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Antonella Campana and Paola Ferretti

Initial premium, aggregate claims and distortion risk measures in XL reinsurance with reinstatements

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Initial premium, aggregate claims and distortion risk measures in XL reinsurance with reinstatements*

Antonella Campana
<campana@unimol.it>
Dept. SEGeS
University of Molise

Paola Ferretti
<ferretti@unive.it>
Dept. of Applied Mathematics and
Advanced School of Economics
University of Venice

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Abstract. With reference to risk adjusted premium principle, in this paper we study excess of loss reinsurance with reinstatements in the case in which the aggregate claims are generated by a discrete distribution. In particular, we focus our study on conditions ensuring feasibility of the initial premium, for example with reference to the limit on the payment of each claim. Comonotonic exchangeability shows the way forward to a more general definition of the initial premium: some properties characterizing the proposed premium are presented.

Keywords: Excess of loss reinsurance; reinstatements; distortion risk measures; initial premium; exchangeability.

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Correspondence to:

Paola Ferretti	Dept. of Applied Mathematics, University of Venice
	San Giobbe, Cannaregio 387
	30121 Venezia, Italy
Phone:	[++39] (041)-234-6923
Fax:	[++39] (041)-234-7444
E-mail:	ferretti@unive.it

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1 Introduction

The excess of loss reinsurance model we study in this paper is related to the model that has been originally proposed and analyzed by Sundt [9] and that it has been subsequently generalized (see [1] and [3]). Given the initial premium P, the limit on the payment of each claim m, the number of reinstatements K (such that the aggregate limit M satisfies $M = (K+1)m$), the aggregate deductible D, the percentages of reinstatement c_i, $(i = 1, \dots, K)$, the total premium income results to be a random variable, say $\delta(P)$, which is so defined

$$\delta(P) = P\left(1 + \frac{1}{m}\sum_{i=0}^{K-1} c_{i+1} L_X(D + im, D + (i+1)m)\right) \tag{1}$$

where $L_X(a, a+b) = \min\{(X-a)_+, b\}$ and $(X-a)_+ = X - a$ if $X > a$, otherwise $(X-a)_+ = 0$ $(a, b \in \mathbb{R}^+)$. From the point of view of the reinsurer, the aggregate claims S paid by the reinsurer for this XL reinsurance treaty, namely

$$S = L_X(D, D + (K+1)m) \tag{2}$$

satisfy the relation

$$S = \sum_{i=0}^{K} L_X(D + im, D + (i+1)m). \tag{3}$$

This reinsurance cover is called an XL reinsurance for the layer m xs d with aggregate deductible D and K reinstatements and provides total cover for the amount S. The initial premium P is supposed to cover the original layer

$$L_X(D, D+m) = \min\{(X-D)_+, m\}. \tag{4}$$

The condition that the reinstatement is paid pro rata means that the premium for the i-th reinstatement is a random variable given by

$$\frac{c_i P}{m} L_X(D + (i-1)m, D + im) \tag{5}$$

where $0 \leq c_i \leq 1$ is the i-th percentage of reinstatement.

As it is well-known, excess of loss reinsurance with reinstatement has been essentially studied in Actuarial Literature in the framework of collective model of risk theory and this choice requires the knowledge of the claim size distribution in the classical evaluation of pure premiums. Conversely, as a rule, only few characteristics of aggregate claims can be explained and the interest for general properties characterizing the involved premiums is still flourishing. In order to contribute to this research, we set our analysis in the framework of risk adjusted premiums (a more general choice than that of Walhin and Paris [10]) and we concentrate our attention on properties exhibited by initial premiums, both with respect to equilibrium condition (section 2), both with respect to feasibility with reference to the limit on the payment of each claim (section 3). Then the attention is moved to the recent framework of generalized risk adjusted premiums (Wu and Zhou [12]) where the main role is played by comonotonic exchangeability assumption: we propose the definition of the related generalized initial premium and we state some regularity properties displayed by it (section 4). Finally section 5 ends the paper with some concluding remarks.

2 Excess of loss reinsurance with reinstatements: some preliminary results

Recently (see [3]) we studied the initial risk adjusted premium P as solution of the equilibrium condition expressing the fact that the value of the total premium income $\delta(P)$ equals the distorted expected value of the aggregate claims S, that is

$$W_{g_1}(\delta(P)) = W_{g_2}(S) \tag{6}$$

where

$$W_{g_j}(X) = \int_0^\infty g_j(S_X(x))dx \tag{7}$$

and the functions g_j $(j = 1, 2)$ are distortion function, i.e. non-decreasing functions $g_j : [0,1] \to [0,1]$ such that $g_j(0) = 0$ and $g_j(1) = 1$; $S_X(x) = P(X > x)$ is the decumulative distribution function of X. This initial premium P is well-defined and it is given by

$$P = \frac{\sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m))}{1 + \frac{1}{m}\sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m))}. \tag{8}$$

We see that (6) gives a sort of global equilibrium: the distortion risk measure associated with the aggregate claims to the layer S must be equal to the distortion risk measure associated with the total premium income $\delta(P)$. If we apply the same scheme to each layer, we derive that the premium P_i, for all $i \in \{0, 1, \ldots, K\}$, must satisfy the following equation

$$W_{g_1}(P_i) = W_{g_2}(L_X(im, (i+1)m)). \tag{9}$$

Given that distortion risk measures obey positive homogeneity, the initial premium P_0 is easily defined by the previous equation; in fact

$$W_{g_1}(P_0) = P_0 = W_{g_2}(L_X(0, m)). \tag{10}$$

The premium for the i-th reinstatement is given by

$$P_i = \frac{c_i P_0}{m} L_X(D + (i-1)m, D + im), \tag{11}$$

then we have

$$\frac{c_i P_0}{m} W_{g_1}(L_X((i-1)m, im)) = W_{g_2}(L_X(im, (i+1)m)). \tag{12}$$

Hitherto we have implicitly assumed the reinstatements percentages c_i to be known. If the reinstatement percentages c_i are not fixed in advance, we obtain

$$c_i = \frac{m W_{g_2}(L_X(im, (i+1)m))}{P_0 W_{g_1}(L_X((i-1)m, im))} \tag{13}$$

where the initial premium P_0 given by (10). We note that both the initial premium P_0 both the percentages c_i do not depend on the number of reinstatements K. The values given by (13) are useful even when the c_i are to be determined in advance. They will give us a point of reference in order to obtain local equilibrium for each reinstatement. If the reinstatements percentages c_i are determined by formula (13), one can easily verify that the following equality holds

$$P = P_0$$

both in the case of same distortion functions $g_1 = g_2$ (see [1]) both in the case of not necessarily equal distortion functions g_j $(j = 1, 2)$.

In fact, by (8) and (13) it is

$$P = \frac{\sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m))}{1 + \frac{1}{P_0} \sum_{i=0}^{K-1} W_{g_2}(L_X((i+1)m, (i+2)m))}, \tag{14}$$

that is,

$$P = \frac{\sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m))}{1 + \sum_{i=0}^{K-1} \frac{W_{g_2}(L_X((i+1)m,(i+2)m))}{W_{g_2}(L_X(0,m))}} \tag{15}$$

and, finally,

$$P = W_{g_2}(L_X(0, m)) = P_0. \tag{16}$$

Therefore local equilibrium condition for each reinstatement ensures global equilibrium as defined in (6).

3 Initial premiums and limit on the payment of each claim

Generally, reinstatement percentages are fixed in advance: this assumption moves our attention to the study of the initial premium defined by local equilibrium condition (9). From now on we make the assumption that the premium for the i-th reinstatement (5) is a two-point random variable distributed as $c_i\, P\, B_{p_i}$ and B_{p_i} denotes a Bernoulli random variable such that $Pr[B_{p_i} = 1] = p_i = 1 - Pr[B_{p_i} = 0]$. Closely related to equation (12) is the definition of the initial premium P_{0i} $(i = 1, \ldots, K)$ for which the local equilibrium condition is satisfied in each layer, that is

$$P_{0i} = \frac{mW_{g_2}(L_X(im, (i+1)m))}{c_i W_{g_1}(L_X((i-1)m, im))} = \frac{mg_2(p_{i+1})}{c_i g_1(p_i)}. \tag{17}$$

Clearly acceptability of the treaty in the market requires that $P_{0i} \leq m$ for each $i = 1, \ldots, K$. In order to determine conditions ensuring the validity of this condition which is not generally true (see Campana [2]), we direct our attention to the analysis of the relation between two consecutive values of the initial premium, that is to the comparison of P_{0i} to P_{0i+1}.

Let us consider the following real-valued function $\mathcal{H}$ defined on $[0, p_{i+1}]$:

$$\mathcal{H}(x) = c_{i+1} g_1(p_{i+1}) g_2(p_{i+1}) - c_i g_1(p_i) g_2(x).$$

Note that $\mathcal{H}(0) = c_{i+1} g_1(p_{i+1}) g_2(p_{i+1}) \geq 0$. Given that

$$\mathcal{H}(p_{i+1}) = g_2(p_{i+1})[c_{i+1} g_1(p_{i+1}) - c_i g_1(p_i)] \leq g_2(p_{i+1}) c_{i+1}[g_1(p_{i+1}) - g_1(p_i)]$$

if the percentages of reinstatement are decreasing, then increasing monotonicity of the distortion function g ensures both non-positivity of $\mathcal{H}(p_{i+1})$ both decreasing monotonicity of $\mathcal{H}$. If the distortion function g_2 is continuous, intermediate value theorem ensures that there exists at least one point $\overline{x}$ in $[0, p_{i+1}]$ such that

$$\mathcal{H}(\overline{x}) = c_{i+1}g_1(p_{i+1})g_2(p_{i+1}) - c_i g_1(p_i)g_2(\overline{x}) = 0.$$

As a consequence, we have that

$$\mathcal{H}(x) \begin{cases} \geq 0, & 0 \leq x \leq \overline{x}; \\ \leq 0, & \overline{x} \leq x \leq p_{i+1}. \end{cases}$$

This means that, given any choice of probabilities p_{i+1} and p_i ($p_{i+1} \leq p_i$), it is possible to state that there exists al least one probability $\overline{x} \in [0, p_{i+1}]$ such that $P_{0i+1} = P_{0i}$. Moreover, we can conclude that it is possible to anticipate the relation underlying two consecutive initial premiums P_{0i} and P_{0i+1} for any choice of a probability p_{i+2} such as follows

$$P_{0i+1} \begin{cases} \leq P_{0i}, & 0 \leq p_{i+2} \leq \overline{x}; \\ \geq P_{0i}, & \overline{x} \leq p_{i+2} \leq p_{i+1}. \end{cases} \tag{18}$$

In this way, it is possible to define a finite sequence of initial premiums P_{0i} ($i = 1, \ldots, K$) such that P_{0i} is increasing (decreasing): it is sufficient to choose a finite sequence of probabilities p_{0i} ($i = 2, \ldots, K+1$) where at any step, $\overline{x} \leq p_i \leq p_{i-1}$, (respectively, $0 \leq p_i \leq \overline{x}$). In this way we obtain the following results.

Proposition 1 *Given an XL reinsurance with K reinstatements and distortion functions g_1 and g_2, where g_2 is continuous and the percentages of reinstatement are decreasing, then there exists a finite sequence of probabilities p_i ($i = 1, \ldots, K+1$) such that the finite sequence of initial risk adjusted premiums P_{0i} ($i = 1, \ldots, K$) is monotone (increasing or decreasing).*

Proposition 2 *Given an XL reinsurance with K reinstatements and distortion functions g_1 and g_2, where g_2 is continuous and the percentages of reinstatement are decreasing, there exists a finite sequence of probabilities p_i ($i = 1, \ldots, K+1$), with $g_2(p_{K+1}) \leq g_1(p_K)$ or $g_2(p_2) \leq g_1(p_1)$, such that $P_{0i} \leq m$, for all $i = 1, \ldots, K$.*

4 Generalized initial premiums and exchangeability

As it is well-known, insurance premium principles and risk measures can be characterized with reference to a set of axioms, that is to a set of desirable properties that they should have for actuarial practice. Among them, recently Goovaerts et al. ([5], [6]), Wu and Zhou (see [12]) propose a generalization of Yaari's risk measure by relaxing his proposal. Comonotonic additivity has been generalized to comonotonic exchangeability in [5], while the link between comonotonic additivity and independent additivity has been addressed in [6]. Additivity for a finite number of comonotonic risks is substituted by countable additivity and countable exchangeability in [12] in order to characterize generalized distortion premium principles. In this context, we refer to a non-empty collection of risks $\mathcal{L}$ on $(\Omega, \mathcal{F}, P)$ such that $\min\{X, a\} - \min\{X, b\}$, aX and $X + a$ are all in $\mathcal{L}$ for any $X \in \mathcal{L}$ and for any $a, b \in \mathbb{R}$. In studying different characterizations of insurance premium principles and risk measures, Goovaerts et al. [5] refer to some realistic situations owing to specific sets of axioms: they refer to the notion of exchangeability, that with reference to a principle H states

A1. Exchangeability

$$H(\mathbf{X^c} + \mathbf{Y^c}) = H(\mathbf{X^{*c}} + \mathbf{Y^c}) \qquad \text{provided that} \qquad H(\mathbf{X}) = H(\mathbf{X}^*) \tag{19}$$

where the random vectors $\mathbf{X^c}$ and $\mathbf{Y^c}$ admit the same marginal distributions of $\mathbf{X}$ and $\mathbf{Y}$, respectively, with the comonotonic dependence structure. The following theorem states a representation result of the so called generalized distortion principle

Theorem 1 *A premium principle $H : \mathcal{L} \to \mathbb{R}^+$ admits for all $X \in \mathcal{L}$ the representation*

$$H(X) = h\left(\int_0^\infty g(S_X(x))dx\right) \tag{20}$$

where h is a continuous non-decreasing function on $\mathbb{R}^+$, if and only if H satisfies exchangeability, monotonicity in stochastic order and continuity.

As comonotonic additivity is the most essential property of distortion integral (and Choquet integral), comonotonic exchangeability by generalizing comonotonic additivity suggests a new definition of distortion premium principle (and Choquet price principle), that is

$$W_g^h(X) = h\left(\int_0^\infty g(S_X(x))dx\right) \tag{21}$$

which is called *generalized risk adjusted premium principle*; the function g is a distortion function and h is a continuous non-decreasing function on $\mathbb{R}^+$.

Now we turn our attention to the analysis of local equilibrium condition (9) and in order to consider the case of generalized risk adjusted premium principle we define and study the following *generalized local equilibrium condition*

$$W_{g_1}^{h_1}(P_i) = W_{g_2}^{h_2}(L_X(im, (i+1)m)) \tag{22}$$

where the functions g_j are distortion functions and h_j are continuous non-decreasing functions on $\mathbb{R}^+$ $(j = 1, 2)$. Accordingly, the definition of *generalized initial premium* naturally follows: it is solution, write P_{0i} $(i = 1, \ldots, K)$, of the generalized local equilibrium condition (22). Clearly, positivity of P_{0i} is assumed. In the following we focus our attention on properties exhibited by the generalized initial premium P_{0i}, with reference to some assumptions made on the functions involved, namely the distortion functions g_j and the composing functions h_j, $(j = 1, 2)$. The analysis of equation (22) when aggregate claims are generated by a discrete distribution may be conducted by introducing the following related function

$$F_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1}) = h_1\left[\frac{c_i x}{m} g_1(p_i)\right] - h_2\left[g_2(p_{i+1})\right]. \tag{23}$$

In this way, the generalized initial premium P_{0i} is such that

$$F_{g_1 g_2}^{h_1 h_2}(P_{0i}, p_i, p_{i+1}) = 0.$$

Then if g_j $(j = 1, 2)$ are continuously differentiable distortion functions and h_j are continuously differentiable and non-decreasing functions on $\mathbb{R}^+$ $(j = 1, 2)$, where $g_1(p_i) \neq 0$ and $h_1' \left[\frac{c_i x}{m} g_1(p_i)\right] \neq 0$, P_{0i} results to be a continuously differentiable function of (p_i, p_{i+1}). Moreover

$$\begin{aligned}
\frac{\partial P_{0i}}{\partial p_i}(p_i, p_{i+1}) &= -\left[\frac{\partial F_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial p_i}\right] \Big/ \left[\frac{\partial F_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial x}\right] = -x \frac{g_1'(p_i)}{g_1(p_i)} \\
\frac{\partial P_{0i}}{\partial p_{i+1}}(p_i, p_{i+1}) &= -\left[\frac{\partial F_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial p_{i+1}}\right] \Big/ \left[\frac{\partial F_{g_1 g_2}^{h_1 h_2}(x, p_i, p_{i+1})}{\partial x}\right] \\
&= \frac{m}{c_i} \frac{h_2'[g_2(p_{i+1})]}{h_1'[\frac{c_i x}{m} g_1(p_i)]} \frac{g_2'(p_{i+1})}{g_1(p_i)}
\end{aligned}$$

where $x = P_{0i}$. Then we can state the following result.

Proposition 3 *Let g_j be continuously differentiable distortion functions and let h_j be continuously differentiable and non-decreasing functions on $\mathbb{R}^+$ $(j = 1, 2)$, where $g_1(p_i) \neq 0$ and $h_1' \left[\frac{c_i x}{m} g_1(p_i)\right] \neq 0$. Then the generalized initial premium P_{0i} $(i = 1, \ldots, K)$ is a continuously differentiable function of (p_i, p_{i+1}) with*

$$\nabla P_{0i}(p_i, p_{i+1}) = \left(-x \frac{g_1'(p_i)}{g_1(p_i)}, \frac{m}{c_i} \frac{h_2'[g_2(p_{i+1})]}{h_1'[\frac{c_i x}{m} g_1(p_i)]} \frac{g_2'(p_{i+1})}{g_1(p_i)} \right)$$

where $x = P_{0i}$.

Note that the generalized initial premium P_{0i} is monotone function of the probabilities: as a function of the probability p_i it is decreasing, while as a function of the probability p_{i+1} it is increasing.

5 Concluding remarks

In the considered XL model of reinsurance, when the reinstatements are paid the total premium income becomes a random variable which is correlated to aggregate claims. Incomplete information on aggregate claims distribution amplifies the interest for general properties characterizing the involved premiums. In this paper our main goal is to study properties exhibited by initial premium: our analysis refers to two settings, that of risk adjusted premiums and that of generalized risk adjusted premiums. We assume that the premium for each reinstatement is a two-point random variable, a particularly interesting hypothesis given that the reinsurance companies often assess treaties under the conjecture that there are only total losses. This happens, for example, when they use the rate on line method to price catastrophe reinsurance. As a matter of fact, conditions ensuring feasibility of the initial premium with respect to comparison with the limit on the payment of each claim and regularity properties displayed by the generalized initial premium are presented. In particular, the regularity results of the generalized initial premium hint that further research may be addressed to the analysis of optimal initial premiums.

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