

# CONTINUOUS WORKOUT MORTGAGES

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# Continuous Workout Mortgages\*

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## Abstract

This paper models Continuous Workout Mortgages (CWMs) in an economic environment with refinancings and prepayments by employing a market-observable variable such as the house price index of the pertaining locality. Our main results include: (a) explicit modelling of repayment and interest-only CWMs; (b) closed form formulae for mortgage payment and mortgage balance of a repayment CWM; (c) a closed form formula for the actuarially fair mortgage rate of an interest-only CWM. For repayment CWMs we extend our analysis to include two negotiable parameters: adjustable “workout proportion” and adjustable “workout threshold.” These results are of importance as they not only help understanding the mechanics of CWMs and estimating key contract parameters. These results are of importance as they not only help in the understanding of the mechanics of CWMs and estimating key contract parameters, but they also provide guidance on how to enhance the resilience of the financial architecture and mitigate systemic risk.

**Keywords:** Continuous Workout Mortgage (CWM), Repayment, Interest-only, House price index, Prepayment intensity, Cap and floor on continuous flow.

**JEL:** C63, D11, D14, D92, G13, G21, R31.

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# 1 Introduction

The ongoing crisis has exposed the vulnerability of the most sophisticated financial structures to systemic risk. This crisis—emanating from mortgage loans to borrowers with high credit risk—has devastated the capital base of financial intermediaries on both sides of the Atlantic. Its impact on the real sector of the economy has given rise to a fear and uncertainty not seen since the Great Depression of the 1930s (see Reinhart and Rogoff, (2008); and Diamond and Rajan, (2008)).

The fragility of the financial intermediaries stems from the rigidity of the traditional mortgage contracts such as the Fixed Rate Mortgages (FRMs), Adjustable Rate Mortgages (ARMs) and their hybrids (see Stiglitz (1988); Campbell and Cocco (2003); Green and Wachter (2005); and Reinhart and Rogoff (2008)). Modigliani (1974, p. 1) reiterates this quite strongly when he states that: *“As long as loan contracts are expressed in conventional nominal terms, a high and variable rate of inflation — or more precisely a significant degree of uncertainty about the future of the price level — can play havoc with financial markets and interfere seriously with the efficient allocation of the flow of saving and the stock of capital.”* This necessitates reforming the financial architecture with facilities that can better absorb shocks in the financial system in order to make it more resilient and mitigate systemic risk.

One such solution studied in this paper is that of Continuous Workout Mortgages (CWMs) as expounded in Shiller (2008a), (2009b), (2008b) and Feldstein (2009). This facility eliminates the expensive workout of a defaulting rigid plain vanilla mortgage contract. CWMs share the price risk of a home with the lender and thus provide automatic adjustments for changes in home prices or incomes, for all home owners. Mortgage balances are thus adjusted and monthly payments are varied automatically with changing home prices. This feature eliminates the incentive to rationally exercise the costly option to default which, by construction, is embedded in every loan contract. Despite sharing

the underlying risk, the lender continues to receive an uninterrupted stream of monthly payments. Moreover, this can occur without multiple and costly negotiations.

Unfortunately, prior to the current crisis CWMs have never been considered. The academic literature (with the exception of Ambrose and Buttimer (2010), i.e., A&B hereafter) therefore has not discussed its mechanics and especially its design. Shiller is the first researcher, who forcefully articulates the exigency of its employment as stated above. CWMs are conceived in his (2008b) and (2009b) studies<sup>1</sup> as an extension of the well-known Price-Level Adjusted Mortgages (PLAMs), where the mortgage contract adjusts to a narrow index of local home prices instead of a broad index of consumer prices. In a recent study Ambrose and Buttimer (2010) numerically investigate properties of Adjustable Balance Mortgages which bear many similarities to CWMs. Our simple and fully analytic model complements that of a more intricate one of Ambrose and Buttimer (2010). We rely on methodology which allows to value optional continuous flows (see e.g. Carr, Lipton and Madan (2000) and Shackleton & Wojakowski (2007)).

For lenders to hedge risks, continuous workout mortgages thus need indicators and markets for home prices and incomes. These markets and instruments already exist.<sup>2</sup> For example, the Chicago Mercantile Exchange (CME) offers options and futures on single-family home prices. An example of this are the Real Estate MacroShares launched on NYSE in May 2009. Furthermore, reduction of moral hazard incentives require inclusion of home-price index of the neighborhood into the monthly payment formula. This is to prevent moral hazard stemming from an individual failing to maintain or damaging the property just to reduce mortgage payments. Likewise, occupational income indicators matter along with the borrower's actual income. This is again to prevent moral hazard stemming from an individual deliberately losing a job just to reduce mortgage payments.

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<sup>1</sup>See Shiller (2003) for home equity insurance.

<sup>2</sup>Creation of new markets for large macroeconomic risks is discussed in Shiller (1993). For derivatives markets for home prices see Shiller (2009a).

We feel that CWMs retain the positive attribute of PLAMs in reducing purchasing power risk (see Leeds (1993)). They, however, improve upon PLAMs by reducing prepayment, default and interest rate risks, by the very nature of the contract where the borrower shares in the risk of the housing market. This implies that borrowers participate in the appreciation of the property during any positive economic event such as an interest rate decrease, which mitigates prepayment risk. Borrowers, nonetheless, participate in the depreciation of the property during any negative economic event, thereby reducing default risk. This is because a CWM aims to keep the mortgage balance always lower than or equal to the value of the property, thus keeping the embedded default option (at or) out of the money at all times. Therefore, during the tenure of the mortgage which is comprised of periods of varying economic cycles (including changing interest rates) CWMs are anticipated to defy associated risks. The only risk which CWMs (like PLAMs) currently cannot cure is liquidity risk. This, however, can be expected to alleviate over time by their securitization and eventual deployment (in sufficient volume).

We implicitly assume the existence of an information infrastructure, where property rights, foreclosure procedures (needed for real estate to serve as collateral) and accurate method of valuing property are well established (see Levine, Loayza and Beck (2000)). We also draw parallels (or similarities) between CWMs and Shared Income Mortgages (SIMs) discussed in Ebrahim, Shackleton and Wojakowski (2009). This helps us derive closed form solutions to price variants of CWMs based on observable inputs of local home prices. Finally, we provide examples, where exogenous variables governing CWMs imputed from real world observations involving individual home prices, house price indices, individual incomes, occupational income indicators, etc.

This paper is structured as follows. Section 2 initiates the pricing of CWM with a simple model involving an interest only contract. This is extended to the amortizing CWM in Section 3 and to CWMs with protective threshold in Section 4. Finally, we provide

concluding comments in Section 7.

## 2 Interest-only Continuous Workout Mortgage

To begin, we provide a simple example where the payment of a CWM is continuously readjusted based on the current house price index of a location. To simplify our setup we focus on interest only mortgages with repayment maturity  $T$ . Property is acquired at time  $t = 0$  for initial cost  $H_0$ . Subsequent values of the house are not observable until the time of its sale. In lieu of this information a house price *index* of the area  $\bar{H}_t$  is observed for all times  $t \geq 0$ . To make things even simpler we assume that: (i) the loan to value ratio is 100%; (ii) the constant, risk free market interest rate is  $r$ ; and (iii)  $i$  is the mortgage rate. If the mortgage were an “ordinary” interest only mortgage, the borrower would repay the continuous interest flow  $iH_0dt$  plus a lump sum principal repayment  $H_0$  at maturity. The risk free discounted present value of such payments equals

$$V = \int_0^T e^{-rt} iH_0 dt + e^{-rT} H_0 = i \frac{H_0}{r} (1 - e^{-rT}) + e^{-rT} H_0 . \quad (1)$$

For an actuarially fair loan this present value should be equal to the initially borrowed amount, i.e.  $V = H_0$ , which gives  $i = r$ .

Now consider the home value weighted by the local home price index  $\bar{H}_t$  and its initial value  $\bar{H}_0$

$$h_t = \frac{H_0}{\bar{H}_0} \bar{H}_t . \quad (2)$$

A continuous workout mortgage can be designed to protect the homeowner against falls in the home value and the consequence of potential default. Starting backwards from maturity  $T$ , implies the repayment  $H_0$  in this contract is replaced with

$$\min \{h_0, h_T\} = h_0 - \max \{0, h_0 - h_T\} = h_0 - (h_0 - h_T)^+ . \quad (3)$$

That is, the initial balance plus a *short* position in a put option on a house price index with a strike price of  $h_0 = H_0$ . Similarly, for the interval  $0 < t < T$ , the intermediate balance of the loan on which interest payments are based can be adjusted to

$$\min \{h_0, h_t\} = h_0 - \max \{0, h_0 - h_t\} = h_0 - (h_0 - h_t)^+ , \quad (4)$$

in lieu of  $H_0$ . Substituting the above relationships the formula giving the expected present value of the loan becomes

$$\bar{V} = V - \frac{i}{r} \int_0^T E \left[ e^{-rt} (rh_0 - rh_t)^+ \right] dt - E \left[ e^{-rT} (h_0 - h_T)^+ \right] . \quad (5)$$

The value  $\bar{V}$  is an expected present value as  $h_t$  and  $h_T$  are random variables contingent on a house price index. Expectations ( $E$ ) are computed under risk-neutral measure with discounting at the riskless rate. Risk neutral valuation is employed because the house price index  $\bar{H}_t$  is traded.

Note that  $\bar{V}$  is less than  $V$  as the consumer obtains extra protection against price falls. The last term is in fact a *protective put* issued at the money. With appropriate assumptions regarding the dynamics of the house price index  $\bar{H}_t$ , its value,  $p(h_0, h_0, T, r, \delta, \sigma)$ , can be expressed using the Black-Scholes (1973) formula. Parameters  $\delta, \sigma$  are, respectively, the service flow and volatility of the house price index (see Appendix A). Likewise, the intermediate term is equal to the ratio of interest rates  $i/r$  times an integral of put options written on a continuum of maturities from  $t = 0$  to  $t = T$ . This integral is in fact a *floor* issued at the money written on *flow*  $rh_t$ . The value of this *floor*,  $P(rh_0, rh_0, T, r, \delta, \sigma)$ , can also be computed analytically employing the closed form expressions developed in Shackleton and Wojakowski (2007) (see Appendix A).

The goal is now to establish the actuarially fair mortgage rate  $i$  for this CWM. This calculation is of utmost importance for prospective originators of continuous workouts. While the mortgage balance adjusts systematically and automatically, the mortgage rate  $i$

should be established only once as a part of the contract, remaining constant until maturity. Using  $V$  from (1) and setting  $\bar{V} = H_0$  and  $h_0 = H_0$  yields

$$H_0 = i \frac{H_0}{r} (1 - e^{-rT}) + e^{-rT} H_0 - \frac{i}{r} P(rH_0, rH_0, T, r, \delta, \sigma) - p(H_0, H_0, T, r, \delta, \sigma), \quad (6)$$

where

$$P(r\bar{H}_0, r\bar{H}_0, T, r, \delta, \sigma) = r \int_0^T p(\bar{H}_0, \bar{H}_0, t, r, \delta, \sigma) dt. \quad (7)$$

Solving (6) for  $i$  gives the mortgage interest rate.

**Proposition 1** *An interest-only Continuous Workout Mortgage (CWM) with protective put and floor has the mortgage interest rate  $i$  equal to*

$$i = r \left[ \frac{H_0 (1 - e^{-rT}) + p(H_0, H_0, T, r, \delta, \sigma)}{H_0 (1 - e^{-rT}) - P(rH_0, rH_0, T, r, \delta, \sigma)} \right]. \quad (8)$$

The above formula confirms our intuition on the constitution of a continuous workout mortgage. If both protective put ( $p$ ) and floor ( $P$ ) are deep out of the money (as can be the case when house prices are expected to strongly appreciate), then  $p, P \rightarrow 0$  and  $i \rightarrow r$ . However, when this is not the case and the market anticipates a possibility of a price fall (as can be the case following a bubble), both put and floor are in the money and  $i > r$  on new deals. This implies that the market prices the insurance offered by the continuous workout as an *ex ante* increase of the mortgage rate  $i$ .

PUT FIGURE 1 AND TABLE 1 ABOUT HERE

Furthermore, the required interest  $i$  increases with risk  $\sigma$  consistent with option pricing theory. Interestingly, increasing risk even to  $\sigma = 30\%$  does not have a great impact on  $i$ , as illustrated in Figure 1 and Table 1. For a typical loan with  $T = 30$  years to maturity, riskless rate  $r = 5\%$  and a service flow  $\delta = 1\%$  the impact amounts to only an increase of 0.37% above 5%. Surprisingly, insuring the house price level is *less* costly as interest rates



become higher. However, such guarantees are more expensive when time to maturity is short and when the service flow is higher. The maturity effect is due to all options (i.e. the put and the floor) in this example being issued at-the-money, thereby insuring price levels of the property at the time of issuance. More time to maturity implies that the house has more chance to appreciate and the guarantee to depreciate. Likewise, the service flow effect is due to intensive property use and lower maintenance, which decreases its value making guarantee more expensive. The  $\delta$  represents a service flow to the occupier that erodes capital gains in the home value  $H$  and therefore cannot be shared with the lender. The higher the interest, the lower the impact of service flow and thus the lower the vulnerability to moral hazard.

Table 1 gives numerical values of annual payments when the risk free interest rate is  $r = 5\%$ . Interest-only continuous workouts are compared to the corresponding fixed-rate mortgages for different values of service flow,  $\delta = 1\%$  and  $\delta = 4\%$ . Note that fixed rate ( $r = 5\%$ ) mortgage payments do not depend on the service flow.

From an analytical perspective, equation (8) is equivalent to the riskless rate  $r$  times a fraction. To facilitate this interpretation, we re-write it as

$$i = r \left[ \frac{H_0 e^{-rT} - H_0 - p(H_0, H_0, T, r, \delta, \sigma)}{P(rH_0, rH_0, T, r, \delta, \sigma) - \frac{rH_0}{r} (1 - e^{-rT})} \right]. \quad (9)$$

The numerator of the fraction represents the present value  $H_0 e^{-rT}$  of the forward sale of the house at price  $H_0$ , minus the initial value of the house  $H_0$ , minus the cost of the put protection  $p(H_0, H_0, T, r, \delta, \sigma)$  to be paid through an increased mortgage interest rate  $i$ . The denominator, meanwhile, represents intermediate period cash flows where interest payments accommodate the cost of insurance. It is equal to the value of the floor  $P(rH_0, rH_0, T, r, \delta, \sigma)$  written on continuous flow  $rH_0$  minus the value of annuity  $\frac{rH_0}{r} (1 - e^{-rT})$ , based again on the same continuous flow  $rH_0$ . Intuitively, this annuity is still paid at rate  $r$  with the interest on difference  $r(H_0 - H_t)$  added back to

$P(rH_0, rH_0, T, r, \delta, \sigma)$  when  $H_t < H_0$ .

From an economic perspective the above fraction is a ratio of two values, each corresponding to a set of transactions. The first set of transactions—collected in the numerator of (9)—occur at the beginning and at the end of the interval  $[0, T]$ . The appropriate marginal discount rate for this protected forward sale is  $r$ . By contrast, transactions within the interval  $(0, T)$ —collected in the denominator of (9)—must earn the marginal rate  $i > r$  to offset the added insurance.<sup>3</sup> In equilibrium, both marginal costs must be equal yielding  $i$ .

**Proposition 2** *An interest-only Continuous Workout Mortgage (CWM) has mortgage interest  $i$  and mortgage interest spread  $\frac{i-r}{r}$  independent of initial property value  $H_0$  equal to*

$$i = r \left[ \frac{1 - e^{-rT} + p(1, 1, T, r, \delta, \sigma)}{1 - e^{-rT} - P(r, r, T, r, \delta, \sigma)} \right] \quad (10)$$

and

$$\frac{i - r}{r} = \frac{p(1, 1, T, r, \delta, \sigma) + P(r, r, T, r, \delta, \sigma)}{1 - e^{-rT} - P(r, r, T, r, \delta, \sigma)}. \quad (11)$$

**Proof.** Use (8) and the homogeneity properties for put and floor

$$p(\alpha S, \alpha K, T, r, \delta, \sigma) = \alpha p(S, K, T, r, \delta, \sigma), \quad P(\alpha s, \alpha k, T, r, \delta, \sigma) = \alpha P(s, k, T, r, \delta, \sigma), \quad (12)$$

where  $\alpha$  is a constant,  $S, K$  are values [\$] and  $s, k$  are flows [\$/yr.]. For  $\alpha = H_0$  the initial property value,  $H_0$ , cancels out from the numerator and denominator of (8). ■

Property (10) is very useful as it allows quoting mortgage interest rate  $i$  to a (potential) borrower without knowing the value of the house to be purchased. This estimation is based on the maturity of the loan, several market parameters and only needs the information that the house value is determined at arms length. With the exception of the riskless

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<sup>3</sup>In the current CWM insurance is added against random decline of terminal property value  $H_T$  as well as against intermediate falls in property value  $H_t$ ,  $t < T$ , that negatively impact on the value of interest payments before maturity.

interest rate  $r$ , the parameters—such as the (average) service flow ( $\delta$ ) and the house price volatility ( $\sigma$ ) of the locality—relate to the dynamics of the house price index and can be estimated from data. Likewise, the second property (11) provides a useful estimation of the mortgage spread.

### 3 Repayment Continuous Workout Mortgage

A major innovation during years of Great Depression was fully amortizing mortgages (see Green and Wachter (2005)). Their repayment flow rate ( $R$ ) is constant in time and their balance  $Q_t$  decreases to become zero at maturity. That is,  $Q_T = 0$ . The amount owed to the lender

$$Q_t = \int_t^T R e^{-r(s-t)} ds = \frac{R}{r} \left(1 - e^{-r(T-t)}\right) \quad (13)$$

is equal to the present value of remaining payments and is essentially an accounting identity. The dynamics of  $Q_t$  is fully deterministic and is described by the ordinary differential equation

$$\frac{dQ_t}{dt} = rQ_t - R \quad (14)$$

(with terminal condition  $Q_T = 0$ ). It implies a growth in the balance  $Q_t$  at rate  $r$  offset by progressive constant mortgage payment flow  $R$ . The mortgage is fully repaid when the interest flow on principal  $rQ_t$  is *lower* than the coupon flow  $R$ , in which case

$$R = \frac{rQ_0}{1 - e^{-rT}}. \quad (15)$$

A Repayment Continuous Workout Mortgage (RCWM) scales down the repayment flow  $R$  when the house price index of the location decreases. To emphasize that the repayment flow of a RCWM changes with time as a function of the initial home value weighted by the local house price index we denote it as  $R(h_t)$ . Furthermore, for a full workout, we

assume that it linearly scales down the mortgage payment flow to

$$R(h_t) = \min \left\{ \rho, \rho \frac{\bar{H}_t}{H_0} \right\} = \rho \min \left\{ 1, \frac{h_t}{h_0} \right\}, \quad (16)$$

where  $\rho$  is the maximal repayment flow of a RCWM elaborated below.

This setup fully protects the homeowner against a decline in the value of property because the repayment flow  $R(h_t)$  decreases whenever the house price index is below its initial value. The repayment flow attains its maximal value and becomes constant (equal to  $\rho$ , where  $\rho$  is a constant to be determined) whenever the house price index is above its initial value. In this case repayments of a RCWM behave in the same way as repayments of a standard fully amortizing mortgage. However, the maximal repayment flow of a RCWM ( $\rho$ ) must be set at a level higher than the repayment flow of a corresponding standard fully amortizing mortgage. That is,  $\rho > R$  (see Proposition 4).

A partial workout will provide *partial protection* with the mortgage payment only partly scaled down to

$$R_\alpha(h_t) = \rho \min \left\{ 1, 1 - \alpha \left( 1 - \frac{h_t}{h_0} \right) \right\} = \rho \left[ 1 - \alpha \left( 1 - \frac{h_t}{h_0} \right)^+ \right], \quad (17)$$

where  $0 \leq \alpha \leq 1$ . Depending on the value of  $\alpha$ , the repayment flow can be adjusted continuously between full protection ( $\alpha = 1$ ) and none at all ( $\alpha = 0$ ). For  $\alpha = 0$  the repayment flow becomes constant equal to  $\rho$ . In this particular case the RCWM reduces to the well-known, classic case of fully amortizing repayment mortgage. The reduction of mortgage payment  $R_\alpha(h_t)$  as a function of current adjusted house price level  $h_t$  and for different “workout proportions” ( $\alpha$ ) is illustrated in Figure 2.

PUT FIGURE 2 ABOUT HERE

Note that due to the stochastic nature of the repayment flow there is (in general) no guarantee that the mortgage will be repaid in full. This is the case when the maturity

of the loan is kept fixed and the values of  $\alpha$  are strictly greater than zero. Indeed, in the worst case scenario of  $h_t$  suddenly dropping to zero repayments of a full workout ( $\alpha = 1$ ) also drop to zero. Mortgage repayments (17) are random, contingent on future values of the house price index. Fortunately, it is possible to accurately price this random stream of cash flows as follows.

The expected present value under the risk-neutral measure gives the mortgage balance

$$Q_t = E_t \left[ \int_t^T R_\alpha(h_s) e^{-r(s-t)} ds \right] = E_t \left[ \int_t^T \rho \left( 1 - \alpha \left( 1 - \frac{h_s}{h_0} \right)^+ \right) e^{-r(s-t)} ds \right]. \quad (18)$$

Since

$$\frac{h_s}{h_0} = \frac{\bar{H}_s}{\bar{H}_0}, \quad (19)$$

we derive

$$Q_t = \int_t^T \rho e^{-r(s-t)} ds - \frac{\rho\alpha}{r\bar{H}_0} E_t \left[ \int_t^T (r\bar{H}_0 - r\bar{H}_s)^+ e^{-r(s-t)} ds \right]. \quad (20)$$

Clearly, the first integral is given by (13) while the expectation in the second integral is computed in closed form from the floor formula on flow  $\{r\bar{H}_s\}_{s=t}^T$  with strike level  $r\bar{H}_0$  for all  $t$  (see Shackleton and Wojakowski (2007))

$$E_t \left[ \int_t^T (r\bar{H}_0 - r\bar{H}_s)^+ e^{-r(s-t)} ds \right] = r \int_t^T p(\bar{H}_t, \bar{H}_0, s-t, r, \delta, \sigma) ds \quad (21)$$

$$= P(r\bar{H}_t, r\bar{H}_0, T-t, r, \delta, \sigma), \quad (22)$$

where  $p$  and  $P$  are the corresponding put and floor (see Appendix A). Combining the above formulas we get the following Proposition.

**Proposition 3** *The mortgage balance of a Repayment Continuous Workout Mortgage (RCWM) at time  $t \in [0, T]$  is equal to*

$$Q_t = \frac{\rho}{r} \left[ 1 - e^{-r(T-t)} - \frac{\alpha}{\bar{H}_0} P(r\bar{H}_t, r\bar{H}_0, T-t, r, \delta, \sigma) \right], \quad (23)$$

where  $\bar{H}_0$  and  $\bar{H}_t$  are values of the house price index at times of origination ( $t = 0$ ) and the semi-closed interval  $t \in (0, T]$ .

Clearly, for  $\alpha = 0$  we recover the full repayment case (13). Furthermore, with  $\alpha > 0$  and for the same  $\rho$ , the mortgage balance of a RCWM is lower. That does not mean, however, that a full repayment mortgage and a RCWM both originating at the same time  $t = 0$  will have the same  $\rho$ . On the contrary, a RCWM has a *higher* mortgage payment flow  $\rho$ . The intuition is that for a given “workout proportion”  $\alpha \in [0, 1]$ , the mortgage payment flow  $\rho$  must be computed at origination to compensate for the guarantee against house price decline provided by a RCWM.

**Proposition 4** *For a given “workout proportion”  $\alpha \in [0, 1]$ , the mortgage payment flow parameter is given by*

$$\rho = \frac{rQ_0}{1 - e^{-rT} - \frac{\alpha}{\bar{H}_0} P(rH_0, rH_0, T, r, \delta, \sigma)}, \quad (24)$$

where  $Q_0$  is the initial value of the loan.

**Remark 5** *The mortgage payment flow  $\rho$  is a constant parameter which is computed at origination ( $t = 0$ ) for the duration of the contract. It should not be confused with mortgage payment  $R_\alpha(h_t)$  given by equation (17) which is a function of randomly changing adjusted house price level  $h_t$ . Mortgage payments  $R_\alpha(h_t)$  decrease when home prices decline.*

**Remark 6** *For  $\alpha = 0$  the mortgage becomes a standard repayment type. Equation (24) yields  $\rho = R$  in accordance with equation (15).*

Equation (24) is of utmost importance for potential originators of continuous workouts with repayment features. This pricing condition helps us evaluate the maximal annual payment for this mortgage. A broker can instantly compute this quantity on a computer screen and make an offer to a customer. More importantly, different levels of protection can be offered to different borrowers.

In fact all brokers in the world use a special version of this formula already. This is because for  $\alpha = 0$  equation (24) yields the value of mortgage payment for the unprotected, standard repayment mortgage, defined by equation (15). Since  $P > 0$  it is clear that the protection commands a higher payment, as  $\alpha > 0$  implies  $\rho > R$ .

PUT TABLE 2 ABOUT HERE

For a 30-year repayment mortgage the continuous workout premium is not very large making RCWMs very attractive. As summarized in Table 2, in a typical situation when interests are 5% and volatility is 15%, the mortgage payment increases from \$6.44 thousand a year for a standard fixed rate repayment mortgage, to \$6.82 thousand a year for a corresponding *full* continuous workout. This is for an initial loan of \$100 thousand and when the service flow rate is 1%. Therefore, the full protection costs only \$383.41 a year, i.e., it only adds about \$31.95 to a monthly repayment of about \$536.34.

PUT FIGURE 3 ABOUT HERE

From Figure 3 we also observe that the service flow rate  $\delta$  does not influence the standard repayment level (no workout,  $\alpha = 0$ , flat dotted line). However, adding a continuous workouts (full  $\alpha = 1$  or partial  $\alpha = \frac{1}{2}$ ) to a property with higher service flow (e.g.  $\delta = 1\%$  or  $\delta = 4\%$ ) increases the monthly insurance premium only slightly. Lines for partial protection ( $\alpha = \frac{1}{2}$ ) lie only minimally below half-way between lines for  $\alpha = 1$  and  $\alpha = 0$ . For  $\delta = 4\%$  and  $\sigma = 30\%$ , for example, the endpoint of the line for partial protection is located at 11.46, only slightly below the midpoint 11.55. For lower volatilities the partial protection premium converges exactly to the midpoint, unless the service flow parameter  $\delta$  is larger than the riskless rate, in which case (not reported on Figure 3) this line is “repelled” down from the midpoint, even for low volatilities.

PUT FIGURE 4 ABOUT HERE

Finally, it is interesting to note that the cost of protection—in terms of basis points that are added to standard repayment by full protection—is higher at lower interest rates. See Figure 4. This spread essentially mimics the behaviour of an at the money put option as a function of the interest rate  $r$ .

Similarly, the evaluation of mortgage balances can be done using (23). This is a very important information when the customer sells the house and prepays the mortgage. This expression is also of practical importance for computation of annual mortgage statements which are typically sent to customers at each year end.

## 4 Repayment CWM with Protective Threshold

A significant increase in home prices leads to a considerable increase in mortgage payments. If such an increase is faster than the growth in household income, it leads to a drain on discretionary income (see Wilcox (2009)). Wilcox (2009)—in a comment on Shiller’s (2009b) paper—is of the view that households would still be willing to accept this risk as they would be able to extract equity from their appreciating homes as discretionary income. He, however, fails to observe that this risk would already be priced *ex ante* by the calculation of an appropriate mortgage rate. The higher the risk assumed by the household, the lower the mortgage rate would be.

There are, in fact, several ways to fine-tune a CWM to make it less or more expensive. One such tuning parameter is  $\alpha$  for repayment CWM which specifies the *proportion* of response of mortgage balance. If the workout proportion  $\alpha$  is set higher than 0 and closer to 1, the household benefits from a lower mortgage balance in poor states of the economy but must assume a higher mortgage repayment flow  $\rho$  parameter on a daily basis.<sup>4</sup> Conversely, if the household sets  $\alpha$  closer to 0 (corresponding to the case of fixed rate standard

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<sup>4</sup>Recall that  $\rho$  defines the maximal annual repayment for a CWM i.e. the payment in good states of the economy, when housing values appreciate.



repayment mortgages) it sets a lower repayment flow. This, however, leads to a higher mortgage balance in poor states of the economy. Thus, by setting lower repayments, the household bears a higher risk of negative equity during times of house price downturns.

A second way is to set a different *protection threshold*, higher or lower than the initial house price index level  $\bar{H}_0$ . If, for example, we set a lower threshold  $\bar{K} < \bar{H}_0$ , the protection kicks-in only after house values fall below  $\bar{K}$ . For such a CWM the household then repays

$$R_\alpha(h_t, k) = \rho \min \left\{ 1, 1 - \alpha \left( 1 - \frac{h_t}{k} \right) \right\} = \rho \left[ 1 - \alpha \left( 1 - \frac{h_t}{k} \right)^+ \right], \quad (25)$$

where

$$k = \frac{H_0}{\bar{H}_0} \bar{K} < H_0. \quad (26)$$

Note that in the special case when the strike price  $\bar{K}$  equals  $\bar{H}_0$  (or, equivalently,  $k = h_0$ ), the repayment rate (25) reverts back to (17). A *selectable* protection threshold thus effectively defines a new family of CW mortgages which we term as Threshold Repayment Continuous Workout Mortgages. To use Figure 2 to illustrate (25) i.e. how the threshold protection works, assume e.g.  $h_0 = 120$  and  $k = 100$ . Automatic workout will start when  $h_t$  falls below the threshold  $k = 100$ .

**Proposition 7** *The mortgage balance of a Threshold Repayment Continuous Workout Mortgage (TRCWM) at time  $t \in [0, T]$  is equal to*

$$Q_t = \frac{\rho}{r} \left[ 1 - e^{-r(T-t)} - \frac{\alpha}{\bar{K}} P(r\bar{H}_t, r\bar{K}, T-t, r, \delta, \sigma) \right], \quad (27)$$

where  $\bar{H}_t$  is the value of the house price index at time  $t \geq 0$  and  $\bar{K}$  is the “protection threshold.”

**Proof.** We have

$$Q_t = E_t \left[ \int_t^T \rho \left[ 1 - \alpha \left( 1 - \frac{h_s}{k} \right)^+ \right] e^{-r(s-t)} ds \right]. \quad (28)$$

Using  $\frac{h_s}{k} = \frac{\bar{H}_s}{\bar{K}}$  we get

$$Q_t = \int_t^T \rho e^{-r(s-t)} ds - \frac{\rho\alpha}{r\bar{K}} \int_t^T E_t \left[ (r\bar{K} - r\bar{H}_s)^+ e^{-r(s-t)} \right] ds . \quad (29)$$

The first integral is given by

$$\int_t^T \rho e^{-r(s-t)} ds = \frac{\rho}{r} \left( 1 - e^{-r(T-t)} \right) , \quad (30)$$

while the expectation in the second integral is computed in closed form from to the *floor* formula on flow  $\{r\bar{H}_s\}_{s=t}^T$  with strike  $r\bar{K}$  evaluated at time  $t$  (see Shackleton and Wójcikowski (2007))

$$\int_t^T E_t \left[ (r\bar{K} - r\bar{H}_s)^+ e^{-r(s-t)} \right] ds = r \int_t^T p(\bar{H}_t, \bar{K}, s-t, r, \delta, \sigma) ds \quad (31)$$

$$= P(r\bar{H}_t, r\bar{K}, T-t, r, \delta, \sigma) , \quad (32)$$

where  $p$  and  $P$  are the corresponding put and floor (see Appendix A). Substituting (32) and (30) into (29) we derive (27). ■

Similarly, setting  $t = 0$  and solving for  $\rho$  in (27) we derive the mortgage payment parameter of a TRCWM.

**Proposition 8** *For a given “workout proportion”  $\alpha \in [0, 1]$ , the mortgage payment flow of a Threshold Repayment Continuous Workout Mortgage (TRCWM) is given by*

$$\rho = \frac{rH_0}{1 - e^{-rT} - \frac{\alpha}{\bar{K}} P(r\bar{H}_0, r\bar{K}, T, r, \delta, \sigma)} , \quad (33)$$

where  $H_0$  is the initial value of the loan,  $\bar{H}_0$  is the initial value of the house price index and  $\bar{K}$  is the “protection threshold.”

**Proof.** For  $t = 0$  equation (27) becomes

$$Q_t = \frac{\rho}{r} \left( 1 - e^{-rT} - \frac{\alpha}{\bar{K}} P(r\bar{H}_0, r\bar{K}, T, r, \delta, \sigma) \right) . \quad (34)$$

Substituting  $Q_0 = H_0$  and solving for  $\rho$  gives (33). ■

PUT FIGURE 5 ABOUT HERE

Figure 5 illustrates how a high threshold  $\bar{K}$  commands a higher maximal repayment  $\rho$  for the household to benefit from higher reduction of mortgage balance  $Q_t$  (scaled for given  $\rho$  and  $r$ ). The higher the threshold  $\bar{K}$ , the earlier the workout kicks in, providing a greater reduction of the mortgage balance in poor states of the economy. In the particular case of no protection ( $\bar{K} \rightarrow 0$ ), a CWM reverts back to the standard repayment mortgage. The mortgage payment  $\rho$  converges to a constant value equal to  $R$  given by (15).

## 5 Refinancing and Prepayment of CWMs

Refinancing constitutes a major risk faced by financial institutions as it leads to termination of profitable businesses at inopportune moments. Lenders also face prepayment risk because they don't know for how long a loan will be outstanding. Financial institutions can mitigate this risk by imposing a penalty to deter potentially mobile borrowers or those who flip property frequently. The penalty in this section is designed so as to mimic those found in alternative mortgages like subprime (see Chomsisengphet and Pennington-Cross (2006)) and is in the spirit of Stanton and Wallace (1998). It can be evaluated using the methodology described below.

Empirical evidence confirms that borrowers do not prepay optimally, i.e. they do not maximize the loss to the lender. Reasons for prepaying typically include a sale of the property due to: a) change of employment & relocations; b) change in family composition (e.g. births, children leaving for university); c) a natural disaster, accident or default of the borrower followed by insurance indemnity payment. These factors are relatively constant in time and are often modelled in the mortgage industry using the constant Conditional

Prepayment Rate (CPR) convention or the Public Securities Association (PSA) benchmark (see e.g. Fabozzi (2005)). These professional conventions do not explicitly model prepayments due to refinancings which typically occur at an increased rate during periods of decreasing interest rates.

Prepayments can be modelled using a hazard rate (prepayment intensity) approach. The random prepayment time  $\tau_p$  is modelled as a Poisson process with intensity  $\lambda$  and is independent of state variables governing the processes for house prices. Formally, information in the model is described by enlarged filtration  $\mathcal{G}$ , generated by the Brownian motion  $Z_t$  and the prepayment indicator process  $\mathbf{1}_{\{t \leq \tau_p\}}$  (see e.g. Bielecki and Rutkowski (2002)).

Financial institutions typically impose a dollar prepayment penalty  $\Pi_{\tau_p}$  at refinancing time  $\tau_p$ . It is often a fixed fraction  $\alpha$  of the principal outstanding  $Q_{\tau_p}$  and is imposed if refinancing occurs before some lock-in date  $T^*$

$$\Pi_{\tau_p} = \alpha Q_{\tau_p} \mathbf{1}_{\tau_p < T^*} = \begin{cases} \alpha Q_{\tau_p} & \text{if } \tau_p < T^* \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

The prepayment penalty is attractive to only serious long term investors who like to match the cost of financing with the income and the growth in the value of the asset instead of just their permanent income in case of alternative mortgages (see Hurst and Stafford (2004); Doms and Krainer (2007)).

In practice, the contract is negotiated so that no prepayment occurs immediately. For interest-only Continuous Workout Mortgages (see Section 2) the value of the mortgage at origination is equal to the expected value of outstanding cash flows

$$M_0 = E \left[ \underbrace{\int_0^T \mathbf{1}_{t < \tau_p} e^{-rt} c_t dt}_{\text{payments}} + \underbrace{\mathbf{1}_{0 < \tau_p \leq T} e^{-r\tau} S_{\tau_p}}_{\text{prepayment}} + \underbrace{\mathbf{1}_{T < \tau_p} e^{-rT} S_T}_{\text{repayment}} \right]. \quad (36)$$

Working out expectations with respect to random prepayment time  $\tau_p$  gives

$$M_0 = E \left[ \int_0^T F(t) e^{-rt} c_t dt + \int_0^T g(t) e^{-rt} S_t dt + F(T) e^{-rT} S_T \right] , \quad (37)$$

where the expectation is to be taken along the remaining house price risk dimension. In the first integral and in the last term above

$$F(t) = \Pr(t < \tau_p) = e^{-\lambda t} \quad (38)$$

is the cumulative probability of the loan surviving beyond time  $t$  and  $g(t)$  is the probability density of the loan being prepaid within time interval  $(t, t + dt]$  i.e.

$$g(t) = \frac{d}{dt} (1 - F(t)) = \lambda e^{-\lambda t} . \quad (39)$$

Defining the adjusted principal of a CWM outstanding at time  $t \in [0, T]$  as

$$Q_t = \min \{h_0, h_t\} = h_0 - (h_0 - h_t)^+ , \quad (40)$$

we can also express the remaining components of (37):

- $c_t$  is the value of continuous payments received until prepayment date or maturity, whichever comes first

$$c_t = iQ_t ; \quad (41)$$

- $S_t$  is the lump sum payment received if prepayment occurs before maturity, and is equal to principal outstanding  $Q_t$  plus applicable prepayment penalty  $\Pi_t$  (if any);

$$S_t = Q_t + \Pi_t ; \quad (42)$$

- $S_T$  is the lump sum payable at the tenure of the CWM (defined in the indenture of the contract) if prepayment did not occur, i.e. any capital outstanding

$$S_T = Q_T . \quad (43)$$

Note that (43) implies that if prepayment did not occur before maturity, the principal must be repaid in full. The other important assumption is that the house price index of the locality is used to estimate the automatic workout if prices depreciate. Alternatively, the property can be re-appraised or a sale will reveal its value  $H_{\tau_p}$  if a prepayment occurs, or  $H_T$  at maturity.

As long as prepayment does not occur early ( $t < \tau_p$ ) the lender receives  $c_t$  i.e., the contractual interest flow which automatically incorporates the continuous workout. Furthermore, if prepayment does not occur before maturity ( $T < \tau_p$ ), the lender receives the principal as a lump sum on maturity. Otherwise, if prepayment occurs at time  $\tau_p \in (0, T)$ , the interest flow ceases immediately and the lender receives a lump sum corresponding to the workout-adjusted principal payback and prepayment penalty (if any).

Using (38) and (39) in (37) the value of the mortgage at origination can be shown to be equal to

$$M_0 = E \left[ \int_0^T e^{-(r+\lambda)t} (c_t + \lambda S_t) dt + e^{-(r+\lambda)T} S_T \right]. \quad (44)$$

**Proposition 9 (CWM valuation equation)** *In an economic environment with prepayments, the relationship linking contractual parameters (contract rate  $i$  and early prepayment penalty  $\alpha$ ) of a Continuous Workout Mortgage to other exogenous parameters is given by*

$$\begin{aligned} M_0 = & H_0 + H_0 (i - r) \frac{1 - e^{-(r+\lambda)T}}{r + \lambda} + \lambda \alpha H_0 \frac{1 - e^{-(r+\lambda)T^*}}{r + \lambda} \quad (45) \\ & - P((i + \lambda) H_0, (i + \lambda) H_0, T, r + \lambda, \delta + \lambda, \sigma) \\ & - P(\lambda \alpha H_0, \lambda \alpha H_0, T^*, r + \lambda, \delta + \lambda, \sigma) \\ & - p(H_0, H_0, T, r + \lambda, \delta + \lambda, \sigma) \end{aligned}$$

where  $T^*$  is the last date the penalty (35) is chargeable, while  $P$  and  $p$  are closed-form expressions for floor and at-the-money put given in Appendix A.

**Proof.** Substituting (41), (42) and (43) into (44) we get

$$M_0 = E \left[ \int_0^T e^{-(r+\lambda)t} [iQ_t + \lambda (Q_t + \Pi_t)] dt + e^{-(r+\lambda)T} Q_T \right]. \quad (46)$$

Grouping terms and introducing the penalty (35) gives

$$M_0 = E \left[ \int_0^T e^{-(r+\lambda)t} [(i + \lambda) Q_t + \lambda \alpha Q_t \mathbf{1}_{t < T^*}] dt + e^{-(r+\lambda)T} Q_T \right]. \quad (47)$$

Distributing the expectation operator over the plus sign yields

$$M_0 = E \left[ \int_0^T e^{-(r+\lambda)t} (i + \lambda) Q_t dt \right] + E \left[ \int_0^{T^*} e^{-(r+\lambda)t} \lambda \alpha Q_t dt \right] + E \left[ e^{-(r+\lambda)T} Q_T \right]. \quad (48)$$

Using expression (40) for current balance, the first expectation can be computed as

$$\begin{aligned} E \left[ \int_0^T e^{-(r+\lambda)t} (i + \lambda) Q_t dt \right] &= H_0 \int_0^T e^{-(r+\lambda)t} dt - E \left[ \int_0^T e^{-(r+\lambda)t} (h_0 - h_t)^+ dt \right] \\ &= (i + \lambda) H_0 \frac{1 - e^{-(r+\lambda)T}}{r + \lambda} \\ &\quad - P((i + \lambda) H_0, (i + \lambda) H_0, T, r + \lambda, \delta + \lambda, \sigma), \end{aligned} \quad (49)$$

where  $P$  is given by the floor formula (see Appendix A). Similarly, the second expectation in (48) is equal to

$$\begin{aligned} E \left[ \int_0^{T^*} e^{-(r+\lambda)t} \lambda \alpha Q_t dt \right] &= \lambda \alpha H_0 \frac{1 - e^{-(r+\lambda)T^*}}{r + \lambda} \\ &\quad - P(\lambda \alpha H_0, \lambda \alpha H_0, T^*, r + \lambda, \delta + \lambda, \sigma). \end{aligned} \quad (50)$$

The third expectation in (48) can be expressed using the Black-Scholes at the money put formula (see Appendix A)

$$E \left[ e^{-(r+\lambda)T} Q_T \right] = H_0 e^{-(r+\lambda)T} - p(H_0, H_0, T, r + \lambda, \delta + \lambda, \sigma). \quad (51)$$

Replacing expectations (49), (50) and (51) in (48) and after some algebra then gives (45). ■

**Remark 10** *The value of the put in (45) can be expressed as the probability of the mortgage loan*

surviving until maturity time  $T$  multiplied by an otherwise identical at-the-money European put option on the property

$$p(H_0, H_0, r + \lambda, \delta + \lambda, \sigma, T) = e^{-\lambda T} p(H_0, H_0, r, \delta, \sigma, T) . \quad (52)$$

The main difficulty in (45) is computing expressions giving the floor value

$$P(X_0, k, \varrho, \phi, \sigma, T) = \int_0^T E \left[ e^{-\varrho t} (k - X_t)^+ \right] dt = \int_0^T e^{-\lambda t} E \left[ e^{-rt} (k - X_t)^+ \right] dt , \quad (53)$$

where  $\varrho = r + \lambda$  and  $\phi = \delta + \lambda$  i.e. sums of floorlets on  $X$  struck at  $k$  over a continuum of maturities  $t \in [0, T]$ . In the last integral in (53) each floorlet is weighted by the probability  $e^{-\lambda t}$  that the loan will “survive” beyond  $t$ . This is because each  $(k - X_t)^+$  will be due only if the loan was not prepaid before  $t$ . These sums can be interpreted as weighted floors on flow  $X_t$ . In our setup flows  $X_t$  are generated by prepaid “fractions” of the property value  $\lambda H_t$  and  $\lambda \alpha H_t$  (see Remark that follows). Weighted floors can be computed thanks to closed-form solution developed in Shackleton and Wojakowski (2007) (see Appendix A).

**Remark 11** *Flows in both expressions for finite floors  $P$  appearing in (45) are written on flows  $iH_t + \lambda H_t$  and  $\lambda \alpha H_t$ . Flow  $iH_t$  results from regular interest payment, while  $\lambda H_t$  and  $\lambda \alpha H_t$  can be interpreted as effective flows resulting from prepayments statistically occurring at rate  $\lambda$ . Initial values of these flows enter the two floor functions  $P$  appearing in (45). A bank securitizing CWMs after bundling them into portfolios will effectively be exposed to both flows  $\lambda H_t$  and  $\lambda \alpha H_t$ . The former flow results from earlier principal repayments due to prepayments at rate  $\lambda$  throughout the lifetime  $[0, T]$  of the product. The latter flow represents the supplementary principal fraction  $\alpha$  accruing at rate  $\lambda$  during period  $[0, T^*]$  when early prepayment penalty is charged.*

In order to eliminate arbitrage the value of the mortgage at origination,  $M_0$ , must be equal to the value of the loan outstanding,  $H_0$ . Solving this condition, i.e.  $M_0 = H_0$ , where  $M_0$  is given by (45) above, enables linking of the key parameters of the contract, such as



the contract rate  $i$  and the prepayment penalty  $\alpha$  and express them as functions of market data and other inputs (such as the prepayment rate  $\lambda$ ) and constraints. We will focus on this problem in sections that follow.

**Remark 12** When  $M_0 = H_0$  and in absence of prepayment penalties ( $\alpha = 0$ ) the CWM valuation equation gives

$$H_0 (i - r) \frac{1 - e^{-(r+\lambda)T}}{r + \lambda} = P((i + \lambda) H_0, (i + \lambda) H_0, T, r + \lambda, \delta + \lambda, \sigma) \quad (54)$$

$$+ p(H_0, H_0, T, r + \lambda, \delta + \lambda, \sigma)$$

*i.e.* in absence of arbitrage the extra income generated by charging the customer the premium  $i - r > 0$  must compensate the cost of continuous workout insurance expressed by the floor  $P$  (intermediate, continuous workouts) and the put  $p$  (automatic workout at maturity).

## 6 Numerical Illustration

To illustrate CWM valuation in a world with prepayment risk, we focus on the case of a CWM for which also  $\alpha > 0$  and  $T^* > 0$ . This case is particularly important for residential mortgages where early prepayment penalties are invariably imposed in practice. Note that unlike the case without prepayment and in addition to terms expressing income from early prepayment penalty, valuation now depends on the sum of floorlets weighted by the probability of the mortgage “surviving” over the time period  $[0, T)$ . This quantity is analytically expressed by the floor function  $P((i + \lambda) H_0, (i + \lambda) H_0, T, r + \lambda, \delta + \lambda, \sigma)$ .

Some key parameters can be expressed explicitly. For example, given parameters such as the prepayment rate  $\lambda$ , prepayment penalty fraction  $\alpha$  and lock-in time  $T^*$  we can explicitly compute the contract rate  $i$  as follows

$$i = r \times \frac{\frac{r}{r+\lambda} H_0 \left(1 - e^{-(r+\lambda)T}\right) - \frac{\lambda\alpha}{r+\lambda} H_0 \left(1 - e^{-(r+\lambda)T^*}\right) + P_{\lambda,T} + \alpha P_{\lambda,T^*} + p}{\frac{r}{r+\lambda} H_0 \left(1 - e^{-(r+\lambda)T}\right) - P_{r,T}} \quad (55)$$

where

$$P_{u,\tau} = P(uH_0, uH_0, \tau, r + \lambda, \delta + \lambda, \sigma) : u = \lambda, r; \quad \tau = T, T^* \quad (56)$$

and

$$p = p(rH_0, rH_0, T, r + \lambda, \delta + \lambda, \sigma) . \quad (57)$$

When  $\lambda = 0$  (no prepayment) and  $T^* = 0$  or  $\alpha = 0$  (no early prepayment penalty) we recover the expression (8). Here too, it is readily seen that in fact  $H_0$  can be eliminated from (55) to obtain expressions similar to (9) and (10). This means that there is no size effect, i.e. the contract rate  $i$  is an intensive parameter, insensitive to the size of the loan and thus to the size of the property.

Equation (55) implies that the contract rate  $i$  can be decreased by higher prepayment penalties  $\alpha$ . A further reduction in the contract rate  $i$  is achieved by extending the lock-in period  $T^*$  towards maturity  $T$ . This is, however, not immediate to be seen from (55). Therefore, to illustrate the sensitivity of the contract rate  $i$  to various inputs we present a numerical example assuming the following case:

market interest rate	$r$	10% p.a.
service flow rate	$\delta$	7.5% p.a.
house price volatility	$\sigma$	1%, 5% & 10% p.a.
term	$T$	30 years
lock-in period	$T^*$	5 years
prepayment penalty	$\alpha$	none & 5% of outstanding balance

We further assume that an individual, initially paying  $i = 10\%$  per annum on a standard mortgage loan without prepayment penalties (base case) is offered to refinance his mortgage. A zero-profit bank offers a set of alternative mortgages, with and without prepayment penalty. Figure 6 summarizes Continuous Workout Mortgage contracts offered to this individual under different house price volatility market scenarios and as a function

of the prepayment rate  $\lambda$ . It combines these two types of CWMs in three market scenarios encompassing low and high volatilities.

INSERT FIGURE 6 ABOUT HERE.

The adopted range of the prepayment rate  $\lambda$  allows to do computations for different types of individuals, according to how likely they are to prepay their mortgage. Our range encompasses 50%, 100%, 200% and 400% multiples of the long-term standard PSA value of 6% annual prepayment rate ( $\lambda = 0.06$ ). This results in values for  $\lambda$  which are 0.03 (less likely to prepay than average), 0.06 (average), 0.12 (twice more likely to prepay as average) and 0.24 (four times more likely to prepay compared to average).

When volatility is low ( $\sigma = 1\%$ ) the probability of a continuous workout being triggered is low. From Figure 6 it is clear that in this case the lender expects to make money on customers prepaying early. These homeowners are charged the prepayment penalty. A zero-profit bank will pass on this gain to customers by offering a lowered interest on these CWM mortgages up front.

When volatility is moderate ( $\sigma = 5\%$ ) the probability of continuous workout being triggered increases. Figure 6 suggests that this will be reflected in the contract rate  $i$  of the CWM which will have to be increased, even for customers not expected to prepay ( $\lambda = 0$ ). However, the contract rate  $i$  can still be reduced by introducing a prepayment penalty.

Higher volatilities ( $\sigma = 10\%$ ) will mean yet more protection offered to customers but at a higher cost reflected in further increases of the CWM contract rate  $i$ . Only for customers with sufficiently high prepayment rate  $\lambda$  the contract rate  $i$  can be reduced.

As intuitively expected, higher volatility and higher prepayment rates will typically mean increased risks to the lender (workout risk, prepayment risk). This will result in charging higher contract rate  $i$  up front. However, for realistic market parameter values,

increases of contract rate  $i$  are moderate rather than dramatic (e.g. increase by up to 0.1% on average, when rates are in the range of 10%). Furthermore, these increases can be further moderated by introducing early prepayment penalties, as is practiced for most mortgage contracts. Figure 7 illustrates this point further by plotting combinations of early prepayment parameters  $\alpha$  and  $T^*$  for different levels of prepayment intensity  $\lambda$ , for which the contract rate  $i$  of a Continuous Workout Mortgage remains “preserved” i.e. equal to  $i = 10\%$ .

INSERT FIGURE 7 ABOUT HERE.

## 7 Concluding Remarks

The current crisis has increased the awareness of the susceptibility of the economy to fragile plain vanilla mortgages. We argue that different variants of Continuous Workout Mortgages can be offered to homeowners as an *ex ante* solution to non-anticipated real estate price declines. “If workouts are part of the original mortgage contract, then they can be done continually, systematically, and automatically, eliminating the delays, irregularities and uncertainties that are seen with the workouts we have observed in the subprime crisis. A great deal of human suffering at a time of economic contraction would be eliminated.” (Shiller (2009b), Article 4, p. 11.)

This paper has attempted to mitigate this fragility by bringing into focus and formalizing the above intuition. CWMs are related to PLAMs (advocated by Franco Modigliani for high inflation regimes) as pointed out by Shiller (2008b), (2008a) and (2009b). CWMs share some positive attributes with PLAMs (such as purchasing power risk) and mitigate other negative attributes (such as default risk, interest rate risk and prepayment risk). The only residual risk attributed to CWMs is liquidity risk. It too can be alleviated by employing it in sufficient amount to warrant its securitization.

We model a variety of Continuous Workout Mortgages (CWMs), including interest-only and repayment contracts. For repayment CWMs we extend our analysis to include two negotiable parameters: adjustable “workout proportion”  $\alpha$  and adjustable “protection threshold”  $\bar{K}$ . The “workout proportion”  $\alpha$  enables the workout protection to change the proportion while the “protection threshold”  $\bar{K}$  can be set so as the workout starts (and stops) working below (above) a specific level.

An interest-only CWM protects both (a) interest payments and (b) terminal repayment against declines of property value. The insurance for this is paid for via interest  $i$  which is increased above  $r$  (the riskless rate in our examples).

A repayment CWM protects against declines in property value. As there is no terminal repayment, this mortgage protects intermediate repayments  $R_\alpha(h_t)$ , which decrease as house price decreases. The premium of this insurance is incorporated into higher maximal repayment flow parameter  $\rho$ , established *ex ante*. Furthermore, a repayment mortgage, and thus a continuous workout repayment mortgage, has the advantage of reducing moral hazard. This is due to the earlier repayment of the loan’s principal.

The main contributions of this paper thus are: (a) simple modelling of variants of CWMs; (b) closed form formulae for key parameters of CWMs. For repayment contracts, we provide closed form expressions for mortgage balances and payments. For interest-only CWMs, we provide closed form expressions for mortgage interest. These formulae are of utmost importance for potential originators of CWMs. Our results help in the understanding the mechanics of CWMs and estimation of key contract parameters for providing the protection level to borrowers in the form of protection threshold. Thus, employment of CWMs should be recommended to improve the resilience of the financial system, mitigate systemic risk and enhance the quality of life of agents in the economy.

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## A Appendix: Floor and Put Formulae

Floors  $P$  on flow  $s$  with strike flow level  $k$  for finite horizon  $T$  can be computed using the following closed-form formula (see Shackleton and Wojakowski (2007)):

$$P(s_0, k, T, r, \delta, \sigma) = E \left[ \int_0^T e^{-rt} (k - s_t)^+ dt \right] \quad (58)$$

$$\begin{aligned} &= A s_0^a (\mathbf{1}_{s_0 < k} - N(-d_a)) - \frac{s_0}{\delta} (\mathbf{1}_{s_0 < k} - e^{-\delta T} N(-d_1)) \\ &+ \frac{k}{r} (\mathbf{1}_{s_0 < k} - e^{-rT} N(-d_0)) - B s_0^b (\mathbf{1}_{s_0 < k} - N(-d_b)), \end{aligned} \quad (59)$$

where

$$\begin{aligned} A &= \frac{k^{1-a}}{a-b} \left( \frac{b}{r} - \frac{b-1}{\delta} \right), \\ B &= \frac{k^{1-b}}{a-b} \left( \frac{a}{r} - \frac{a-1}{\delta} \right), \end{aligned} \quad (60)$$

and

$$a, b = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \pm \sqrt{\left( \frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}, \quad (61)$$

whereas the cumulative normal integrals  $N(\cdot)$  are labelled with parameters  $d_\beta$

$$d_\beta = \frac{\ln s_0 - \ln k + \left( r - \delta + \left( \beta - \frac{1}{2} \right) \sigma^2 \right) T}{\sigma \sqrt{T}} \quad (62)$$

(different to the standard textbook notation) for elasticity  $\beta$  which takes one of four values  $\beta \in \{a, b, 0, 1\}$ .

Standard Black-Scholes (1973) put on  $S$  with strike value of  $K$  can be computed using

$$p(S_0, K, T, r, \delta, \sigma) = K e^{-rT} N(-d_0) - S_0 e^{-\delta T} N(-d_1) \quad (63)$$

where  $d_0$  and  $d_1$  can be computed using formula (62) in which *values*  $S_0$  and  $K$  can (formally) be used in place of *flows*  $s_0$  and  $k$ .

Both floor (58) and put (63) formulae assume that the underlying flow  $s$  or asset  $S$

follows the stochastic differential equation

$$\frac{ds_t}{s_t} = \frac{dS_t}{S_t} = (r - \delta) dt + \sigma dZ_t \quad (64)$$

with initial values  $s_0$  and  $S_0$ , respectively. Clearly, (64) describes a geometric Brownian motion under risk-neutral measure where  $Z_t$  is the standard Brownian motion,  $\sigma$  is the volatility,  $r$  is the riskless rate and  $\delta$  is the service flow. We assume that (64) describes the dynamics of the repayment flow  $s$ . Similarly, (64) also defines the dynamics of the value  $S$  of the real estate property. Alternatively, the model could be specified under the actual probability measure, extending Shiller and Weiss (1999) to allow for computation of floors, for example along the lines of Shackleton and Wojakowski (2007).

## B Tables

Volatility $\sigma$	Interest-only mortgage payments			Repayment mortgage payments		
	Standard	CWM		Standard	CWM	
	$r = \text{const.}$	$i > r$ ( $\delta = 1\%$ )   ( $\delta = 4\%$ )		$\rho = R (\alpha = 0)$	$\rho > R (\alpha = 1)$ ( $\delta = 1\%$ )   ( $\delta = 4\%$ )	
0.025	5.	5.	5.02	6.44	6.44	6.46
0.05	5.	5.01	5.15	6.44	6.45	6.59
0.075	5.	5.03	5.36	6.44	6.48	6.78
0.1	5.	5.1	5.62	6.44	6.55	7.
0.125	5.	5.21	5.9	6.44	6.67	7.24
0.15	5.	5.37	6.2	6.44	6.82	7.5
0.175	5.	5.57	6.53	6.44	7.01	7.78
0.2	5.	5.82	6.87	6.44	7.23	8.08
0.225	5.	6.1	7.24	6.44	7.47	8.4
0.25	5.	6.41	7.63	6.44	7.75	8.74
0.275	5.	6.76	8.04	6.44	8.05	9.1
0.3	5.	7.13	8.47	6.44	8.37	9.48
0.325	5.	7.53	8.93	6.44	8.72	9.88
0.35	5.	7.96	9.41	6.44	9.09	10.3
0.375	5.	8.42	9.91	6.44	9.49	10.8
0.4	5.	8.91	10.4	6.44	9.92	11.2
0.425	5.	9.42	11.	6.44	10.4	11.7
0.45	5.	9.96	11.6	6.44	10.9	12.3
0.475	5.	10.5	12.2	6.44	11.4	12.8
0.5	5.	11.1	12.8	6.44	11.9	13.4

Table 1: Annual payments when the current interest rate is  $r = 5\%$ . Interest-only and repayment continuous workouts compared to the corresponding fixed-rate (interest-only and repayment) mortgages for different values of service flow  $\delta = 1\%$  and  $\delta = 4\%$ . Fixed rate ( $r = 5\%$ ) mortgage payments do not depend on the service flow  $\delta$ . Initial loan balance,  $Q_0$ , is normalized to \$100 thousand.

Volatility $\sigma$	Interest-only mortgage payments			Repayment mortgage payments		
	Standard	CWM		Standard	CWM	
	$r = \text{const.}$	$i > r$		$\rho = R (\alpha = 0)$	$\rho > R (\alpha = 1)$	
		$(\delta = 1\%)$	$(\delta = 4\%)$		$(\delta = 1\%)$	$(\delta = 4\%)$
0.025	10.	10.	10.	10.52	10.52	10.52
0.05	10.	10.	10.01	10.52	10.53	10.53
0.075	10.	10.01	10.03	10.52	10.54	10.56
0.1	10.	10.03	10.09	10.52	10.56	10.62
0.125	10.	10.07	10.18	10.52	10.6	10.72
0.15	10.	10.14	10.33	10.52	10.67	10.86
0.175	10.	10.24	10.52	10.52	10.78	11.06
0.2	10.	10.38	10.76	10.52	10.92	11.29
0.225	10.	10.55	11.03	10.52	11.09	11.56
0.25	10.	10.77	11.35	10.52	11.31	11.87
0.275	10.	11.03	11.71	10.52	11.56	12.21
0.3	10.	11.32	12.1	10.52	11.85	12.58
0.325	10.	11.66	12.52	10.52	12.18	12.98
0.35	10.	12.03	12.97	10.52	12.53	13.42
0.375	10.	12.44	13.45	10.52	12.93	13.88
0.4	10.	12.88	13.97	10.52	13.35	14.37
0.425	10.	13.36	14.51	10.52	13.81	14.89
0.45	10.	13.87	15.08	10.52	14.3	15.44
0.475	10.	14.42	15.69	10.52	14.82	16.01
0.5	10.	15.	16.32	10.52	15.37	16.62

Table 2: Annual payments when the current interest rate is  $r = 10\%$ . Interest-only and repayment continuous workouts compared to the corresponding fixed-rate (interest-only and repayment) mortgages for different values of service flow  $\delta = 1\%$  and  $\delta = 4\%$ . Fixed rate ( $r = 10\%$ ) mortgage payments do not depend on the service flow  $\delta$ . Initial loan balance,  $Q_0$ , is normalized to \$100 thousand.

## C Figures

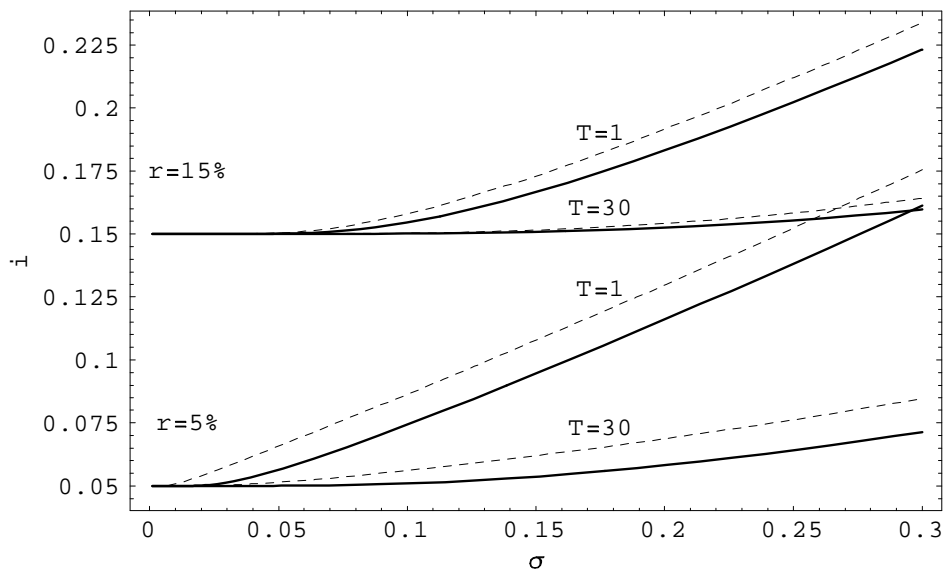


Figure 1: **Interest-only Continuous Workout Mortgage.** Required interest  $i$  as a function of risk  $\sigma$ . Tenure is  $T = 30$  years or  $T = 1$  years to maturity, riskless rates are  $r = 5\%$  and  $r = 15\%$  and service flow rate is either  $\delta = 1\%$  (thick lines) or  $4\%$  (dashed lines).

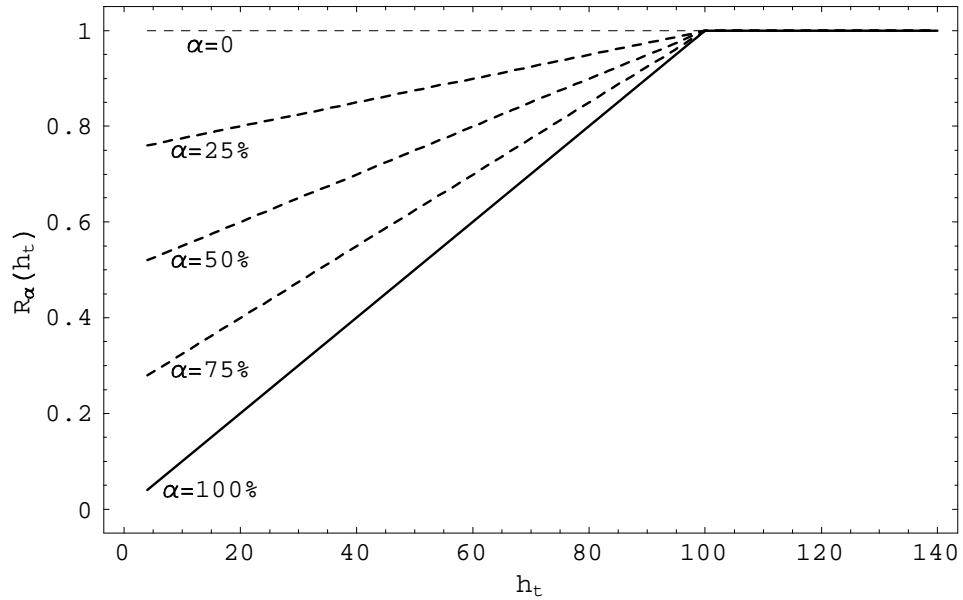


Figure 2: **Repayment Continuous Workout Mortgage (with Threshold).** Reduction of mortgage payment  $R_{\alpha}(h_t)$  as a function of current adjusted house price level  $h_t$  and different “workout proportions”  $\alpha$ . Full workout is  $\alpha = 100\%$  (thick solid line), no workout (standard repayment mortgage, thin dashed line) is  $\alpha = 0$ . Maximal annual repayment (mortgage repayment parameter) is normalized to one i.e.  $\rho = 1$  and  $h_0 = k = 100$ . To illustrate the “threshold” variant of workouts assume  $k = 100$  and a higher starting price level e.g.  $h_0 = 120$ .

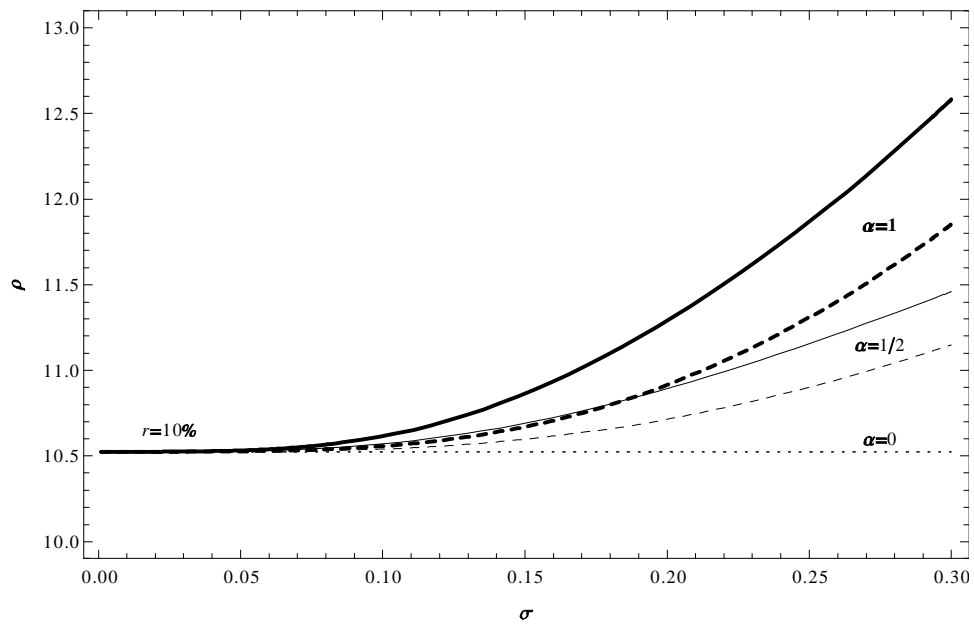


Figure 3: **Repayment Continuous Workout Mortgage.** Mortgage payment  $\rho$  as a function of risk  $\sigma$ . Partial workout ( $\alpha = \frac{1}{2}$ ) is positioned between full workout ( $\alpha = 1$ , thick lines) and standard repayment (no workout,  $\alpha = 0$ , dotted flat line). Tenure is  $T = 30$  years to maturity, riskless rate is  $r = 10\%$ . Service flow rates are  $\delta = 1\%$  (solid lines) or  $4\%$  (dashed lines).

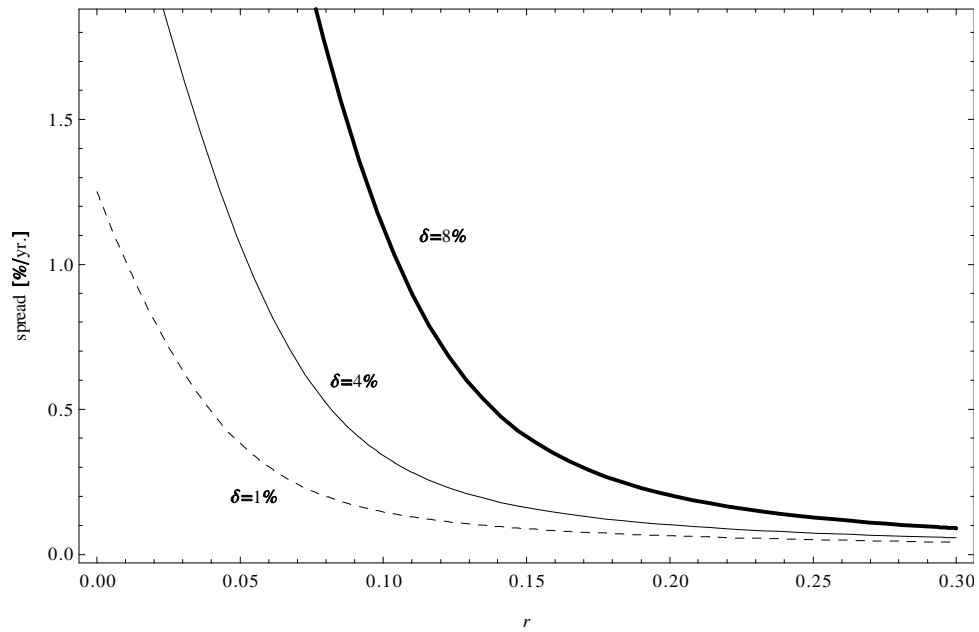


Figure 4: **Repayment Continuous Workout Mortgage.** Mortgage payment spread  $\rho_\alpha - rH_0(1 - \exp(-rT))^{-1}$  as a function of interest rate  $r$  for full workout ( $\alpha = 1$ ). Tenure is  $T = 30$  years to maturity, service flow rates are  $\delta = 1\%$  (dashed line),  $4\%$  (solid line) and  $8\%$  (bold line).  $H_0$  is normalized to \$100 thousand.



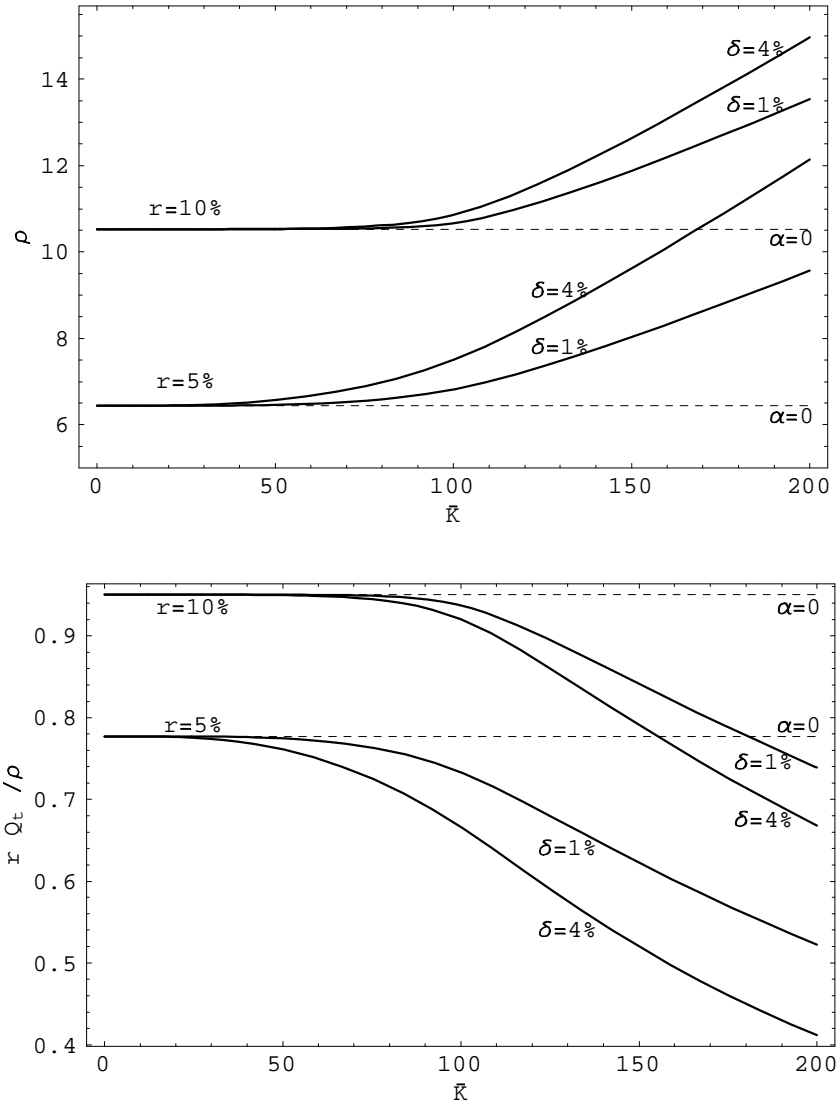


Figure 5: **Threshold Repayment Continuous Workout Mortgage (TRCWM)** with threshold parameter  $\bar{K}$ . Increase in the mortgage payment flow  $\rho$  (above) compared to reduction of mortgage balance  $rQ_t/\rho$  (below), as a function of “workout threshold”  $\bar{K}$ . Interest rate  $r = 5\%$  and  $10\%$ , service flow rate  $\delta = 1\%$  and  $4\%$ , volatility  $\sigma = 15\%$ , term  $T = 30$  years to repayment, initial house price and index  $\bar{H}_0 = 100$ .

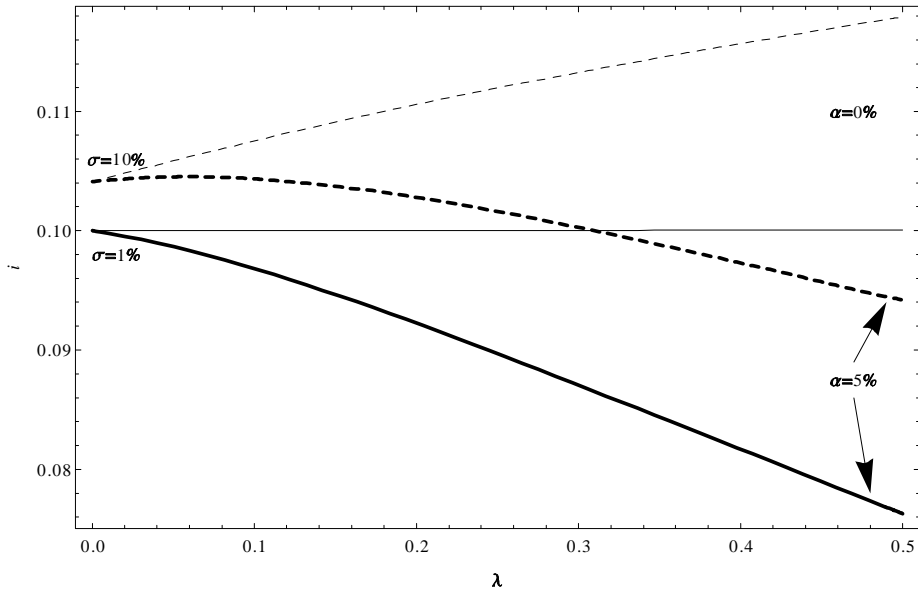


Figure 6: **Continuous Workout Mortgage contract rate  $i$  as a function of the prepayment rate  $\lambda$ .**

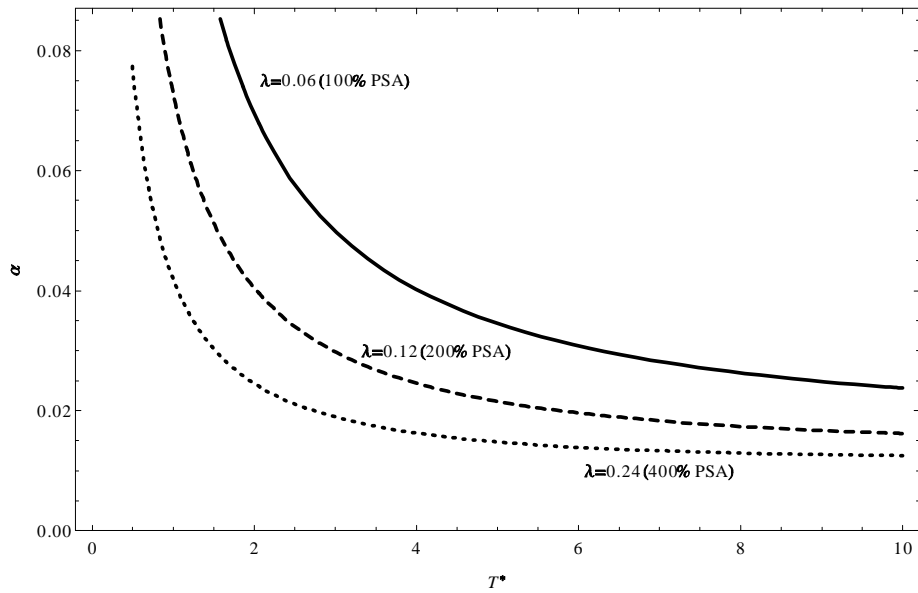


Figure 7: **Continuous Workout Mortgage contract-rate-preserving combinations (at  $i = 10\%$ ) of early prepayment parameters  $\alpha$  and  $T^*$  for different levels of prepayment intensity  $\lambda$ .**