BREAKDOWN POINT THEORY FOR IMPLIED PROBABILITY BOOTSTRAP

By

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Breakdown point theory for implied probability bootstrap*

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Abstract

This paper studies robustness of bootstrap inference methods under moment conditions. In particular, we compare the uniform weight and implied probability bootstraps by analyzing behaviors of the bootstrap quantiles when outliers take arbitrarily large values, and derive the breakdown points for those bootstrap quantiles. The breakdown point properties characterize the situation where the implied probability bootstrap is more robust than the uniform weight bootstrap against outliers. Simulation studies illustrate our theoretical findings.

1 Introduction

Since Hansen (1982), the generalized method of moments (GMM) has been a standard tool for empirical analysis in econometrics. The GMM provides a unified framework for statistical inference in econometric models that are specified by some moment conditions (see, e.g., Hall, 2005, for a review on the GMM). However, recent research indicates that there are considerable problems with the GMM, particularly in its finite sample performance, and that approximations based on the asymptotic theory can yield poor results (see, e.g., the special issue of the *Journal of Business and Economic Statistics*, vol. 14).

To refine the approximations for the distributions of the GMM estimator and related test statistics, bootstrap methods have been developed. A key issue to apply bootstrap methods to the GMM context is that one typically needs to impose the overidentified moment conditions to the bootstrap resamples. Hall and Horowitz (1996) suggested to use the uniform weight bootstrap with recentered moment conditions, and established higher-order refinements of their bootstrap inference over the asymptotic approximations. On the other hand, Brown and Newey (2002) suggested to use a weighted bootstrap

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based on the implied probabilities from the moment conditions (see also Hall and Presnell, 1999). These implied probabilities can be computed based on the GMM (Back and Brown, 1993), empirical likelihood (Owen, 1988), or generalized empirical likelihood (Smith, 1997, and Newey and Smith, 2004). This implied probability bootstrap also provides a refinement over the asymptotic approximations.[1]

The purpose of this paper is to study robustness of the uniform weight and implied probability bootstraps, based on the breakdown point theory in the literature of robust statistics (see Hampel, 1971, and Donoho and Huber, 1983, for general definitions of breakdown point, and Singh, 1998, Salibian-Barrera, Van Aelst and Willems, 2007, and Camponovo, Scaillet and Trojani, 2010a, for the use of breakdown point theory in bootstrap contexts). The need for robust statistical procedures has been stressed by many authors and is now widely recognized; see, e.g., Hampel, Ronchetti, Rousseeuw and Stahel (1986), Maronna, Martin and Yohai (2006), and Huber and Ronchetti (2009). To be more precise, by extending the approach of Singh (1998), we analyze behaviors of bootstrap quantiles of the uniform weight bootstrap and implied probability bootstrap (using Back and Brown's, 1993, weight) when outliers take arbitrary large values, and compare the breakdown points for these bootstrap quantiles. Our breakdown point analysis characterizes the situation where the implied probability bootstrap is more robust than the uniform weight bootstrap against outliers. Therefore, researchers can decide which bootstrap approach should be adopted for each application. In particular, when all elements of the moment functions diverge to infinity as (the norm of) outliers diverge, the implied probability bootstrap is typically more robust than the uniform weight bootstrap. The literature of robustness study in the GMM context is relatively thin and is currently under development. Ronchetti and Trojani (2001) extended the robust estimation techniques for (just-identified) estimating equations to overidentified moment condition models. Gagliardini, Trojani and Urga (2005) develop a robust GMM test for structural breaks. Hill and Renault (2010) propose a GMM estimator with asymptotically vanishing tail trimming for robust estimation of dynamic moment condition models. Kitamura, Otsu and Evdokimov (2010) and Kitamura and Otsu (2010) studied local robustness against perturbations controlled by the Hellinger distance for point estimation and hypothesis testing, respectively, in moment condition models. Our breakdown point analysis studies global robustness of bootstrap methods when outliers take arbitrarily large values.

The rest of the paper is organized as follows. Section 2 investigates a benchmark example, inference for a trimmed mean, to understand the basic idea of our breakdown point analysis. Section 3 generalizes the results obtained in Section 2 to a moment condition model. Section 4 illustrates the theoretical results by simulations. Section 5 concludes.

[1]An important feature of implied probabilities is that they provide semiparametrically efficient estimators for the distribution function and it moments under the moment conditions (Back and Brown, 1993, and Brown and Newey, 1998). Antoine, Bonnal and Renault (2007) employed implied probabilities to construct an asymptotically efficient estimator for parameters in the moment conditions.

2 Benchmark example

We start our analysis by introducing the benchmark example analyzed in Singh (1998) about the trimmed mean. Consider a random sample $\{X_i\}_{i=1}^n$ of size n from $X \in \mathbb{R}$. Suppose that we wish to approximate the distribution of the 10% trimmed mean $T(0.1)$ (i.e., 5% trimming for each side) by a bootstrap method, when $n \geq 20$. Let $X_{(1)} \leq \ldots \leq X_{(n)}$ be the ordered sample. Since $n \geq 20$, $T(0.1)$ is always free from the largest observation $X_{(n)}$, which is treated as an outlier. On the other hand, consider the trimmed mean $T^{\#}(0.1)$ using the (uniform weight) bootstrap resample. Since the bootstrap resample can contain $X_{(n)}$ more than once, $T^{\#}(0.1)$ is not necessarily free from $X_{(n)}$. Letting $B(n,p)$ be a binomial random variable with parameters n and p, the probability that $T^{\#}(0.1)$ is free from $X_{(n)}$ is

$$p^{\#} = P\left(B\left(n, \frac{1}{n}\right) \leq 1\right).$$

Therefore, if $X_{(n)} \to +\infty$, then $100\left(1-p^{\#}\right)\%$ of resamples of $T^{\#}(0.1)$ will diverge to $+\infty$. In other words, the bootstrap quantile $Q_t^{\#}$ of $T^{\#}(0.1)$ will diverge to $+\infty$ for all $t > p^{\#}$.

Consider the situation where we have auxiliary information

$$E\left[g\left(X_i\right)\right] = 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a scalar-valued function. Let $\bar{g} = \frac{1}{n}\sum_{i=1}^n g(X_i)$. Back and Brown (1993) showed that under this auxiliary information, the distribution function of X can be efficiently estimated by using the implied probabilities:

$$\pi_i = \frac{1}{n} - \frac{1}{n}\frac{\left(g\left(X_i\right)-\bar{g}\right)\bar{g}}{\frac{1}{n}\sum_{i=1}^n g\left(X_i\right)^2}, \tag{1}$$

for $i = 1, \ldots, n$.[2,3] The second term in π_i can be interpreted as a penalty term for the deviation from auxiliary information: if $|g(X_i)|$ becomes larger, then $(g(X_i)-\bar{g})\bar{g}$ tends to be positive and the weight π_i tends to be smaller. Let $T^*(0.1)$ be the trimmed mean using a bootstrap sample based on the implied probabilities $\{\pi_i\}_{i=1}^n$. Then, the probability that $T^*(0.1)$ is free from the largest observation $X_{(n)}$ is written as

$$P\left(B\left(n, \pi_{(n)}\right) \leq 1\right).$$

Thus, in terms of the bootstrap quantiles, the implied probability bootstrap becomes more robust than the uniform weight bootstrap when $P\left(B\left(n, \pi_{(n)}\right) \leq 1\right) > p^{\#}$ (or $\pi_{(n)} \leq \frac{1}{n}$).

[2] For the breakdown point analysis, we focus on Back and Brown's (1993) implied probability because of its simplicity. It is interesting to extend the analysis to other implied probabilities such as the generalized empirical likelihood-based implied probabilities discussed by Brown and Newey (2002).

[3] Our breakdown point analysis assumes that all implied probabilities are non-negative. This assumption is typically justified when the sample size is sufficiently large. However, in finite samples, it is possible to have negative implied probabilities. In the simulation study below, we adopt a shrinkage-type modification suggested by Antoine, Bonnal and Renault (2007) to avoid negative implied probabilities.

To adapt the conventional breakdown point theory to our setup, we need to analyze the limiting behavior of $g\left(X_{(n)}\right)$ as $X_{(n)} \to +\infty$. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be the extended real line. Suppose that

$$g\left(X_{(n)}\right) \to g_* \in \bar{\mathbb{R}} \quad \text{as } X_{(n)} \to +\infty.$$

Then, the limit of $\pi_{(n)}$ is obtained as

$$\pi_* = \begin{cases} \frac{1}{n} - \frac{1}{n} \frac{\left(1 - \frac{1}{n} - \frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*} + \frac{1}{n}\right)}{\frac{\bar{v}_-}{g_*^2} + \frac{1}{n}} & \text{if } g_* \in \mathbb{R} \setminus \{0\} \\ \frac{1}{n} + \frac{1}{n} \frac{\bar{g}_-^2}{\bar{v}_-} & \text{if } g_* = 0 \\ \frac{1}{n^2} & \text{if } |g_*| = +\infty \end{cases},$$

where $\bar{g}_- = \frac{1}{n} \sum_{i=1}^{n-1} g\left(X_{(i)}\right)$ and $\bar{v}_- = \frac{1}{n} \sum_{i=1}^{n-1} g\left(X_{(i)}\right)^2$. The limit of the probability $P\left(B\left(n, \pi_{(n)}\right) \leq 1\right)$ is

$$p^* = P\left(B\left(n, \pi_*\right) \leq 1\right).$$

If $g_* \in \mathbb{R} \setminus \{0\}$, then the sign of $\left(1 - \frac{1}{n} - \frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*} + \frac{1}{n}\right)$ determines robustness of the implied probability bootstrap. If $\left(1 - \frac{1}{n} - \frac{\bar{g}_-}{g_*}\right)\left(\frac{\bar{g}_-}{g_*} + \frac{1}{n}\right)$ is positive (or negative), then $p^* > p^\#$ (or $p^* < p^\#$) and the implied probability bootstrap is more (or less) robust than the uniform weight bootstrap. If $g_* = 0$, then $p^\# > p^*$ is always satisfied and the uniform weight bootstrap is more robust than the implied probability bootstrap. On the other hand, if $|g_*| = +\infty$, then $p^* > p^\#$ is always satisfied and the implied probability bootstrap becomes more robust. These findings are summarized as follows.

Proposition 1. *Consider the setup of this section. If $X_{(n)} \to +\infty$, the followings hold true.*

(i) *The uniform weight bootstrap quantile $Q_t^\#$ of $T^\#(0.1)$ will diverge to $+\infty$ for all $t > p^\#$.*

(ii) *The implied probability bootstrap quantile Q_t^* of $T^*(0.1)$ will diverge to $+\infty$ for all $t > p^*$.*

(iii) *If $g_* = 0$, then $p^\# > p^*$ is always satisfied. If $|g_*| = +\infty$, then $p^\# < p^*$ is always satisfied.*

For the divergent case, $|g_*| = +\infty$, we can numerically compare $p^\# = P\left(B\left(n, \frac{1}{n}\right) \leq 1\right)$ and $p^* = P\left(B\left(n, \frac{1}{n^2}\right) \leq 1\right)$. For example, when the sample size is $n = 20$, we have $p^\# = 0.736$ and $p^* = 0.999$. This means that for the uniform weight bootstrap, a single outlier implies the divergence of more than 26% of resamples of $T^\#(0.1)$, while for the implied probability bootstrap, a single outlier implies the divergence of less than 1% of resamples of $T^*(0.1)$. Section 4.1 illustrates this proposition by simulations.

Although the results obtained in this section is insightful, there are several limitations: (i) the statistic of interest is a trimmed mean, (ii) X is scalar, and (iii) $g(\cdot)$ is a scalar-valued function and does not contain parameters. The next section discusses how to generalize the insights obtained in this section.

3 Breakdown point theory

We now introduce our setup. Let $\{X_i\}_{i=1}^n$ be a random sample of size n from $X \in \mathbb{R}^d$. Consider the situation where we have the following overidentified moment conditions:

$$E\left[g\left(X_i,\theta_0\right)\right] = E\begin{bmatrix} g_1\left(X_i,\theta_0\right) \\ g_2\left(X_i,\theta_0\right) \end{bmatrix} = 0,$$

where g_1 and g_2 are scalar-valued functions and $\theta_0 \in \mathbb{R}$ is a scalar parameter. In this case, Back and Brown's (1993) implied probabilities are defined as

$$\pi_i = \frac{1}{n} - \frac{1}{n}\left\{g\left(X_i,\hat{\theta}\right) - \bar{g}\left(\hat{\theta}\right)\right\}'\left[\frac{1}{n}\sum_{i=1}^{n} g\left(X_i,\hat{\theta}\right) g\left(X_i,\hat{\theta}\right)'\right]^{-1} \bar{g}\left(\hat{\theta}\right),$$

for $i = 1, \ldots, n$, where $\hat{\theta}$ is an estimator of θ_0 and $\bar{g}\left(\hat{\theta}\right) = \frac{1}{n}\sum_{i=1}^{n} g\left(X_i,\hat{\theta}\right)$. For simplicity and technical tractability (basically to obtain an explicit formula for π_i), we focus on the case of $\dim(g) = 2$. Remark 3 below discusses some extensions for the case of $\dim(g) > 2$. Let $\left\{X_{(i)}\right\}_{i=1}^n$ be the ordered sample, where $\left\|X_{(1)}\right\| \leq \ldots \leq \left\|X_{(n)}\right\|$ and $\|\cdot\|$ is the Euclidean norm. Suppose we are interested in a real-valued object $T_n = T_n\left(X_1,\ldots,X_n;\theta_0\right)$, where $T_n \to +\infty$ as $\left\|X_{(n)}\right\| \to +\infty$.[4] We also assume that as $\left\|X_{(n)}\right\| \to +\infty$,

$$\begin{pmatrix} g_1\left(X_{(n)},\hat{\theta}\right) \\ g_2\left(X_{(n)},\hat{\theta}\right) \end{pmatrix} \to \begin{pmatrix} g_{1*} \\ g_{2*} \end{pmatrix} \in \bar{\mathbb{R}}^2, \quad \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)},\hat{\theta}\right) \\ \frac{1}{n}\sum_{i=1}^{n-1} g_2\left(X_{(i)},\hat{\theta}\right) \\ \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)},\hat{\theta}\right)^2 \\ \frac{1}{n}\sum_{i=1}^{n-1} g_2\left(X_{(i)},\hat{\theta}\right)^2 \\ \frac{1}{n}\sum_{i=1}^{n-1} g_1\left(X_{(i)},\hat{\theta}\right) g_2\left(X_{(i)},\hat{\theta}\right) \end{pmatrix} \to \begin{pmatrix} \bar{g}_{1-} \\ \bar{g}_{2-} \\ v_{11} \\ v_{22} \\ v_{12} \end{pmatrix} \in \mathbb{R}^5.$$

Note that the second condition requires that the limits are finite and restricts the form of g and/or the limit of $\hat{\theta}$. In this case, the limit of $\pi_{(n)}$ is obtained as

$$\pi_* = \begin{cases} \frac{1}{n} - c_* & \text{if } g_{1*} \in \mathbb{R} \text{ and } g_{2*} \in \mathbb{R} \\ \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{2-}^2}{v_{22}} & \text{if } |g_{1*}| = +\infty \text{ and } g_{2*} \in \mathbb{R} \\ \frac{1}{n^2} + \frac{1}{n}\frac{\bar{g}_{1-}^2}{v_{11}} & \text{if } g_{1*} \in \mathbb{R} \text{ and } |g_{2*}| = +\infty \\ \frac{1}{n^2} + \frac{1}{n}\frac{\left(\bar{g}_{1-} - \bar{g}_{2-}\right)^2}{v_{11}+v_{22}-2v_{12}} & \text{if } |g_{1*}| = +\infty \text{ and } |g_{2*}| = +\infty \end{cases},$$

[4]For brevity, as in Singh (1998), we consider only the case where $T_n \to +\infty$ as $\left\|X_{(n)}\right\| \to +\infty$. Nevertheless, following the definition of finite sample breakdown point in Donoho and Huber (1983), the results in our paper extend straightforward to the case $T_n \to +\infty$ as $X_i \to x_* \in \mathbb{R}$. In the case of a statistic with bounded support, the definition can be modified to ask the statistic to converge to its boundaries (see, e.g., Genton and Lucas, 2003).

where

$$c_* = \frac{1}{n}\left\{\left(v_{11}+\frac{1}{n}g_{1*}^2\right)\left(v_{22}+\frac{1}{n}g_{2*}^2\right)-\left(v_{12}+\frac{1}{n}g_{1*}g_{2*}\right)^2\right\}^{-1}$$

$$\times\left\{\begin{array}{c}\left(v_{22}+\frac{1}{n}g_{2*}^2\right)\left\{\left(1-\frac{1}{n}\right)g_{1*}-\bar{g}_{1-}\right\}\left(\bar{g}_{1-}+\frac{1}{n}g_{1*}\right)\\ -\left(v_{12}+\frac{1}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{1}{n}\right)g_{2*}-\bar{g}_{2-}\right\}\left(\bar{g}_{1-}+\frac{1}{n}g_{1*}\right)\\ -\left(v_{12}+\frac{1}{n}g_{1*}g_{2*}\right)\left\{\left(1-\frac{1}{n}\right)g_{1*}-\bar{g}_{1-}\right\}\left(\bar{g}_{2-}+\frac{1}{n}g_{2*}\right)\\ +\left(v_{11}+\frac{1}{n}g_{1*}^2\right)\left\{\left(1-\frac{1}{n}\right)g_{2*}-\bar{g}_{2-}\right\}\left(\bar{g}_{2-}+\frac{1}{n}g_{2*}\right)\end{array}\right\}.$$

Therefore, we obtain the following result.

Proposition 2. *Consider the setup of this section. If $\left\|X_{(n)}\right\| \to +\infty$, the followings hold true.*

(i) *The uniform weight bootstrap quantile $Q_t^{\#}$ from the resamples $T_n^{\#}$ of T_n will diverge to $+\infty$ for all $t > p^{\#} = P\left(B\left(n, \frac{1}{n}\right) = 0\right)$.*

(ii) *The implied probability bootstrap quantile Q_t^* from the resamples T_n^* of T_n will diverge to $+\infty$ for all $t > p^* = P\left(B\left(n, \pi_*\right) = 0\right)$.*

Note that this proposition is presented for the non-robust statistic T_n (i.e., a single outlier yields divergence of T_n). More generally, if the statistic T_n is robust to k outliers (the trimmed mean $T(0.1)$ with $n = 20$ in Section 2 corresponds to the case of $k = 1$), then this proposition holds for $p^{\#} = P\left(B\left(n, \frac{1}{n}\right) \leq k\right)$ and $p^* = P\left(B\left(n, \pi_*\right) \leq k\right)$.

Proposition 2 shows that in this setting the implied probability bootstrap is not necessarily more robust than the uniform weight bootstrap. In particular, the robustness properties of the implied probability bootstrap depend also on the terms $\bar{g}_{1-}$, $\bar{g}_{2-}$, v_{11}, v_{22}, and v_{12}. Nevertheless, note that $\bar{g}_{1-}$, $\bar{g}_{2-}$ are the components of the empirical moment without considering the outlier, and consequently they are typically close to 0. Therefore, if either $|g_{1\star}| = +\infty$, $|g_{2\star}| = +\infty$ or both $|g_{1\star}| = +\infty$ and $|g_{2\star}| = +\infty$, then the implied probability bootstrap becomes typically more robust than the uniform weight bootstrap.

If we impose more assumptions, this proposition can be presented by using the notion of the quantile breakdown point (Singh, 1998). Let b_T be the upper breakdown point of T_n, i.e., nb_T is the smallest number of observations whose Euclidean norm need to go to $+\infty$ in order to force T_n to go to $+\infty$. In our context, the upper breakdown point of quantiles $\{Q_t\}_{t\in(0,1)}$, where $Q_t := Q_t(X_1, \ldots, X_n)$, can be defined as

$$UB_t = \inf\left\{b \in \left[\frac{1}{n}, b_T\right] : nb \in \mathbb{N} \text{ and } Q_t(X_1, \ldots, X_n) \to +\infty\right\},$$

where b is the fraction of observations $X_{(n)}, X_{(n-1)}, \ldots, X_{(nb+1)}$ such that $\left\|X_{(j)}\right\| \to +\infty$, for all $j = nb+1, \ldots, n$.[5] Consider the situation where as $\left\|X_{(j)}\right\| \to +\infty$,

$$\begin{pmatrix} g_1\left(X_{(j)}, \hat{\theta}\right) \\ g_2\left(X_{(j)}, \hat{\theta}\right)\end{pmatrix} \to \begin{pmatrix} g_{1*} \\ g_{2*}\end{pmatrix} \in \bar{\mathbb{R}}^2,$$

[5]The same argument applies to the lower breakdown point which focuses on the lower tail (i.e., $Q_t(X_1, \ldots, X_n) \to -\infty$).

and for any $j \neq j'$, $g_1\left(X_{(j)}, \hat{\theta}\right)/g_1\left(X_{(j')}, \hat{\theta}\right) \to 1$, and $g_2\left(X_{(j)}, \hat{\theta}\right)/g_2\left(X_{(j')}, \hat{\theta}\right) \to 1$, where $n-k+1 \leq j, j' \leq n$, $1 \leq k \leq n$. In this case, for $j = n-k+1, \ldots, n$, the limit of $\pi_{(j)}$ is obtained as:

$$\pi_{*,k} = \begin{cases} \frac{1}{n} - c_{*,k} & \text{if } g_{1*} \in \mathbb{R} \text{ and } g_{2*} \in \mathbb{R} \\ \frac{k}{n^2} + \frac{1}{n}\frac{\bar{g}_{2-,k}^2}{v_{22,k}} & \text{if } |g_{1*}| = +\infty \text{ and } g_{2*} \in \mathbb{R} \\ \frac{k}{n^2} + \frac{1}{n}\frac{\bar{g}_{1-,k}^2}{v_{11,k}} & \text{if } g_{1*} \in \mathbb{R} \text{ and } |g_{2*}| = +\infty \\ \frac{k}{n^2} + \frac{1}{n}\frac{\left(\bar{g}_{1-,k} - \bar{g}_{2-,k}\right)^2}{v_{11,k} + v_{22,k} - 2v_{12,k}} & \text{if } |g_{1*}| = +\infty \text{ and } |g_{2*}| = +\infty \end{cases},$$

where $\bar{g}_{1-,k}$, $\bar{g}_{2-,k}$, $v_{11,k}$, $v_{22,k}$, and $v_{12,k}$ are the limits of $\frac{1}{n}\sum_{i=1}^{n-k} g_1\left(X_{(i)}, \hat{\theta}\right)$, $\frac{1}{n}\sum_{i=1}^{n-k} g_2\left(X_{(i)}, \hat{\theta}\right)$, $\frac{1}{n}\sum_{i=1}^{n-k} g_1\left(X_{(i)}, \hat{\theta}\right)^2$, $\frac{1}{n}\sum_{i=1}^{n-k} g_2\left(X_{(i)}, \hat{\theta}\right)^2$, and $\frac{1}{n}\sum_{i=1}^{n-k} g_1\left(X_{(i)}, \hat{\theta}\right) g_2\left(X_{(i)}, \hat{\theta}\right)$, respectively, which are assumed to be finite, and

$$\begin{aligned} c_{*,k} &= \frac{1}{n}\left\{\left(v_{11,k} + \frac{k}{n}g_{1*}^2\right)\left(v_{22,k} + \frac{k}{n}g_{2*}^2\right) - \left(v_{12,k} + \frac{k}{n}g_{1*}g_{2*}\right)^2\right\}^{-1} \\ &\times \left\{\begin{array}{c} \left(v_{22,k} + \frac{k}{n}g_{2*}^2\right)\left\{\left(1 - \frac{k}{n}\right)g_{1*} - \bar{g}_{1-,k}\right\}\left(\bar{g}_{1-} + \frac{k}{n}g_{1*}\right) \\ -\left(v_{12,k} + \frac{k}{n}g_{1*}g_{2*}\right)\left\{\left(1 - \frac{k}{n}\right)g_{2*} - \bar{g}_{2-,k}\right\}\left(\bar{g}_{1-,k} + \frac{k}{n}g_{1*}\right) \\ -\left(v_{12,k} + \frac{k}{n}g_{1*}g_{2*}\right)\left\{\left(1 - \frac{k}{n}\right)g_{1*} - \bar{g}_{1-,k}\right\}\left(\bar{g}_{2-,k} + \frac{k}{n}g_{2*}\right) \\ +\left(v_{11,k} + \frac{k}{n}g_{1*}^2\right)\left\{\left(1 - \frac{k}{n}\right)g_{2*} - \bar{g}_{2-,k}\right\}\left(\bar{g}_{2-,k} + \frac{k}{n}g_{2*}\right) \end{array}\right\}. \end{aligned}$$

Then, following proposition holds.

Proposition 3. *Let T_n be a statistic with breakdown point $b_T \in (0,1)$. Under the setup of this section, the followings hold true.*

(i) *(Singh, 1998, Theorem 1) The upper breakdown point of the uniform weight bootstrap quantile $Q_t^{\#}$ is*

$$UB_t^{\#} = \frac{1}{n}\min\left\{k \in \{1, \ldots, n\} : P\left(B\left(n, \frac{k}{n}\right) \geq nb_T\right) \geq 1 - t\right\}.$$

(ii) *The upper breakdown point of the implied probability bootstrap quantile Q_t^* is*

$$UB_t^* = \frac{1}{n}\min\left\{k \in \{1, \ldots, n\} : P\left(B\left(n, k\pi_{*,k}\right) \geq nb_T\right) \geq 1 - t\right\}.$$

We close this section by remarks on the main result.

Remark 1. [Implication for confidence interval and hypothesis testing] The upper breakdown point of the bootstrap quantile describes the minimal fraction of outliers in the original sample such that the bootstrap quantile diverges to infinity. It turns out that when this occurs, inference based on the bootstrap distribution becomes meaningless. For example, if the researcher wish to construct a bootstrap confidence interval for a parameter of interest, the breakdown of the bootstrap quantiles implies non-informative confidence intervals for the parameter of interest. Thus, the quantile breakdown point can be considered as a diagnostic tool to check robustness of bootstrap-based inference by describing up to which fraction of contaminations the bootstrap distribution still provides some reliable information.

Remark 2. [Statistics by recentered moments] For the uniform weight bootstrap, the bootstrap statistic $T_n^{\#}$ is typically computed by using recentered moments, i.e., $g^{\#}(X_i, \theta) = g\left(X_i^{\#}, \theta\right) - \frac{1}{n}\sum_{i=1}^n g\left(X_i^{\#}, \theta\right)$ (Hall and Horowitz, 1996). This recentering is required to satisfy the moment conditions by bootstrap resamples. On the other hand, the implied probability bootstrap does not require such recentering since the bootstrap resamples always satisfy the moment conditions at $\hat{\theta}$ by construction. Therefore, it is possible that the breakdown point of the bootstrap statistic $T_n^{\#}$ (say $b_T^{\#}$) is different from the breakdown point b_T of T_n and T_n^*. In this case, b_T in Proposition 3 (i) should be replaced by $b_T^{\#}$.

Remark 3. [Higher dimension case] Our breakdown point analysis can be extended to the case of $\dim(g) > 2$. However, if each element of $g\left(X_{(n)}, \hat{\theta}\right)$ takes a different limit as $\left\|X_{(n)}\right\| \to +\infty$, we need to explicitly evaluate the limit of the inverse $\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right) g\left(X_i, \hat{\theta}\right)'\right]^{-1}$ and the result becomes more complicated and less intuitive. To obtain a comprehensible result, it would be reasonable to consider the case where all elements of $g\left(X_{(n)}, \hat{\theta}\right)$ take only two limiting values. In this case, we can split $g\left(X_{(n)}, \hat{\theta}\right)$ into two sub-vectors and apply the partitioned matrix inverse formula for $\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \hat{\theta}\right) g\left(X_i, \hat{\theta}\right)'\right]^{-1}$.

Remark 4. [Time series data] In order to suitably capture the dependence of the data generating process in a time series framework, the bootstrap requires some modifications. Consequently, besides the conventional uniform bootstrap also the implied probability bootstrap analyzed in our study cannot be directly applied to the time series case. Combining the ideas of Kitamura (1997) and Brown and Newey (2002), in a recent study Allen, Gregory and Shimotsu (2010) propose an extension of the implied probability bootstrap to the time series case by developing an empirical likelihood block bootstrap for time series. We expect that the breakdown point analysis of our paper can be adapted to such a modified bootstrap method (see Camponovo, Scaillet and Trojani (2010b) for the use of breakdown point analysis for the bootstrap in time series context).

4 Simulations

4.1 Benchmark example

To evaluate our theoretical results in finite samples, we first conduct a simulation study for the benchmark example introduced in Section 2. The setup is as follows. Let $\{X_1, \ldots, X_{20}\}$ be a scalar iid sample of size $n = 20$ from $X \sim N(0, 1)$. As in Section 2, we wish to estimate the distribution of the 10% trimmed mean $T(0.1) = \frac{1}{18}\sum_{i=2}^{19} X_{(i)}$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ is the ordered sample. In order to study the robustness of the bootstrap methods, we consider two data generating processes: (i) no contamination ($X_i \sim N(0, 1)$, $1 \leq i \leq 20$), and (ii) contamination ($X_i \sim N(0, 1)$, $1 \leq i \leq 19$, and $X_{20} = 3, 5, 10, 20, 100$). We compare the uniform weight bootstrap quantile $Q_t^{\#}$ and the implied probability bootstrap quantile Q_t^* to estimate the quantile Q_t of the trimmed mean $T(0.1)$. For the implied probability bootstrap, we consider four moment functions: $g_1(X) = X$, $g_2(X) = X^2 - 1$, $g_3(X) = |X|^{-1/2} - E\left[|X|^{-1/2}\right]$, and $g_4(X) = |X|^{-1/2}\operatorname{sgn}(X)$. Note that as $X \to +\infty$,

$$g_1(X) \to +\infty, \quad g_2(X) \to +\infty, \quad g_3(X) \to -E\left[|X|^{-1/2}\right], \quad g_4(X) \to 0.$$

Thus, Proposition 1 says that for g_1 and g_2, the implied probability bootstrap is more robust than the uniform weight bootstrap. On the other hand, the conclusion is undetermined for g_3, and the implied probability bootstrap becomes less robust for g_4.

To ensure that all the implied probabilities are non-negative, we employ a shrinkage approach suggested by Antoine, Bonnal and Renault (2007), i.e., $\tilde{\pi}_i = \frac{1}{1+\epsilon_n}\pi_i + \frac{\epsilon_n}{1+\epsilon_n}\frac{1}{n}$ with $\epsilon_n = -n\min\{\min_{1\le i\le n}\pi_i, 0\}$. As pointed out by Antoine, Bonnal and Renault (2007) this approach preserves the order of the implied probabilities, has no impact when the imply probabilities are already non-negative, and assigns zero probability only to the observation associated to the smallest probability when negative.

Table 1 reports the Monte Carlo medians of the bootstrap quantiles $Q_t^{\#}$ and Q_t^{*} for t =0.9, 0.95, and 0.99, based on 200 draws, over $1,000$ replications. Without contamination, both methods provide very similar results. In the presence of contamination, the results basically confirm the theoretical predictions in Proposition 1. For large values of contamination, the uniform weight bootstrap quantiles $Q_t^{\#}$ dramatically increase. In particular, for the case of $X_{20} = 100$, $Q_{.95}^{\#}$ becomes larger than 11 whereas $Q_{.95}$ is 0.4806. In contrast, the implied probability bootstrap quantiles using g_1 and g_2 show desirable stability. For the case of g_3 (or g_4), the implied probability bootstrap quantiles are slightly closer (or further) to the target Q_t than the uniform weight bootstrap quantiles. Overall the simulation for the benchmark case suggests that our breakdown point analysis reasonably characterizes (lack of) robustness of the bootstrap methods in finite samples.

4.2 Hall and Horowitz's (1996) example

The second example, considered in a simulation study by Hall and Horowitz (1996), is based on a simplified version of an asset pricing model. Let $\{X_i\}_{i=1}^n$ be a random sample, where

$$X_i = \begin{pmatrix} Z_{1i} \\ Z_{2i} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.2^2 & 0 \\ 0 & 0.2^2 \end{pmatrix}\right). \tag{2}$$

Then, we consider following moment conditions

$$E[g(X_i, \theta_0)] = E\left[\begin{pmatrix} 1 \\ Z_{2i} \end{pmatrix}(\exp(\mu - \theta_0(Z_{1i} + Z_{2i}) + 3Z_{2i}) - 1)\right] = 0,$$

where $\theta_0 = 3$ is the parameter of interest, and μ is a known constant. As a statistic of interest, we consider the overidentifying restriction test statistic (called Hansen's J-statistic)

$$T_n = \min_\theta n\hat{g}(\theta)'\left[\frac{1}{n}\sum_{i=1}^n g\left(X_i, \tilde{\theta}\right) g\left(X_i, \tilde{\theta}\right)'\right]^{-1} \hat{g}(\theta),$$

where $\hat{g}(\theta) = \frac{1}{n}\sum_{i=1}^{n} g(X_i, \theta)$, and $\tilde{\theta} = \arg\min_\theta \hat{g}(\theta)' \hat{g}(\theta)$ is a preliminary estimator. This statistic is used to check the validity of the overidentifying restrictions. We compare the uniform weight and the implied probability bootstrap quantiles, $Q_t^{\#}$ and Q_t^*, respectively, to estimate the quantile Q_t of the J-statistic T_n. To compute the implied probabilities, we use $\left\{g\left(X_i, \hat{\theta}\right)\right\}_{i=1}^{n}$, the moment functions evaluated at the two-step GMM estimator $\hat{\theta} = \arg\min_\theta n\hat{g}(\theta)' \left[\frac{1}{n}\sum_{i=1}^{n} g\left(X_i, \tilde{\theta}\right) g\left(X_i, \tilde{\theta}\right)'\right]^{-1} \hat{g}(\theta)$.

Before introducing the simulation results, we apply our breakdown point analysis derived in Section 3 to this setup. Let $g(X, \theta_0) = (g_1(X, \theta_0), g_2(X, \theta_0))'$. As $Z_1 \to +\infty$, we have $g_1((Z_1, Z_2), \theta_0) \to -1$ and $g_2((Z_1, Z_2), \theta_0) \to -Z_2$. On the other hand, as $Z_1 \to -\infty$, we have $g_1((Z_1, Z_2), \theta_0) \to +\infty$ and $g_2((Z_1, Z_2), \theta_0) \to +\infty$ (if Z_2 is positive) or $-\infty$ (if Z_2 is negative). Therefore, Proposition 2 indicates that the implied probability bootstrap will be more robust than the uniform weight bootstrap for negative outliers in Z_1. For positive outliers in Z_1, Proposition 2 does not provide a definitive answer about which bootstrap method is more robust. On the other hand, as $|Z_2| \to +\infty$, although $|g_2((Z_1, Z_2), \theta_0)| \to +\infty$, $g_1((Z_1, Z_2), \theta_0) \to \exp(\mu - X_1) - 1$. Therefore, the implied probability bootstrap will be more robust than the uniform weight bootstrap.

We now illustrate the above theoretical predictions by Monte Carlo simulations. We consider a sample of size $n = 100$, and two data generating processes: (i) no contamination (X_i distributed as in equation (2), $i = 1, \ldots, 100$), and (ii) contamination (X_i distributed as in equation (2), $i = 1, \ldots, 99$, while $Z_{1,100} = -3, -1, 1, 3$, and $Z_{2,100} = -3, -1, 1, 3$). Table 2 and Table 3 report the median of the uniform weight and implied probability bootstrap quantiles, based on 100 draws, over $1,000$ replications. Also in this case, to ensure that all the implied probabilities are non-negative we apply the shrinkage approach introduced in Antoine, Bonnal and Renault (2007). Without contaminations both the uniform weight and the implied probability bootstrap approach provide accurate approximations to the target distribution. In the presence of contaminations, we can see that the breakdown point analysis provides useful descriptions of the (lack of) robustness of the bootstrap methods. From the tables, we observe that positive outliers in Z_1 and positive or negative outliers in Z_2 do not deteriorate the reliability of the bootstrap methods; but negative outliers in Z_1 dramatically decreases the accuracy of the uniform weight bootstrap compared to the implied probability bootstrap. For example, in the case of $X_{1n} = -3$, $Q_{.9}^{\#}$ is larger than 58 whereas $Q_{.9}$ is 2.7587. On the other hand, $Q_{.9}^*$ is 5.8244. Therefore, for this example, our breakdown point analysis recommends to use the implied probability bootstrap to guard against outliers.

5 Conclusion

This paper studies robustness of the uniform weight and implied probability bootstrap inference methods for moment condition models. In particular, we analyze the breakdown point properties of the quantiles for those bootstrap methods. Simulation studies illustrate the theoretical findings. Our breakdown point analysis can be an informative guideline for applied researchers who wish to decide which bootstrap method should be applied. It is interesting to apply our breakdown point analysis to more specific setups (e.g., instrumental variable regression models to evaluate the effects of outliers in dependent, endogenous, and instrumental variables). Also it is important to extend our analysis to dependent data setups, where different bootstrap methods need to be employed. These extensions are currently under investigation by the authors.

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		No Con.	Contamination				
			3	5	10	20	100
True	$Q_{.9}$	0.2862	0.3951	0.3959	0.3959	0.3959	0.3959
	$Q_{.95}$	0.3718	0.4794	0.4806	0.4806	0.4806	0.4806
	$Q_{.99}$	0.5259	0.6304	0.6318	0.6318	0.6318	0.6318
Uniform	$Q^{\#}_{.9}$	0.3057	0.4717	0.5753	0.8759	1.4678	5.9124
	$Q^{\#}_{.95}$	0.3828	0.5692	0.7199	1.1642	2.2249	11.1105
	$Q^{\#}_{.99}$	0.5150	0.7438	0.9712	1.6452	3.1368	16.4478
Implied (g_1)	$Q^{*}_{.9}$	0.2738	0.2987	0.2991	0.3012	0.3033	0.3014
	$Q^{*}_{.95}$	0.3537	0.3932	0.4054	0.4171	0.3944	0.3847
	$Q^{*}_{.99}$	0.4870	0.5594	0.6015	0.6447	0.6699	0.5394
Implied (g_2)	$Q^{*}_{.9}$	0.3110	0.3591	0.3170	0.3100	0.3100	0.3127
	$Q^{*}_{.95}$	0.3929	0.4418	0.3987	0.3905	0.3922	0.3956
	$Q^{*}_{.99}$	0.5327	0.5990	0.5526	0.5383	0.5300	0.5311
Implied (g_3)	$Q^{*}_{.9}$	0.2748	0.3915	0.4500	0.6377	1.1405	5.5538
	$Q^{*}_{.95}$	0.3519	0.4832	0.5670	0.8751	1.4840	5.8935
	$Q^{*}_{.99}$	0.4713	0.6440	0.7913	1.3181	2.4231	11.2638
Implied (g_4)	$Q^{*}_{.9}$	0.2978	0.4725	0.5806	0.9102	1.5299	5.9821
	$Q^{*}_{.95}$	0.3817	0.5724	0.7236	1.2005	2.2758	11.1675
	$Q^{*}_{.99}$	0.5235	0.7483	0.9828	1.6872	3.2835	16.6013

Table 1: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of $T(0.1)$ based on 20,000 realizations of $T(0.1)$. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied (g_a)" report the implied probability bootstrap quantiles using the moment function g_a for a =1, 2, 3 and 4.

		No Con.	Contamination in $Z_{1,100}$			
			-3	-1	1	3
True	$Q_{.9}$	3.1668	2.7587	2.3036	3.4342	3.4452
	$Q_{.95}$	4.7095	3.8809	3.1513	5.1804	5.1977
	$Q_{.99}$	9.6176	13.9351	13.6717	10.4805	10.4859
Uniform	$Q^{\#}_{.9}$	3.1211	58.1749	9.1001	3.1438	3.1087
	$Q^{\#}_{.95}$	4.8497	74.9338	16.9474	4.8567	4.8215
	$Q^{\#}_{.99}$	9.6616	92.2483	36.6108	9.6870	9.6094
Implied	$Q^{*}_{.9}$	3.1279	5.8244	4.7028	3.1489	3.1453
	$Q^{*}_{.95}$	4.7993	9.8464	7.3681	4.7236	4.7080
	$Q^{*}_{.99}$	9.9595	25.4335	16.8625	9.6343	9.6143

Table 2: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the J-statistic distribution based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.

		No Con.	Contamination in $Z_{2,100}$			
			-3	-1	1	3
True	$Q_{.9}$	3.1668	3.1032	3.1048	3.1943	3.2773
	$Q_{.95}$	4.7095	4.4612	4.5147	4.7413	4.7080
	$Q_{.99}$	9.6176	8.3591	9.0743	9.3187	8.6588
Uniform	$Q^{\#}_{.9}$	3.1211	3.2885	3.1263	3.2061	3.0363
	$Q^{\#}_{.95}$	4.8497	4.7722	4.6984	4.8426	4.4571
	$Q^{\#}_{.99}$	9.6616	8.5820	9.2276	9.6337	7.9688
Implied	$Q^{*}_{.9}$	3.1279	3.3369	3.1345	3.1840	3.0562
	$Q^{*}_{.95}$	4.7993	4.8033	4.6676	4.7596	4.5438
	$Q^{*}_{.99}$	9.9595	8.4371	8.7645	9.4534	7.9938

Table 3: **Quantiles of the uniform weight and implied probability bootstrap.** "No Con." means "No Contamination". The rows labelled "True" report the simulated quantiles of the J-statistic distribution based on 20,000 realizations. The rows labelled "Uniform" report the uniform weight bootstrap quantiles. The rows labelled "Implied" report the implied probability bootstrap quantiles.