Department of Applied Mathematics, University of Venice

WORKING PAPER SERIES

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Cumulative prospect theory and second order stochastic dominance criteria: an application to mutual funds performance

Working Paper n. 157/2007
October 2007

ISSN: 1828-6887

Cumulative prospect theory and second order stochastic dominance criteria: an application to mutual funds performance

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Abstract. In this note using the rules of stochastic dominance of the second order and the recent cumulative prospect theory for classified, according to their performance, a set of common funds. The criteria used are closely linked to the preferences of decision maker and refer to either hypothesis of aversion and of seeking to risk both hypothesis on the sign of derived second of the function which characterizes the losses and gains.

Keywords: Stochastic dominance rules, preferences, cumulative prospect theory, mutual funds performance, total and partial ranking.

JEL Classification Numbers: G11, C44, C63.

MathSci Classification Numbers: 60E15.

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1 Introduction

The maximum return criterion is employed when there is no risk at all. According to this rule, the investor simply chooses the alternative with the highest rate of return. In order to compare the risk and return of alternative investments decision criteria are needed.

The expected utility paradigm analyze risk and return jointly and has been the foundation for most of modern theories. Since the expected utility rules do not accurately predict empirically observed performances one can try to extend the stochastic dominance results. Knowing certain qualitative features of the utility function of the decision maker, one can use the corresponding stochastic dominance conditions to eliminate dominate alternative (mutual funds), which can be useful for management practitioners.

This study is in such a direction and uses the Cumulative Prospect Theory and two recently developed investment criteria proposed in [6] and called Prospect Stochastic Dominance (PSD) and Markowitz Stochastic Dominance (MSD), respectively.

The original and cumulative prospect theories (see [4] and [9] respectively for a presentation of these methodologies) suggest that decision maker encodes outcomes in term of gains and losses. Furthermore, the prospect theory replaces the traditional utility function by a value function defined over changes of wealth with respect to some reference point and includes a probability weight function that reflects the subjective probability distortions shown by most investors.

There is significative evidence to support the cumulative prospect theory (CPT) although clearly no one would claim that it is a definitive theory. Nevertheless, it does predict many of key features of risky decision-making behavior. Recently, prospect theory gained much popularity and there is stream of papers that build economic and financial model based on this theory. There have been many empirical and experimental attempts to test prospect theory, most of which support the theory (for a comprehensive survey see [2]).

The prospect stochastic dominance criterion (PSD) is a rule, based on CPT, that determines the dominance of one investment alternative over another for all prospect theory S-shaped value functions. The Markowitz stochastic dominance rule (MSD) is a rule that determines the dominance of one investment alternative over another for all reverse S-shaped value functions, as suggested by Markowitz in [7]. Both the PSD and MSD belong to second order stochastic dominance criteria. As well known a stochastic dominance rule gives a partial ordering of a set of alternative investments. On the other hand, the cumulative prospect theory allows to obtain a complete ranking of risk alternatives.

The structure of this note is as follows. In section 2 we briefly describe the main properties of original and cumulative prospect theory. In section 3 we discuss the main classes of preferences proposed in literature. Section 4 presents the second order stochastic dominance criteria used in empirical analysis. In section 5 we briefly analyze some intuitive explanations, graphical representation, necessary and sufficient conditions of stochastic dominance criteria. Section 6 presents both some tricks that can help to reduce the number of comparison and the algorithms that permit to easily implement the stochastic dominance rules. Section 7 present the empirical analysis carried out on the Italian mutual funds market. Finally, some concluding remarks are presented in Section 8.

2 Review of Cumulative Prospect Theory (CPT)

Expected utility theory has been the foundation for most of modern theories on uncertainty and risk. However, it has been challenged by more and more empirical results pointing at violation of it. For example, it has been found that people tend to think of possible outcome usually relative to a certain reference point (often the status quo) rather than the final status. Moreover, they have different risk attitudes towards gains (i.e. outcomes above the referencee point) and losses (i.e. outcomes below the reference point) and they care generally more about potential losses than potential gains (loss aversion). These observations call for alternative theories that are psychologically more appealing. The original and cumulative prospect theory standout as one of the most well-accepted alternative to the expected utility paradigm. The main characteristics of prospect theory can be summarized as follows:

i) Investors make decision based on change of wealth rather on total wealth, in contrast to what advocated by expected utility theory.

ii) Investors maximize the expectation of a value function, $v(x)$, where x is the change in wealth rather than total wealth. $v(x)$ is S-shaped: $v'(x) > 0 \; \forall \, x \neq 0$, $v''(x) > 0$ for $x < 0$, and $v'' < 0$ for $x > 0$. The parameters of the function may change with the wealth, but the S-shaped property of function $v(x)$ is a common run for all initial wealth levels.

iii) Investors subjectively distort probabilities. They make decisions based on the subjective distribution function F^*, which is given by $F^* = T(F)$, where F is the objective distribution and T is a monotonically increasing transformation with $T(0) = 0$ and $T(1) = 1$ (this is the main modification of cumulative prospect theory in comparison to original prospect theory).

The main formal features of cumulative prospect theory introduced by Tversky and Kahneman in [9] are now summarized. This theory assumes a shape of value function which reflects the psychological phenomenon of diminishing sensitivity as one moves away from the “references point”. This point divides gains from losses and is taken to be zero in this contribution.

Consider a prospect (mutual fund) X with outcomes

$$x_1 \leq \ldots x_k \leq 0 \leq x_{k+1}, \ldots \leq x_n$$

having probability

$$p_1, \ldots, p_k, p_{k+1}, \ldots, p_n.$$

Cumulative prospect theory predicts that people will choose prospects according to the value given by

$$V(X) = \sum_{i=1}^{k} \pi_i \lambda v(x_i) + \sum_{j=k+1}^{n} \pi_j v(x_j) \tag{1}$$

where $\lambda > 0$ is a loss-aversion parameter and the π's are decision weights that are calculated based on the "cumulative" probabilities associated with the outcomes.

In particular, prospect theory assumes a probability weighting function $w^+ : [0,1] \to [0,1]$ for gains, and a probability weighting function $w^- : [0,1] \to [0,1]$ for losses. The decision weights π's in equation (1) are defined as follows.

If $k \geq 1$ then

$$\pi_1 = w^-(p_1)$$

and

$$\pi_i = w^-(p_1 + \ldots + p_i) - w^-(p_1 + \ldots + p_{i-1}) \quad \text{for } 2 \leq i \leq k$$

If $k < n$ then

$$\pi_n = w^+(p_n)$$

and

$$\pi_j = w^+(p_n + \ldots + p_j) - w^+(p_n + \ldots + p_{j+1}) \quad \text{for } n-1 \geq j > k.$$

Tversky and Kahneman in [9] estimated the following parameter form

$$v(x) = \begin{cases} x^{0.88} & \text{for } x > 0 \\ -(-x)^{0.88} & \text{for } x \leq 0, \end{cases} \tag{2}$$

$\lambda = 2.25$ and

$$w^+(p) = \frac{p^{0.61}}{(p^{0.61} + (1-p)^{0.61})^{1/0.61}} \qquad w^-(p) = \frac{p^{0.69}}{(p^{0.69} + (1-p)^{0.69})^{1/0.69}}. \tag{3}$$

Obviously, the decision maker can adopt other trends both for value function and for risky weighting function: a variety of specific functional forms have been suggested and are reported in Tables 1 and 2.

With reference to Table 1 note that each function is increasing for $x \geq 0$; moreover some parameters require constraints. Inevitably there is not room to include every value

Table 1: Summary of functional form for value function.

Name	Abbreviation	Equation
Linear	Lin	$v(x) = x$
Logarithmic	Log	$v(x) = \ln(a + x)$
Power	Pwr	$v(x) = x^a$
Quadratic	Quad	$v(x) = ax - x^2$
Exponential	Expo	$v(x) = 1 - e^{-ax}$
Bell	Bell	$v(x) = bx - e^{-ax}$
Hara	hara	$v(x) = -(b + x)^a$

function* and every risky weighting function†, so that the more notable forms are selected and reported in the Tables 1 and 2.

With reference to Table 2 note that most of the labels relate to the authors that seem to have first reported them. The qualitative features of most of these functions are:

- concavity of $w(p)$ for small p, close to 0, and
- convexity of $w(p)$ for large p, close to 1.

Table 2: Summary of functional form for risky weighting function.

Name	Abbreviation	Equation
Linear	Lin	$w(p) = p$
Power	Pwr	$w(p) = p^r$
Goldstein-Einhorn	GE	$w(p) = \frac{sp^r}{ps^r(1-p)^r}$
Tversky-Kahneman	TK	$w(p) = \frac{p^r}{(p^r+(1-p)^r)^{1/r}}$
Wu-Gonzales	WG	$w(p) = \frac{p^r}{(p^r+(1-p)^r)^s}$
Prelec I	PrlI	$w(p) = e^{-(-\ln p)^r}$
Prelec II	PrlII	$w(p) = e^{-s(-\ln p)^r}$

3 Characterization of preferences

There is an ongoing debate in literature regarding the shape of the utility (or value functions). The investors always prefer more money than less money, i.e the utility function

*Excluded value functions include: $v(x) = e^{kx^\beta} - 1$, the Log-Quadratic $v(x) = \ln(x+1) + a[\ln(x+1)]^2$, the Sumex $v(x) = a\,e^{bx} + c\,e^{dx}$, and the Linear times Exponential $v(x) = (ax + b)e^{cx}$.

†Excluded risky weighting functions include: the Exponential-Power $w(p) = exp[-\frac{r}{s}(1 - p^s)]$, the Hyperbolic-logarithm $w(p) = (1 - r\ln p)^{-r/s}$ and a linear form with discontinuous end points.

u is non-decreasing with $u' \geq 0$. The first degree stochastic dominance rule (FSD) is appropriate for all investor with $u' \geq 0$ (with strict inequality at some range).

The FSD rule is a criterion that tell us whether one investment dominates another investment where the only available information is that $u \in \mathcal{U}_1$ ($\mathcal{U}_1$ is the class of all monotonic non decreasing utility functions). The non-decreasing assumption give partial information on u and its precise shape is unknown.

There is much evidence that most investors are risk averters: their utility function has non-negative first derivative and non positive second derivative ($u' \geq 0$, $u'' \leq 0$). We can define $\mathcal{U}_2$ as the class of nondecreasing concave utility functions. Obviously, $\mathcal{U}_2 \subset \mathcal{U}_1$. A risk averter will not play a fair game; moreover, risk averters will be ready to pay a risk premium to insure their wealth.

The investor with utility function in $\overline{\mathcal{U}_2} = \{u : u' \geq 0, u'' \geq 0\}$ (and to avoid trivial cases there is at least one utility function with strict inequality) is called risk seeking. $\overline{\mathcal{U}_2}$ is the class of nondecreasing convex utility function.

In the economic and finance literature it is very uncommon to claim that risk-seeking prevail in the whole domain of outcomes. Friedman and Savage (see [3]) claim that the fact that investors buy insurance, buy lottery tickets, and buy both insurance and lottery tickets simultaneously imply that the investor utility function must have two concave regions with a convex region in between.

Markowitz (see [7]) points out several severe problems with the Friedman and Savage utility function. However, he shows that the problems are solved if the first inflection point of Friedman and Savage utility function is exactly in correspondence to investor's current wealth. Thus, Markowitz introduces the idea that decisions are based on change of wealth. Hence, the Markowitz utility function can be also thought of as a value function. By analyzing some hypothetical games, Markowitz suggests that individuals are risk averse for losses and risk seeking for gains, as long as the possible outcomes are not very extreme. For extreme outcomes, Markowitz argues that individuals become risk averse for gains and risk seeking for losses. Thus, Markowitz suggests a utility/value function which is characterized by three inflexion points. Note that the central part of this function has a reverse S-shape. In our study we confine the analysis to this region, i.e. to the reverse S-shape range.

The most investigated class of value functions is the prospect theory S-shape function. Based on several experimental results with bets which are either negative or positive outcomes, Kahneman and Tversky claim that the value function is concave for gains and convex for losses, yielding an S-shape function.

Note that previous preferences concern only the sign of the first and second derivative of the utility (value) function. Obviously we can build a stochastic indicator of the decision maker preferences which is progressively weaker. For example we can require that the utility function belong to the set

$$\mathcal{U}_3 = \{u : u' \geq 0, u'' \leq 0,\ u''' \geq 0\} \tag{4}$$

and more generally to belong to

$$\mathcal{U}_n = \{u \in \mathcal{U}_{n-1} : (-1)^n u^{(n)} \leq 0\}, \tag{5}$$

but economic meaning of highest derivatives is not evident.

A widely accepted hypothesis on preferences is the decreasing absolute risk aversion (DARA) hypothesis (see [1]). If we define $\rho = -u''/u'$ (whenever this quantity exists), the DARA hypothesis implied $\rho' \leq 0$ and we have

$$\mathcal{U}_{DARA} = \{u \in \mathcal{U}_2 : u' \neq 0,\ \rho' \leq 0\}. \tag{6}$$

It is easy to verify that $\mathcal{U}_{DARA} \subset \mathcal{U}_3$.

Another property of decision maker preferences is prudence which entails a positive third derivative and characterizes the so called precautionary saving (see [5]). For measuring the degree of prudence we can introduce the absolute prudence index $\phi = -u'''/u''$ and show that decreasing absolute prudence is a necessary and sufficient condition that guarantees that the saving of wealthier people is less sensitive to the risk associated to feature incomes. Observe that the decreasing absolute prudence (DAP) hypothesis entails that the fourth derivatives of the utility functions are negative, a feature called temperance. Finally, note that DAP can be used in conjunction with DARA hypothesis giving the standard risk aversion set

$$\mathcal{U}_{SRA} = \{u \in \mathcal{U}_3 : u' \neq 0, u'' \neq 0,\ \rho' \leq 0, \phi' \leq 0\}. \tag{7}$$

4 Second order stochastic dominance criteria

In this section we present traditional and recent stochastic dominance rules; for a formal proof you can see [6].

Let A and B be two distinct prospects with cumulative distributions F and G, respectively.

FSD (First-Degree Stochastic Dominance) The alternative A dominates the alternative B with respect to first-degree stochastic dominance rule ($A >_{FSD} B$) if and only if

$$F(x) \leq G(x) \quad \forall\, x \in \mathbb{R} \Leftrightarrow \mathbb{E}_F u(x) \geq \mathbb{E}_G u(x) \quad \forall\, u \in \mathcal{U}_1, \tag{8}$$

where there is a strict inequality for some $x = x_0$ (first inequality), and a strict inequality for some $u_0 \in \mathcal{U}_1$ (second inequality).

SSD (Second-Degree Stochastic Dominance) The alternative A dominates the alternative B with respect to second-degree stochastic dominance rule ($A >_{SSD} B$) if and only if

$$\int_{-\infty}^{x} [G(t) - F(t)] \geq 0 \qquad \forall\, x \in \mathbb{R} \Leftrightarrow \mathbb{E}_F u(x) \geq \mathbb{E}_G u(x) \quad \forall\, u \in \mathcal{U}_2, \tag{9}$$

where there is a strict inequality for some $x = x_0$, and a strict inequality for some $u_0 \in \mathcal{U}_2$.

RSSSD (Risk Seeking Second-Degree Stochastic Dominance) The alternative A dominates the alternative B with respect to risk seeking second stochastic dominance rule ($A >_{RSSSD} B$) if and only if

$$\int_{x}^{\infty} [G(t) - F(t)] \geq 0 \qquad \forall\, x \in \mathbb{R} \Leftrightarrow \mathbb{E}_F u(x) \geq \mathbb{E}_G u(x) \quad \forall\, u \in \overline{\mathcal{U}}_2, \tag{10}$$

where there is a strict inequality for some $x = x_0$, and a strict inequality for some $u_0 \in \overline{\mathcal{U}}_2$ ($\overline{\mathcal{U}}_2$ is the class of all nondecreasing convex utility function, i.e. $u \in \overline{\mathcal{U}}_2$ if $u' \geq 0$ and $u'' \geq 0$).

Denote by $\mathcal{V}_{KT}$ the class of all prospect theory value functions (where the subscript KT denotes Kahneman and Tversky): functions which are S-shaped with an inflexion point at $x = 0$. Thus, $v \in \mathcal{V}_{KT}$ if $v' \geq 0\ \forall x \neq 0$, $v'' \geq 0$ for $x < 0$, and $v'' \leq 0$ for $x > 0$. We call v a value function, rather than a utility function, to be consistent with the terminology of Kahneman and Tversky.

PSD (Prospect Stochastic Dominance) Let A and B be two distinct prospects with cumulative distributions F and G, respectively. Then A dominates B ($A >_{PSD} B$) for all S-shape utility/value functions, $v \in \mathcal{V}_{KT}$, if and only if

$$\int_{y}^{0} [G(t) - F(t)] \geq 0 \qquad \forall\, y \leq 0 \qquad , \qquad \int_{0}^{x} [G(t) - F(t)] \geq 0 \qquad \forall\, x \geq 0 \tag{11}$$

(Once again, we require a strict inequality for some pair (y_0, x_0) and for some $v_0 \in \mathcal{V}_{KT}$).

Denote by $\mathcal{V}_M$ the class of all Markowitz utility functions: functions which are reverse S-shaped with an inflexion point at $x = 0$. Thus, $v \in \mathcal{V}_{KT}$ if $v' \geq 0\ \forall x \neq 0$, $v'' \leq 0$ for $x < 0$, and $v'' \geq 0$ for $x > 0$. As Markowitz's function, like the prospect theory value function, depends on change of wealth, we denote it by $\mathcal{V}_M$ rather than $\mathcal{U}_M$.

MSD (Markowitz Stochastic Dominance) Let A and B be two distinct prospects with cumulative distributions F and G, respectively. Then A dominates B ($A >_{MSD} B$) for all reverse S-shape utility/value functions, $v \in \mathcal{V}_M$, if and only if

$$\int_{-\infty}^{y} [G(t) - F(t)] \geq 0 \qquad \forall\, y \leq 0 \qquad , \qquad \int_{x}^{\infty} [G(t) - F(t)] \geq 0 \qquad \forall\, x \geq 0 \tag{12}$$

(Once again, we require a strict inequality for some pair (y_0, x_0) and for some $v_0 \in \mathcal{V}_M$).

5 Intuitive explanation, graphical representation, necessary and sufficient conditions of stochastic dominance criteria

Let A and B be two distinct prospects with cumulative distributions F and G, respectively. The intuition of FSD is straightforward: FSD dominance requires that the two distributions being compared not cross but they can tangent each other. If A dominates B by FSD rule then F must be below G (in weak sense) for the whole range of x. If F and G intercept then there is an utility function $u^* \in \mathcal{U}_1$ such that $\mathbb{E}_F[u^*(x)] > \mathbb{E}_G[u^*(x)]$ and there is another utility function $u^{**} \in \mathcal{U}_1$ such that $\mathbb{E}_F[u^{**}(x)] < \mathbb{E}_G[u^{**}(x)]$.

A sufficient condition for FSD is

***Sufficient rule I*:** $F >_{FSD} G$ if

$$\min_x[F(x)] \leq \max_x[G(x)]. \tag{13}$$

There are many more necessary rules for FSD but the following three are the most important ones.

***Necessary rule I*:** $\mathbb{E}_F(x) > \mathbb{E}_G(x)$ is a necessary condition for FSD.

***Necessary rule II*:** If $A >_{FSD} B$ then the geometric mean[‡] of A must be larger than the geometric mean of B.

Necessary rule III (Left tail condition): If $A >_{FSD} B$ then

$$\min_x[F(x)] \geq \min_x[G(x)]. \tag{14}$$

The condition (14) means that the distribution G of B starts to accumulate area before distribution F. This is called the "left tail" problem because a thicker left tail is a sufficient condition for no dominance.

The intuitions of SSD is relatively straightforward too. The condition for SSD dominance of A over B is that $\int_{-\infty}^{x}[G(t) - F(t)dt \geq 0]$, $\forall\, x$. Graphically, this mean that the area enclosed between G and F from $-\infty$ to any value x is positive. Thus, the SSD rule states that for any negative area ($F > G$) there is larger preceding positive area ($G > F$).

As in the case of FSD there exist many necessary and sufficient rules for risk aversion (SSD).

Sufficient rule I: The FSD rule is a sufficient condition for SSD.

***Sufficient rule II*:** $\min_x[F(x)] < \max_x[G(x)]$ is a sufficient rule for SSD.

Necessary rule I: $\mathbb{E}_{F(x)} \geq \mathbb{E}_{G(x)}$ is a necessary condition for dominance of A over B in $\mathcal{U}_2$.

[‡]It is well known that the geometric mean is defined only for positive numbers; note that the rates of return r_k can be negative or positives number, but $x_k = 1 + r_k$, the terminal wealth of 1 euro invested is positive.

Necessary rule II: if $A >_{FSD} B$ then the geometric mean of A must be larger than the geometric mean of B.

Necessary rule III (Left tail condition): The left tail of G must be thicker than the left tail of F, i.e. $\min_x[G(x)] > \min_x[F(x)]$.

The intuition of RSSSD rule is important in particular because both PSD and MSD preference contain risk seeking regions. As in SSD, also with RSSSD we calculate the area enclose between the two cumulative distributions. However, this time the area accumulation is done from the upper bound ($+\infty$) to x. One is tempted to believe that if F dominates G by RSSSD rule then G dominates F by SSD. This is not true as simple examples show. By RSSSD criterion for reach negative area enclosed between F an G there must be a larger positive area located to the right of it.

Sufficient rule I : The FSD rule is a sufficient condition for RSSSD.

Necessary rule I: $\mathbb{E}_{F(x)} \geq \mathbb{E}_{G(x)}$ is a necessary condition for dominance of A over B in $\overline{\mathcal{U}}_2$.

Necessary rule II (Right tail condition): The right tail of F must be thicker than the right tail of G, i.e. $\max_x[F(x)] > \max_x[G(x)]$.

Graphically, the previous necessary condition means that the last area (largest x) enclosed between G and F is positive.

The intuition of PSD relates directly to the preceding explanation about SSD and RSSSD. On the other hand, the MSD rule combines the two conditions $\int_x^{\infty}[G(t)-F(t)]dt \geq 0,\ \forall x > 0$ and $\int_{-\infty}^{y}[G(t) - F(t)]dt \leq 0, \forall y < 0$, therefore the MSD rule requires both the left tail condition and the right tail condition. Note that MSD is generally not " the opposite" of PSD. In other words, if F dominates G by PSD, this not necessary mean that G dominates F by MSD. This is easy to see, because having a higher mean is a necessary condition for dominance for both criteria. Therefore, if F dominates G by PSD, and F has a higher mean than G, then G cannot dominate F by MSD. However, if the two distribution have the same mean, then PSD and MSD are opposite (see for example [6]).

6 Algorithms for empirical distributions

In this section we present both some tricks that can reduce enormously the number of comparisons and some SD algorithms in the framework of uniform discrete distributions.

6.1 The reduction of comparisons

To obtain the efficient sets (non dominated portfolios) corresponding to the each stochastic dominance criterion, we need to know the precise shape of various distributions under comparison. In practice, stochastic dominance criteria are commonly applied to empirical distributions (e.g. ex-post rates of return of mutual funds, or other available portfolios).

In such cases, if we have n observations, say n monthly rates of return, then each observation is usually assigned an equal probability of $1/n$. Thus, for each fund we have an empirical distribution which can be used from decision maker also as an estimate of future distributions.

Given an admissible set containing T alternative mutual funds (risky projects), each stochastic dominance rule requires $P_2^T = T!/(T-2)!$ pairwise comparisons in order to determinate an efficient subset. We emphasize permutation rather than combinations, because for each pair of distributions F and G we have to examine whether F dominates G and if such dominance not exist we also have to check whether G dominates F. Moreover, for each pair a large number of calculations have to be performed. Since for practical purposes these comparisons often involve a considerable amount of resources in terms of computing time, a number of tricks have been proposed in literature in order to reduce the comparisons.

A necessary condition for all stochastic dominance criteria is that the superior investment must have a higher or equal mean. Therefore, the number of necessary comparison can be reduced to $C_2^T = T(T-1)/2$ by computing the mean rate of return and ordering the funds in decreasing order. By the same token, we can use the other necessary conditions (for example the left or right tails which are easy to check), to further reduce the number of comparisons.

Furthermore, knowing that First-Degree Stochastic Dominance implies all second order stochastic dominance, it is suggested that FSD be conduced first.

Finally, remember that all the stochastic dominance criteria are transitive rules and this property can reduces, once again, the number of necessary comparisons. Indeed, suppose for example that we have 100 investments. Then, after ranking these investments by their means, we have a maximum of $100 \cdot 99/2 = 4\,950$ comparisons to perform. Suppose that the investment with maximum mean I_{100} dominates, say, 30 out of the available 100 investments. Then these 30 investments will be relegated into the inefficient set. Suppose that I_j is relegated into the inefficient set. Should we keep I_j in the meantime in order to check whether I_j dominates I_k (where $k \neq j \neq 100$)? There is no need because, if I_j dominates I_k, due to the transitivity, also I_{100} dominates I_k. Thus, it is guaranteed that the inferiority of I_k will be discovered by I_{100}, hence I_j can be safely relegated into the inefficient set. This means that after comparing I_{100} to all the other investments (99 comparisons), if 30 investments are dominated by I_{100}, the maximum number of comparisons left will be $\binom{69}{2}$=2 346. Using the transitivity property the total number of comparisons in the above hypothetical example is reduced from 4 950 to $2\,346 + 99 = 2\,445$. By the same token, the number of comparisons can be further reduced when more investments are relegated into the inefficient set by investments other than I_{100}.

6.2 Algorithms for SD rules

In practice, stochastic dominance criteria are applied to empirical distributions, for example to ex-post rates of return of mutual funds. Denote by A and B be two distinct mutual funds with cumulative distributions F and G, respectively. Reorder the observations of A and B from the lowest to the highest value such that

$$\begin{aligned} A:&\ x_1 \leq x_2 \leq \ldots \leq x_n \\ B:&\ y_1 \leq y_2 \leq \ldots \leq y_n \end{aligned} \tag{15}$$

(If there are two identical observations, write them one after the other and assign $1/n$ probability to each one).

The FSD algorithm: A (or F) dominates B (or G) by FSD if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots n$ and there is at least one strict inequality.

As the FSD, the SSD algorithm require to rank all observation. Then define the "cumulate" observations:

$$\begin{aligned} xx_1 = x_1, \qquad xx_2 = x_1 + x_2, \ldots xx_i = \sum_{j=1}^{i} x_j, \ldots, xx_n = \sum_{j=1}^{n} x_j, \\ yy_1 = y_1, \qquad yy_2 = y_1 + y_2, \ldots yy_i = \sum_{j=1}^{i} y_j, \ldots, yy_n = \sum_{j=1}^{n} y_j. \end{aligned} \tag{16}$$

The SSD algorithm: A (or F) dominates B (or G) by SSD if and only if $xx_i \geq yy_i$ for all $i = 1, 2, \ldots n$ and there is at least one strict inequality.

As in SSD, also with RSSSD we calculate the area enclosed between the two distribution. However, this time the area accumulation is computed from the end observation x_n to any value. Therefore, the new definition of cumulative observation is:

$$\begin{aligned} \overline{xx}_1 = x_n \qquad \overline{xx}_2 = x_n + x_{n-1}, \ldots, \overline{xx}_i = x_n + x_{n-1} + \ldots x_{n-i+1}, \\ \ldots, \overline{xx}_n = x_n + x_{n-1} + \ldots + x_1, \\ \overline{yy}_1 = y_n, \qquad \overline{yy}_2 = y_n + y_{n-1}, \ldots, \overline{yy}_i = y_n + y_{n-1} + \ldots y_{n-i+1}, \\ \ldots, \overline{yy}_n = y_n + y_{n-1} + \ldots + y_1. \end{aligned} \tag{17}$$

The RSSSD algorithm: A (or F) dominates B (or G) by SSD if and only if $\overline{xx}_i \geq \overline{yy}_i$ for all $i = 1, 2, \ldots, n$ and there is at least one strict inequality.

With the preceding explanation about SSD and $RSSSD$ algorithms the formulation of PSD and MSD is immediate; however note that we must consider separately the case of observations less or equal zero and greater than zero.

7 The empirical analysis

In jugging the quality of an investment decision making rule two factors have to be token into account: the severity of the underlying assumption and the efficiency evaluated in term of the relative size of resultant efficient set. Based only on the first factor, the FSD is the best criterion, because the only assumption required is $u' \geq 0$. However this rule seem to be very ineffective in that the size of resultant efficient set is often very similar to the size of the initial feasible set. Generally, the larger the number of assumption, the smaller the induced efficient set.

Empirical studies of the SD criteria generally focus on two main issues: the effectiveness of the various SD rules and the performance of mutual funds relative to an unmanaged portfolio (benchmark). Our empirical analysis concerns the application of second order stochastic dominance criteria previously described and cumulative prospect theory to rates of return of a set of Italian mutual funds. The performance of 32 mutual funds are been considered including also the Stock Exchange Mibtel Index. The funds considered belong to different classes, have different total capital and refer to different management companies[§]. The size of each efficient set are computed for weekly, monthly, quarterly and semi-annual rate of return. The data regard the Monday net prices in a period of 30 months. The comparisons related to stochastic dominance criteria are summarized in Table 3, where the last row presents the results relative to the famous mean-variance rule and the column "all time" indicates the number of funds (and the funds) which belong to the efficient set for any kind of sampling. The FSD rule (not reported on the Table), which as well known

Table 3: Size of efficient sets from an initial feasible set of 33 portfolios

	weekly	monthly	quarterly	semi-annual	all time
Observations	130	30	10	5	
SSD	17	20	16	3	1(11)
RSSSD	3	3	3	1	1(11)
PSD	11	15	14	15	4(11,12,18,33)
MSD	3	2	3	15	2(29,33)
M-V	11	15	13	10	7(11,13,15,24,28,29,30)

is virtually assumption free, is very ineffective especially when the number of observations

[§]The mutual funds considered are: 1.Arca 27, 2. Azimut Borse Int., 3. Centrale Global 4. Epta-International 5. Fideuram Azione 6. Fondicri Int. 7. Genercomit Int., 8. Investire Int., 9. Prime Global, 10. SanPaolo Int., 11. Centrale Italia, 12. Epta aziaoni Italia, 13. Fondicri Sel. Italia, 14. Genercomit Azioni Italia, 15. Gesticredit Borsit, 16. Imi Italy, 17. Investire Azion., 18. Oasi Azionario Italia, 19. Azimut Europa, 21. Gesticredit Euro Az., 21. Imi Europe, 22. Investire Europa, 23. Sanpaolo H. Europe, 24. Arca BB, 25 Azimut Bil., 26. Eptacapital, 27. Genercomit, 28. Investire Bil., 29. Arca TE, 30. Fideuram Performance, 31. Fondo Centrale, 32. Genercomit Espansione, 33. Mibtel index.

is relatively large (the efficient set coincides with the initial set for weekly and monthly observations).

On the other hand, the SSD rule is very effective, hence indicating that the assumption of risk aversion substantially reduce the size of the efficient set.

The RSSSD and MSD rules relegate many funds to the inefficient set. In most cases these assumptions reduce the size of the efficient set quite dramatically.

Another important finding is that PSD rule is effective: the size of efficient sets is suitable for a optimal choice. Note also that the mean-variance efficient set is very similar in size to the PSD efficient set.

The same data set has been analyzed as well in the context of the cumulative prospect theory, assuming both the value function (with $\lambda = 1$ and $\lambda = 2.25$) and the functional form of risky weighting reported proposed by Tversky and Kahneman and described in relations (2) and (3). Table 4 reports for each sampling the funds which have obtained the five best performances. Note that in our analysis the λ parameter produces two different

Table 4: The first five positions of performance ranking evaluated with CPT

	weekly	monthly	quarterly	semi-annual
$\lambda = 1$	11,16,33,15, 13	11,16,33,13,17	11,16,33,13,17	11,16,13,33,17
$\lambda = 2.25$	29,24,28,27,26	11,28,29,24,27	11,16,13,17,33	11,16,13,33,17

orders even if for quarterly and semi-annual data the ranking is very similar. However for $\lambda = 2.25$ the losses have a more relevant importance so that the cardinality of sample can affect the ranking.

8 Concluding remarks

This contribution represents a first attempt to rank a set of mutual funds by using cumulative prospect theory and new second order stochastic dominance criteria. Even if the empirical comparison is examined for weekly, monthly, quarterly, semi-annual rates of return, the results refer to a unique data set; therefore we cannot jump to hasty conclusions. Anyway, the outcomes obtained suggest that the RSSSD and MSD rules relegate too many portfolios to the inefficient set, while the PSD criterion is very effective, hence indicating that the assumptions of CPT are valid and can be used in order to make optimal selections.

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