

Department of Applied Mathematics, University of Venice

WORKING PAPER SERIES



Marco Corazza and Silvio Giove

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Working Paper n. 170/2008
November 2008

ISSN: 1828-6887

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Fuzzy Interval Net Present Value

MARCO CORAZZA

<corazza@unive.it>

Dept. of Applied Mathematics
Ca' Foscari University of Venice

and

Advanced School of Economics

SILVIO GIOVE

<sgiove@unive.it>

Dept. of Applied Mathematics
Ca' Foscari University of Venice

(November 2008)

Abstract. In this paper we conjugate the operative usability of the net present value with the capability of the fuzzy and the interval approaches to manage uncertainty. Our fuzzy interval net present value can be interpreted, besides the usual present value of an investment project, as the present value of a contract in which the buyer lets the counterpart the possibility to release goods/services for money amounts that can vary, at time instants that can also vary. The buyer can reduce the widths of these variations by paying a cost. So, it is “natural” to represent the good/service money amounts and the time instants by means of triangular fuzzy numbers, and the cost of the buyer as a strictly-increasing function of the level $\alpha \in [0, 1]$ associated to the generic cut of the fuzzy interval net present value. As usual, the buyer is characterized by an utility function, depending on α and on the cost, that he/she has to maximize. As far the interest rates regard, we assume that the economic operators are only able to specify a variability range for each of the considered period interest rate. So, we represent the interest rates by means of interval numbers. Besides proposing our model, we formulate and solve the programming problems which have to be coped with to determine the extremals of the cut of the fuzzy interval net present value, and we deal with some questions related to the utility function of the buyer.

Keywords: Net present value, fuzzy set theory, interval number theory, α -cut, utility function.

JEL Classification Numbers: C61, G19.

MathSci Classification Numbers: 90C99.

Correspondence to:

Marco Corazza Dept. of Applied Mathematics, Ca' Foscari University of Venice
Dorsoduro 3825/E
30123 Venezia, Italy
Phone: [+39] (041)-234-6921
Fax: [+39] (041)-522-1756
E-mail: corazza@unive.it

1 Introduction

The correct evaluation of investment projects is an ever-present problem in everyday firm life and in several economic research fields. In order to deal with this problem, many approaches and tools have been developed, from the classical deterministic net present value (NPV)¹ and internal interest rate (IIR) to the refined stochastic real options.

Of course, a central question concerns the capability of the considered approaches/tools to manage the ignorance of the future. Whereas the classical models simply do not consider this aspect (with conceivable consequences), on the other hand the majority of the stochastic approaches face it in a too much technical way for real-life applications. Because of that, several academic and professional researches presented and present intermediate tools.

In this paper we propose a model of this latter kind, in which we conjugate the operative usability of the NPV with the capability of the fuzzy and the interval approaches to manage uncertainty. It is important to notice that we have chosen to not deal with uncertainty by means of probability and stochastic tools because, coherently with the real life, these tools are usually beyond the knowledge and the computational ability of standard economic operators.

The literature concerning this topic is rich enough. Some of the first models were presented in [2], in which fuzzy cash amounts, a fuzzy interest rate, and a fuzzily defined number of time periods are used. These models are mainly investigated from a theoretical point of view. More operative approaches are proposed in [3] and [4]. In the former, a triangular fuzzy NPV is used for fuzzifying the Myers-Cohn model of property-liability insurance pricing. In the latter, an NPV based on triangular fuzzy cash amounts and triangular fuzzy interest rates is used for specifying dynamic replacement model. In [5], a fuzzy NPV-based procedure is considered as an effective tool for the selection of aviation technologies. In [1], a triangular fuzzy NPV is introduced as intermediate step for the determination of the related fuzzy IIR. In [7], trapezoidal fuzzy cash amounts, crisp interest

¹A commonly used formulation of the NPV is $\sum_{j=0}^n S_j / \prod_{k=0}^j (1 + i_k)$, where n is the number of (equal length) time periods, S_j is the cash amount at time instant j , and i_k is the interest rate during the time period $(k - 1, k]$.

rates, and probabilistic methods are jointly used for developing a fuzzy-probabilistic NPV algorithm for the evaluation of investment projects.

Of course, this review does not intend to be exhaustive but only exemplificative.

With respect to the cited contributions, and the referred ones therein, our model differs from a technical standpoint, from the economic interpretability, and from its usability.

In short, the NPV we detail in section 2 can be interpreted, besides the usual present value of a generic investment project, as the present value of a contract in which the buyer lets the counterpart the possibility to release goods/services for money amounts that can vary in prefixed intervals, at time instants that can also vary in prefixed intervals. In her/his turn, the buyer can reduce the widths of both these kinds of intervals by paying a proper cost. Given this interpretation of the NPV, also coherently with the cited contributions and with several other fuzzy modelings, it is “natural” enough to represent the good/service money amounts and the time instants by means of triangular fuzzy numbers, and the cost of the buyer as a strictly-increasing function of the level $\alpha \in [0, 1]$ associated to the generic cut of the fuzzy interval NPV.

Of course, the buyer – as any economic operator – is characterized by an utility function that, in our case, depends on the features of the NPV of the contract and negatively depends on the cost. So, recalling that all these quantities depend on α , the buyer has to determine the optimal value of such an α which maximizes her/his utility.

As far the interest rates regard, recalling that the economic operators involved in the investment project/contract are ignorant of the future, we reasonably assume that they are only able to specify a suitable variability range for each of the considered period interest rate. This is why we represent the interest rates by means of interval numbers. Moreover, it is to notice that, to the best of our knowledge, ours is the first model in which the time instants in which one releases the cash amounts/goods/services are fuzzified, and the interest rates are represented in terms of interval numbers.

The remainder of the paper is organized as follows. In the next session we propose our Fuzzy Interval Net Present Value (FINE) model and some of its main features. In section 3

we formulate and solve the nonlinear optimization programming problems which have to be coped with in order to determine the extremals of the generic cut of the FINE. In section 4 we briefly deal with some questions related to the utility function of the buyer. Finally, in section 5 we list some open items.

2 The FINE model

Let us start by specifying the discrete-time frame associated to the considered investment project/contract:

$$t_0, \tilde{t}_1, \dots, \tilde{t}_j, \dots, \tilde{t}_n,$$

where

t_0 is the current time instant, which we represent by a crisp number; $\tilde{t}_1, \dots, \tilde{t}_j, \dots, \tilde{t}_n$ are the future time instants in which the cash amounts/goods/services are released, that we represent by triangular fuzzy numbers;² as known, the membership functions of this typology of numbers are

$$\mu_{\tilde{t}_j}(t) = \begin{cases} 0 & \text{if } t < t_{j,1} \\ (t - t_{j,1}) / (t_{j,2} - t_{j,1}) & \text{if } t_{j,1} \leq t < t_{j,2} \\ (t_{j,3} - t) / (t_{j,3} - t_{j,2}) & \text{if } t_{j,2} \leq t < t_{j,3} \\ 0 & \text{if } t \geq t_{j,3} \end{cases}, \text{ with } j = 1, \dots, n,$$

in which $t_{j,1}, t_{j,2}$ and $t_{j,3}$ are real numbers such that $t_{j,1} \leq t_{j,2} \leq t_{j,3}$ (a commonly used notation for this kind of numbers is $\tilde{t}_j = (t_{j,1}, t_{j,2}, t_{j,3})$);

n is the number of time periods, which we represent by a crisp number.

In particular, we assume that, as in standard NPV models, the time instants in which the cash amounts/goods/services are released constitute a non decreasing sequence with respect to the deponent, that is we assume that

$$t_0 \leq t_{1,1} \text{ and } t_{j,3} \leq t_{j+1,1}, \text{ with } j = 1, \dots, n - 1. \quad (1)$$

²Throughout the paper, we use the overwriting $\tilde{}$ for indicating fuzzy numbers. Moreover, for notational convenience, thereafter we represent the crisp number t_0 in terms of the (degenerate) triangular fuzzy number $\tilde{t}_0 = (t_0, t_0, t_0)$.

Let us continue by specifying the fuzzy investment project/contract:

$$\tilde{\mathbf{A}} = \left\{ (-S_0, \tilde{t}_0), (\tilde{S}_1, \tilde{t}_1), \dots, (\tilde{S}_j, \tilde{t}_j), \dots, (\tilde{S}_n, \tilde{t}_n) \right\},$$

where

$S_0 > 0$ is the current cash amount/good/service value, which we represent by a crisp number;

$\tilde{S}_1, \dots, \tilde{S}_j, \dots, \tilde{S}_n$ are the future positive cash amount/good/service values, which we represent by positive triangular fuzzy number.³

Finally, let us assume the following time period frame associated to the interest rate structure:

$$(\tau_1, \tau_2], (\tau_2, \tau_3], \dots, (\tau_{j-1}, \tau_j], \dots, (\tau_n, \tau_{n+1}], \quad (2)$$

where

$\tau_1 = t_0, \tau_2 \in (t_{1,3}, t_{2,1}], \dots, \tau_j \in (t_{j-1,3}, t_{j,1}], \dots, \tau_n \in (t_{n-1,3}, t_{n,1}], \tau_{n+1} > t_{n,3}$ are the crisp time instants in which the interest rates can change.⁴

It is important to notice that, by construction, such time instants constitute a strictly-increasing sequence with respect to the deponent, that is that

$$\tau_1 < \tau_2 < \dots < \tau_j < \dots < \tau_n. \quad (3)$$

Of course, it is possible to choose more articulated time period frames than the one specified by (2) and (3). In case of such a choice, one should obtain a model which should be more difficult to algebraically treat than ours, but that at the while should not give significant improvements from the economic meaning standpoint. This is why we opt for the proposed time period frame.

Given this notation, we formulate as follows our FINE model:

³A triangular fuzzy number $\tilde{x} = (x_1, x_2, x_3)$ is defined positive if $x_1 > 0$.

⁴It is to notice that, in general, this (crisp) discrete-time frame differs from the (fuzzy) one associated to the investment project/contract.

$$FINE(\tilde{\mathbf{A}}) = -S_0 + \sum_{j=1}^n \tilde{S}_j \left(1 + \hat{i}_j\right)^{-(\tilde{t}_j - \tau_j)} \prod_{k=1}^{j-1} \left(1 + \hat{i}_k\right)^{-(\tau_{k+1} - \tau_k)}, \quad (4)$$

where

$\hat{i}_1, \dots, \hat{i}_j, \dots, \hat{i}_n$ are the current and the future positive interest rates, which we represent by positive interval numbers;⁵ as known, the membership functions of this typology of numbers are

$$\mu_{\hat{i}_j}(i) = \begin{cases} 0 & \text{if } i < i_{j,1} \\ 1 & \text{if } i_{j,1} \leq i \leq i_{j,2} \\ 0 & \text{if } i > i_{j,2} \end{cases}, \text{ with } j = 1, \dots, n,$$

in which $i_{j,1}$ and $i_{j,2}$ are real numbers such that $i_{j,1} \leq i_{j,2}$ (a commonly used notation for this kind of numbers is $\hat{i}_j = (i_{j,1}, i_{j,2})$).⁶

3 Determination of the α -cuts for the FINE model

Once the NPV of $\tilde{\mathbf{A}}$ is formulated in terms of fuzzy and interval numbers, we go on to the determination of the shape of this number. In order to carry out that, we have to determine the extremals of the α -cuts, with $\alpha \in [0, 1]$, of (4), that is the *minimum* and the *maximum* of the sets:

$$X_{FINE(\tilde{\mathbf{A}}), \alpha} = \left\{ x : \mu_{FINE(\tilde{\mathbf{A}})}(x) \geq \alpha \right\},$$

that we respectively indicate by $FINE(\tilde{\mathbf{A}})(\alpha)_L$ and $FINE(\tilde{\mathbf{A}})(\alpha)_R$.

In their turn, $FINE(\tilde{\mathbf{A}})(\alpha)_L$ and $FINE(\tilde{\mathbf{A}})(\alpha)_R$ depend on the values the arguments of (4) take in their corresponding α -cuts. Therefore, in order to determine the extremals of $X_{FINE(\tilde{\mathbf{A}}), \alpha}$ we have to cope with the following nonlinear optimization programming problems:

⁵Throughout the paper, we use the overwriting $\hat{\cdot}$ for indicating interval numbers.

⁶An interval number $\hat{x} = (x_1, x_2)$ is defined positive if $x_1 > 0$.

$$\begin{aligned}
& \min / \max && -S_0 + \sum_{j=1}^n S_j (1 + i_j)^{-(t_j - \tau_j)} \prod_{k=1}^{j-1} (1 + i_k)^{-(\tau_{k+1} - \tau_k)} \\
& S_1, \dots, S_n, i_1, \dots, i_n, t_1, \dots, t_n && , \\
& \text{s.t.} && \begin{cases} S_{j,L}(\alpha) \leq S_j \leq S_{j,R}(\alpha), \text{ with } j = 1, \dots, n, \\ i_{j,L}(\alpha) \leq i_j \leq i_{j,R}(\alpha), \text{ with } j = 1, \dots, n, \\ t_{j,L}(\alpha) \leq t_j \leq t_{j,R}(\alpha), \text{ with } j = 1, \dots, n, \end{cases}
\end{aligned} \tag{5}$$

where

$S_{j,L} = S_{j,1} + (S_{j,2} - S_{j,1})\alpha$ is the *minimum* of the α -cut of the triangular fuzzy number \tilde{S}_j ;
 $S_{j,R} = S_{j,3} + (S_{j,3} - S_{j,2})\alpha$ is the *maximum* of the α -cut of the triangular fuzzy number \tilde{S}_j ;

$i_{j,L} = i_{j,1}$ is the *minimum* of the α -cut of the interval number \hat{i}_j ;

$i_{j,R} = i_{j,2}$ is the *maximum* of the α -cut of the interval number \hat{i}_j ;

$t_{j,L} = t_{j,1} + (t_{j,2} - t_{j,1})\alpha$ is the *minimum* of the α -cut of the triangular fuzzy number \tilde{t}_j ;

$t_{j,R} = t_{j,3} + (t_{j,3} - t_{j,2})\alpha$ is the *maximum* of the α -cut of the triangular fuzzy number \tilde{t}_j .

It is important to notice that the *minimum* programming problem is related to the determination of $FINe(\tilde{\mathbf{A}})(\alpha)_L$, and that the *maximum* programming problem is related to the determination of $FINe(\tilde{\mathbf{A}})(\alpha)_R$.

After some calculations and some arrangements, the following partial derivatives of the target function (TF) of (5) with respect to S_j , i_j , and t_j are obtained:

$$\begin{aligned}
\frac{\partial}{\partial S_j} \text{TF} &= (1 + i_j)^{-(t_j - \tau_j)} \prod_{k=1}^{j-1} (1 + i_k)^{-(\tau_{k+1} - \tau_k)}, \text{ with } j = 1, \dots, n; \\
\frac{\partial}{\partial i_j} \text{TF} &= -(t_j - \tau_j) (1 + i_j)^{-(t_j - \tau_j - 1)} S_j \prod_{k=1}^{j-1} (1 + i_k)^{-(\tau_{k+1} - \tau_k)} - \\
&\quad - (\tau_{j+1} - \tau_j) (1 + i_j)^{-(\tau_{j+1} - \tau_j - 1)} \sum_{k=j+1}^n S_k (1 + i_k)^{-(t_k - \tau_k)} \cdot \\
&\quad \cdot \prod_{l=1, l \neq j}^{k-1} (1 + i_l)^{-(\tau_{l+1} - \tau_l)}, \text{ with } j = 1, \dots, n; \\
\frac{\partial}{\partial t_j} \text{TF} &= -(1 + i_j)^{-(t_j - \tau_j)} \ln(1 + i_j) S_j \prod_{k=1}^{j-1} (1 + i_k)^{-(\tau_{k+1} - \tau_k)}, \text{ with } j = 1, \dots, n.
\end{aligned}$$

Given the positions (1), the results (3), and the positivity of the cash amount/good/service values and of the interest rates, it is easy to verify that $\frac{\partial}{\partial S_j} \text{TF} > 0$ for all j , that $\frac{\partial}{\partial i_j} \text{TF} < 0$ for all j , and that $\frac{\partial}{\partial t_j} \text{TF} < 0$ for all j .

At this point, given the strict monotony of the TF of (5) with respect to S_j , i_j and t_j , and given the latter results, for each $\alpha \in [0, 1]$ we can determine as follows the respective points of *optimum* of the programming problems (5):

<i>Minimum programming problem</i>	<i>Maximum programming problem</i>
$S_{j,\min}^* = S_{j,L}(\alpha) =$ $= S_{j,1} + (S_{j,2} - S_{j,1}) \alpha,$ with $j = 1, \dots, n;$	$S_{j,\max}^* = S_{j,R}(\alpha) =$ $= S_{j,3} + (S_{j,3} - S_{j,2}) \alpha,$ with $j = 1, \dots, n;$
$i_{j,\min}^* = i_{j,R}(\alpha) = i_{j,2},$ with $j = 1, \dots, n;$	$i_{j,\max}^* = i_{j,L}(\alpha) = i_{j,1},$ with $j = 1, \dots, n;$
$t_{j,\min}^* = t_{j,R}(\alpha) =$ $= t_{j,3} + (t_{j,3} - t_{j,2}) \alpha,$ with $j = 1, \dots, n;$	$t_{j,\max}^* = t_{j,L}(\alpha) =$ $= t_{j,1} + (t_{j,2} - t_{j,1}) \alpha,$ with $j = 1, \dots, n;$

Of course, exploiting the usual extension principle (for details see [6]):

$$\begin{aligned}
\text{FINE}(\tilde{\mathbf{A}})(\alpha)_L &= -S_0 + \sum_{j=1}^n S_{j,\min}^* (1 + i_{j,\min}^*)^{-(t_{j,\min}^* - \tau_j)} \\
&\quad \cdot \prod_{k=1}^{j-1} (1 + i_{k,\min}^*)^{-(\tau_{k+1} - \tau_k)}
\end{aligned}$$

and

$$\begin{aligned}
\text{FINE}(\tilde{\mathbf{A}})(\alpha)_R &= -S_0 + \sum_{j=1}^n S_{j,\max}^* (1 + i_{j,\max}^*)^{-(t_{j,\max}^* - \tau_j)} \\
&\quad \cdot \prod_{k=1}^{j-1} (1 + i_{k,\max}^*)^{-(\tau_{k+1} - \tau_k)}.
\end{aligned}$$

It is to notice that, if $\alpha = 1$, then $S_{j,\min}^* = S_{j,\max}^* = S_{j,2}$, $i_{j,\min}^* = i_{j,1} < i_{j,\max}^* = i_{j,2}$ and $t_{j,\min}^* = t_{j,\max}^* = t_{j,2}$. So, the width of the 1-cut of $\text{FINE}(\tilde{\mathbf{A}})$ depends only on the extremals of the 1-cut of \hat{i}_j .

4 About the buyer's utility function

As premised, in case (4) is the FINE of a contract like the one described in the introduction, the utility function of the buyer depends on the features of this FINE and negatively depends on the cost the buyer has to pay for reducing the widths of the α -cuts related to \tilde{S}_j and \tilde{t}_j , that is for reducing the width of the α -cut related to $FINE(\tilde{\mathbf{A}})$. Recalling that all these quantities depend on α , in order to determine the optimal value of α itself which maximizes the buyer's utility, at first we have to detail such a function.

Coherently with the rationality of the generic economic operator, for each α we assume that the buyer's utility function positively depends on some return measure of the considered α -cut (that we indicated by $r(\alpha)$), negatively depends on some variability measure of the considered α -cut (that we indicated by $v(\alpha)$), and negatively depends on the cost $c(\alpha)$. In particular, for these quantities we propose the following specifications:

$$r(\alpha) = \frac{FINE(\tilde{\mathbf{A}})(\alpha)_L + FINE(\tilde{\mathbf{A}})(\alpha)_R}{2},$$

$$v(\alpha) = FINE(\tilde{\mathbf{A}})(\alpha)_R - FINE(\tilde{\mathbf{A}})(\alpha)_L$$

and

$$c(\alpha) = b\alpha \in [0, b], \text{ with } b > 0.$$

So, we can indicate the buyer's utility function as $U(r(\alpha), v(\alpha), c(\alpha))$.

It is important to notice that $v(\alpha)$ and $c(\alpha)$ are, respectively, a strictly-decreasing function and a strictly-increasing function of the level α , whereas the monotony of $r(\alpha)$ depends of the shape of $FINE(\tilde{\mathbf{A}})$. In fact, it is easy to verify that $r(\alpha)$ is a strictly-decreasing/constant/strictly-increasing function of α according to the occurrence that $FINE(\tilde{\mathbf{A}})$ is a right-skewed/symmetric/left-skewed fuzzy interval number with respect to $r(1)$ (see figure 1).⁷

⁷A fuzzy interval number \tilde{x} is defined right-skewed/symmetric/left-skewed with respect to $r(1)$ if $r(\alpha) < / = / > r(1)$ for all $\alpha \in [0, 1)$.

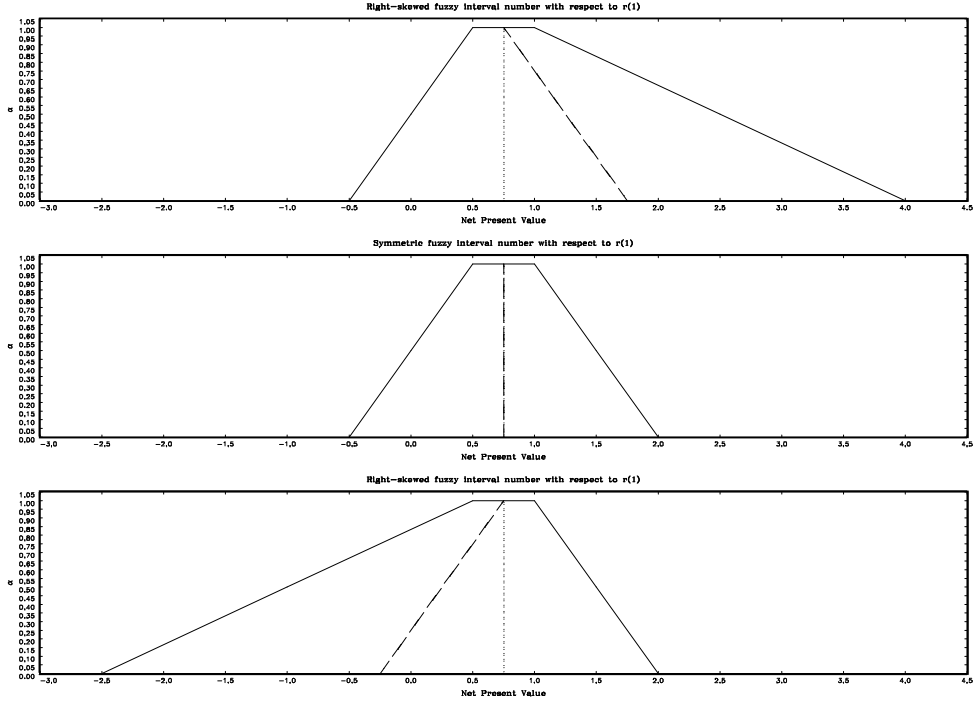


Figure 1: From up to down: a right-skewed/symmetric/left-skewed fuzzy interval number with respect to $r(1)$. In all the figures, the continuous line, the dashed line, and the dotted line represent, respectively, the shape of the considered fuzzy interval number, of its $r(\alpha)$, and of its $r(1)$.

At this point, in order to determine the optimal value of α which maximizes the buyer's utility we have to deal with the following nonlinear programming problem:

$$\begin{aligned} \max_{\alpha} \quad & U(r(\alpha), v(\alpha), c(\alpha)) \\ & \begin{cases} c(\alpha) < FINE(\tilde{\mathbf{A}})(\alpha)_L \\ 0 \leq \alpha \leq 1 \end{cases} \end{aligned} \quad (6)$$

It is to notice that, due to the first constraint, the non emptiness of the feasible region of (6) is not ensured.

5 Some open items

Given to its youth, our approach leaves open interesting items that we intend to investigate in future researches. Among the ones we consider the most significant, we list the following:

- specifying some analytical/numerical procedure by which to determine the optimal value of the level α which maximizes the buyer's utility;
- extending our approach in order to be able to apply it, besides to investment projects/contracts, to whatever financial operation;
- given two, or more, fuzzy financial operations/contracts, specifying some methodology by which to order them on the basis of suitable comparisons of their respective FINE.

References

- [1] I.E.G. Banholzer, A.L. Sanches, E.O. Pamplona and J.A.B. Montevechi, An interpretation for the internal rate of returns through fuzzy calculation, *Proceedings of the XII Congress of International Association for Fuzzy-Set Management and Economy* (2005), 33–51
- [2] J.J. Buckley, The fuzzy mathematics of finance, *Fuzzy Sets and Systems* **21** (1987), 257–273.
- [3] J.D. Cummins and R.A. Derrig, Fuzzy financial pricing of property-liability insurance [and Discussions and Authors' reply], *North American Actuarial Journal* **4** (1997) 21–44.
- [4] A.O. Esogbue and W.E. II Hearnese, On replacement models via a fuzzy set theoretic framework, *IEEE Transaction on Systems, Man, and Cybernetics – Part C: Applications and Reviews* **28** (1998) 549–560.
- [5] O. Houseman, A. Tiwari and R. Roy, A methodology for the selection of new technologies in the aviation industry, *Decision Engineering Report Series of the Cranfield University* (2004) i–iii and 1–36.
- [6] A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets. Volume I*, Academic Press, New York, NY, 1975.
- [7] C.-T. Tsao, Assessing the probabilistic fuzzy net present value for a capital investment choice using fuzzy arithmetic, *Journal of the Chinese Institute of Industrial Engineers* **22** (2005) 106–118.