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Abstract. In this paper a set of desirable properties for measures of positive dependence of ordered n-tuples of continuous random variables ($n \geq 2$) is proposed and a class of multivariate positive dependence measures is introduced. We consider the comonotonicity dependence structure as the strong dependency structure and so the class consists of measures that take values in the range $[0, 1]$ and are defined in such a way that they equal 1 in case the random vector is comonotonic and equal 0 in case it is independent.

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Introduction

The concept of dependency in a bivariate and in a multivariate setting has been largely studied in recent literature and the development of the theory of copulas has determined a great impact in the study of measures of dependence. This is of interest in economics, insurance, in finance and risk management and in many areas of applied probability and statistics. Several notions of positive dependence were introduced in the literature to model the fact that large values of a component of a multivariate random vector are "probabilistically associated" with large values of the others. As it is well known that the most widely applied dependence measures, the Pearson's product moment correlation coefficient which captures the linear dependence between couples of random variables is a weak measure in many setups and consider only pairwise dependency. So there are a variety of ways to consider and to measure dependence between random variables and this fact confirm the importance and the high interest in the concept. For a review see e.g [1], [4] and[5]. These concepts have been introduced on the class of bivariate random vectors and many of these orders can be further extended to comparison of general multivariate distributions that have the same marginals.

The focus of this paper is on a class of measures of multivariate positive dependence defined by means of some comparison between a joint distribution function and a distribution representing independence. Properties of these measures are also investigated in a axiomatic framework.

The paper is organized as follows. In Section 2, notations will be fixed and various concepts and results that are essential to the development of the present paper are reviewed. After a brief overview about some properties of copula, we introduce the copula approach for studying dependence between random variables and Section 3 presents the main results.

1 Notations and preliminaries

We are going to review some basic definitions and properties about dependence concepts and dependence orders which we will use later. We will be concerned with random vectors that take on values in $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ will be denoted by $\mathbf{x}, \mathbf{y}$ or more explicitly, as $\mathbf{x} = (x_1, \ldots, x_n)$ or $\mathbf{y} = (y_1, \ldots, y_n)$.

For two n-vectors x and y, the notation $\mathbf{x} \leq \mathbf{y}$ will be used for the componentwise order which is defined by $x_i \leq y_i$ for all $i = 1, \ldots, n$.

Throughout this paper, all the random variables considered are defined on a common probability space $(\Omega, \mathcal{F}, P)$. The set Ω consists of all possible results or outcomes and its generic element is denoted by ω and the σ-field $\mathcal{F}$ is a collection of subset of Ω.

The distribution function of a random vector $\mathbf{X} = (X_1, \ldots, X_n))$ is the function $F : \mathbb{R}^n \to [0, 1]$ defined as follows:

$$F(\mathbf{x}) = P\left[X_1 \leq x_1, \ldots, X_n \leq x_n\right]$$

A distribution function is non-decreasing and right-continuous on $\mathbb{R}^n$ while the survival function is a non-increasing, right-continuous function so defined:

$$\bar{F}(\mathbf{x}) = P\left[X_1 > x_1, \ldots, X_n > x_n\right]$$

If we want to consider the distribution of $X_i, i = 1, \ldots, n$ singularly, we need the marginal distribution function $F_i(x_i) = P\left[X_1 \leq x_1\right]$.

Fréchet spaces are important in studying dependence between random vectors, paying attention to probability distributions with fixed univariate marginals and so the elements of a Fréchet space only differ in their dependence structure and not in the marginals behaviors. If $F_1, \ldots, F_n$ are univariate distribution functions the Fréchet space $\mathcal{R}(F_1, \ldots, F_n)$ consists of all the n-dimensional random vectors with $F_1, \ldots, F_n$ as marginal distributions.

Some random variables $X_1, \ldots, X_n$ are mutually independent if and only if

$$F(\mathbf{x}) = \prod_{i=1}^{n} F_i(x_i),$$

where $\mathbf{x} \in \mathbb{R}^n$ and F is the distribution function of the random vector $X = (X_1, \ldots, X_n)$. In particular if $\mathbf{X}$ is a random vector we denote by $\mathbf{X}^{\perp}$ the random vector whose marginal distributions coincide with the marginals of $\mathbf{X}$, but whose components are independent. Now we introduce the concept of comonotonicity by defining comonotonicity of a set of vectors in $\mathbb{R}^n$.

Definition 1.1 *The set $A \subset \mathbb{R}^n$ is comonotonic if for any $\mathbf{x}, \mathbf{y}$ in A, either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{y} \leq \mathbf{x}$ holds.*

So a set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $\mathbf{x}, \mathbf{y}$ in A, the inequality $x_i < y_i$ for some i, implies that $\mathbf{x} \leq \mathbf{y}$. Next we define a comonotonic random vector $\mathbf{X}$ through its support where a support of a random vector $\mathbf{X}$ is the smallest closed set $A \subseteq \mathbb{R}^n$ for which $P\left[\mathbf{X} \in A\right] = 1$.

Definition 1.2 *The random vector $\mathbf{X}$ is comonotonic if it has a comonotonic support.*

The following result provides some equivalent conditions of comonotonicity.

Proposition 1.1 *If $\mathbf{X} = (X_1, \ldots, X_n)$ is a n-dimensional random vector the following statements are equivalent:*

  i) *The random vector $\mathbf{X}$ is comonotonic;*

  ii) *For any i, j with $1 \leq i, j \leq n$ the inequality*

$$[X_i(\omega_1) - X_i(\omega_2)][X_j(\omega_1) - X_j(\omega_2)] \geq 0$$

  holds almost surely for ω_1 and ω_2 in Ω;

  iii) *If F is the distribution function of the random vector $\mathbf{X}$ for all $\mathbf{x} = (x_1, \ldots, x_n)$ we have $F(\mathbf{x}) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$;*

  iv) *There exists a random variable $\mathbf{Z}$ and n increasing functions t_i, we obtain $\mathbf{X} =_d (t_1(\mathbf{Z}), \ldots, t_n(\mathbf{Z}))$.*

The symbol "$=_d$"in the above result means "is equally distributed as".

So in order to find the probability of all the outcomes of n comonotonic vectors being less than x_i, one simply takes the probability of the least likely of these n events. Comonotonicity is indeed a very strong positive dependency structure. A possibility to introduce some weakest form of positive dependence dependence concept is to consider the set of all random vectors which are larger than $X^\perp$ with respect to some dependence order. We consider in particular the positive quadrant dependence orders.

Definition 1.3 *Let* $\mathbf{X}, \mathbf{Y}$ *are two n-dimensional random vectors with distribution functions* F *and* G *and survival functions* $\bar{F}$ *and* $\bar{G}$ *respectively. We say that* $\mathbf{X}$ *is smaller then* $\mathbf{Y}$ *in the positive lower orthant dependence order and we denote*

$$\mathbf{X} \preceq_{puod} \mathbf{Y} \Leftrightarrow F(\mathbf{t}) \leq G(\mathbf{t}), \quad for\ all \quad \mathbf{t} \in \mathbb{R}^n$$

and we say that $\mathbf{X}$ *is smaller then* $\mathbf{Y}$ *in the positive upper orthant dependence order and we denote*

$$\mathbf{X} \preceq_{plod} \mathbf{Y} \Leftrightarrow \bar{F}(\mathbf{t}) \leq \bar{G}(\mathbf{t}), \quad for\ all \quad \mathbf{t} \in \mathbb{R}^n$$

The positive orthant orders measure the amount of positive dependence of the underlying random vectors. Now we consider some notions of positive dependence for multivariate random vectors.

Definition 1.4 *If* $\mathbf{X}$ *a n-dimensional random vector we say that* $\mathbf{X}$ *is a positive lower orthant dependent(PLOD) if* $\mathbf{X} \geq_{plod} \mathbf{X}^\perp$ *and is a positive upper orthant dependent (PUOD) if* $\mathbf{X} \geq_{puod} \mathbf{X}^\perp$.

In this paper we use the term "positive dependence " for a random vector which is positive lower orthant dependent. Then if $\mathbf{X}$ is a n-dimensional random vector in $\mathcal{R}(F_1, \ldots, F_n)$ with distribution function F, $\mathbf{X}$ is positive dependent if and only if

$$F(\mathbf{x}) \geq \prod_{i=1}^{n} F_i(x_i)$$

We denote by F^I the distribution function of the random vector in $\mathcal{R}(F_1, \ldots, F_n)$ with independent components,

$$F^I(\mathbf{x}) = \prod_{i=1}^{n} F_i(x_i) \tag{1}$$

and F^C the distribution function of the random vector in $\mathcal{R}(F_1, \ldots, F_n)$ with comonotonic components.

$$F^C(\mathbf{x}) = \min\{F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\} \tag{2}$$

Then for PLOD random vectors it is possible to prove the following result:

Theorem 1.1 *If* $\mathbf{X}$ *be a PLOD n-dimensional random vector in* $\mathcal{R}(F_1, \ldots, F_n)$ *with distribution functions* F *then for all* $\mathbf{x} \in \mathbb{R}^n$

$$F^I(\mathbf{x}) \leq F(\mathbf{x}) \leq F^C(\mathbf{x}).$$

The concept of copula introduced by Sklar in 1959 is now common in the statistical literature, but only recently its potential for applications has become clear. Copulas permit to represent joint distribution functions by splitting the marginal behavior, embedded in the marginal distributions, from the dependence captured by the copula itself. A copula can also be seen as a joint probability distribution with uniform marginals. Elimination of marginals through copulas helps to model and understand dependence structure between variables more effectively, as the dependence has nothing to do with the marginal behavior. First of all we are focusing our attention to the n-increasing property for multivariate functions.

If f is a function $\mathbb{R}^n \to \mathbb{R}$, a B is the n-box $B = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ with $a_i < b_i$, $i = 1, \ldots, n$, then the f-volume of B is

$$V_f(B) = \sum_{i_1=1}^{2} \ldots \sum_{i_n=1}^{2} (-1)^{i_1+\ldots+i_n} f(x_{1i_1}, \ldots, x_{ni_n})$$

where $x_{i1} = a_i$, $x_{i2} = b_i$.

If there exists $i \in 1, \ldots, n$ such that $a_i = b_i$ then $V_f(B) = 0$.

Equivalently the f-volume of B is the n-order difference of f on B is :

$$V_f(B) = \Delta_{(a_1\ldots,a_n),(b_1\ldots,b_n)} f(x_1, \ldots, x_n) = \Delta_{a_1,b_1} \ldots \Delta_{a_n,b_n} f(x_1, \ldots, x_n)$$

where we define the n first order differences of f as

$$\Delta_{a_k,b_k} f(x_1, \ldots, x_n) = f(x_1 \ldots x_{k-1}, b_k, x_{k+1}, \ldots, x_n) - f(x_1 \ldots x_{k-1}, a_k, x_{k+1}, \ldots, x_n).$$

Moreover by convention if $k = n$ we set

$$\Delta_{(a_1\ldots,a_n),(b_1\ldots,b_n)} f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).$$

Definition 1.5 *A function $f: \mathbb{R}^n \to \mathbb{R}$ is n-increasing if for all n-box $B = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq [0,1]^n$ where $a_i \leq b_i, i = 1, \ldots, n$, $V_f(B) \geq 0$.*

If f has nth-order derivatives, n-increasing is equivalent to $\frac{\partial^n}{\partial x_1 \ldots \partial x_n} f \geq 0$.

Definition 1.6 *A n-copula is a function $C : [0,1]^n \to [0,1]$ that satisfies:*

(a) $C(x_1, \ldots, x_n) = 0$ *if $x_i = 0$ for any $i = 1, \ldots, n$.*

(b) $C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$ *for each $i = 1, \ldots, n$ and all $x_i \in [0,1]$.*

(c) *C is n-increasing.*

Conditions (a) and (b) are known as *boundary conditions* whereas condition (c) is known as *monotonicity*. In fact the definition of *n*- increasing function is the multivariate extension of the concept of "increasing" for a univariate function when we interpret "increasing" as "increasing as a distribution function".

Various properties of copulas have been studied in literature, but most part of the research concentrates on the bivariate case, since multivariate extensions are generally not easily to be done.

The following theorem (Sklar 1959), which partially explains the importance of copulas in statistical modeling, justifies the role of copulas as dependence functions

Theorem 1.2 (Sklar's Theorem) *Let F be an n-dimensional distribution function with marginal distribution $F_1, \ldots, F_n$. Then there exists an n-dimensional copula $C(u_1, \ldots, u_n)$ such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
If the functions $F_1, \ldots, F_n$ are all continuous then C is unique. Conversely if C is a n-dimensional copula and $F_1, \ldots, F_n$ are distribution functions, then the function F defined above is a n-dimensional distribution function with marginal distribution $F_1, \ldots, F_n$.

A rigorous mathematical description of copulas and their features is available in [8].
If we consider the *independent copula*

$$C^i(u_1, \ldots, u_n) = u_1 u_2, \ldots, u_n$$

and the *comonotonic copula*

$$C^c(u_1, \ldots, u_n) = min\{u_1, u_2, \ldots, u_n\}$$

and we call a n-copula C a PLOD copula if

$$C(u_1, \ldots, u_n) \geq C^i(u_1, \ldots, u_n)$$

for all $(u_1, \ldots, u_n) \in \mathbb{R}^n$ it is easy to prove that if C is a n-dimensional copula

$$C^i(u_1, \ldots, u_n) \leq C(u_1, \ldots, u_n) \leq C^c(u_1, \ldots, u_n)$$

for all $(u_1, \ldots, u_n) \in \mathbb{R}^n$.
If θ is the permutation $(1, \ldots, n) \mapsto (i_1, \ldots, i_n)$ and C is a n-dimensional copula we denote by C^θ the copula defined by $C(u_1, \ldots, u_n) = C(u_{i_1}, \ldots, u_{i_n})$.

2 Main Results

The purpose of this paper is to investigate the notion of dependence between multidimensional distributions with the same marginal distributions by proposing some complete dependence orderings. In particular Joe defined a set of axioms that a dependence ordering of distributions should have in order that higher in the ordering means more positive dependence. These properties may be generalized to cover the case of n-variate distribution

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functions. We suggest a list of desirable properties that any multivariate positive dependence notion should fulfill.

By a *measures of positive dependence* we means a function that attaches to every n-tuple of continuous and PQD random variable $\mathbf{X} = (X_1, \ldots, X_n)$ where $n \geq 2$, a real number $m(\mathbf{X})$ satisfying the following:

P1. **(Normalization)** $m(\mathbf{X})$=0 if and only if and $\mathbf{X}$ has independent component and $m(\mathbf{X})$=1 if and only the random vector $\mathbf{X}$ is comonotonic;

P2. **(Monotonicity)** If $\mathbf{X} \preceq_{plod} \mathbf{Y}$ then $m(\mathbf{X}) \leq m(\mathbf{Y})$;

P3 **(Permutation Invariance)** $m(X_{i_1}, X_{i_2}, \ldots, X_{i_n}) = m(X_1, X_2, \ldots, X_n)$ for all permutations $i_1, \ldots, i_n$ of $\{1, \ldots, n\}$.

P4 **(Functional Invariance)** $m((X_1, X_2, \ldots, X_n)) = m((a(X_1), X_2, \ldots, X_n))$ for all increasing continuous function a;

P5 **(Continuity)** If $\mathbf{X}_n \to \mathbf{X}$ in distribution then $m(\mathbf{X}_n) \to m(\mathbf{X})$.

The following result show that the copula accounts for all the dependence between random variables.

Theorem 2.1 *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an n-dimensional random vector with copula C and $a_1, \ldots, a_n$ be increasing and continuous functions. Then the random vector $(a_1(X_1), \ldots, a_n(X_n))$ has the copula C.*

Invariance properties of copulas suggests that they facilitate the study of scale-free measures of dependence and so we reformulate the axioms that characterize measures of positive dependence in terms of copulas.
If $n \geq 2$, we indicate by $\mathcal{C}^+(n)$ the set of PLOD n-dimensional copulas .

By a *measures of positive dependence* we means a function $m : \mathcal{C}^+(n) \mapsto \mathbb{R}$ satisfying the following:

A1. $m(C)$=0 if and only if $C = C^i$ and $m(C)$=1 if and only $C = C^c$;

A2. If $C_1 \leq C_2$ pointwise then $m(C_1) \leq m(C_2)$;

A3 $m(C^\theta) = m(C)$ for all permutations θ of $\{1, \ldots, n\}$;

A4 If $C_n \to C$ pointwise then $m(C_n) \to m(C)$.

If θ is the permutation $(1, \ldots, n) \mapsto (i_1, \ldots, i_n)$ we denote by f_θ the function defined by $f_\theta(x_1, \ldots, x_n) = f(x_{i_1}, \ldots, x_{i_n})$.

Definition 2.1 *A Borel measure μ is permutation invariant if for any Borel set B and any permutation θ, $\mu(f_\theta(B)) = \mu(B)$.*

In this contribution the strength of dependence between n variables is compared using the notion of comonotonicity and we consider measures of positive dependence that quantify the degree of comonotonicity in a random vector.

Theorem 2.2 *If μ is a permutation invariant finite Borel measure on $[0,1]^n$ such that $0 < \int (C^c - C^i)d\mu < \infty$ then*

$$m(C) = \frac{\int (C - C^i)d\mu}{\int (C^c - C^i)d\mu}$$

is a measure of positive dependence defined on $\mathcal{C}^+(n)$.
Proof

Thanks to the previous observations, $m(C)=0$ if and only if $C = C^i$ and $m(C)=1$ if and only $C = C^c$. The monotonicity property follows easily from the definition of the measure m. Moreover we note that $\int (C^\theta - C^i)d\mu = \int (C - C^i)d(\mu \circ f_\theta) = \int (C - C^i)d\mu$ and then $m(C^\theta) = m(C)$ and the measure is permutation invariant. Since every sequence of copulas converging to a copula pointwise, does so uniformly (see [6]) then the continuity property follows.

Finally we characterize measures of multivariate dependence that preserve convex sums of copulas (see [3] for the case of bivariate measure of concordance).

Theorem 2.3 *A measure of positive dependence m defined on $\mathcal{C}^+(n)$ is such that*

$$m(tA + (1-t)B) = tm(A) + (1-t)m(B) \quad for \quad t \in (0,1) \quad and \quad A, B \in \mathcal{C}^+(n)$$

if and only if there exists a permutation invariant finite Borel measure μ on $[0,1]^n$ such that $0 < \int (C^c - C^i)d\mu < \infty$ and

$$m(C) = \frac{\int (C - C^i)d\mu}{\int (C^c - C^i)d\mu}.$$

Proof

It is easy to prove that a function defined on $\mathcal{C}^+(n)$ by

$$m(C) = \frac{\int (C - C^i)d\mu}{\int (C^c - C^i)d\mu}$$

where μ is a permutation invariant finite Borel measure on $[0,1]^n$ such that $0 < \int (C^c - C^i)d\mu < \infty$ is a multivariate measure of positive dependence that preserves the convex sum of copulas.
Now we suppose that m is a measure of positive dependence defined on $\mathcal{C}^+(n)$ such that

$$m(tA + (1-t)B) = tm(A) + (1-t)m(B) \quad for \quad t \in (0,1) \quad and \quad A, B \in \mathcal{C}^+(n)$$

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and $m(C^c) = 1$. Letting $\mathcal{C} = \{C - C^i : C \in \mathcal{C}^+(n)\}$ and Γ the functional defined by $\Gamma(C - C^i) = m(C)$. The functional Γ is bounded and linear and defined on a compact and convex subset of the space $C([0,1]^n)$ of continuous functions on $[0,1]^n$ with respect to $\|\cdot\|_\infty$ and then it can be extended to a bounded linear functional on $C([0,1]^n)$. By the Riesz Representation Theorem, there exists a Borel measure μ on $[0,1]^n$ such that

$$\Gamma(A) = \int A d\mu$$

for every $A \in C([0,1]^n)$. Then $m(C) = \int (C - C^i) d\mu$ for any $C \in \mathcal{C}^+(n)$.
Moreover as in the proof of Theorem 3.3 we can prove that μ is permutation invariant.

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