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Abstract. This paper presents the role of copula functions in the theory of aggregation operators and an axiomatic characterization of Archimedean aggregation functions. In this context we are focusing our attention about several properties of aggregation functions, like supermodularity and Schur-concavity.

Keywords: Aggregation function, supermodularity, Schur- concavity, copula, Archimedean copulae.

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Introduction

Aggregation has for purpose the simultaneous use of different pieces of information provided by several sources, in order to come to a conclusion or a decision so aggregation functions transform a finite number of inputs, called arguments, into a single output. They are applied in many different domains and in particular aggregation functions play important role in different approaches to decision making, where values to be aggregated are typically preference or satisfaction degrees. Many functions of different type have been considered in connection with different situations and various properties of these functionals can be imposed by the nature of the considered aggregation problem. A class of aggregation functions can also be introduced axiomatically by means of a set of properties.
This note develops a new unified approach to copula-based modelling and characterizations of aggregation functions. The concept of copula introduced by Sklar in 1959 is now common in the statistical literature, but only recently its potential for applications has become clear. Copulae permit to represent joint distribution functions by splitting the marginal behavior, embedded in the marginal distributions, from the dependence captured by the copula itself. So the copula approach is particularly useful when we investigate the interaction between different arguments of aggregation functions. In fact the problem of modelling interaction between attributes remains a difficult question in the theory of aggregation functions.
A number of families of copulae exists in literature but here we focus on the study of the class of Archimedean copulae, that has proven useful for modelling dependence in a variety of settings and that forms a dense subclass of the class of associative copulae.
The paper is structured as follows. In the next section we establish the notations and we present some mathematical properties of the aggregation functions. In section 2 we define copulae and discuss some mathematical properties. Section 3 presents the copula approach to aggregation problem while in section 4 we propose an axiomatic characterization of Archimedean aggregation functions. Section 5 concludes with a discussion of perspectives for future developments.

1 Aggregation functions: some basic definitions and properties

The aggregation operators are mathematical objects that have the function of reducing a set of numbers into a unique representative number.
For example, the arithmetic mean as an aggregation function is defined by

$$AM(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

We introduce some properties which could be desirable for the aggregation of criteria.
If we consider the behavior of the aggregation in the best and in the worst case we expect

that an aggregation satisfies the following boundary conditions :

$$A(0,\ldots,0)=0 \qquad \text{and} \qquad A(1,\ldots,1)=1$$

These conditions mean that if we observe only completely bad (or satisfactory) criteria the total aggregation has to bee completely bad (or satisfactory). We consider aggregation functions that satisfy the boundary conditions.
Increasingness is another property, which is often required for aggregation and commonly accepted for functions used to aggregate preferences.
So, as it has been shown in [6], we can define an aggregation operator as a function

$$A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

that satisfies:

- $A(x)=x$
- $A(0,\ldots,0)=0$ and $A(1,\ldots,1)=1$
- $A(x_1,\ldots,x_n)\le A(y_1,\ldots,y_n)$ if $(x_1,\ldots,x_n)\le(y_1,\ldots,y_n)$

Associativity is also an interesting property for aggregation functions. The associativity property concerns the “ clustering ” character of an aggregation function. For two arguments the property can be written:

Definition 1.1 *A binary aggregation function A is associative if, for all $x_1, x_2, x_3, \in [0,1]$, we have $A(A(x_1,x_2),x_3)=A(x_1,A(x_2,x_3))$*

This property can be extended to n arguments as follows:

Definition 1.2 *An aggregation function A is associative if $A(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in[0,1]^n$ and*

$$A(x_1,\ldots,x_k,x_{k+1},\ldots,x_n)=A(A(x_1,\ldots,x_k),A(x_{k+1},\ldots,x_n))$$

for all integers $0\le k\le n$, with $n\ge 1$, and all $\mathbf{x}\in[0,1]^n$.

In many situations when attributes are equally important a simmetry property is required.

Definition 1.3 *An aggregation function A is symmetric if*

$$A(x_1,\ldots,x_k,\ldots,x_n)=A(x_{\sigma(1)},\ldots,x_{\sigma(k)}\ldots,x_{\sigma(n)}))$$

for all permutation σ of $\{1,\ldots,n\}$ and all $\mathbf{x}\in[0,1]^n$.

2 Supermodularity and Schur-concavity

In the next section we are considering also the class of continuous functions and we are focusing our attention to the supermodular property of aggregation operators. So we need the concept of supermodular functions. We endowed $\mathbb{R}^n$ with the usual product order. With this order $\mathbb{R}^n$ become a lattice and we denote the supremum and the infimum of $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ respectively. As it is well known supermodular functions play a central role in modelling concordance between random vectors (see [4]).

Definition 2.1 *A function $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ is supermodular if*

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}) \tag{1}$$

when $\mathbf{x}, \mathbf{y}, \mathbf{x} \vee \mathbf{y}, \mathbf{x} \wedge \mathbf{y} \in D$.

Supermodularity may also be defined in terms of *increasing differences*, as it has been shown in [7].

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has increasing differences if, for any $t \geq t'$, $g(x) = f(x,t) - f(x,t')$ is an increasing function of x. A function $f : \mathbb{R}^S \to \mathbb{R}$ has increasing differences if for any s, t and $\mathbf{x}$, the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$,

$$\hat{f}(\hat{x}_s, \hat{x}_t) = f(\mathbf{x}_{-s,t}, \hat{x}_s, \hat{x}_t),$$

obtained by allowing only x_s and x_t to vary from $\mathbf{x}$, has increasing differences.

Topkis(1998, Corollary 2.6.1) shows that a mapping $f : \mathbb{R}^S \to \mathbb{R}$ is supermodular if and only if it displays increasing differences and so the following proposition characterizes supermodular functions.

Proposition 2.1 *Let f be a function $D \subseteq \mathbb{R}^n \to \mathbb{R}$. f is supermodular if and only if when* $\mathbf{x}, \mathbf{y} \in D$

$$f(\mathbf{x} + \mathbf{h} + \mathbf{k}) - f(\mathbf{x} + \mathbf{k}) \geq f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \tag{2}$$

for all $\mathbf{h}, \mathbf{k}$ with $\mathbf{h}, \mathbf{k} \geq 0$, $\mathbf{h} \perp \mathbf{k}$ such that $\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{k}, \mathbf{x} + \mathbf{h} + \mathbf{k} \in D$.

Increasing difference transfers the supermodularity condition to one involving the linear structure of $\mathbb{R}^n$.

It is worth noting that supermodularity condition is only an "inter-attribute" relation. Intuitively increasing differences say that there must be "complementarity" between attributes. So a supermodular aggregation function has the interpretation of a "complementarity" condition of the inputs to be aggregated and a tendency of a collection of high scores to reinforce each other.

We also consider the concept of majorization arising as a measure of diversity of the components of an n-dimensional vector. Majorization gives a mean for comparing two vectors in a elegant way that arises surprisingly often in fields such as computer and economics science. This notion has been comprehensively treated by [5].

We aim to formalize the idea that the components of a vector $\mathbf{x}$ are less "spread out" or "more balanced" than the components of $\mathbf{y}$. For a vector $\mathbf{x} \in \mathbb{R}^n$ we denote its elements ranked in descending order as

$$x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}. \tag{3}$$

Thus $x_{(1)}$ is the largest of the x_i's, while $x_{(n)}$ is the smallest.

Definition 2.2 *The vector $\mathbf{y}$ is said to majorize the vector $\mathbf{x}$, which is denoted as $\mathbf{x} \preceq \mathbf{y}$, if*

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \quad for \quad k = 1, 2 \ldots, n-1 \quad and \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{4}$$

Majorization is a partial ordering among vectors, which applies only to vectors having the same sum. For example it can be easily shown that all vectors of sum s majorize the uniform vector $u = (\frac{s}{n}, \ldots \frac{s}{n})$. Intuitively, the uniform vector is the vector with minimal differences between elements, so all vectors majorize it. Formally, this follows from the fact that for any vector x of sum s,

$$\sum_{i=1}^{k} x_{(i)} \geq \frac{k}{n} s. \tag{5}$$

Now, we introduce a property for a real multivariate function, because key to the power of majorization is the companion notion of monotonicity associated with it:

Definition 2.3 *Let f be a function $D \subseteq \mathbb{R}^n \to \mathbb{R}$. f is Schur-convex if $f(\mathbf{x}) \leq f(\mathbf{y})$ when $\mathbf{x} \preceq \mathbf{y}$ and$\mathbf{x}, \mathbf{y} \in D$. f is Schur-concave if $-f$ is Schur-convex.*

Schur-convex functions thus preserve majorization. In other words, Schur-convexity (resp. Schur-concavity) corresponds to monotone increasingness (resp. decreasingness) for majorization (viewed as a pre-order on subsets of $\mathbb{R}^n$).

Let $\{\sigma_i, i = 1, \ldots, n!\}$ be a given enumeration of all $n!$ permutations of $\{i = 1, \ldots, n\}$. A subset A of $\mathbb{R}^n$ is said to be *symmetric* if for any $\mathbf{x}$ in A, the element $\sigma_i(\mathbf{x})$ also belongs to A for *each* $i = 1, \ldots, n!$. Moreover, for any subset A of $\mathbb{R}^n$, a mapping $\phi : A \to \mathbb{R}$ is said to be *symmetric* if A is symmetric and for any $\mathbf{x}$ in A, we have $\phi(\sigma_i(\mathbf{x})) = \phi(\mathbf{x})$ for *each* $i = 1, \ldots, n!$. If the mapping $\phi : A \to \mathbb{R}$ is Schur-convex (resp. Schur-concave) with symmetric A, then ϕ is necessarily symmetric since $\sigma_i(\mathbf{x}) \preceq \mathbf{x} \preceq \sigma_i(\mathbf{x})$ implies $\phi(\sigma_i(\mathbf{x})) = \phi(\mathbf{x})$ for each $i = 1, \ldots, n!$.
We note that a Schur-convex or Schur-concave function must be a symmetric function. Moreover a symmetric convex function is Schur-convex [5].
It is important to note that a Schur-concave aggregation function considers the attribute as symmetric and prefers attributes that are less "spread out".

3 Copula theory

Now we recall the copula definition and briefly introduce some properties that lie within the scope of this article. For more details we refer to [8] or [4].
For simplicity purposes we consider only the bivariate case.

Definition 3.1 *A 2-dimensional copula function (or briefly a copula) is a function C, whose domain is $[0,1]^2$ and whose range is $[0,1]$ with the following properties:*

i) $C(x,0) = C(0,x) = 0$ *and* $C(x,1) = C(1,x) = x$ *for all* $x \in [0,1]$

ii) C *is supermodular function*

Hence any bivariate distribution function whose margins are standard uniform distributions is a copula.
It is easy to see that function $\Pi(u,v) = uv$ satisfies conditions (i) and (ii) and hence is a copula. The copula Π, called the *product copula*, has an important statistical interpretation. The following Sklar's Theorem (Sklar 1959), which partially explains the importance of copulae in statistical modeling, justifies the role of copulae as dependence functions.

Theorem 3.1 (Sklar's theorem) . *Let H be a 2-dimensional distribution function with margins F and G. Then there exists a 2-copula C such that for all $\mathbf{x}$ in $\overline{\mathbf{R}}^2$,*

$$H(x_1, x_2) = C(F(x_1), G(x_2)). \tag{6}$$

If F and G are continuous, then C is unique. Conversely, if C is a 2-copula and F and G are distribution functions, then the function H defined by (6) is a 2-dimensional distribution function with margins F and G.

As a consequence of Sklar's Theorem, if X and Y are random variables with a joint distribution function H and margins F and G, respectively, then for all x, y in $\overline{\mathbf{R}}$,

$$max(F(x) + G(y) - 1, 0) \leq H(x,y) \leq min(F(x), G(y)) \tag{7}$$

or (since $H(x,y) = C(F(x), G(y))$)

$$W(u,v) = max(u + v - 1, 0) \leq C(u,v) \leq min(u,v) = M(u,v). \tag{8}$$

Since M and W are copulae, the above bounds are joint distribution functions, and are called the *Fréchet-Hoeffding bounds* for joint distribution functions H with margins F and G.
The copulae M, W and Π have important statistical interpretations. Let X and Y be continuous random variables, then:

(i) the copula of X and Y is $M(u,v)$ if and only if each of X and Y is almost surely an increasing function of the other;

(ii) the copula of X and Y is $W(u,v)$ if and only if each of X and Y is almost surely a decreasing function of the other;

(iii) the copula of X and Y is $\Pi(u,v) = uv$ if and only if X and Y are independent.

The concept of copula can be easily extended to n dimensions with $n \geq 2$.
An n-dimensional copula is an n-dimensional cumulative distribution function (CFD), denoted by $C_n(u_1, \ldots, u_n)$, whose support is the n-dimensional hypercube $[0,1]^n$ and whose univariate marginal distributions are uniform on $[0,1]$.
Implicit in this definition are the properties that $C_n(u_1, \ldots, u_n) = 0$ if $u_j = 0$ for any $j \leq n$ and that the marginal distributions are given by $C_n(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$ for each $j \leq n$ and all $u_j \in [0,1]$. C_n is also n-increasing, the standard multivariate extension of the concept of "increasing" for a univariate function.

In this work, we focus on the class of *Archimedean copulae* because their properties fit the needs of the aggregation problem.
The family of Archimedean copulae is particularly interesting because they can be defined by means of a single function. Let ϕ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(1) = 0$. The *pseudo-inverse* of ϕ is the function $\phi^{[-1]}$ with Dom $\phi^{[-1]} = [0,\infty]$ and Ran $\phi^{[-1]} = [0,1]$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) \leq t \leq \infty. \end{cases}$$

Note that $\phi^{[-1]}$ is continuous and non-increasing on $[0,\infty]$, and strictly decreasing on $[0,\phi(0)]$. The function C defined by

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)) \quad u,v \in [0,1]^2$$

is a copula if and only if ϕ is convex.
Copulae of the form described above are called *Archimedean* copulae and the function ϕ is called a *generator* of the copula.
Because of their simple forms, the ease with which they can be constructed and their many nice properties, Archimedean copulae frequently appear in discussions of multivariate distributions.

4 The copula approach

The subject of assessing probabilistic dependence between one-dimensional distribution functions to construct a joint distribution function is an important task in probability theory and statistics. The copula function captures the dependence relationships among the individual random variables as any multivariate distribution can be represented in terms of its marginals through a given copula structure.

The aim of this section is to present the copula approach for studying aggregation problems. We consider a class of aggregation functions that can be expressed in terms of marginal functions using the method of copulae. The following proposition introduces an analogy between probability distributions and this class of aggregation functions.
We define the following marginal functions of a binary aggregation function A:

$$A_1(x) = A(x, 0) \quad A_2(y) = A(0, y) \quad \text{and} \quad A^1(x) = A(x, 1) \quad A^2(y) = A(1, y)$$

Proposition 4.1 *If A is a supermodular continuous binary aggregation function then there exists a copula C, such that*

$$A(x, y) = A_1(x) + A_2(y) + C(A^1(x), A^2(y))$$

Proof.

If A is a supermodular continuous and increasing function, then, as it has been shown in [9], there exist functions f_1, f_2, g such that

$$A(x, y) = f_1(x) + f_2(y) + \int_0^y \int_0^x g(s, t) ds dt,$$

where the function $H(x, y) = \int_0^y \int_0^x g(s, t) ds dt$ is a supermodular function.
Since

$$A_1(x) = f_1(x) + f_2(0) \quad \text{and} \quad A_2(y) = f_1(0) + f_2(y)$$

we obtain that $f_1(x) + f_2(y) = A_1(x) + A_2(y) - (f_1(0) + f_2(0)) = A_1(x) + A_2(y)$. By lemma 2.1.5 of [8]

$$|A(x, y) - A(x', y')| \leq |A^1(x) - A^1(x')| + |A^2(y) - A^2(y')|$$

for all $x, x', y, y' \in [0, 1]$. Then, $H(x, y) = H(A^1(x), A^2(y))$ and by lemmas 2.3.4 and 2.3.5 of [9] there exists a copula C such that $H(x, y) = C(A^1(x), A^2(y))$.

If $A_1(x) = A_2(y) = 0$, then $A(x, y) = C(A^1(x), A^2(y))$ and so $A(x, y)$ is a conjunctive aggregation function.
In fact, we remember that aggregation functions can be roughly divided into three classes, each possessing very distinct behavior: *conjunctive* functions, *disjunctive* and *compensative* functions.

Definition 4.1 $A : E^n \to \mathbb{R}$ *is*

conjunctive *if* $A(\mathbf{x}) \leq min\ x_i$ *for all* $\mathbf{x} \in E^n$*,*

disjunctive *if* $max\ x_i \leq A(\mathbf{x})$ *for all* $\mathbf{x} \in E^n$*,*

compensative *if* $min\ x_i \leq A(\mathbf{x}) \leq max\ x_i$ *for all* $\mathbf{x} \in E^n$*, that is averaging operations.*

In our case, from Sklar's Theorem it follows that copulae generalize the notion of conjunctiveness. Moreover, basic conjunctions are the minimum operation, the product and the linear operation $max(a + b - 1)$.

5 Archimedean aggregation functions

We introduce the class of Archimedean aggregation functions.

Definition 5.1 *If ϕ is a continuous, strictly decreasing and convex function from $[0,1]$ to $[0,\infty]$ such that $\phi(1)=0$, the aggregation fuction is defined by*

$$A(x_1,x_2)=\phi^{[-1]}(\phi(x_1)+\phi(x_2)) \quad x_1,x_2\in[0,1]^2$$

We present in this section several propositions concerning some properties of Archimedean aggregation functions.

Proposition 5.1 *A bivariate Archimedean aggregation function A is symmetric, associative, Schur-convex and satisfies the following conditions:*

i) $A(x,0)=A(0,x)=0$ *and* $A(x,1)=A(1,x)=x$ *for all* $x\in[0,1]$

ii) $A(x,x)<x$

Proposition 5.2 *A bivariate associative aggregation function which satisfies conditions i) and ii) of Proposition 5.1 is Archimedean.*
Proof.

To be Archimedean, $A(x_1,x_2)$ must be symmetric and Schur-convex. Really, we must see that it is Schur-convex, because a Schur-convex function must be a symmetric function. So, we wonder whether $(x_1,x_2)\preceq(y_1,y_2)$ implies $A(x_1,x_2)\le A(y_1,y_2)$. This is true, because non decreasing is a property of aggregation functions.
Another main result can be formulated as follows:

Proposition 5.3 *The set $A_{\mathbf{a}}$ of all associative aggregation functions is the closure of both the set $A_{\mathbf{s}}$ of all strict aggregation functions and the set $A_{\mathbf{ns}}$ of all non-strict Archimedean aggregation functions.*

This means in particular that each associative aggregation function can be approximated with arbitrary precision by some strict as well as by some non-strict Archimedean aggregation function. Notice that $A_{\mathbf{s}}$ and $A_{\mathbf{ns}}$ are disjoint sets whose union, i.e., the set of Archimedean aggregation functions, is a proper subset of $A_{\mathbf{a}}$.

6 Some examples

Now we show that associative binary aggregation functions like $A(x,y)=f^{-1}(f(x)f(y))$ and nilpotent ones like $A(x,y)=f^{-1}((f(x)+f(y)-1)\vee 0)$ on the unit interval $[0,1]$ are Archimedean. In fact, if the following conditions hold:
(1) $A(1,x)=x$, (2) $A(0,x)=0$ and (3) $A(x,x)<x$ for all $x\in(0,1)$,

then the proposition 5.2 holds.
For example, we consider $A(x,y) = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$, or $A(x,y) = ((x^a + y^a - 1) \vee 0)^{\frac{1}{a}}$. Similarly, if we define $A(x,y) = f^{-1}((f(0)^2/f(x)f(y))$, where the aggregation function on $[0,1]$ is Archimedean if the conditions, like in the previous case, hold.

7 Concluding remarks

An interesting generalization of copulae is the notion of *semicopula*, namely a binary operation on $[0,1]$ that satisfies the boundary condition $\forall x \in [0,1]$ $C(x,1) = C(1,x) = x$ and the property of increasingness in each place, that is $C(x,y) \leq C(x',y')$ for all $x \leq x'$ and $y \leq y'$. But, as it has been shown in [3], the first generalization of copulae has been the concept of *quasi-copula*. In detail, a quasi-copula $Q : [0,1]^2 \to [0,1]$ satisfies the conditions of semicopula and it is also 1-*Lipschitz*: $|C(x,y) - C(x',y')| \leq |x - x'| + |y - y'|$ for all $x, x', y, y' \in [0,1]$.
The study of quasi-copula as an aggregation operator is an open problem.
The concept of copula can be extended to n dimensions. Moreover, for any n-copula:

$$T_{\mathbf{L}}(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq T_{\mathbf{M}}(x_1, \ldots, x_n).$$

The upper function $T_{\mathbf{M}}$ is an n-copula for any $n \in \mathbb{N}$, the lower function $T_{\mathbf{L}}$ is not an n-copula for any $n > 2$. It could be an aggregation function.
In fact, the copula approach to aggregation functions can be generalized to n dimensions. It is interesting to note that the property of associativity serves two purposes. First it allows us to semplify the aggregation in the following way:

$$\text{if} \quad A(x_1, \ldots, x_n) = a, \quad \text{then} \quad A(x_1, \ldots, x_n, x_{n+1}) = A(a, x_{n+1}).$$

Second, it helps us to simplify the definition of a function by permitting the use of binary definition of this operator to define the function for multiarguments.
However, this generalization remains an open problem in such context, above all for Archimedean aggregation functions.
Moreover, there is a close link between supermodularity and Schur-concavity, but this is another open problem.

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