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What do distortion risk measures tell us on excess of loss reinsurance with reinstatements ?

What do distortion risk measures tell us on excess of loss reinsurance with reinstatements?*

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Abstract. In this paper we focused our attention to the study of an excess of loss reinsurance with reinstatements, a problem previously studied by Sundt [5] and, more recently, by Mata [4] and Hürlimann [3]. As it is well-known, the evaluation of pure premiums requires the knowledge of the claim size distribution of the insurance risk: in order to face this question, different approaches have been followed in the actuarial literature. In a situation of incomplete information in which only some characteristics of the involved elements are known, it appears to be particularly interesting to set this problem in the framework of risk adjusted premiums. It is shown that if risk adjusted premiums satisfy a generalized expected value equation, then the initial premium exhibits some regularity properties as a function of the percentages of reinstatement.

Keywords: Excess of loss reinsurance; reinstatements; distortion risk measures; expected value equation.

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1 Introduction

In recent years the study of excess of loss reinsurance with reinstatements has become a major topic, in particular with reference to the classical evaluation of pure premiums which is based on the collective model of risk theory.

The problem, previously studied by Sundt [5] and, more recently, by Mata [4] and Hürlimann [3], requires the evaluation of pure premiums given the knowledge of the claim size distribution of the insurance risk: in order to face this question, different approaches have been followed in the actuarial literature. Sundt [5] based the computation on the Panjer recursion numerical method and Hürlimann [3] provided distribution-free approximations to pure premiums.

In a situation of incomplete information in which only some characteristics of the involved elements are known, it appears to be particularly interesting to set this problem in the framework of risk adjusted premiums.

We start from the methodology developed by Sundt [5] to price excess of loss reinsurance with reinstatements for pure premiums and, with the aim of relaxing the basic hypothesis made by Walhin and Paris [6] who calculated the initial premium P under the Proportional Hazard transform premium principle, we address our analysis to the study of the role played by risk adjusted premium principles. The particular choice in the proposal of Walhin and Paris of the PH-transform risk measure strengthens our interest in the study of risk adjusted premiums which belong to the class of distortion risk measures defined by Wang [7].

In the mathematical model we studied (for more details see Campana [1]), when the reinstatements are paid ($0 \leq c_i \leq 1$ is the i-th percentage of reinstatement) the total premium income $\delta(P)$ becomes a random variable which is correlated to the aggregate claims S. Since risk measures satisfy the properties of linearity and additivity for comonotonic risks (see [2]) and layers are comonotonic risks, we can define the function

$$
\begin{aligned}
F(P, c_1, c_2, \ldots, c_K) \quad &= P \left[1 + \tfrac{1}{m} \sum_{i=0}^{K-1} c_{i+1} \, W_{g_1}(L_X(im, (i+1)m)) \right] - \\
&\quad - \sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m))
\end{aligned}
\tag{1}
$$

where g_1 and g_2 are distortion functions. This function gives a measure of the distance between the total premium income $\delta(P)$ and the distortion risk measures of the aggregate claims S. It is shown that if risk adjusted premiums satisfy the expected value equation, that is the previous distance is null, then the initial premium exhibits some regularity properties as a function of the percentages of reinstatement.

The paper is organized as follows. In Section 2 we first review some basic settings for describing the excess of loss reinsurance model and we remind some definitions and preliminary results in the field of non-proportional reinsurance covers. Section 3 is devoted to the problem of detecting the total initial premium: we present the study of the case in which the reinstatements are paid in order to consider the total premium income as a random variable which is correlated to the aggregate claims. The analysis is set in the framework of distortion risk measures: some basic definitions and results in this field are recalled. Section 4 presents the main results related to the problem of measuring the total initial premium as a function of the percentages of reinstatement, dependence that it is generally neglected in the literature. Some concluding remarks in Section 5 end the paper.

2 Excess of loss reinsurance with reinstatements: problem setting

The excess of loss reinsurance model we study in this paper is related to the model that has been proposed and analyzed by Sundt [5]. Some notations, abbreviations and conventions used throughout the paper are the following.

An excess of loss reinsurance for the layer m in excess of d, written m xs d, is a reinsurance which covers the part of each claim that exceeds the deductible d but with a limit on the payment of each claim, which is set equal to m; in other words, the reinsurer covers for each claim of size Y the amount

$$L_Y(d, d+m) = \min\{(Y-d)_+, m\}$$

where $(a)_+ = a$ if $a > 0$, otherwise $(a)_+ = 0$.

We consider an insurance portfolio: N is the number of claims occurred in the portfolio during the reference year and Y_i is the i-th claim size $(i = 1, 2, \ldots, N)$. The aggregate claims to the layer is the random sum given by

$$X = \sum_{i=1}^{N} L_{Y_i}(d, d+m).$$

It is assumed that $X = 0$ when $N = 0$. An excess of loss reinsurance, or for short an XL reinsurance, for the layer m xs d with aggregate deductible D and aggregate limit M covers only the part of X that exceeds D but with a limit M:

$$L_X(D, D+M) = \min\{(X-D)_+, M\}.$$

This cover is called an XL reinsurance for the layer m xs d with aggregate layer M xs D.

Generally it is assumed that the aggregate limit M is given as a whole multiple of the limit m, i.e. $M = (K+1)m$: in this case we say that there is a limit in the number of the losses covered by the reinsurer. This reinsurance cover is called an XL reinsurance for the layer m xs d with aggregate deductible D and K reinstatements and provides total cover for the following amount

$$L_X(D, D+(K+1)m) = \min\{(X-D)_+, (K+1)m\}. \tag{2}$$

Let P be the initial premium: it covers the original layer, that is

$$L_X(D, D+m) = \min\{(X-D)_+, m\}. \tag{3}$$

It can be considered as the 0-th reinstatement.

The condition that the reinstatement is paid pro rata means that the premium for the i-th reinstatement is a random variable given by

$$\frac{c_i P}{m} L_X(D+(i-1)m, D+im) \tag{4}$$

where $0 \leq c_i \leq 1$ is the i-th percentage of reinstatement. If $c_i = 0$ the reinstatement is free, otherwise it is paid.

The related total premium income is a random variable, say $\delta(P)$, which is so defined

$$\delta(P) = P \left(1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} L_X(D + im, D + (i+1)m) \right). \tag{5}$$

From the point of view of the reinsurer, the aggregate claims S paid by the reinsurer for this XL reinsurance treaty, namely

$$S = L_X(D, D + (K+1)m) \tag{6}$$

satisfy the relation

$$S = \sum_{i=0}^{K} L_X(D + im, D + (i+1)m). \tag{7}$$

3 Initial premium, aggregate claims and distortion risk measures

The total premium income $\delta(P)$ results to be a random variable which is correlated to the aggregate claims S in the case in which the reinstatements are paid. Then it follows that it is not obvious how to calculate the initial premium P.

Despite its importance in practice, only recently some Authors have moved their attention to the study of techniques to calculate the initial premium: more precisely, Sundt [5] proposed the methodology to calculate the initial premium P under pure premiums and premiums loaded by standard deviation principle.

Looking at the pure premium principle for which the expected total premium income should be equal to the expected aggregate claims payments

$$E[\delta(P)] = E[S] \tag{8}$$

it is quite natural to consider the case in which premium principles belong to more general classes: with the aim of plugging this gap, we focus our attention to the class of distortion risk measures. Our interest is supported by Walhin and Paris [6] who calculated the initial premium P under the Proportional Hazard transform premium principle. Even if their analysis is conducted by a numerical recursion, the choice of the PH-transform risk measure as a particular concave distortion risk measure strengthens our interest.

Furthermore, in an excess of loss reinsurance with reinstatements the computation of premiums requires the knowledge of the claim size distribution of the insurance risk: with reference to the expected value equation of the XL reinsurance with reinstatements (8), Sundt [5] based the computation on the Panjer recursion numerical method and Hürlimann [3] provided distribution-free approximations to pure premiums.

Note that both the Authors assumed only the case of equal reinstatements, a particular hypothesis on basic elements characterizing the model.

In this paper we set our analysis in the framework of distortion risk measures: the core of our proposal is represented by the choice of a more general equation characterizing the excess of loss reinsurance with reinstatements, in such a way that it is possible to obtain some general properties satisfied by the initial premium as a function of the percentages of reinstatement. In order to present the main results, we recall some basic definitions and results.

3.1 Distortion risk measures

A risk measure is defined as a mapping from the set of random variables, namely losses or payments, to the set of real numbers. In actuarial science common risk measures are premium principles; other risk measures are used for determining provisions and capital requirements of an insurer in order to avoid insolvency (see e.g. Dhaene et al. [2]).

In this paper we consider the distortion risk measure introduced by Wang [7]:

$$W_g(X) = \int_0^\infty g(H_X(x))dx \tag{9}$$

where the distortion function g is defined as a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$. As it is well-known, the *quantile risk measure* and the *Tail Value-at-Risk* are examples of risk measures belonging to this class. In the particular case of a power g function, i.e. $g(x) = x^{1/\rho}$, $\rho \geq 1$, the corresponding risk measure is the *PH-transform risk measure*, that is the choice made by Walhin and Paris [6].

Distortion risk measures satisfy the following properties (see Wang [7] and Dhaene et al. [2]):

P1. Additivity for comonotonic risks

$$W_g(S^c) = \sum_{i=1}^n W_g(X_i) \tag{10}$$

where S^c is the sum of the components of the random vector $\mathbf{X^c}$ with the same marginal distributions of $\mathbf{X}$ and with the comonotonic dependence structure.

P2. Positive homogeneity

$$W_g(a\,X) = a\,W_g(X) \quad \text{for any non-negative costant } a; \tag{11}$$

P3. Translation invariance

$$W_g(X + b) = W_g(X) + b \quad \text{for any costant } b; \tag{12}$$

P4. Monotonicity

$$W_g(X) \leq W_g(Y) \tag{13}$$

for any two random variables X and Y where $X \leq Y$ with probability 1.

In the particular case of a concave distortion measure, the related distortion risk measure satisfying properties *P1-P4* is also sub-additive and it preserves stop-loss order. As it is well-known, examples of concave distortion risk measures are the *Tail Value-at-Risk* and the *PH-transform risk measure*, whereas *quantile risk measure* is not a concave risk measure.

4 Risk adjusted premiums

With reference to equation (8) we consider the new expected value condition

$$W_{g_1}(\delta(P)) = W_{g_2}(S) \tag{14}$$

where g_1 e g_2 are distortion functions.

It expresses the fact that the value of the total premium income $\delta(P)$ equals the distorted expected value of the aggregate claims S. Note that non necessarily the distortion functions g_1 and g_2 are the same.

We consider the previous equilibrium condition an equation on the initial premium P: if it admits a solution which is unique, we call *initial risk adjusted premium P* the solution.

Proposition 1 *Given an XL reinsurance with K reinstatements and distortion functions g_1 and g_2, the initial risk adjusted premium P which is the unique solution of equation*

$$W_{g_1}(\delta(P)) = W_{g_2}(S)$$

satisfies the following properties:

i) *P is a decreasing function of each percentage of reinstatement c_i ($i = 1, \ldots, K$);*

ii) *P is a convex, supermodular, quasiconcave and quasiconvex function of the percentages of reinstatement $c_1, c_2, \ldots, c_K$.*

Proof

By hypothesis, the initial risk adjusted premium P is the solution, if it exists and it is unique, of equation (14), that is of the equilibrium condition in which the distorted expected premium income equals the distorted expected claim payments.

Since the layers $L_X(im, (i+1)m)$, $i = 1, 2, \ldots, K+1$, are comonotonic risks from (7) we find

$$W_{g_2}(S) = \sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m)). \tag{15}$$

From (5) by assuming the absence of aggregate deductible (i.e. $D = 0$) we have

$$W_{g_1}(\delta(P)) = P\left(1 + \frac{1}{m}\sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m))\right). \tag{16}$$

Therefore, the initial premium P is well-defined and it is given by

$$P = f(c_1, c_2, \cdots, c_K) = \frac{\sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m))}{1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m))}. \tag{17}$$

Clearly the function f is a decreasing function of any percentage of reinstatement c_i (where $i = 1, \ldots, K$).

Moreover, if we set

$$A = \sum_{i=0}^{K} W_{g_2}(L_X(im, (i+1)m)),$$

the gradient vector $\nabla f(\underline{c})$ is

$$\nabla f(\underline{c}) = \left(\frac{\partial f}{\partial c_l}(\underline{c}) \right) = \left(\frac{-A \, W_{g_1}(L_X((l-1)m, lm))}{m \left[1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m)) \right]^2} \right)$$

for each $l = 1, \ldots, K$.

Convexity follows by the strict positivity and concavity of the function

$$1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m)).$$

Moreover, the Hessian matrix $H_f(\underline{c})$ of the function f is given by

$$H_f(\underline{c}) = \left(\frac{\partial^2 f}{\partial c_l \partial c_n}(\underline{c}) \right) = \left(\frac{2A \, W_{g_1}(L_X((l-1)m, lm)) \, W_{g_1}(L_X((n-1)m, nm))}{m^2 \left[1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m)) \right]^3} \right)$$

for each $l, n = 1, \ldots, K$. More compactly it can be expressed as

$$H_f(\underline{c}) = \left(\; W_{g_1}(L_X((l-1)m, lm)) \, W_{g_1}(L_X((n-1)m, nm)) \; \right) B$$

for each $l, n = 1, \ldots, K$, where

$$B = \frac{2A}{m^2 \left[1 + \frac{1}{m} \sum_{i=0}^{K-1} c_{i+1} W_{g_1}(L_X(im, (i+1)m)) \right]^3}.$$

Clearly, $H_f(\underline{c})$ is nonnegative definite.

Given that any cross partial derivative of the matrix $H_f(\underline{c})$ is nonnegative, the function g is supermodular.

Finally, the initial risk adjusted premium P is both quasiconcave and quasiconvex function of the percentages of reinstatement $c_1, c_2, \ldots, c_K$ because it is ratio of affine functions.

∎

Remark 1. Note that the regularity properties exhibited by the initial risk adjusted premium P are not influenced by equality/inequality conditions between the two distortion functions g_1 and g_2. Moreover, any hypothesis on concavity/convexity of distortion risk measures may be omitted because they are unnecessary to prove the smooth shape of the initial premium P as a function of $c_1, c_2, \ldots, c_K$.

Remark 2. The reinsurance companies often assess treaties under the assumption that there are only total losses. This happens, for example, when they use the rate on line method to price catastrophe reinsurance. Then it follows that the aggregate claims are generated by a discrete distribution and we have that (for more details see Campana ([1])

$$P = f(c_1, c_2, \cdots, c_K) = \frac{m \sum_{i=0}^{K} g_2(p_{i+1})}{1 + \sum_{i=0}^{K-1} c_{i+1} \, g_1(p_{i+1})} \tag{18}$$

where the premium for the i-th reinstatement (4) is a two-point random variable distributed as $c_i \, P \, B_{p_i}$ and B_{p_i} denotes a Bernoulli random variable such that

$$Pr[B_{p_i} = 1] = p_i = 1 - Pr[B_{p_i} = 0].$$

Then

$$\nabla f(\underline{c}) = \left(\frac{\partial f}{\partial c_l}(\underline{c}) \right) = \left(\frac{-m \sum_{i=0}^{K} g_2(p_{i+1})}{\left[1 + \sum_{i=0}^{K-1} c_{i+1} g_1(p_{i+1}) \right]^2} \, g_1(p_l) \right)$$

and

$$H_f(\underline{c}) = \left(\frac{\partial^2 f}{\partial c_l \partial c_n}(\underline{c}) \right) = \left(\frac{2m \sum_{i=0}^{K} g_2(p_{i+1})}{\left[1 + \sum_{i=0}^{K-1} c_{i+1} g_1(p_{i+1}) \right]^3} \, g_1(p_l) g_1(p_n) \right)$$

for each $l, n = 1, \ldots, K$.

5 Concluding remarks

In Actuarial Literature excess of loss reinsurance with reinstatement has been essentially studied in the framework of collective model of risk theory for which the classical evaluation of pure premiums requires the knowledge of the claim size distribution. Generally, in practice, there is incomplete information: only few characteristics of the aggregate claims can be computed. In this situation the interest for general properties characterizing the involved premiums is flourishing.

Setting this problem in the framework of risk adjusted premiums, it is shown that if risk adjusted premiums satisfy a generalized expected value equation, then the initial premium exhibits some regularity properties as a function of the percentages of reinstatement. In this way it is possible to relax the particular choice made by Walhin and Paris [6] of the

PH-transform risk measure and to extend the analysis of excess of loss reinsurance with reinstatements to cover the case of not necessarily equal reinstatements.

The obtained results suggest that further research may be addressed to the analysis of optimal premium plans.

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