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Abstract. This contribution presents a network of interdependent firms in which the spatial diffusion of the business relations is described by an entropy spatial interaction model. This network is used in a credit risk model in order to take into account the counterparty risk and describe the resulting contagion effects.

Keywords: credit risk contagion, networks, counterparty risk, entropy spatial models.

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1 Introduction

The main aim of this contribution is to present a model for the description of the dependence among defaults of firms in an interrelated economy. The dependence we are interested in is the one implied by the so called counterparty risk, defined by [6] as the risk that the default of a firm's counterparty might affect its own default probability.

The problem is widely studied and it is of interest for the analysis of the credit risk of portfolios of dependent positions, such as, for example, portfolios of bank loans to enterprises. Different approaches have been proposed to tackle this issue.

A first class of models introduces dependence through the presence of common factors, which determine the behavior of the economy and are linked to the business cycle. In this way it is possible to reduce the dimension of the correlation matrix which has to be considered, turning the attention from the correlation among positions to the correlation among factors, which are usually considerably fewer than the number of positions. An advantage of this widely accepted and widely applied approach is that conditionally on the factors the positions are mutually independent.

Nevertheless, empirical evidence gathered in a number of studies shows that the dependence among defaults cannot be fully explained using only factor models (see, for example, [6], [4], [2]). In particular, default events exhibit clustering behaviors, especially in recession periods, and firms seem to show correlation in their defaults due not only to a dependence on some common risk factors, but also to some firm-specific risks. The counterparty risk introduces an additional source of dependence and a mechanism of contagion through which the financial distress can spread in the economy.

Recent empirical studies support the presence of contagion effects in the diffusion of defaults (see, for example, [4] and the references therein). Taking into consideration this evidence, we propose to extend a factor model that describes the value of a firm in order to include an idiosyncratic term which takes into account the business connections with other firms in the economy.

Different contributions in the literature rely on the introduction of network-like structures to model the dependence among positions in a portfolio of risks, see for example [3], [4], [7]. To model the connections among firms in the economy we propose to introduce a weighted network based on a spatial interaction model, with directed arcs which can accounts both for the presence of business relations and for their intensity.

The structure of the paper is as follows. In Section 2, we introduce the use of networks to model business relations. Section 3 describes our network model including spatial interaction, while in Section 4 we describe the credit contagion model. Section 5 concludes.

2 Networks for business relations

Many phenomena where interaction plays a crucial role are modeled using networks. There are different types of networks which allow to describe different behavior of a system of interacting or connected elements. The structure of a network can be analyzed by considering some important properties such as the connectivity of each vertex and the strength of each interaction.

The graphical representation of networks enables us to easily grasp the essence of the interactions. The elements of the system are usually represented using nodes while (directed) edges describe the (directed) relationships which occur among nodes. Each arc can be assigned a weight, which is usually interpreted as the intensity of the relation.

Many recent contributions in the credit risk literature introduce network structures to explicitly model the interactions among the positions in a portfolio of firms, in order to describe the microeconomic foundations of dependence and thus capture the effect of a contagion mechanism.

Some of these approaches try to adapt some models from physics in order to describe the economic interactions; see, for example, [7], in which the authors apply a lattice gas model to describe couplings among counterparties. Other contributions in the literature propose to use weighted networks to give a stylized description of the interaction among counterparties and introduce some simplifying hypotheses on the structure of the dependence to make the model analytically tractable; see, for example, [4], [3]. In [10] the authors apply queueing theory to analyze the behavior of networks with looping lending relationships. Other recent applications of network theory to the analysis of credit risk problems include the description of interbank payment flows; see e.g. [8]. Furthermore, in [5] some business connection variables such as the intensity of the relationships and the distance are considered as relevant explanatory variables to describe the recovery rates of bank loans in Germany.

3 A business relation network with spatial interaction

In order to study the propagation of the defaults in a system of firms and its effects on the assessment of credit risk, we build a model in which the business connections are modeled using a directed weighted network where the nodes represent the firms, directed edges between firms represent the business connections and the weight associated to each edge gives a measure of the intensity of the connection.

The existence of the directed arc (k, i), connecting the origin node k to the destination node i, means that the (possible) default of firm k will cause some financial distress to firm i. The precise way in which we model this distress will be discussed in next session, but it relies on the existence of a client-supplier relation.

More precisely, we choose as an indicator of the presence of a connection with k which may cause distress to i the fact that firm k is a client of firm i. The basic idea is that if a client which represents a significant percentage of the turnover of firm i, let us say a percentage above a given threshold, defaults, this causes difficulties to firm i and may result in a serious distress for the firm.

A natural choice for the measure of the intensity of the connection is given by the the percentage of sales on the turnover of the firm; therefore, the weight associated to the edge (k, i), w_{ki}, is given by the percentage of sales to client k on the turnover of firm i. An example of a network of business connections is shown in figure 1, which depicts the incoming and outgoing arcs for two generic nodes k and i which are connected; notice that the presence of the arc (k, i) indicates that firm k is a client of firm i and the weight w_{ki}

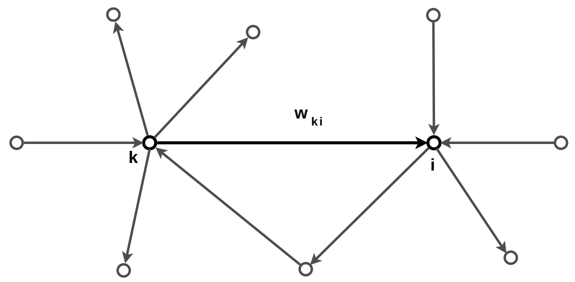

Figure 1: Example of network of business connections.

gives the per cent sales to k on the turnover of i. Of course, different and more general choices could be made, but this definition of the business connections is straightforward and fairly natural.

In our model the edges in the network are determined by considering a spatial dimension and resorting to an entropy spatial interaction model constrained to origins which takes into account both the economic weight of the different firms and their distance; on spatial interaction models see, for example, [12].

As it is known, entropy spatial interaction models can be obtained by maximizing the entropy of the system under the available information on the distance matrix between the origin and destination nodes and on the weights assigned to these origins and destinations, which account for the size of the outgoing (for origins) or incoming (for destinations) flows.

Let us indicate with n the number of firms in the system, and let us consider a generic firm $i \in \{1, 2, \ldots, n\}$. Let us denote by π_{ki} the probability that firm k, with $k \in \{1, 2, \ldots, n\}$, is connected to firm i, i.e. that it is a client of firm i, and by d_{ki} the distance between firms k and i. Moreover, let $s(k)$ denote the economic sector of firm k and let W_k be a weight representing the relative attractiveness of firm k; in the spatial interaction model the weight W_k can naturally be defined as the overall turnover of firm k.

Let us observe that the probability π_{ki} that firm k is a client of firm i coincides with the probability that the directed arc (k, i) exists in the network and can be decomposed into the product of the probability $P_{s(i)s(k)}$ that a client of i pertains to sector $s(k)$ and the probability Q_{ki} that firm k is a client of firm i conditional to the fact that the client of i pertains to sector $s(k)$

$$\pi_{ki} = P_{s(i)s(k)}\, Q_{ki}. \tag{1}$$

In order to compute these probabilities we assume that the probability $P_{s(i)s(k)}$ that a client of i pertains to sector $s(k)$ (i.e. to the same sector of firm k) is the same for all firms in the same sector, and that it coincides with the percentage of the output of sector $s(i)$ sold to sector $s(k)$, so that it can be computed as follows

$$P_{s(i)s(k)} = \mathrm{Prob}\{\, \text{client of } i \ \in \ s(k) \,\} = \frac{A_{s(i)s(k)}}{\sum_{j=1}^{S} A_{s(i)j}}, \tag{2}$$

where S is the number of sectors and the $S \times S$ matrix $A = \left(A_{s(i)s(k)}\right)$ is the input-output table of the economy, in which the element $A_{s(i)s(k)}$ represents the output of sector $s(i)$ sold to sector $s(k)$.

Moreover, in order to compute the conditional probability Q_{ki} we apply an entropy spatial interaction model constrained to origins which makes this probability depend on the relative attractiveness of the different firms and their distance as follows

$$Q_{ki} = \text{Prob}\{\, k \text{ is a client of } i \mid \text{client of } i \in \ s(k)\,\} = W_k \frac{e^{-\alpha_{s(i)} d_{ki}}}{\sum_{j \in S_{s(k)}} W_j e^{-\alpha_{s(i)} d_{ji}}} \tag{3}$$

where $S_s(k)$ denotes the set of firms in sector $s(k)$ and $\alpha_{s(i)}$ is a positive real parameter, dependent on the sector $s(i)$, which determines the relevance of the effect of distance on the business relations.

It can be seen that the sum of the probabilities π_{ki} on all $k \in \{\,1, 2, \ldots, n\,\}$ is equal to 1 for all nodes $i \in \{\,1, 2, \ldots, n\,\}$.

Once determined which arcs exist in the network, we need to compute their weights, where the weight w_{ki} of arc (k, i) is given by the per cent sales to client k on the turnover of firm i. Formally, let us indicate with $I_{(k,i)}$ the indicator function of arc (k, i), which takes value 1 if the arc (k, i) exists and value 0 otherwise. We make the assumption that the weight w_{ki} on the arc (k, i) is proportional to the output of sector $s(i)$ sold to sector $s(k)$ and to the weight W_k that determines the relative attractiveness of firm k, i.e. to the turnover of firm k, as follows

$$w_{ki} = \frac{A_{s(i)s(k)}}{\sum_{j=1}^{S} A_{s(i)j}} \, \frac{W_k \, I_{(k,i)}}{\sum_{j \in S_{s(k)}} W_j \, I_{(j,i)}}. \tag{4}$$

The business relation network defined in such a way will be used in next section to model the contagion of defaults in the system.

4 The credit contagion model

The network of firms described in the previous section can be applied to model the counterparty risk, which gives rise to a contagion effect and may provoke clustering of defaults during crisis periods.

To this aim we use the network of business relations to model the connections in the system of firms applying a discrete time model analogous to that proposed in [1]; this model describes the asset value of firms by taking into account the counterparty risk of the firms connected by arcs in the network.

The asset value of firm i at time t, $V_i(t)$, is modeled as the sum of three components: a macroeconomic component F, modeled using a factor model which takes into account the influence of the business cycle (on factor models see, for example, [11], [9]), a microeconomic component M which introduces a contagion effect due to the business connections with other firms and a residual idiosyncratic term ε:

$$V_i(t) = F_i(t) + M_i(t) + \varepsilon_i(t) \qquad t = 0, 1, \ldots \tag{5}$$

The value of the macroeconomic component $F_i(t)$ is described by the following factor model:

$$F_i(t) = \sum_{j=1}^{J} \beta_j^{s(i)} Y_j(t) \qquad t = 0, 1, \ldots, \tag{6}$$

where $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_J(t))$ is the vector of the values at time t of the driving factors, $s(i)$ represents the economic sector of firm i and β_j^s is the weight of factor j for the firms of sector s. The driving factors Y_j, with $j \in \{1, 2, \ldots, J\}$, follow a vector stochastic process modeled according to the nature of the macroeconomic factors considered.

The microeconomic component, $M_i(t)$, takes into account the effects of past distresses on the health of firm i and is defined as follows

$$M_i(t) = \mu_{s(i)} \sum_{\tau=1}^{\infty} \lambda_{s(i)}^{\tau} \left[p(t-\tau) - \sum_{k \in C_i(t-\tau)} \delta_k(t-\tau)\, w_{ki}(t-\tau) \right], \tag{7}$$

where $\mu_s \in \mathbb{R}_+$ is a real parameter dependent on the economic sector of the firm, λ_s, with $0 \leq \lambda_s < 1$, is a dampening factor which determines the distress memory of firms in sector s, $p(t)$ is the average default rate of the economy at time t, $C_i(t)$ is the set of clients of firm i, i.e. the subset of nodes in the network which are origins of arcs entering node i, $\delta_k(t)$ is the indicator function of default of client k which takes value 1 if client k defaults at time t and w_{ki} is the weight of arc (k, i), given by the percentage of the sales to client k on the turnover of firm i.

The quantity within brackets represents a measure of the distress undergone by firm i at time $(t - \tau)$ due to the defaults observed among its clients. This distress measure is the difference between the average default rate of the economy, $p(t - \tau)$, and the percentage of turnover of firm i sold to clients which defaulted in period $(t - \tau)$; therefore it has a negative value if the firm suffered a rate of defaults of clients higher than the average rate of the economy, and a positive value in the opposite case. Notice that this distress measure affects the health of the firm with a one-period delay and its effects are dampened according to an exponential decay in time.

Furthermore, we have a residual idiosyncratic term $\varepsilon_i(t)$, which is assumed to be normally distributed with zero mean and standard deviation σ_{ε_i}. As is usual in factor models, we assume that the residual idiosyncratic terms $\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_N(t)$ are both mutually independent and independent of the factors $Y_1, Y_2, \ldots, Y_J$.

As in a structural approach, in our model a firm defaults when the value of its asset falls below a given threshold.

From equations (7) and (5) we can see that the default of a client lowers the value of the microeconomic component and thus the asset value of the firm; as a result, the probability of crossing the default threshold for the firm increases.

This is the mechanism through which the contagion of defaults spreads in the system. In the network of firms, the contagion of defaults starts with the default of a node and spreads along the directed edges which connect the firms, proceeding in the direction of the edges.

5 Concluding remarks

In this contribution we tackle the issue of modeling the microeconomic relations among the firms in a network economy in order to describe the contagion mechanism which spreads the financial distress in the system when a firm defaults.

To this aim we define a weighted network based on a spatial interaction model to describe the business relations that take place among the firms and we include the effect of the microeconomic interactions as an additional idiosyncratic term in a factor model that describes the value of the firms in the network. A default occurs when the value of a firm cross a given default threshold and has repercussions on the financial health of the connected firms in the subsequent periods, thus representing a possible source of contagion of defaults.

References

[1] D. Barro, A. Basso (2005), Counterparty risk: a credit contagion model for a bank loan portfolio, *The ICFAI Journal of Financial Risk Management*, **2**(4), 34-52.

[2] S. Das, D. Duffie, N. Kapadia, L. Saita (2007), Common Failings: How corporate defaults are correlated, *Journal of Finance*, **62**(1), 93-118.

[3] D. Egloff, M. Leippold, P. Vanini (2007), A Simple Model of Credit Contagion, *Journal of Banking and Finance*, **31**, 2475-2492.

[4] K. Giesecke, S. Weber (2004), Cyclical correlations, credit contagion and portfolio losses, *Journal of Banking and Finance*, **28**(12), 3009-3036.

[5] J. Grunert, M. Weber (2005), Recovery rates of bank loans: empirical evidence from Germany, *Working Paper n. 124*, Department of Banking and Finance, University of Mannheim.

[6] R. Jarrow, F. Yu (2001), Counterparty risk and the pricing of defaultable securities, *Journal of Finance*, **56**(5), 555-576.

[7] P. Neu, R. Khun (2004), Credit risk enhancement in a network of interdependent firms, *Physica A*, **342** Issues 3-4, 639-655.

[8] M. Pröpper, I. van Lelyveld, R. Heijmans (2008), Towards a network description of interbank payment flows, *Working Paper* No. 177/2008, De Nederlandsche Bank.

[9] D. Saunders, C. Xiorus, S. Zenios (2007), Credit risk optimization using factor models, *Annals of Operations Research*, **152**(1), 49-77.

[10] H. Schellhorn, D. Cossin (2007), Credit risk in a network economy, *Management Science* **53**, 1604-1617.

[11] P.J. Schonbucher (2000), Factor models for portfolio credit risk, *Bonn Econ Discussion Paper*, University of Bonn.

[12] A.G. Wilson, *Entropy in urban and regional modelling*, Pion Limited, London (1970).