Department of Applied Mathematics, University of Venice

WORKING PAPER SERIES

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Working Paper n. 156/2007
October 2007

ISSN: 1828-6887

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Abstract. This paper deals with non-monotonic Choquet integral, a generalization of the regular Choquet integral. The discrete non-monotonic Choquet integral is considered under the viewpoint of aggregation. In particular we give an axiomatic characterization of the class of non-monotonic Choquet integrals.We show how the Shapley index, in contrast with the monotonic case, can assume positive values if the criterion is in average a *benefit*, depending on its effect in all the possible coalition coalitions, and negative values in the opposite case of a *cost* criterion.

Keywords: aggregation operators, nonadditive measures, non-monotonic measures, Choquet integral.

JEL Classification Numbers: 62C05, 91B08.

MathSci Classification Numbers: C60.

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Introduction

The Choquet integral has been applied to many areas of decision making and in particular in the context of decision under uncertainty. The Choquet integral have been extensively studied in the field of multicriteria decision aid, fuzzy sets, aggregation theory.
Nevertheless every non-monotonic measure is unable to represent the *neutralization* effect, a situations where increasing two criteria by alone has a positive effect, but the contemporary increasing of both of them has a negative effect (the same can happen also for more than two criteria). This case is different, and *stronger*, in some sense, than the so called *redundance* effect represented by monotonic sub-additive measures. The specialized literature usually considers interchangeable the terms *redundance* and *conflicting*, both referring to the sub-additive case. But in this paper we reserve the term *conflicting* for the *neutralization* case, represented by non-monotonic measures. Note that, in boolean logic, the neutralization effect corresponds to the Exclusive Or connective (EXOR). Even if not often observed, the neutralization case can be of interest in real world MADA problems (Multi Attribute Decision Aid). Considering for instance an environmental decision making problem, where a Decision Maker (DM) has to rank some contaminated sites, considering both toxic sampled data and socio-economics indices. It can be unclear if an economic index is a *criterion* or a *benefit*, given that it can be preferable to decontaminate a site economically well developed, if the DM intends to preserve the economical activity, or the contrary if he/she wants to favor the sustainable development for a depressed area. An other typical MADA problem quite cited in the literature concerns the candidate selection for a particular jobs [5]. Even here, a very high specialization in a single item only, can be considered unfavorable with respect to a lower specialization but equally distributed in all the items. A justification for this choice is that a beginner can be better instructed in all the items than a deeply expert in a single item only, This is due to the fact that usually he maintains a very high *auto-estimation*, conflicting with the correct behavior for a beginner. Moreover again, optimizing an expenditure basket, we usually do not assign all the budget to a single good. This can be worst that buy anything, thus saving the total money amount, but the null option is worst again that a budget allocation to differentiated items. All such situations appear when one (or more) criterion in a coalition acts as a criterion, in an other one as a benefit. Of course, such a situation cannot be represented by non-monotonic measures.
The contributions in the nowadays literature in the field of non-monotonic measure is quite scarce, we limit to quote few papers about. De Waegenaere and Wakker [4] consider non-monotonic Choquet integrals in the study of intertemporal preferences, while Murofushi et al. [9] furnish a theoretical contribution. Radojeviĉ uses non-monotonic measures [10] in the context of logical measures. Also the paper of Rébillé [11] gives interesting contributions. No many other papers exist in the field of non-monotonic measure, especially for MSDA applications.
In this paper we propose an axiomatic characterization by means of rather natural properties of the class of nonmonotone Choquet integral and we show that non-monotonic Choquet integrals are meaningful as aggregation operators. The structure of the paper is as follows.

In Section 1 we introduce some properties of aggregation functions, we recall the definition of capacity and of discrete Choquet integral and we introduce the axiomatic characterization. Section 2 deals with the representation in the dual space while in Section 3 we consider the Shapley index in the non-monotonic case. The paper finishes with some conclusions.

1 A characterization of non-monotonic Choquet integral

Aggregation has for purpose the simultaneous use of different pieces of information provided by several sources, in order to come to a conclusion or a decision so aggregation functions transform a finite number of inputs, called arguments, into a single output. They are applied in many different domains and in particular aggregation functions play important role in different approaches to decision making, where values to be aggregated are typically preference or satisfaction degrees. Many functions of different type have been considered in connection with different situations and various properties of these functionals can be imposed by the nature of the considered aggregation problem. A class of aggregation functions can also be introduced axiomatically by means of a set of properties.
We denote by E a non empty real interval. If the integer n represents the number of values to be aggregated an aggregation operator is a function $A : E^n \to E$. To motivate the use of the Choquet integral as an aggregation operator, we present some basic mathematical properties of the aggregation functions.

- **Monotonicity** For all $\mathbf{x}, \mathbf{y} \in E^n$ if $x_i \leq y_i\,(i = 1, \ldots, n)$ then $A(\mathbf{x}) \leq A(\mathbf{y})$
- **Idempotence** If $x \in E$ then $A(x, \ldots, x) = x$
- **Positive Homogeneity** If $\mathbf{x} \in E^n$ and $a \in \mathbb{R}, a > 0$ then $A(a\mathbf{x}) = aA\mathbf{x}$

Moreover we define x_{-i} the element of $\mathbb{R}^{n-1}$ that is obtained from x by eliminating component i, and let (x_{-i}, y_i) obtained from x by replacing x_i with y_i.
Now we present the concept of comonotonicity.

Definition 1.1 *If x, y are elements of $\mathbb{R}^n$ then x, y are said comonotonic if $x_i < x_j$ implies that $y_i \leq y_j$.*

Two vectors x, y are comonotonic if they have the same ranking of their components or there exists a permutation σ defined on such that x, y are solution of the equation :

$$t_{\sigma(1)} \geq t_{\sigma(2)} \geq \ldots\ldots \geq t_{\sigma(n)}$$

Our representation result depend on the following axioms related to the concept of comonotonicity..

- **Comonotonic Monotonicity** If $\mathbf{x}, \mathbf{y} \in E^n$ are comonotonic and $x_i \leq y_i$ $(i = 1, \ldots, n)$ then $A(\mathbf{x}) \leq A(\mathbf{y})$

- **Comonotonic Separability** If $\mathbf{x}, \mathbf{y} \in E^n$are comonotonic then for every i, $A(x_{-i}, x_i) \geq A(y_{-i}, x_i)$ iff $A(x_{-i}, y_i) \geq A(y_{-i}, y_i)$

The comonotonic separability axiom is obviously a variation of an additive separability axiom and it has been applied successfully in decision making under risk and uncertainty . It states that preferences between alternatives depend only on the components that differ between the vectors under consideration, as long as these alternatives maintain the attributes' ordering.
In order to introduce a nonadditive approach to aggregation operators we introduce a nonadditive integral operator and so we consider the integral as a particular averaging operator Due to the context we restrict ourself to the discrete case and and for a comprehensive treatments of nonadditive integration we refer to [2]. As is well known the Choquet integral has been extensively applied in the context of decision under uncertainty.
In this context the Choquet integral may be viewed as a way of aggregating utility across different states in order to arrive to a decision criterion while in multicriteria decision making the nonadditive integral operator is a tool for aggregating over different criteria. The use of variants of the Choquet integral allows some flexibility in the way criteria are combined. In particular in this paper we consider a non-monotonic Choquet integral as in
As in [9] and [4] we define a non-monotonic Choquet measure on 2^N and a Choquet integral for a n-dimensional vector. We consider a finite index set $N = \{1, \ldots, n\}$ and we use the following definitions.

Definition 1.2 *A set function* $\upsilon : 2^N \to \mathbb{R}$ *with* $\upsilon(\emptyset) = 0$ *is called a non-monotonic nonadditive measure.* υ *is normalized if* $\upsilon(N) = 1$*. If* $A \subseteq B \subseteq N$ *implies that* $\upsilon(A) \leq \upsilon(B)$ *the function is said to be a nonadditive measure.*

We note that if $S \subseteq N$, $\upsilon(S)$ can be viewed as the importance of the set of elements S. Note also that non-monotonic nonadditive measures encompass probability measures, belief functions and capacity.
We introduce now the discrete Choquet integral on N viewed as aggregation function.

Definition 1.3 *Let* υ *a non-monotonic nonadditive measure* $\upsilon : 2^N \to \mathbb{R}$ *,* $x \in \mathbb{R}^n$*and*

$$\pi_{\sigma(j)} = \upsilon(\{\sigma(1), \ldots\ldots, \sigma(j)\}) - \upsilon(\{\sigma(1), \ldots\ldots, \sigma(j-1)\}) \tag{1}$$

where σ*is defined by (2.2). The non-monotonic Choquet integral of* x *is*

$$C_\mu(x) = \int x d\sigma = \sum_{j=1}^{n} \pi_j x_j \tag{2}$$

The Choquet integral has important properties for aggregation.

Proposition 1.1 *Let* A *an aggregation function defined on* E^n*.*

- *i) If A is a nonmonotone Choquet integral it is positive homogeneous, comonotonic monotone and comonotonic additive. If the measure is normalized then A is idempotent.*
- *ii) A is a nonmonotone Choquet integral if satisfy positive homogeneity and comonotonic separability.*
- *iii) A is a nonmonotone Choquet integral if is continuous and comonotonic additive.*

Proof The proof of part i) is immediate by the definition of Choquet integral.
Now it is important to note that a comonotonic separable aggregation function A is comonotonic additive that is if $\mathbf{x}, \mathbf{y} \in E^n$ are comonotonic then $A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$.
In fact if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E^n$ are comonotonic and $A(\mathbf{x}) \geq A(\mathbf{y})$ then by comonotonic separability $A(\mathbf{x} + \mathbf{z}) \geq A(\mathbf{y} + \mathbf{z})$. Then if we consider two comonotonic vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E^n$ there exist $c, d \in E$ such that $A(\mathbf{x}) = \mathbf{c} = (c, \ldots, c)$ and $A(\mathbf{y}) = \mathbf{d} = (d, \ldots, d)$. By the fact that each comonotonic vector is comonotonic with each other vector we can prove $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{c}+\mathbf{y}) = A(\mathbf{c}+\mathbf{d}) = A(\mathbf{c})+A(\mathbf{d}) = A(\mathbf{x})+A(\mathbf{y})$. Now we can conclude since a homogeneous function that satisfies comonotonic additivity is a nonmonotone Choquet integral by theorem 1 of [6] and a continuous function that satisfies comonotonic additivity is a nonmonotone Choquet integral by corollary 2.2 of [11].

It is elementary verified that if A satisfies the hypothesis of proposition 1.1 and it is also monotone then it is a Choquet integral.

2 Möbius transform

Each monotonic nonadditive measure $\mu(T)$ biunivocally corresponds to its *dual* set $\alpha(S) : 2^n \to R$, obtained by the Möbius transform, [7]:

$$\alpha(S) = \sum_{T \subseteq S} (-1)^{s-t} \mu(T), \forall T \subseteq S \tag{3}$$

The inverse transform is:

$$\mu(S) = \sum_{T \subseteq S} \alpha(S), \forall T \subseteq S \tag{4}$$

From the monotonicity conditions it follows that not any 2^n real values is the dual set of a monotonic measure μ, since the following constraints need to be satisfied, $\forall S \subseteq N, \forall i \in S$:

$$\alpha(\emptyset) = 0, \sum_{T \in N} \alpha(T) = 1, \sum_{T : i \in T \subseteq S} \alpha(T) \geq 0, \tag{5}$$

Moreover, for $n = 1$ we have immediately that $\mu(i) = \alpha(i), \forall i = 1, .., n$. The sign of the values α has a clear interpretation: a positive value of $\alpha(S)$ means that the coalition S is *synergic*, that is, the effect of such a coalition in the aggregated function in greater than then the sum of the effects of the singletons belonging to S. The contrary holds if $\alpha(S) < 0$, while if $\alpha(S) = 0$ there is no interactions among the criteria in S, and the Choquet integral collapses into the WA aggregation function (Weighted Averaging).

The Choquet integral (2), see [5], can be written in the dual space as:

$$C_\mu(x) = \sum_{T \subseteq N} \alpha(T)(\bigwedge_{i \in T} x_i) \tag{6}$$

This formulation is very useful in most of practical cases. If the measure is additive the values $\alpha(S)$ satisfy $-1 \le \alpha(S) \le 1$, and the *tendency to synergy* increases as most as $\alpha(S)$ tends to 1 (the contrary holds if $\alpha(S) < 0$). But in the case of non-monotonic measure, this constraints holds no more. Nevertheless, we can assert the following Proposition.

Proposition 2.1 *For every nonadditive non-monotonic measure $\mu(S)$, $S \subseteq N$, the dual values $\alpha(S)$ satisfy the following bounds:*

$$1 - 2^s \le \alpha(S) \le 2^s - 1 \tag{7}$$

where $s = card(S)$.

Proof From (3), the value $\alpha(S)$ is obtained by summing the measures included in T, $\mu(T)$, $T \subseteq S$, multiplied by -1 if $(s - t)$ is odd. Those measures are in the number of 2^n. For the upper bound, let us observe that, being null the measure of the empty set, it is sufficient to set $\mu(T) = 1$ if $(s - t)$ is even, and $\mu(T) = -1$ if $(s - t)$ is odd; this choice is admissible since the measure in **not** necessarily monotone. In so doing, all the 2^n terms in the summation are equal to one (except the null term corresponding to the empty set), and thus we obtain immediately the upper bound. For the lower one, we proceed in the same way, interchanging the sign between the terms with odd and even cardinality.

Clearly, also for non-monotonic measure, if $0 < \alpha(S) < 1$ we can say that we have a synergic *effect* and if $-1 < \alpha(S) < 0$ we have a *redundance* effect. But if $\alpha(S) < -1$ (and $\ge 1 - 2^n$) we have a *conflicting* effect, lower than the redundance one, see note (1). And if $\alpha(S) > 1$ the coalition exhibits a *reinforcement* effect, higher than the synergic effect. This effect can be observed if a set of criteria, considered as cost (benefit) by alone, that is, with negative *weights*, when considered together, appear as benefit. Of course, this fact can exist only with non-monotonic measures. To distinguish it from the *synergic* behavior, we name it as *super-synergic* effect.

3 Shapley index for criterion characterization: cost or benefit?

One of the most important problem when dealing with non additive measures is the concept of *relative importance* of a criterion. The following *Shapley* index, [5], [7], measures the average importance of a criterion, taking all the interactions with the other criteria into account. In this sense, it is an extension of the concept of *weight* for the WA approach. For a give criterion, the Shapley index averages all the marginal gains considering all the possible coalitions including the criterion. It is then normalizes between 0 and 1. If the Shapley index equals 0 it means that, in average, the "weight" of the criterion is null (it has no importance), while if it equals 1 it has a very high (relative) importance. In fact, it can happen that the measure of the *singleton* is very low, but when the criterion is merged inside a coalition including other ones, its marginal contribution becomes very high. The formal definition of the Shapley index is the following one:

Definition 3.1 *Let μ be a nonadditive monotonic measure, $\mu : 2^n \to [0,1]$, the Shapley index $v(i)$ for the $i-th$ criterion is given by:*

$$v(i) = \sum_{T \subseteq N \backslash i} \frac{(n-t-1)!t!}{n!} [\mu(T \cup i) - \mu(T)] \tag{8}$$

where $t = card(T)$.

The Shapley index varies between 0 (the criterion exhibits a *null* importance) and 1 (the criterion exhibits a *complete* importance). Indeed[1], this index in nothing else that the average marginal gain obtained to the insertion of the considered criterion in a coalition, and, given the monotonicity, each marginal contribution is in between 0 and 1.
But what happen for a *non-monotonic* measure? The Shapley index can be similarly defined, but in this case it belongs to $[-1, 1]$. The meaning of the sign is clear. For positive values, the corresponding criterion has to be considered, in average, as a *benefit*, conversely, for negative values, it represents a *cost*. In fact, given that the definition remains the same as for monotonic measures, we can assert the following Proposition.

Proposition 3.1 *For a non-monotonic measures μ, the Shapley values $v(i)$ of each criterion is in between -1 and $+1$.*

Proof From the definition (8), it holds for the $i-th$ criterion:

[1]The Shapley index can be computed also to every subset of criteria, instead that a sigle criterion only, obtaining the so called *Shapley interaction index*. An other index is the *Banzhaf* index. We do not address here such items, the interested reader can refer to [7].

$$v(i) = \sum_{T \subseteq N \backslash i} \frac{(n-t-1)!t!}{n!}[\mu(T \cup i) - \mu(T)] =$$

$$\frac{1}{n} \sum_{T \subseteq N \backslash i} \frac{((n-1)-t)!t!}{(n-1)!}[\mu(T \cup i) - \mu(T)] =$$

$$\frac{1}{n} \sum_{t=0}^{n-1} \{ \frac{1}{k(t)} \sum_{j=1}^{k(t)} [\mu(T \cup i) - \mu(T)] \}$$

being $k(t) = \frac{((n-1)-t)!t!}{(n-1)!}$, combinations of n objects taken t at a time, the number of subsets of $N \backslash i$ with cardinality t. Such a value is minimum as soon as $\mu(T \cup i) = 0, \mu(T) = 1$, that is if:

$$\mu(T) = 0, \forall T \subseteq N \backslash i$$
$$\mu(S) = 1, \forall S \subseteq N : i \in S$$

In this case, $\mu(T \cup i) - \mu(T) = -1$, $\forall T, i \notin T$. Exactly $k(t)$ coalitions exist with cardinality t, then we have $\sum_{j=1}^{k(t)} [\mu(T \cup i) - \mu(T)] = \sum_{j=1}^{k(t)} (-1) = -k(t)$, so $\frac{1}{k(t)} \sum_{j=1}^{k(t)} [\mu(T \cup i) - \mu(T)] = -1$ and then:

$$min_{\mu} v(i) = \frac{1}{n} \sum_{t=0}^{n-1} \{ \frac{1}{k(t)} \sum_{j=1}^{k(t)} [\mu(T \cup i) - \mu(T)] \} = -1 \tag{9}$$

The upper bound (+1) can be computed in the same way.
Given the construction of the index, its interpretation is obvious. If it equals +1, the corresponding "weight" has the possible maximum value, and the criterion is, in average with all the possible coalitions, a "complete" benefit. The contrary holds if $v(i)$ equals -1, that is, the $i-th$ criterion is a "complete" cost criterion. While if $v(i) = 0$ the importance of the criterion is null (in average). Of course, in all the other possible intermediate situation, if the value of the criterion is close to zero, if positive it is a benefit with low importance, if negative it is a cost (again, with low importance). The same if $v(i)$ is close to -1 or to 1. Thus we can answer to the question we posed at the beginning; a criterion cannot *a priori* be classified as a *benefit* or a *cost*. Its effect depends on its marginal contribution (positive or negative) to all the coalitions including itself. If, in average, this marginal contribution is positive, we can say that it is a benefit, on the contrary, it represents a cost.

Consider the following example. We have 3 criteria, a, b, c and the following measure is defined over every of the $2^n = 2^3 = 8$ possible coalition:

$\mu(\emptyset) = 0, \mu(a) = 0.4, \mu(b) = 0.5, \mu(c) = 0.6, \mu(a,b) = 1, \mu(a,c) = -0.1, \mu(b,c) = -0.8, \mu(a,b,c) = 0$

Clearly, the measure in non monotone. A simple computation shows:

$$v(1) = 0.4 + \frac{1}{2}(1-1) = 0.4$$
$$v(2) = 0.5 + \frac{1}{2}(1-0.8) = 0.4$$
$$v(3) = 0.6 + \frac{1}{2}(-0.8-1) = -0.3$$

Thus the third criterion, being considered as a benefit is taken by alone (its weight is 0.6) has a negative Shapley index, meaning that, in averaging, it acts as a cost, decreasing the value of the coalitions including it. Such an information can be useful when dealing with contrasting criteria, since the sentence "this criterion is a benefit" maybe supported by its "relative importance" can be uncorrect and due to a myopic view; the "importance" of a criterion, and thus its "true nature" as benefit or as cost needs to be evaluated considering **all** the possible coalition including it.

4 Concluding remarks

In this paper we described the the Choquet integral with non-monotonic measures. We give an axiomatic characterization, and showed how also this operator can be computed in the dual space. Next we showed how the Shapley index can be used also to measure *how much* a criterion can be in average intended as a benefit or a cost. As a future improvement we intend to define an algorithmic procedure to identify the value of the measures using an implicit approach based on suitably designed questionnaire, following the idea proposed by [3] and recently applied in a real world application, see ([1]).

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