{1}_{\{l_1=n\}} \right] \leq \sum_{n=1}^{\infty} \mathbb{E} \left[\Pi_n^{2p} \right]^{\frac{1}{2}} \mathbb{P}(l_1 \geq n)^{\frac{1}{2}} \\ &\leq 6C_{b,p}^{(1)} \sum_{n=1}^{\infty} n^p \exp \left\{ -\frac{1}{2} [n/\zeta] H \left(\frac{1}{n} \mid (1 - \gamma_b) \varphi_b \right) \right\} < \infty. \end{aligned}$$

In the last inequality we used Corollary 3.5. \square

Next, we want to prove that the 12/5-moment of $L(\rho, \infty)$ is finite. We start with two intermediate results.

Lemma 4.4 Consider VRJP on $\{0, 1\}$, which starts at 1, and with initial weights $a_0 = c$ and $a_1 = 1$. Define

$$\xi(t) := \inf \left\{ s : L(1, s) = t \right\}.$$

We have

$$\sup_{t \geq 1} \mathbb{E} \left[\left(\frac{L(0, \xi(t))}{t} \right)^3 \right] = c^3 + 3c^2 + 3c. \quad (4.28)$$

Proof. We have $L(0, \xi(t+dt)) = L(0, \xi(t)) + \chi\eta$, where χ is a Bernoulli which takes value 1 with probability $L(0, \xi(t))dt$, and η is exponential with mean $1/t$. Given $L(0, \xi(t))$, the random variables χ and η are independent. Hence

$$\mathbb{E} \left[L(0, \xi(t+dt)) \right] - \mathbb{E} \left[L(0, \xi(t)) \right] = \frac{\mathbb{E} [L(0, \xi(t))]}{t} dt,$$

i.e. $\mathbb{E} [L(0, \xi(t))]$ is solution of the equation $y'(t) = y(t)/t$, with initial condition $y(1) = c$ (see [8]). Hence

$$\mathbb{E} [L(0, \xi(t))] = ct.$$

Similarly

$$\begin{aligned} &\mathbb{E} \left[L(0, \xi(t+dt))^2 \right] \\ &= \mathbb{E} \left[L(0, \xi(t))^2 \right] + 2\mathbb{E} \left[L(0, \xi(t)) \mathbb{E} \left[\chi \mid L(0, \xi(t)) \right] \right] \mathbb{E} [\eta] + \mathbb{E} \left[\chi^2 \mid L(0, \xi(t)) \right] \mathbb{E} [\eta^2] \\ &= \mathbb{E} \left[L(0, \xi(t))^2 \right] + (2/t) \mathbb{E} \left[L(0, \xi(t))^2 \right] dt + (2/t^2) \mathbb{E} \left[L(0, \xi(t)) \right] dt \\ &= \mathbb{E} \left[L(0, \xi(t))^2 \right] + (2/t) \mathbb{E} \left[L(0, \xi(t))^2 \right] dt + (2c/t) dt. \end{aligned}$$

Thus $\mathbb{E} \left[L(0, \xi(t))^2 \right]$ satisfies the equation $y' = (2/t)y + (2c/t)$, with $y(1) = c^2$. Then,

$$\mathbb{E} \left[L(0, \xi(t))^2 \right] = -c + (c^2 + c) t^2.$$

Finally, reasoning in a similar way, we get that $\mathbb{E}\left[L(0, \xi(t))^3\right]$ satisfies the equation $y' = (3/t)y + 6(c^2 + c)$, with $y(1) = c^3$. Hence,

$$\mathbb{E}\left[L(0, \xi(t))^3\right] = -3(c^2 + c)t + (c^3 + 3c^2 + 3c)t^3.$$

Divide both sides by t^3 , and use the fact that $c > 0$ to get (4.28). \square
For any subtree E of \mathcal{G}_b , $b \geq 1$, define

$$\delta(a, E) := \sup \left\{ t : \int_0^t \mathbb{1}_{\{X_s \in E\}} ds \leq a \right\}.$$

The process $X_{\delta(t, E)}$ is called the **restriction** of \mathbf{X} to E . The next property of VRJP was stated by Davis and Volkov in [8].

Proposition 4.5 (Restriction principle) *Consider VRJP \mathbf{X} defined on a tree \mathcal{J} rooted at ρ . Assume this process is recurrent, i.e. visits each vertex infinitely often, a.s.. Consider a subtree $\tilde{\mathcal{J}}$ rooted at ν , where $|\nu| = \min\{|\mu| : \mu \in \text{Vert}(\tilde{\mathcal{J}})\}$. The process $X_{\delta(t, \tilde{\mathcal{J}})}$ is VRJP defined on $\tilde{\mathcal{J}}$. Moreover, for any subtree \mathcal{J}^* disjoint from $\tilde{\mathcal{J}}$, we have that $X_{\delta(t, \tilde{\mathcal{J}})}$ and $X_{\delta(t, \mathcal{J}^*)}$ are independent.*

Proof. This principle follows directly from the Poisson construction and the memoryless property of the exponential distribution. \square

Definition 4.6 *Recall that $P(x, y)$, with $x, y \in \text{Vert}(\mathcal{G}_b)$ are the Poisson processes used to generate \mathbf{X} on \mathcal{G}_b . Let \mathcal{J} be a subtree of \mathcal{G}_b . Consider VRJP \mathbf{V} on \mathcal{J} which is generated by using $\{P(u, v) : u, v \in \text{Vert}(\mathcal{J})\}$, which is the same collection of Poisson processes used to generate the jumps of \mathbf{X} from the vertices of \mathcal{J} . We say that \mathbf{V} is the **extension** of \mathbf{X} in \mathcal{J} . The processes V_t and $X_{\delta(t, \mathcal{J})}$ coincide up to a random time, that is the total time spent by \mathbf{X} in \mathcal{J} .*

A ray σ is a subtree of \mathcal{G}_b containing exactly one vertex of each level of \mathcal{G}_b . Label the vertices of this ray using $\{\sigma_i, i \geq 0\}$, where σ_i is the unique vertex at level i which belongs to σ . Denote by \mathfrak{S} the collection of all rays of \mathcal{G}_b .

Lemma 4.7 *For any ray σ , consider VRJP $\mathbf{X}^{(\sigma)} := \{X_t^{(\sigma)}, t \geq 0\}$, which is the extension of \mathbf{X} to σ . Define*

$$\begin{aligned} T_n^{(\sigma)} &:= \inf\{t > 0 : X_t^{(\sigma)} = \sigma_n\}, \\ L^{(\sigma)}(n, t) &:= 1 + \int_0^t \mathbb{1}_{\{X_s^{(\sigma)} = \sigma_n\}} ds, \quad \forall i \in \mathbb{N}. \end{aligned}$$

We have that

$$\mathbb{E}\left[L^{(\sigma)}(0, T_n^{(\sigma)})^3\right] \leq (37)^n. \quad (4.29)$$

Proof. By the tower property of conditional expectation,

$$\mathbb{E} \left[(L^{(\sigma)}(0, T_n^{(\sigma)}))^3 \right] = \mathbb{E} \left[(L^{(\sigma)}(1, T_n^{(\sigma)}))^3 \mathbb{E} \left[\left(\frac{L^{(\sigma)}(0, T_n^{(\sigma)})}{L^{(\sigma)}(1, T_n^{(\sigma)})} \right)^3 \middle| L^{(\sigma)}(1, T_n^{(\sigma)}) \right] \right]. \quad (4.30)$$

At this point we focus on the process restricted to $\{0, 1\}$. This restricted process is VRJP which starts at 1, with initial weights $a_1 = 1$, and $a_0 = 1 + \tilde{Y}$, where $\tilde{Y} = h_1(\sigma_0, \sigma_1)$ and $\sigma_0 = \rho$. By applying Lemma 4.4, and using the fact that $h_1(\sigma_0, \sigma_1)$ is exponential with mean 1, we have

$$\mathbb{E} \left[\left(\frac{L^{(\sigma)}(0, T_n^{(\sigma)})}{L^{(\sigma)}(1, T_n^{(\sigma)})} \right)^3 \middle| L^{(\sigma)}(1, T_n^{(\sigma)}) \right] \leq \mathbb{E} [3 + 3\tilde{Y} + 3(1 + \tilde{Y})^2 + (1 + \tilde{Y})^3] = 37. \quad (4.31)$$

Then

$$\begin{aligned} \mathbb{E} \left[(L(0, T_n))^3 \right] &= \mathbb{E} \left[\mathbb{E} \left[\left(\frac{L^{(\sigma)}(0, T_n^{(\sigma)})}{L^{(\sigma)}(1, T_n^{(\sigma)})} \right)^3 \middle| L^{(\sigma)}(1, T_n^{(\sigma)}) \right] (L^{(\sigma)}(1, T_n^{(\sigma)}))^3 \right] \\ &\leq 37 \mathbb{E} \left[(L^{(\sigma)}(1, T_n^{(\sigma)}))^3 \right]. \end{aligned} \quad (4.32)$$

The Lemma follows by recursion and restriction principle. \square

As $L(\rho, T_n) \leq L^{(\sigma)}(0, T_n^{(\sigma)})$, we have

$$\mathbb{E} [L(\rho, T_n)^3] \leq (37)^n. \quad (4.33)$$

Lemma 4.8 $\mathbb{E} \left[(L(\rho, \infty))^{12/5} \right] < \infty$.

Proof. Recall that \mathcal{C}_ν is the connected cluster of good vertices which contains ν and Λ_ν is the subtree composed by ν , its descendants and the edges connecting them. Define

$$B_k := \{ \exists \nu : |\nu| = k \text{ and } \mathcal{C}_\nu \cap \text{Vert}(\Lambda_\nu) \text{ is infinite} \}.$$

Consider a set of b^{k-2} vertices at level k , each having a different ancestor at level $k-2$. Label these vertices using v_j , with $j \in \{1, 2, \dots, b^{k-2}\}$. The sets $\mathcal{C}_{v_i} \cap \text{Vert}(\Lambda_{v_i})$, with $i \in \{1, 2, \dots, b^{k-2}\}$, are independent. The probability that a given vertex is good is given by

$$\int_0^\infty \frac{1}{2+z} e^{-z} dz = 0.36133 \dots$$

By Lemma 2.2,

$$\mathbb{P}(\mathcal{C}_\nu \cap \text{Vert}(\Lambda_\nu) \text{ is finite}) = (.63867 + .36133\beta_b) =: \varpi.$$

As $\beta_b < 1$ then $\varpi < 1$. We have

$$\mathbb{P}(B_k) \leq \mathbb{P} \left(\bigcap_{i=1}^{b^{k-2}} \{ \mathcal{C}_{v_i} \cap \text{Vert}(\Lambda_{v_i}) \text{ is finite} \} \right) \leq (\varpi)^{b^{k-2}} \quad (4.34)$$

Fix $k \in \mathbb{N}$. Consider a collection of rays $m(i) = m(i, k)$, with $i \in \{1, \dots, b^k\}$, with the property that different rays connect the root to different vertices at level k . Clearly

$$L(\rho, \infty) \mathbb{1}_{B_k} \leq \sum_{i=1}^{b^k} L^{(m(i))}(0, T_k^{(m(i))}) \mathbb{1}_{B_k}. \quad (4.35)$$

Using (4.35), Holder's inequality (with $p = 5/4$) and (4.34) we have

$$\begin{aligned} \mathbb{E} \left[(L(\rho, \infty))^{12/5} \right] &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left(\sum_{i=1}^{b^k} L^{(m(i))}(0, T_k^{(m(i))}) \mathbb{1}_{B_k} \right)^{12/5} \right] \\ &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left(\sum_{i=1}^{b^k} L^{(m(i))}(0, T_k^{(m(i))}) \right)^3 \right]^{4/5} (\varpi)^{b^{k-2}/5} \\ &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[1 + \left(\sum_{i=1}^{b^k} L^{(m(i))}(0, T_k^{(m(i))}) \right)^3 \right] (\varpi)^{b^{k-2}/5} \\ &\leq \sum_{k=1}^{\infty} \left(1 + b^{2k} \sum_{i=1}^{b^k} \mathbb{E} \left[(L^{(m(i))}(0, T_k^{(m(i))}))^3 \right] \right) (\varpi)^{b^{k-2}/5} \quad (\text{by Jensen}) \\ &\leq \sum_{k=1}^{\infty} (1 + b^{3k} (37)^k) (\varpi)^{b^{k-2}/5} < \infty. \end{aligned}$$

□

Lemma 4.9 For $\nu \neq \rho$, there exists a random variable Δ_ν which is $\sigma\{P(u, v) : u, v \in \text{Vert}(\Lambda_\nu)\}$ -measurable, such that

- i) $L(\nu, \infty) \leq \Delta_\nu$, and
- ii) Δ_ν and $L(\rho, \infty)$ are identically distributed.

Proof. Let $\tilde{\mathbf{X}} := \{\tilde{X}_t, t \geq 0\}$ be the extension of \mathbf{X} on Λ_ν . Define

$$\Delta_\nu := 1 + \int_0^\infty \mathbb{1}_{\{\tilde{X}_t = \nu\}} dt.$$

By construction, this random variable satisfies i) and ii) and is $\sigma\{P(u, v) : u, v \in \text{Vert}(\Lambda_\nu)\}$ -measurable. □

Theorem 4.10 $\mathbb{E} [(\tau_1)^{11/5}] < \infty$.

Proof. Suppose we relabel the vertices that have been visited by time τ_1 , using $\theta_1, \theta_2, \dots, \theta_\Pi$, where vertex ν is labeled θ_k if there are exactly $k - 1$ distinct vertices that have been visited

before ν . Notice that Δ_ν and $\{\theta_k = \nu\}$ are independent, because they are determined by disjoint non-random sets of Poisson processes (Δ_ν is $\sigma\{P(u, v) : u, v \in \text{Vert}(\Lambda_\nu)\}$ -measurable). As the variables Δ_ν , with $\nu \in \text{Vert}(\mathcal{G}_b)$, share the same distribution, for any $p > 0$, we have

$$\mathbb{E}[\Delta_{\theta_k}^p] = \mathbb{E}[\Delta_\nu^p] = \mathbb{E}[L(\rho, \infty)^p].$$

By Jensen's and Holder's (with $p = 12/11$) inequalities, Lemma 4.9 i) and ii), and Lemma 4.8, we have

$$\begin{aligned} \mathbb{E}[(\tau_1)^{11/5}] &\leq \mathbb{E}\left[\left(\sum_{k=1}^{\Pi} \Delta_{\theta_k}\right)^{11/5}\right] \leq \mathbb{E}\left[\Pi^{(11/5)-1} \sum_{k=1}^{\Pi} (\Delta_{\theta_k})^{11/5}\right] \\ &= \mathbb{E}\left[\sum_{k=1}^{\infty} \Delta_{\theta_k}^{11/5} \Pi^{6/5} \mathbb{1}_{\{\Pi \geq k\}}\right] \leq \sum_{k=1}^{\infty} \mathbb{E}\left[\Delta_{\theta_k}^{12/5}\right]^{11/12} \mathbb{E}\left[\Pi^{72/5} \mathbb{1}_{\{\Pi \geq k\}}\right]^{1/12} \\ &\leq C_b^{(3)} \sum_{k=1}^{\infty} \mathbb{E}\left[\Pi^{144/5}\right]^{1/24} \mathbb{P}(\Pi \geq k)^{1/24} \quad (\text{by Cauchy-Schwartz and Lemma 4.8}) \\ &\leq C_b^{(4)} \sum_{k=1}^{\infty} \mathbb{P}(\Pi \geq k)^{1/24}, \quad (\text{by Lemma 4.3}), \end{aligned}$$

for some positive constants $C_b^{(3)}$ and $C_b^{(4)}$. It remains to prove the finiteness of the last sum. We use the fact

$$\lim_{k \rightarrow \infty} k^{48} \mathbb{P}(\Pi \geq k) = 0. \quad (4.36)$$

The previous limit is a consequence of the well-known formula

$$\sum_{k=1}^{\infty} k^{48} \mathbb{P}(\Pi \geq k) = \mathbb{E}[\Pi^{49}], \quad (4.37)$$

and the finiteness of $\mathbb{E}[\Pi^{49}]$ in virtue of Lemma 4.3.

$$\sum_{k=1}^{\infty} \mathbb{P}(\Pi \geq k)^{1/24} = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(k^{48} \mathbb{P}(\Pi \geq k)\right)^{1/24} < \infty.$$

□

Theorem 4.11 $\sup_{x \in [1,2]} \mathbb{E}[(\tau_1)^{11/5} \mid L(\rho, T_1) = x] < \infty$.

Proof. Label the vertices at level 1 by $\mu_1, \mu_2, \dots, \mu_b$. Let $\tau_1(\mu_i)$ be the first cut time of the extension of \mathbf{X} on Λ_{μ_i} . This extension is VRJP on Λ_{μ_i} with initial weights 1, hence we can apply Theorem 4.10 to get

$$\mathbb{E}[(\tau_1(\mu_i))^{11/5}] < \infty. \quad (4.38)$$

It is easy to adapt the proof of Lemma 4.8 to prove that $\mathbb{E} [L(\rho, \infty)^{12/5} \mid L(\rho, T_1) = x]$ is finite (we leave this task to the reader). Hence, it remains to realize that for $x \in (1, 2)$

$$\begin{aligned} \mathbb{E} \left[(\tau_1)^{11/5} \mid L(\rho, T_1) = x \right] &\leq \mathbb{E} \left[\left(L(\rho, \infty) + \max_i \tau_1(\mu_i) \right)^{11/5} \mid L(\rho, T_1) = x \right] \\ &\leq \mathbb{E} \left[\left(L(\rho, \infty) + \sum_{i=1}^b \tau_1(\mu_i) \right)^{11/5} \mid L(\rho, T_1) = x \right] \\ &\leq (b+1)^{11/5-1} \mathbb{E} \left[\left(L(\rho, \infty) \right)^{11/5} \mid L(\rho, T_1) = x \right] + (b+1)^{11/5} \mathbb{E} \left[\left(\tau_1(\mu_1) \right)^{11/5} \right] < \infty, \end{aligned}$$

where we used Jensen's inequality and the independence of $\tau(\mu_i)$ and T_1 . \square

5 Splitting the path into one-dependent pieces

Define $Z_i = L(X_{\tau_i}, \infty)$, with $i \geq 1$. This process is a Markov chain. To see this, it is sufficient to notice that the entire process above the i -th cut level, given $Z_i = x$ has a distribution depending only on x . Moreover, given $Z_i = x$, the random vectors $(\tau_{i+1} - \tau_i, l_{i+1} - l_i)$ and $(\tau_i - \tau_{i-1}, l_i - l_{i-1})$, are independent.

Proposition 5.1

$$\sup_{i \in \mathbb{N}} \sup_{x \in [1, 2]} \mathbb{E} \left[(\tau_{i+1} - \tau_i)^{11/5} \mid Z_i = x \right] < \infty \quad (5.39)$$

$$\sup_{i \in \mathbb{N}} \sup_{x \in [1, 2]} \mathbb{E} \left[(l_{i+1} - l_i)^{11/5} \mid Z_i = x \right] < \infty. \quad (5.40)$$

Proof. We only prove (5.39), the proof of (5.40) being similar. Define $C := \{X_t \neq \rho, \forall t > T_1\}$ and fix a vertex ν . Notice that by the self-similarity property of \mathcal{G}_b , we have

$$\mathbb{E} \left[(\tau_{i+1} - \tau_i)^{11/5} \mid \{Z_i = x\} \cap \{X_{\tau_i} = \nu\} \right] = \mathbb{E} \left[(\tau_1)^{11/5} \mid \{L(\rho, T_1) = x\} \cap C \right].$$

By the proof of Lemma 2.2, we have that

$$\inf_{1 \leq x \leq 2} \mathbb{P}(C \mid L(\rho, T_1) = x) \geq (1 - \beta_b) \frac{b}{b+2} > 0. \quad (5.41)$$

Hence

$$\begin{aligned} &\sup_{x: x \in [1, 2]} \mathbb{E} \left[(\tau_1)^{11/5} \mid L(\rho, T_1) = x \right] \\ &\geq \sup_{x: x \in [1, 2]} \mathbb{E} \left[(\tau_1)^{11/5} \mid \{L(\rho, T_1) = x\} \cap C \right] \mathbb{P}(C \mid L(\rho, T_1) = x) \\ &\geq (1 - \beta_b) \frac{b}{b+2} \sup_{x: x \in [1, 2]} \mathbb{E} \left[(\tau_1)^{11/5} \mid \{L(\rho, T_1) = x\} \cap C \right] \\ &\geq (1 - \beta_b) \frac{b}{b+2} \sup_{x: x \in [1, 2]} \mathbb{E} \left[(\tau_{i+1} - \tau_i)^{11/5} \mid \{Z_i = x\} \cap \{X_{\tau_i} = \nu\} \right] \end{aligned}$$

Hence

$$\begin{aligned} & \mathbb{E}\left[(\tau_{i+1} - \tau_i)^{11/5} \mid \{Z_i = x\} \cap \{X_{\tau_i} = \nu\}\right] \\ & \leq \frac{b+2}{b(1-\beta_b)} \sup_{1 \leq x \leq 2} \mathbb{E}\left[(\tau_1)^{11/5} \mid \{L(\rho, T_1 = x)\}\right]. \end{aligned}$$

□

Next we prove that $\{Z_i\}$, $i \geq 1$ satisfies the Doeblin condition.

Lemma 5.2 *There exists a probability measure $\phi(\cdot)$ and $0 < \lambda \leq 1$, such that for every Borel subset B of $[1, 2]$, we have that*

$$\mathbb{P}(Z_{i+1} \in B \mid Z_i = z) \geq \lambda \phi(B) \quad \forall z \in [1, 2]. \quad (5.42)$$

Proof. As Z_i is homogeneous, it is enough to prove (5.42) for $i = 1$. Fix $x, y, z \in (1, 2)$ with $x < y$ and consider the function

$$e^{-(b+z)(u-1)} - (b+1)e^{-(b+2)}e^{-(u-1)}.$$

For fixed $z \in (1, 2)$, this function is non-increasing in u . As $x < y$ we have

$$e^{-(b+z)(x-1)} - e^{-(b+z)(y-1)} \geq (b+1)e^{-(b+2)} \left(e^{-(x-1)} - e^{-(y-1)} \right). \quad (5.43)$$

Fix $\varepsilon \in (0, 1)$ and let $I_\varepsilon(z) := (z - \varepsilon, z + \varepsilon)$. We want to bound from below the probability of the event $\{Z_2 \in (x, y), Z_1 \in I_\varepsilon(z)\}$. Consider the following event. Suppose that

- a) $T_1 < 1$, then
- b) the process spends at X_{T_1} an amount of time enclosed in $(z - 1 - \varepsilon, z - 1 + \varepsilon)$, then
- c) it jumps to a vertex at level 2, spends there an amount of time t where $t + 1 \in (x, y)$, and
- d) it jumps to level 3 and never returns to X_{T_2} .

In the event just described, levels 1 and 2 are the first two cut levels, and $\{Z_2 \in (x, y), Z_1 \in I_\varepsilon(z)\}$ holds. The probability that a) holds is exactly e^{-b} . Given $T_1 = s - 1$, the time spent in X_{T_1} before the first jump is exponential with parameter $(b + s)$. Hence b) occurs with probability larger than

$$\inf_{s \in [1, 2]} \left(e^{-(b+s)(z+\varepsilon)} - e^{-(b+s)(z-\varepsilon)} \right).$$

Given a) and b), the process jumps to level 2 and then to level 3 with probability larger than $(b/(b+2))(b/(b+z+\varepsilon))$. The conditional probability, given a) and b), that the time gap between these two jumps lays in $(x-1, y-1)$ is larger than

$$\inf_{u \in I_\varepsilon(z)} \left(e^{-(b+u)(x-1)} - e^{-(b+u)(y-1)} \right).$$

At this point, a lower bound for the conditional probability that the process never returns to X_{T_2} is

$$\frac{b}{b+y}(1-\alpha_b) \geq \frac{b}{b+2}(1-\alpha_b).$$

We have

$$\begin{aligned} & \mathbb{P}\left(Z_2 \in (x, y), Z_1 \in I_\varepsilon(z)\right) \\ & \geq e^{-b} \frac{b^3}{(b+2)^2(b+z+\varepsilon)} \inf_{s \in [1,2]} \left(e^{-(b+s)(z+\varepsilon)} - e^{-(b+s)(z-\varepsilon)} \right) \\ & \quad \inf_{u \in I_\varepsilon(z)} \left(e^{-(b+u)(x-1)} - e^{-(b+u)(y-1)} \right) (1-\alpha_b) \\ & \geq (1-\alpha_b) e^{-b} \frac{b^3(b+1)}{(b+2)^2(b+z+\varepsilon)} e^{-(b+2)} \left(e^{-(x-1)} - e^{-(y-1)} \right) \inf_{s \in [1,2]} \left(e^{-(b+s)(z+\varepsilon)} - e^{-(b+s)(z-\varepsilon)} \right), \end{aligned} \tag{5.44}$$

where in the last inequality we used (5.43). Notice that there exists a constant $C_b^{(4)} > 0$ such that

$$\inf_{\varepsilon \in (0,1)} \inf_{z, s \in [1,2]} \frac{1}{\varepsilon} \left(e^{-(b+s)(z+\varepsilon)} - e^{-(b+s)(z-\varepsilon)} \right) \geq C_b^{(4)}. \tag{5.45}$$

Hence, for $\varepsilon \in (0, 1)$, there exists a positive constant $C_b^{(5)}$ such that the right hand-side of (5.44) is larger than

$$C_b^{(5)} \left(e^{-(x-1)} - e^{-(y-1)} \right) \varepsilon. \tag{5.46}$$

On the other hand,

$$\sup_{\varepsilon \in (0,1)} \frac{1}{\varepsilon} \mathbb{P}\left(Z_1 \in I_\varepsilon(z)\right) \leq C_b^{(6)}, \tag{5.47}$$

for some positive constant $C_b^{(6)}$. Finally combining (5.46) and (5.47), we get

$$\begin{aligned} \mathbb{P}\left(Z_2 \in (x, y) \mid Z_1 = z\right) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\mathbb{P}\left(Z_1 \in I_\varepsilon(z)\right)} \mathbb{P}\left(Z_2 \in (x, y), Z_1 \in I_\varepsilon(z)\right) \\ &\geq \lambda \frac{\left(e^{-(x-1)} - e^{-(y-1)} \right)}{\left(1 - e^{-1} \right)}. \end{aligned}$$

Where $\lambda = (C_b^{(5)}/C_b^{(6)}) (1 - e^{-1})$. Finally extending the 2 measures from the field of the finite union of intervals to the Borel sigma-field we get our result. \square

The proof of the following Proposition can be found in [2].

Proposition 5.3 *There exists random times $\{N_k, k \geq 1\}$ such that the sequence $\{Y_{N_k}, k \geq 1\}$ is composed by independent and identically distributed random variables with distribution $\phi(\cdot)$. Furthermore there exists a constant $\varrho \in (0, 1)$ such that $N_i - N_{i-1}, i \geq 2$, are i.i.d. with a geometric distribution,*

$$\mathbb{P}(N_2 - N_1 = j) = (1 - \varrho)^{j-1} \varrho.$$

Lemma 5.4 $\sup_{i \in \mathbb{N}} \mathbb{E}[(\tau_{N_{i+1}} - \tau_{N_i})^2] < \infty$.

Proof. It is enough to prove $\mathbb{E}[(\tau_{N_2} - \tau_{N_1})^2] < \infty$. In virtue of Jensen's inequality, we have that

$$\begin{aligned} \mathbb{E}[(\tau_k - \tau_m)^{11/5}] &= \mathbb{E}\left[\left(\sum_{j=1}^{k-m} \tau_{m+j} - \tau_{m+j-1}\right)^{11/5}\right] \\ &\leq (k-m)^{11/5} \mathbb{E}[(\tau_2 - \tau_1)^{11/5}]. \end{aligned} \quad (5.48)$$

Using Holder with $p = 11/10$, we have

$$\begin{aligned} \mathbb{E}[(\tau_{N_2} - \tau_{N_1})^2] &= \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \mathbb{E}[(\tau_k - \tau_m)^2 \mathbb{1}_{\{N_1=m, N_2=k\}}] \\ &\leq \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \mathbb{E}[(\tau_k - \tau_m)^{11/5}]^{10/11} \mathbb{P}(N_1 = m, N_2 - N_1 = k - m)^{1/11} \\ &\leq \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} (k-m)^2 \mathbb{E}[(\tau_2 - \tau_1)^{11/5}]^{10/11} \varrho^{2/11} (1-\varrho)^{(k-2)/11} \\ &\leq \varrho^{2/11} \mathbb{E}[(\tau_2 - \tau_1)^{11/5}]^{10/11} \sum_{k=2}^{\infty} k^3 (1-\varrho)^{(k-2)/11} < \infty, \end{aligned}$$

where we used the fact that $0 < \varrho < 1$. □

With a similar proof we get the following result.

Lemma 5.5 $\sup_{i \in \mathbb{N}} \mathbb{E}[(l_{N_{i+1}} - l_{N_i})^2] < \infty$.

Definition 5.6 A process $\{Y_k, k \geq 1\}$, is said to be **one-dependent** if Y_{i+2} is independent of $\{Y_j, \text{ with } 1 \leq j \leq i\}$.

Lemma 5.7 Let $\Upsilon_i := (\tau_{N_{i+1}} - \tau_{N_i}, l_{N_{i+1}} - l_{N_i})$, for $i \geq 1$. The process $\Upsilon := \{\Upsilon_i, i \geq 1\}$ is one-dependent. Moreover $\Upsilon_i, i \geq 1$, are identically distributed.

Proof. Given $Z_{N_{i-1}}$, Υ_i is independent of $\{\Upsilon_j, j \leq i-2\}$. Thus, it is sufficient to prove that Υ_i is independent of $Z_{N_{i-1}}$. To see this, it is enough to realize that given Z_{N_i} , Υ_i is independent of $Z_{N_{i-1}}$, and combine this with the fact that Z_{N_i} and $Z_{N_{i-1}}$ are independent. The variables Z_{N_i} are i.i.d., hence $\{\Upsilon_i, i \geq 2\}$, are identically distributed. □

The Strong Law of Large Numbers holds for one-dependent sequences of identically distributed variables bounded in \mathcal{L}^1 . To see this, just consider separately the sequence of random variables with even and odd indices and apply the usual Strong Law of Large Numbers to each of them.

Hence, for some constants $0 < C_b^{(\tau)}, C_b^{(s)} < \infty$, we have

$$\lim_{i \rightarrow \infty} \frac{\tau_{N_i}}{i} \rightarrow C_b^{(\tau)}, \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{l_{N_i}}{i} \rightarrow C_b^{(s)}, \quad a.s.. \quad (5.49)$$

Proof of Theorem 1. If $\tau_{N_i} \leq t < \tau_{N_{i+1}}$, then by the definition of cut level, we have

$$l_{N_i} \leq |X_t| < l_{N_{i+1}}.$$

Hence

$$\frac{l_{N_i}}{\tau_{N_{i+1}}} \leq \frac{|X_t|}{t} < \frac{l_{N_{i+1}}}{\tau_{N_i}}.$$

Let $K_b^{(1)} = C_b^{(7)}/C_b^{(8)}$ which are the constants in (5.49). Then

$$\limsup_{t \rightarrow \infty} \frac{|X_t|}{t} \leq \lim_{i \rightarrow \infty} \frac{l_{N_{i+1}}}{\tau_{N_i}} = \lim_{i \rightarrow \infty} \frac{l_{N_{i+1}}}{i+1} \frac{i}{\tau_{N_i}} = K_b^{(1)}, \text{ a.s..}$$

Similarly, we can prove that

$$\liminf_{t \rightarrow \infty} \frac{|X_t|}{t} \geq K_b^{(1)}, \text{ a.s..}$$

The proof of the central limit theorem (1.2) is very similar and uses the fact that $\tau_{N_i} - \tau_{N_{i-1}}$ and $l_{N_i} - l_{N_{i-1}}$, $i \geq 2$, are one-dependent stationary processes bounded in $\mathcal{L}^{2+\delta}$, for some $\delta > 0$. Apply, for example, Theorem 11 of [14], which yields an invariance principle for each of these processes. Then follow the last Section of [11] to end the proof. \square

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