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Abstract. In actuarial literature the properties of risk measures or insurance premium principles have been extensively studied . We propose a characterization of a particular class of coherent risk measures defined in [1]. The considered premium principles are obtained by expansion of TVar measures, consequently they look like very interesting in insurance pricing where TVar measures is frequently used to value tail risks.

Keywords: Risk measures, premium principles, capacity, distortion function, TVar.

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Introduction

Premium principles are the most important risk measures in actuarial sciences and frequently the insurers are also interested to measure the upper tails of distribution functions [8]. There are different methods that actuaries use to develop premium principles [3].
In this paper we propose an axiomatic approach based on a minimal set of properties which characterizes an insurance premium principle as a Choquet integral with respect to a distorted probability. As it is well known distortion risk measures are introduced in the actuarial literature by Wang [6] and are related to the coherent risk measures. Two particular examples of Wang risk measures are given by V_α and $TVaR_\alpha$, (Value at Risk at level α and Tail Value at Risk at level α respectively). The distortion function giving rise to the V_α is not concave, so that V_α is not a coherent measure, while the distortion function giving rise to the $TVaR_\alpha$, is concave so that $TVaR_\alpha$ is a coherent measure $TVaR_\alpha$ [3]. The importance in actuarial science and in finance of $TVaR_\alpha$, as measure of the upper tail of a distribution function is well known and we refer to [3].
In this paper we consider a rather general set of risks and for the premium principles we ask some natural assumptions,(**A1**)- (**A4**). We obtain for all the premium principles of this class an integral representation by a non additive convex measure and then an integral representation by concave distortion functions so that the considered premium principles are a convex combination of coherent risk measures as $TVaR_\alpha$, $\alpha \in [0, 1]$.
The paper is organized as follows. In section 2 we provide some necessary preliminaries and we introduce the properties that characterize the premium principles considered in this paper. In section 3 we recall some basic facts of Choquet expected utility and we introduce a modified version of Greco Theorem [4]. In section 4 we present distortion risk measures. Finally in section 5 we obtain the integral representation result premium principles and the characterization as convex combination of $TVaR_\alpha, \alpha \in [0, 1]$.

1 Insurance premium principles

In actuarial applications a risk is represented by a nonnegative random variable. We consider an insurance contract in a specified time period $[0, T]$. Let Ω be the state space and $\mathcal{F}$ the event σ-field at the time T. Let $\mathbf{P}$ be a probability measure on $\mathcal{F}$. We consider an insurance contract described by a non-negative random variable X, $X : \Omega \to \mathbb{R}$ where $X(\omega)$ represents its payoff at time T if state ω occurs. We denote by F_X the distribution function of X and by S_X the survival function. Frequently an insurance contract provides a franchise and then it is interesting to consider the values ω such that $X(\omega) > a$: in this case the contract pays for $X(\omega) > a$ and nothing otherwise. Then it is useful to consider also the random variable

$$(X - a)_+ = \max(X(\omega) - a, 0) \tag{1}$$

We consider a set, L of nonnegative random variables with the following property:

i) $aX, \quad (X-a)_+, \quad (X-(X-a)_+) \in L \qquad \forall X \in L$, and $a \in [0,+\infty)$.

We observe that such set L is not necessary a vector space.

We denote the insurance prices of the contracts of L by a functional H where

$$H : L \to \widetilde{\mathbb{R}} \tag{2}$$

and $\widetilde{\mathbb{R}}$ is the extended real line. We consider some properties that it is reasonable assume for an insurance functional price H:

(**P1**) $H(X) \geq 0 \quad$ for all $X \in L$.

(**P2**) If $c \in [0,+\infty)$ then $H(c) = c$.

(**P3**) $H(X) \leq \sup_{\omega\in\Omega} X(\omega) \quad$ for all $X \in L$.

(**P4**) $H(aX+b) = aH(X)+b \quad$ for all $X \in L$ such that $aX+b \in L$ with $a, b \in [0,+\infty)$.

(**P5**) If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ for $X, Y \in L$ then $H(X) \leq H(Y)$.

(**P6**) $H(X+Y) \leq H(X) + H(Y) \quad$ for all $X, Y \in L$ such that $X+Y \in L$.

We observe that the properties (**P4**) and (**P6**) imply the following property:

(**P7**) $H(aX+(1-a)Y) \leq aH(X)+(1-a)H(Y) \quad$ for all $X, Y \in L$ and $a \in [0,1]$ such that $aX+(1-a)Y \in L$.

The last property is the convexity property and it means that diversification does not increase the total risk. In the insurance context this property allows to pool of risks effects.

Now we present some assumptions which frequently are in the applications.

(**A1**) $H(X) = H(X-(X-a)_+) + H((X-a)_+) \quad$ for all $X \in L$ and $a \in [0,+\infty)$.

This condition splits into two comonotonic parts a risk X (see for example [2]), and permits to identify the part of premium charged for the risk with the reinsurance premium charged by the reinsurer.

(**A2**) If $E(X-a)_+ \leq E(Y-a)_+ \quad$ for all $a \in [0,+\infty)$ then $H(X) \leq H(Y) \quad$ for all $X, Y \in L$.

In other words the functional price H respects the stop-loss order. We remember that stop-loss order considers the weight in the tail of distributions; when other characteristics are equals, stop-loss order select the risk with less heavy tails [9].

(**A3**) The price, $H(X)$, of the insurance contract X depends only on its distribution F_X.

Frequently this hypothesis is assumed in literature, see for example [8] and [5]. The hypothesis (**A3**) says that it is not the particular scenario that determine the price of a risk, but the probability distribution of X assigns the price to X. So risks with identical distributions have the same price.
Finally, we present a continuity property that is usual, in characterizing certain premium principles.
(**A4**) $\lim_{n\to+\infty} H(X-(X-n)_+) = H(X)$ for all $X, Y \in L$.

2 Choquet pricing of insurance risks

The development of premium functionals based on Choquet integration theory has gained considerable interest in recent years when there is ambiguity on the loss distribution or when there is correlation between the individual risks in this case in fact the traditional pricing functionals may be inadequate to determine the premiums that cover the risk.
Capacities are set functions defined on 2^Ω to real values which generalize the notion of probability distribution. Formally a capacity is a normalized monotone function, for the definition and properties see for example [2], [3].
As is well known the Choquet integral has been extensively applied in the context of decision under uncertainty and in risk applications.

Definition 1 *Let υ a capacity $\upsilon : 2^\Omega \to \mathbb{R}^+$ and X a random non-negative variable defined on $(\Omega, \mathcal{F})$ then the Choquet integral of X respect to υ is defined as*

$$\int_\Omega X d\upsilon = \int_0^{+\infty} \upsilon\{\omega : X(\omega) > x\} dx \tag{3}$$

We give now the representation theorem for the functional H which satisfies some properties of the list above. This result is a new version of the well-known Greco theorem (see [4]), and our new assumptions perfectly match with an actuarial point of view.

Theorem 2.1 *Let L be a set of nonnegative random variables such that L has property i) and we suppose that $I_\Omega \in L$ where I_Ω is the indicator function of Ω. We consider a premium principle $H : L \to \widetilde{\mathbb{R}}$ such that:*

a) $H(I_\Omega) < +\infty$*,*

b) H satisfies the hypotheses(**P5**)*,* (**A1**) *and* (**A4**)*.*

Then there exists a capacity $\upsilon : 2^\Omega \to \mathbb{R}$ *such that for all* $X \in L$

$$H(X) = \int_\Omega X d\upsilon = \int_0^{+\infty} \upsilon\{\omega : X(\omega) > x\} dx \tag{4}$$

Proof By the Representation Theorem and Proposition 1.2 of [4] there exists a capacity υ such that $H(X) = \int_\Omega X d\upsilon$.

The following result consider the convex capacities.

Corollary 1 *Let L be a set of nonnegative random variables such that L has property i). Let H a premium principle that satisfies the hypotheses of Theorem 1, and verifies the property* (**P6**)*, then exists a convex capacity in* (4)*.*

Proof The thesis follows in fact it is well known that υ is convex if and only if H is subadditive [7].

Remark
We observe that:

i) From Theorem 1 follows the property (**P1**) and (**P3**) for H.

ii) The property (**P4**) for $b = 0$ is also according Theorem 1. From Theorem 1 we have property (**P2**) then (**P4**) follows for every b and then (**P7**) follows also.

3 Distortion risk measures

In this section we report some well known risk measures and present the distortion functions measure for some of them. Distortion premium principles have been extensively studied in the past several years, see for example [7], [9], [10]
If X is a random variable the quantile reserve at 100αth percentile or Value at Risk is

$$V_\alpha(X) = \inf\{x \in \mathbb{R} \,|\, F_X(x) \geq \alpha\} \qquad \alpha \in (0,1) \tag{5}$$

A single quantile risk measure of a fixed level α does not an information about the thickness of the upper tail of the distribution function of X, so that other measures are considered. In particular we consider the Tail Value at Risk at level α, $TVaR_\alpha(X)$, is defined as:

$$TVaR_\alpha(X) = \frac{1}{(1-\alpha)} \int_\alpha^1 V_\alpha(X) d\alpha \qquad \alpha \in (0,1) \tag{6}$$

It is known, that given a non negative random variable X, for any increasing function f, with $f(0) = 0$ and $f(1) = 1$, we can define a premium principle

$$H(X) = \int_0^{+\infty} (1 - f(F_X(t))dt = \int_0^{+\infty} g(S_X(t))dt = \int_\Omega X dv \tag{7}$$

where g is a distortion function, $g(x)= 1 - f(1-x)$ and $v = go\mathbf{P}$.

Remark
All distortion premium principles have the properties (**P1**), (**P2**), (**P3**) and (**P4**). If g is concave (f convex) then H satisfies the property (**P6**) also, then (**P7**) follows.
It is well known that the quantile Value at Risk, (5), is a distorted risk measure, while TailVar is a convex distorted risk measure. In fact, $TVaR_\alpha$ can be obtained by (7) where f is the function defined as follows:

$$f(u) = \begin{cases} 0 & u < \alpha, \\ \frac{(u-\alpha)}{(1-\alpha)} & u \geq \alpha \end{cases} \tag{8}$$

4 Representation of a class of premium functionals

Now we provide a a characterization of the class of continuous increasing and convex functions.

Proposition 4.1 *If f is a continuous increasing convex function, defined on $[0,1]$ then exists a probability measure μ on $[0,1]$ such that*

$$f(x) = \int_0^1 \frac{(x-\alpha)_+}{(1-\alpha)} d\mu(\alpha) \tag{9}$$

for $\alpha \in [0,1]$.

Proof Given f a continuous increasing convex function with f(0)=0 then exists a non negative measure ν on $[0,1]$ such that

$$f(x) = \int_0^x (x-\alpha) d\nu(\alpha). \tag{10}$$

We can write

$$f(x) = \int_0^1 (x - \alpha)_+ d\nu(\alpha). \tag{11}$$

Then a probability measure μ on $[0, 1]$ exists such that

$$f(x) = \int_0^1 \frac{(x - \alpha)_+}{(1 - \alpha)} d\mu(\alpha), \qquad \alpha \in [0, 1]. \tag{12}$$

Theorem 4.1 *Let L be a set of nonnegative random variables such that L has property i). We suppose that $I_\Omega \in L$ where I_Ω is the indicator function of Ω and $I_{X>a} \in L$ for any $X \in L$ and any $a > 0$. We consider a premium principle $H : L \to \widetilde{\mathbb{R}}$ such that:*

a) $H(I_\Omega) < +\infty$,

*b) H satisfies the hypotheses (**A1**)- (**A4**).*

Then there exists a probability measure m on $[0, 1]$ such that:

$$H(X) = \int_0^1 TVaR_\alpha(X) dm(\alpha) \tag{13}$$

Proof From Theorem 1 we can conclude that there exists a capacity such that for all $X \in L$

$$H(X) = \int_\Omega X dv \tag{14}$$

Since H is the comonotonic additive see [2], and H verifies (**A2**) then H is subadditive, i.e. H has the property (**P6**) ([7]). It follows from Corollary 1 that the capacity v in (14) is convex. Since $I_{X>a} \in L$ for any $X \in L$ and any $a > 0$ from Corollary 3.1 of [10] follows that there exists a convex increasing function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$ i.e. f such that

$$H(X) = \int_0^{+\infty} (1 - f(F_X(t)) dt \tag{15}$$

From Proposition 1 follows that a probability measure $m(\alpha)$ exists such that f can be represented

$$f(x) = \int_0^1 \frac{(x - \alpha)_+}{(1 - \alpha)} dm(\alpha) \tag{16}$$

for $\alpha \in [0, 1]$, and $f(1) = 1$.

Then interchanging the integrals for the Fubini Theorem for every $X \in L$:

$$
\begin{aligned}
H(X) &= \int_0^{+\infty}(1-f(F_X(t))dt = \int_0^{+\infty}[1-\int_0^1 \frac{(F_X(t)-\alpha)_+}{(1-\alpha)}dm(\alpha)]dt = \\
&= \int_0^{+\infty} dt \int_0^1 [1-\frac{(F_X(t)-\alpha)_+}{(1-\alpha)}]dm(\alpha) = \\
&= \int_0^1 dm(\alpha)\int_0^{+\infty} dt[1-\frac{(F_X(t)-\alpha)_+}{(1-\alpha)}] = \\
&= \int_0^1 dm(\alpha) TVaR_\alpha \qquad (17)
\end{aligned}
$$

Then the results obtained for the class of insurance functional prices seems interesting both because the class of functionals is determined from few natural properties and these functional prices follow closely linked together to a well known risk measure as $TVaR_\alpha$, $\alpha \in [0,1]$. Moreover we point out that the most important properties for a functional price follow easily from the obtained representation.

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