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Dual representation of choice and aspirational preferences [1]

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Abstract

We consider choice over a set of monetary acts (random variables) and study a general class of preferences. These preferences favor diversification, except perhaps on a subset of sufficiently disliked acts, over which concentration is instead preferred. This structure encompasses a number of known models in this setting. We show that such preferences can be expressed in dual form in terms of a family of measures of risk and a target function. Specifically, the choice function is equivalent to selection of a maximum index level such that the risk of beating the target function at that level is acceptable. This dual representation may help to uncover new models of choice. One that we explore in detail is the special case of a bounded target function. This case corresponds to a type of satisficing and has descriptive relevance. Moreover, the model results in optimization problems that may be efficiently solved in large-scale.

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1 Introduction

The notion of an aspiration level rests at the core of Simon's [48] concept of bounded rationality. Namely, due to limited cognitive resources and incomplete information, real-world decision makers may plausibly follow heuristics in the face of risky choice. Satisficing behavior, in which the decision maker accepts the first encountered alternative that meets a sufficiently high aspiration level, may prevail.

There is indeed ample empirical evidence that aspiration levels, or "targets," play a central role in the decision-making of many individuals. Mao [36], for instance, concludes after interviewing many executives, that "risk is primarily considered to be the prospect of not meeting some target rate of return." Other studies (e.g., Roy [45], Lanzilloti [31], Fishburn [20], Payne et al. [41, 42], March and Shapira [37]) reach similar conclusions regarding the importance of targets in managerial decisions. Diecidue and van de Ven [17] provide many more references and propose a model combining expected utility with loss and gain probabilities, which leads to a number of predictions consistent with empirical data. Recently, Payne [40] showed that many decision makers would be willing to accept a decrease in a gamble's expected value in order to reduce just the probability of not beating a target.

The goal of this paper is to provide some formalism for the role of aspiration levels in rational choice and to introduce a new model in this domain. We consider the case of a decision maker choosing from a set of monetary acts (i.e., random variables), and define preferences over these acts. The structure of these preferences is fairly general: in addition to the usual properties of a weak order, a mild continuity property, and monotonicity, the only requirement we impose is the way the decision maker treats mixtures of acts. In particular, we assume the decision maker prefers to diversify among acts, except possibly on a subset of sufficiently unfavorable choices, for which concentration is preferred. We call these preferences *aspirational preferences*. In the case when the concentration favoring set is empty, diversification is always preferred. Our definition of diversification favoring is a standard one of convex preferences.

Our main theoretical result is a dual representation for choice under such preferences. This result states that we can equivalently express choice in terms of a family of measures of risk and a target function: in particular, the choice function for such preferences is equivalent to a maximum index such that the risk of beating the target function at that index is acceptable. On the set of acts for which diversification is preferred, the risk measures are *convex risk measures* (Föllmer and Schied [21, 22]). In this setting of monetary acts, a number of popular models of choice, including expected utility theory and several generalizations, are aspirational preferences and thus have this representation.

This characterization with risk measures has a number of important implications. First, it yields potentially new interpretations for existing models of choice. For instance, Föllmer and Schied [21] show that convex risk measures have a dual description in terms of robust expected value against a malicious adversary who is manipulating the underlying probabilities. It then follows that expected utility (for example) under a specific distribution can be equivalently expressed in a "robust form" in which the underlying distribution is not precisely specified. In addition, structural properties, such as stochastic dominance, can be established directly from known properties of convex risk measures. Moreover, from an optimization standpoint, the risk representation is important: indeed, over acts where the choice function is quasi-concave (prefers diversification), the level sets that one must search over are in fact acceptance sets for the representing risk measure family.

More generally, representation with risk measures and a target function potentially opens doors for new choice models. One example of this that we explore in detail is the case when the target function is bounded. In this case, the lower and upper limits of the target function correspond to a minimal requirement and a satiation level, respectively. Acts that fail to attain the minimal requirement in any state are least preferred, whereas acts that attain at least the satiation level in all states are most preferred. When the two levels coincide, the target represents a single aspiration level. In this case of a bounded target function, we say the decision maker has *strongly aspirational preferences* and call the choice function a *strong aspiration measure (SAM)*.

Choice under SAM seems to have noteworthy descriptive power. In particular, we find that SAM (with just a single target) can address the classical examples of Allais [1] and Ellsberg [19]. This model also addresses more recent paradoxes, such as an Ellsberg-like example due to Machina [34], and predicts well choice patterns in a set of experiments related to gain-loss separability in Wu and Markle [50].

Thus, from the theory side, we find that a notion of targets is, to some extent, implicitly embedded in several rational choice models and draw connections between preferences and risk measures. Pushing the limit of this general representation to the case when the target becomes a central focus of the decision maker, we obtain the SAM model. From the empirical side, application of SAM to some descriptive settings seems to support the idea that aspiration levels play an important role in the decision making of individuals.

We refer the reader to some recent, related work. Cerreia-Vioglio et al. [11] axiomatize, in an Anscombe-Aumann setting, a model of *uncertainty averse preferences*. These preferences are convex and monotone and the authors provide a general representation result. Drapeau and Kupper [18] develop a similar representation on fairly general topological spaces. Their work focuses on robustness interpretations and the representation is presented primarily in terms of "acceptance sets," whereas ours is in the form of risk measures and target functions. We believe this latter form has important implications: the SAM model is, to our knowledge, new, and is motivated from this form. Also differentiating us is that we do not require convex preferences everywhere.

The allowance of a limited amount of risk-seeking merits a brief discussion. Risk-seeking behavior is traditionally relegated purely to the realm of the descriptive.[1] In a target-driven model like SAM, however, there is some rationale for a decision maker to possibly prefer concentration when they have high aspirations. Indeed, we show a consequence of the SAM model is that, under fairly mild conditions, concentration is a *necessary* feature in order to be able to distinguish among acts with *expected value* below the target. This property is another fairly direct result from the dual representation with convex risk measures. Intuitively, we may reasonably expect decision makers to be willing to shun diversification and take risks when aspirations are ambitious relative to available choices.

[1]Though, there is some debate on this point; one can trace some discussions back to Friedman and Savage [24] and Markowitz [38], for instance.

Finally, we briefly contrast our work with models of target-based utility. There are interesting theoretical connections between expected utility theory and probability of beating a benchmark on a higher dimensional state space (see, for instance, Bordley and LiCalzi [6] and Castagnoli and LiCalzi [9]). In terms of practical use, some issues arise with probability of beating a benchmark as a choice function. First, such a model favors neither diversification nor concentration. This presents significant hurdles in application to optimization settings (e.g., portfolio choice). Second, probability of beating a benchmark is insensitive to the magnitude of gains and losses. In contrast, choice under SAM is in general not insensitive to the size of gains and losses and can be efficiently optimized in problems with many decision variables. For tractability in optimization, the important feature is that the choice function is be either quasi-concave or quasi-convex; we are not aware of other models that incorporate mixed attitudes towards both risk and ambiguity that meet this criterion (for example, prospect theory in general results in objective functions that are neither quasi-convex nor quasi-concave).

Our outline is as follows. In Section 2, we describe the choice setting and define aspirational preferences, then present the main representation result. Later in this section we briefly discuss the interpretation in terms of robustness and characterize some general properties related to stochastic dominance. Section 3 is devoted to the SAM model and some of its properties. Section 4 applies this model to the decision theory paradoxes mentioned earlier, and Section 5 discusses optimization of the model with a portfolio choice example. Finally, Section 6 offers some concluding remarks.

2 Aspirational preferences and dual representation

In this section, we introduce the basic choice problem we consider, then define aspirational preferences. We then establish a generic representation result, in terms of risk measures, for these preferences.

2.1 Model preliminaries and aspirational preferences

Our setup falls into the framework of Savage [46] in the special case of monetary acts. Specifically, we consider a set S of states of the world, endowed with a sigma-algebra Σ. An element $s \in S$ is an individual state of the world, while an element $A \in \Sigma$ is an event. There is a set of consequences X. An element $x \in X$ is an individual consequence. For our analysis, we focus exclusively on the case $X = \mathbb{R}$ and interpret an individual consequence $x \in X$ as a monetary outcome. An act is a measurable function from S to $\mathbb{R}$: its inverse applied to any interval $I \subset \mathbb{R}$ is an event, i.e., belongs to Σ. We denote acts by f, g, h. An act f is *constant* if there is an individual consequence $x \in X$ such that $f(s) = x$ for all $s \in S$. In this case we write $f = x$. The set of all acts on (S, Σ) is denoted by $L_0(S, \Sigma)$, while the set all *bounded* acts on (S, Σ) is denoted by $L_\infty(S, \Sigma)$. A subset $G \subseteq L_\infty(S, \Sigma)$ is said to be closed if it belongs to the topology endowed on $L_\infty(S, \Sigma)$ by the sup-norm $||f|| = \sup_{s \in S} |f(s)|$.

For $f, g \in L_0(S, \Sigma)$ we say that f state-by-state dominates g if and only if $f(s) \geq g(s)$ for all $s \in S$. In this case we write (with some abuse of notation) that $f \geq g$. We say that a mapping ρ from a subset G of $L_0(S, \Sigma)$ to $\mathbb{R}$ is *nondecreasing* when for all $f, g \in G$, if $f \geq g$ then $\rho(f) \geq \rho(g)$.

For $f, g \in L_0(S, \Sigma)$ and $\lambda \in [0,1]$, the convex combination $h = \lambda f + (1-\lambda) g$ is defined state-by-state, i.e., $h(s) = \lambda f(s) + (1-\lambda) g(s)$.

While not needed generally, we will often consider cases when (S, Σ) is endowed with a *probability measure* $\mathbb{P}$. A probability measure is a mapping from Σ to $[0,1]$, such that $\mathbb{P}(S) = 1$, $\mathbb{P}(\emptyset) = 0$, and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for all $A, B \in \Sigma$ such that $A \cap B = \emptyset$. The triple $(S, \Sigma, \mathbb{P})$ is called a *probability space*. A probability space is said to be *atomless* when there exists no $s \in S$ such that $\mathbb{P}(\{s\}) > 0$. Moreover, given a probability measure $\mathbb{P}$ on (S, Σ) and an act $f \in L_0(S, \Sigma)$, we call the function $x \to \mathbb{P}(\{s \in S : f(s) \leq x\})$, $x \in \mathbb{R}$, the *cumulative distribution function of f with respect to* $\mathbb{P}$. We denote by $\mathcal{P}$ the set of all probability measures on (S, Σ).

In a setup like this, any probabilities are entirely "subjective" in the sense that they are part of the decision maker's preference structure and not fixed in advance by the model. Strictly speaking, the framework does not formally separate uncertainty resulting from "risk" versus that from "ambiguity," as is done in an Anscombe-Aumann [2] setting. To us, this does not seem like a severe limitation, especially for real-world applications.[2] Moreover, nothing in our setup precludes the possibility of some objective, probabilistic structure in the model, such as in the experimental settings we later consider. In these cases, decision makers would be hard pressed to disagree on the various "objective" probabilities, and notions of risk versus ambiguity should be clear from context.

We consider the situation of a decision maker who wants to choose an act from a closed and convex subset $F \subseteq L_\infty(S, \Sigma)$. Note that elements in F are assumed to be bounded.[3]

We model decision maker's preferences using a preference relation $\succeq$ on F. For acts $f, g \in F$, the decision maker weakly prefers f to g if and only if $f \succeq g$. As usual, $\succ$ and $\sim$ are defined by $[f \succ g \Leftrightarrow (f \succeq g) \text{ and } \neg(g \succeq f)]$ and $[f \sim g \Leftrightarrow (f \succeq g) \text{ and } (g \succeq f)]$. We call a function $\rho : F \to \mathbb{R} \cup \{-\infty, \infty\}$ a *functional representation* of $\succeq$ when for all $f, g \in F$, $f \succeq g$ if and only if $\rho(f) \geq \rho(g)$.

The conditions for existence of a functional representation are well established. In the interests of formally defining the preference relation, we make this explicit. Ultimately, our focus will be on the form of this representation and its dual structure with risk measures. For our purposes, as explained below, we weaken the standard assumption (e.g., Debreu [15]) of continuity.

Property 1 (Weak order and upper semi-continuity). *Let $\succeq$ be a weak order on F that satisfies:*

(i) For all $f \in F$, the set $\{g \in F : g \succeq f\}$ is closed in F (upper semi-continuity).

(ii) There exists $H \subset F$ that is order-embeddable[4] into $[0,1]$ such that for all $f, g \in F$ with $f \succ g$, there exists $h \in H$ such that $f \succeq h \succeq g$.

[2] Gilboa [26], for instance, mentions that objective probabilities are somewhat "controversial."

[3] The representation result that follows does not require this. One could consider more general acts in $L_0(S, \Sigma)$. However, some care would be required in handling the technical conditions for the underlying topology. For simplicity, we focus on the case of bounded acts in our basic setup.

[4] H is order-embeddable into $[0,1]$ when there exists a order-preserving function $j : (H, \succeq) \to [0,1]$. j is order-preserving when it is increasing and for $f, g \in H$ with $f \succ g$ we have $j(f) > j(g)$.

The assumption of a weak order is standard, while (i) and (ii) in Property 1 are technical conditions needed to obtain a functional representation of $\succeq$. Properties (i) and (ii) are weaker then the standard continuity conditions of Debreu [15], but are sufficient to obtain a functional representation of $\succeq$.[5] Indeed, for the purposes of this paper, the classical continuity assumption is too restrictive. This is because we also want to include in our framework preference relations in the spirit of Simon [48], e.g., $f \succeq g$ if and only if $\rho(f) \geq \rho(g)$, where $\rho : F \to \mathbb{R}$ satisfies $\rho(f) > 0$ on $\{f \geq 0\}$ (satisfactory payoffs) and $\rho(f) < 0$ on $\{f < 0\}$ (unsatisfactory payoffs). Following a generalization of Rader [43], we impose upper semi-continuity (Property (i)) on $\succeq$. This is weaker than continuity and allows preference relations as mentioned. Bosi and Mehta [7] shows that Property (i) is not only sufficient for a functional representation, but also necessary if the preference relation is separable; that is, Property (ii) holds.

We then require the following.

Property 2 (Monotonicity)**.** *For all $f, g \in F$, if $f \geq g$ then $f \succeq g$.*

Monotonicity says that an act that dominates another state-wise is preferred among the two. This is a classical assumption on preferences, and it says that decision makers do not prefer less to more.

The final property we impose is how preferences favor mixtures of positions.

Property 3 (Mixing)**.** *There exists a partition of F into three disjoint subsets F_{++}, F_{--} and F_0, termed the diversification favoring, concentration favoring and neutral sets of acts, respectively, such that for all $f \in F_{++},\ g_1,\ g_2 \in F_0,\ h \in F_{--}$, we have*

$$f \succ g_1 \sim g_2 \succ h,$$

and the following conditions hold:

(i) Diversification favoring set*: For all $f, g \in F_{++}$, $h \in F$, if $f \succeq h$, $g \succeq h$ then*

$$\lambda f + (1 - \lambda) g \succeq h \qquad \forall \lambda \in [0, 1].$$

(ii) Concentration favoring set*: For all $f, g \in F_{--}$, $h \in F$, if $h \succeq f$, $h \succeq g$ then*

$$h \succeq \lambda f + (1 - \lambda) g \qquad \forall \lambda \in [0, 1].$$

Property (i) is simply the definition of convex preferences, and it states that diversification among acts in F_{++} never results in a position that is worse. This is a classical notion of risk aversion. Property (ii) says the opposite: diversification does not help for acts in the set F_{--}.

In our general setup, we are allowing for both risk aversion and risk seeking. Note, however, that any risk seeking behavior that is permitted is localized on a set of less preferable acts. Intuitively, the decision maker will consider avoiding diversification only if available choices are sufficiently unfavorable.

As a very simple example of a situation that may merit concentration, and as a prelude to some of the discussion that will follow on target-oriented choice, consider a decision maker who wishes to attain a

[5] Debreu [15] assumes, in addition to Property (i), that $\{g \in F : f \succeq g\}$ is closed in F.

particular payoff level. Without loss, we set this desired level to zero. Now consider a pair of acts $f, g \in F$, $f, g \leq 0$, that are unfavorable in that each meets this desired payoff in only a single state, and these states are different across f and g. In other words, there exist $s', s'' \in S$, $s' \neq s''$, with $f(s) = 0$ ($g(s) = 0$) if and only if $s = s'$ ($s = s''$). For any convex combination of f and g, $h = \lambda f + (1 - \lambda) g$ for $\lambda \in (0, 1)$, we have $h < 0$. If the decision maker diversifies, they will always attain a mixed position that never attains the desired payoff level. Here, it may be reasonable to prefer at least one of f or g, which each have some chance of attaining the desired level, individually over any mixture.

While this is a very simple (and admittedly contrived) example, it illustrates an important point. Namely, there may be situations, particularly in models of choice that focus on aspiration levels, in which some risk seeking behavior is sensible. Note, however, that we do not impose that F_{--} be nonempty. We allow for risk seeking, but do not require it.

Definition 1. *A decision maker with preference relation $\succeq$ satisfying Properties 1-3 is said to have* aspirational preferences *on F with partition F_{++}, F_{--}, F_0.*

The partition in Property 3 is not uniquely characterized by the preference relation $\succeq$ and thus the partition is part of our definition of aspirational preferences. We note the following in cases when two different partitions of F exist such that $\succeq$ satisfies Property 3.

Proposition 1. *Assume that $\succeq$ is a preference relation on F and Property 3 holds for two different partitions F_{++}, F_{--}, F_0 and G_{++}, G_{--}, G_0. Then either $G_{++} \cup G_0 \subseteq F_{++}$ and $F_{--} \cup F_0 \subseteq G_{--}$, or $F_{++} \cup F_0 \subseteq G_{++}$ and $G_{--} \cup G_0 \subseteq F_{--}$. It follows that for all $f, g \in F_0$ (and all $f, g \in G_0$), $\lambda \in [0, 1]$,*

$$\lambda f + (1 - \lambda) g \succeq g \sim f \quad \text{or} \quad g \sim f \succeq \lambda f + (1 - \lambda) g.$$

Moreover, for all $f, g \in F_{++} \cap G_{--}$ (and all $f, g \in F_{--} \cap G_{++}$), $\lambda \in [0, 1]$, if $f \succeq g$ then

$$f \succeq \lambda f + (1 - \lambda) g \succeq g.$$

Proposition 1 says that while a preference relation $\succeq$ can satisfy Property 3 on more than one partition of F, there is a "natural partition," which can be constructed as follows: let H_{++} (H_{--}) be the biggest set of diversification (concentration) favoring acts among all existing partitions and set $F_{++} = H_{++} \setminus (H_{++} \cap H_{--})$, $F_{--} = H_{--}$ and $F_0 = F \setminus (F_{++} \cup F_{--})$.

2.2 Risk measures and representation of aspirational preferences

We now show how to represent aspirational preferences in terms of a more classical definition of a "risk measure." Risk measures are motivated from the perspective of "minimal capital requirements" to make positions acceptable. We clarify below how this can be interpreted in our framework. Given the fairly specific motivation for risk measures, it is intriguing that they link so closely to the seemingly more general choice model discussed above.

Following Föllmer and Schied [21], we first formally define the concept of a risk measure.

Definition 2. *A function $\mu : F \to \mathbb{R}$ is a* risk measure *over F if it satisfies the following for all $f,\ g \in F$:*

1. Monotonicity*: If $f \geq g$, then $\mu(f) \leq \mu(g)$.*
2. Translation invariance*: If $x \in F$ is a constant act, then $\mu(f + x) = \mu(f) - x$.*

A risk measure $\mu(f)$ may be interpreted as the constant act to be added to f in order to make f acceptable by some standard. Namely, $\mu(f + \mu(f)) = \mu(f) - \mu(f) = 0$, i.e., adding the individual consequence $x = \mu(f)$ to the act f, one obtains a new act $g = f + x$ with "zero risk" (i.e., $\mu(g) = 0$), and acts with non-positive risk can be considered as acceptable. In other words, an act is acceptable if it does not require any additional, guaranteed money. One can formalize the concept of acceptable acts as follows.

Definition 3. *Let $\mu : F \to \mathbb{R}$ be a risk measure. The subset $\mathcal{A}_\mu$ of F defined by*

$$\mathcal{A}_\mu = \{f \in F : \mu(f) \leq 0\}$$

is called the acceptance set *associated to the risk measure μ and $f \in \mathcal{A}_\mu$ is an* acceptable act.

The two properties of risk measures have clear implications for the acceptance set: if one act state-wise dominates an acceptable act, then it must be acceptable as well. In addition, if we add a constant act to another act, then the additional money required in order to make the second act acceptable is reduced accordingly. We refer the reader to Föllmer and Schied [22] and the many references therein for more on risk measures and the properties of the corresponding acceptance sets.

The class of convex risk measures has garnered much attention. Formally, we say a risk measure is *convex* if, for any $f,\ g \in F$, $\lambda \in [0, 1]$,

$$\mu(\lambda f + (1 - \lambda)g) \quad \leq \quad \max\{\mu(f), \mu(g)\} \tag{1}$$

and *concave* if

$$\mu(\lambda f + (1 - \lambda)g) \quad \geq \quad \min\{\mu(f), \mu(g)\}. \tag{2}$$

Notice that the preference relation $\succeq_\mu$ induced by a risk measure μ will follow $f \succeq_\mu g$ if and only if $\mu(f) \leq \mu(g)$. It is not hard to see that (1) is equivalent to the preference relation $\succeq_\mu$ being diversification favoring (i.e., convex preferences), and (2) is equivalent to $\succeq_\mu$ being concentration favoring. Historically, convex risk measures are defined with μ satisfying convexity directly, not quasi-convexity as in (1); this is equivalent.

Proposition 2. *A risk measure μ that is diversification favoring is equivalent to the function μ being convex, i.e., for all $f,\ g \in F$, $\lambda \in [0, 1]$, $\mu(\lambda f + (1 - \lambda)g) \leq \lambda\mu(f) + (1 - \lambda)\mu(g)$. Likewise, concentration favoring is equivalent to the function μ being concave.*

Proposition 2 is also shown in Cerreia-Vioglio et al. [10], who argue that quasi-convexity, rather than convexity, of the risk measure function is the natural way to describe a preference for diversification.

We are now ready for the representation result. In what follows, we use the convention $\sup \emptyset = -\infty$.

Theorem 1. *A preference relation $\succeq$ is aspirational on F with partition F_{++}, F_{--}, F_0 if and only if there exists a corresponding functional representation $\rho : F \to \mathbb{R} \cup \{-\infty, \infty\}$ and $\hat{k} \in \mathbb{R} \cup \{-\infty, \infty\}$ such that ρ is upper semi-continuous, nondecreasing, quasi-concave on F_{++}, quasi-convex on F_{--}, and $\rho(g) = \hat{k}$ for all $g \in F_0$, $\rho(f) > \hat{k} > \rho(h)$ for all $f \in F_{++}$, $h \in F_{--}$. Moreover, ρ has representation*

$$\rho(f) \quad = \quad \sup\left\{k \in \mathbb{R} : \ \mu_k(f) \leq 0\right\}, \tag{3}$$

where $\{\mu_k\}$ is a family of risk measures, nondecreasing in k, convex if $k > \hat{k}$, concave if $k < \hat{k}$, and with closed acceptance sets $\mathcal{A}_{\mu_k}$. Conversely, given aspirational preferences represented by ρ, the underlying risk measure family is given by

$$\mu_k(f) \quad = \quad \inf\{a \in \mathbb{R} \ : \ \rho(f + a) \geq k\}. \tag{4}$$

We call the function ρ from Theorem 1 an *aspiration measure* (AM). Note that ρ is unique up to a nondecreasing transformation.[6] Inspection of the proof of Theorem 1 shows that the acceptance set of the risk measure μ_k corresponds to $\mathcal{A}_{\mu_k} = \{f \in F \ : \ \rho(f) \geq k\}$.

If the risk measure $\mu(f)$ is to be interpreted as the constant act to be added to f in order to have zero risk, it is natural to require $\mu(0) = 0$. A risk measure satisfying this is said to be *normalized*. In the general representation above, no such requirement is made on the family μ_k. The adjusted family of risk measures $\tilde{\mu}_k(f) = \mu_k(f) - \mu_k(0)$ is, however, normalized.[7] While the risk measures $\tilde{\mu}_k$ inherit convexity and concavity properties from μ_k, the family $\{\tilde{\mu}_k\}$ is not necessarily nondecreasing in k.

Noting this, we can equivalently express choice under aspirational preferences in dual form as

$$\rho(f) \quad = \quad \sup\left\{k \in \mathbb{R} \ : \ \tilde{\mu}_k(f - \tau(k)) \leq 0\right\}, \tag{5}$$

where $\tilde{\mu}_k$ is normalized and $\tau(k) = \mu_k(0)$ is a constant act for all k. We call $\tau(k)$ a *target act* and the function $\tau : \mathbb{R} \to F, \ k \to \tau(k)$ is the *target function*. Clearly, τ is nondecreasing.

The representation (5) provides an interpretation of choice under aspirational preferences. Such choice can be interpreted as searching over index levels k, such that at k, the risk $\tilde{\mu}_k$ associated with $f - \tau(k)$ is acceptable. The AM $\rho(f)$ then represents the maximal level at which the risk of f beating the target act at that level is acceptable. When the sup is attained, we have the fixed point relationship for ρ:

$$\tilde{\mu}_{\rho(f)}(f - \tau(\rho(f))) \quad = \quad 0.$$

Since the family $\{\tilde{\mu}_k\}$ is not necessarily nondecreasing, one must exercise some care in interpreting choice intuitively here. For $k^\star > k$, $\tau(k^\star) \geq \tau(k)$ and thus $f - \tau(k^\star) \leq f - \tau(k)$. An AM, therefore, assigns higher value to acts whose risk associated with falling short of the target remain acceptable when measured against

[6] This can be seen as follows. Let ρ and $\tilde{\rho}$ be two functional representations for an aspirational preference relation $\succeq$. Define $T : \rho(F) \to \mathbb{R}$ as $T(\rho(f)) = \tilde{\rho}(f)$ for all $f \in F$. For $f, g \in F$ we have: $\rho(f) \geq \rho(g) \Rightarrow f \succeq g \Rightarrow \tilde{\rho}(f) \geq \tilde{\rho}(g) \Rightarrow T(\rho(f)) \geq T(\rho(g))$. Therefore, T is a nondecreasing transformation.

[7] $\tilde{\mu}_k$ is clearly a risk measure: For $f, g \in F$, $f \geq g$, we have $\tilde{\mu}_k(f) = \mu_k(f) - \mu_k(0) \leq \mu_k(g) - \mu_k(0) = \tilde{\mu}_k(g)$; for $f \in F$ and $x \in X$, $\tilde{\mu}_k(f + x) = \mu_k(f + x) - \mu_k(0) = \mu_k(f) - x - \mu_k(0) = \mu_k(f) - \mu_k(0) - x = \tilde{\mu}_k(f) - x$.

higher targets. In some cases, $\tilde{\mu}$ may be a single normalized risk measure, in which case only the target increases with k (we will see this is true for CARA utility maximizers). On the flip side (which will be the case in our applications of aspirational preferences in Sections 4 and 5), the target function may be a constant τ and the risk measure family $\{\mu_k\}$ may already be normalized. In this case, the target remains constant but the risk increases in k. In general, the overall risk according to $\mu_k(f) = \tilde{\mu}_k(f - \tau(k))$ is nondecreasing in k, as is $\tau(k)$, even if $\tilde{\mu}_k$ is not.

This dual interpretation applies to a number of choice models. We now provide some examples. When not stated explicitly, we will assume that (S, Σ) is endowed with a probability measure $\mathbb{P}$ and expectations are taken with respect to $\mathbb{P}$.

Example 1. Expected utility theory

Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing, concave utility function satisfying $u(0) = 0$. The choice function $\rho(f) = \mathbb{E}[u(f)]$ is an AM. Here, the diversification favoring set F_{--} and the neutral set F_0 can be assumed to be empty, i.e., we can take $F_{++} = F$ as diversification is always (weakly) preferred. The representing risk family is expressed as

$$\begin{aligned} \mu_k(f) &= \inf\{a \;:\; \mathbb{E}[u(f+a)] \geq k\} \\ &= -\sup\{a \;:\; \mathbb{E}[u(f-a)] \geq k\}. \end{aligned}$$

Here, $-\mu_k(f)$ represents a maximum purchase price one would pay to assume the position f while still attaining a level k in expected utility. When u is invertible, we have $\tau(k) = \mu_k(0) = u^{-1}(k)$, i.e., the target act is the constant act with utility k. Föllmer and Schied [22] study the class of *shortfall risk measures* induced by convex, increasing loss functions $l : \mathbb{R} \to \mathbb{R}$:

$$\mu_v^{\text{short}}(f) = \inf\{a \;:\; \mathbb{E}[l(-f-a)] \leq v\}.$$

This is clearly the same object as μ_k here, with $l(y) = -u(-y)$ and $v = -k$.

For CARA utility, $u(y) = 1 - \exp(-y/R)$ for some $R > 0$, and we have

$$\begin{aligned} \mu_k(f) &= \inf\{a \;:\; \mathbb{E}[1 - \exp(-(f+a)/R)] \geq k\} \\ &= \underbrace{R \log \mathbb{E}\left[e^{-\frac{f}{R}}\right]}_{\tilde{\mu}_k(f)} \underbrace{- R\log(1-k)}_{\tau(k)}, \end{aligned}$$

on $k \in (-\infty, 1)$. For this choice function, note that the normalized family $\tilde{\mu}_k$ is independent of the index (utility) level k: all variation over the index is embedded within the target function $\tau(k)$.

Example 2. Maxmin EUT, Choquet utility, and variational preferences

Maxmin expected utility (MEU), developed by Gilboa and Schmeidler [27], has a similar dual representation. Here, we have

$$\mu_k(f) = \inf_{a \in \mathbb{R}} \left\{ a \;:\; \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[u(f+a)] \geq k \right\},$$

where $\mathcal{Q} \subseteq \mathcal{P}$ is a set of probability measures on (S, Σ), and we consider the case when u is nondecreasing and concave. In this case, $F = F_{++}$ again. It is known (Gilboa and Schmeidler [27]) that Choquet expected utility (CEU) axiomatized by Gilboa [26] and Schmeidler [47] falls into the class of MEU in the important case when the decision maker is ambiguity averse.

Generalizing the MEU model, Maccheroni et al. [33] axiomatized a model of *variational preferences*. Here, choice is represented by the function

$$\rho(f) \;=\; \inf_{\mathbb{Q}\in\mathcal{P}} \left\{\mathbb{E}_{\mathbb{Q}}\left[u(f)\right] + c(\mathbb{Q})\right\},$$

where u is a differentiable, nondecreasing utility function, and c is a nonnegative convex function on $\mathcal{P}$ with $\inf_{\mathbb{Q}\in\mathcal{P}} \alpha(\mathbb{Q}) = 0$. In the case of risk aversion, i.e., u concave, this falls into our setup with $F_{++} = F$.

The generating family of convex risk measures is

$$\mu_k(f) \;=\; \inf_{a\in\mathbb{R}} \left\{ a \;:\; \inf_{\mathbb{Q}\in\mathcal{P}} \left\{\mathbb{E}_{\mathbb{Q}}\left[u(f+a)\right] + c(\mathbb{Q})\right\} \geq k \right\}.$$

For both MEU and variational preferences, assuming u is invertible, the target function is $\tau(k) = u^{-1}(k)$.

Example 3. Acceptability, satisficing, and riskiness indices

Several recent papers have attempted to formalize definitions of risk and related measures of performance. Aumann and Serrano [4] axiomatize a definition of risk. Their axioms lead to an index of riskiness $r(f)$ defined as

$$r(f) \;=\; \inf \left\{ a > 0 \;:\; \mathbb{E}\left[e^{-f/a}\right] \leq 1 \right\},$$

and has the interpretation of being the smallest risk tolerance level for a CARA decision maker such that at that level the decision maker would accept the act f (i.e., the expected utility of f is nonnegative). It is not hard to see that $1/r(f)$ yields the function

$$\rho(f) \;=\; \sup \left\{ k > 0 \;:\; k^{-1} \log \mathbb{E}\left[e^{-k\,f}\right] \leq 0 \right\},$$

where $k^{-1} \log \mathbb{E}\left[\exp(-k\,f)\right]$ is the *entropic risk measure* at level $k > 0$. This is a convex risk measure (e.g., [22]) and therefore such a ρ is an AM. Foster and Hart [23] derive an "operational" definition of risk that has a similar representation with logarithmic utility replacing the exponential.

Both of these definitions of risk fall into the class of *satisficing measures* introduced by Brown and Sim [8], which take the form of a function:

$$\rho(f) \;=\; \sup \left\{ k > 0 \;:\; \mu_k(f - \tau) \leq 0 \right\},$$

where $\{\mu_k\}_{k>0}$ is a family of normalized convex risk measures and $\tau \in L_0(S, \Sigma)$ is a competing benchmark. The *acceptability indices* of Cherny and Madan [13] fall into this framework with $\tau = 0$ when the risk measures μ_k are also positive homogeneous (or "coherent" according to the definition of Artzner et al. [3]).

In all of these cases, note that one has convex preferences for any act, so $F_{--} = \emptyset$. Note also that these indices may be attempting to provide an explicit quantity (e.g., a risk value) rather than being used directly as a choice function. However, a nonnegative combination of an AM with the expected value function remains an AM, so a "risk-reward" tradeoff in this sense still represents choice under an AM. Alternatively, one may imagine that the set F represents a set of available acts meeting certain conditions (e.g., sufficiently high expected return in a portfolio choice setting). In this case, one of these indices could plausibly be used as a choice function.

2.3 An ambiguity interpretation

The dual representation of this choice model in terms of convex (and concave) risk measures leads to an interpretation of choice that explicitly accounts for ambiguity. Indeed, it is known (Föllmer and Schied [22]) that any convex risk measure μ in such a setting can be represented as

$$\mu(f) \quad = \quad \sup_{\mathbb{Q} \in \mathcal{P}} \left\{ -\mathbb{E}_{\mathbb{Q}}[f] - \alpha(\mathbb{Q}) \right\}, \tag{6}$$

where $\mathcal{P}$ is the set of all probability measures on (S, Σ) and $\alpha : \mathcal{P} \to \mathbb{R}$ is a convex function, $\alpha(\mathbb{Q}) = \sup_{f \in \mathcal{A}_\mu} \mathbb{E}_{\mathbb{Q}}[-f]$. It is easy to see that if μ is normalized, i.e., $\mu(0) = 0$, then $\inf_{\mathbb{Q} \in \mathcal{P}} \alpha(\mathbb{Q}) = 0$.

We can now express choice above using this form. Specifically, if $f \in F_{++}$, then we have

$$\begin{aligned} \rho(f) \quad &= \quad \sup \left\{ k > \hat{k} \ : \ \mu_k(f) \leq 0 \right\} \\ &= \quad \sup \left\{ k > \hat{k} \ : \ \sup_{\mathbb{Q} \in \mathcal{P}} \left\{ -\mathbb{E}_{\mathbb{Q}}[f] - \alpha_k(\mathbb{Q}) \right\} \leq 0 \right\} \\ &= \quad \sup \left\{ k > \hat{k} \ : \ \inf_{\mathbb{Q} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{Q}}[f - \tau(k)] + \tilde{\alpha}_k(\mathbb{Q}) \right\} \geq 0 \right\}, \end{aligned}$$

where we are using the normalized representation for the family of generating risk measures: $\tau(k)$ is the target function and $\tilde{\alpha}_k$ is a convex function corresponding to the normalized convex risk measure $\tilde{\mu}_k$ according to the representation (6).

This perspective implies a robustness interpretation of choice over acts in the diversification favoring set. Namely, the decision maker with preferences represented by ρ ranks acts in F_{++} according to an index level. At a particular index level k, the decision maker looks at the expected value of the act in excess of the target, where the probability measure is chosen by a malicious adversary. This adversary wishes to make the expected value of $f - \tau(k)$ as small as possible, but has to pay a penalty $\tilde{\alpha}_k(\mathbb{Q}) \geq 0$ for choosing measure $\mathbb{Q}$ and therefore selects a distribution that minimizes the expected value plus the penalty.

As the index level grows, both the target and the combined effect of $\tau(k)$ and $\tilde{\alpha}_k$ increase. Therefore, robustly satisfying the (penalized) expected value constraint becomes more difficult, as the adversary's power grows. The choice function $\rho(f)$ thus represents the maximum index level k such that the condition

$$\mathbb{E}_{\mathbb{Q}}[f - \tau(k)] + \tilde{\alpha}_k(\mathbb{Q}) \quad \geq \quad 0$$

holds for all probability measures $\mathbb{Q} \in \mathcal{P}$. In this sense, ρ over the diversification favoring set corresponds to a notion of *robustness* or *security*: acts with larger ρ can beat a higher target in expectation in a more robust sense.

This robustness interpretation holds even if the choice function ρ uses an explicit probability measure in its "primal" form. For instance, consider EUT as described earlier with utility function u and probability measure $\mathbb{P}$ as in Example 1. Here, $F_{++} = F$ and the choice function is $\rho(f) = \mathbb{E}_{\mathbb{P}}[u(f)]$. Following the representation of shortfall risk due to Föllmer and Schied [22], we can express this choice in the "robust" form:

$$\rho(f) \quad = \quad \sup \left\{ k \ : \ \inf_{\mathbb{Q} \in \mathcal{P}} \{ \mathbb{E}_{\mathbb{Q}}[f] + \alpha_k^u(\mathbb{Q}) \} \geq 0 \right\},$$

where α_k^u is the penalty function:

$$\alpha_k^u(\mathbb{Q}) \quad = \quad \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ -k + \mathbb{E}_{\mathbb{P}} \left[l^* \left(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \right\}$$

if $\mathbb{Q}$ is absolutely continuous to $\mathbb{P}$ and $+\infty$ otherwise. Here, l^* is the conjugate of $-u(-y)$, i.e., $l^*(z) = \sup_{y \in \mathbb{R}} \{ zy + u(-y) \}$.

Notice that α_k^u is nonincreasing in k. Thus, we can interpret an EUT maximizer as a decision maker who prefers acts to the extent to which they have nonnegative expectation in a robust sense. Here, the notion of robustness is tilted somehow according to the "subjective prior" $\mathbb{P}$, embedded within α_k^u. In the case of a CARA utility function, for example, one obtains $l^*(z) = z \log z - z + 1$, which leads to a penalty $\alpha_k^u(\mathbb{Q})$ that depends on the relative entropy from $\mathbb{P}$ to $\mathbb{Q}$.

In this sense, robustness is thus a hidden feature of choice under the diversification favoring part of aspirational preferences. Drapeau and Kupper [18] study more generally similar robust representations.

When there are acts for which concentration is preferred, the dual representation is over concave risk measures and has a somewhat different interpretation. Noting that if $\hat{\mu}$ is a convex risk measure, then $\mu(f) = -\hat{\mu}(-f)$ is a concave risk measure, we can equivalently express choice over acts in F_{--} as

$$\rho(f) \quad = \quad \sup \left\{ k < \hat{k} \ : \ \mu_k(-f) \geq 0 \right\},$$

where μ_k is a nonincreasing family of convex risk measures on $k < \hat{k}$. Using the dual representation of convex risk measures and normalizing as above, we thus obtain the representation

$$\rho(f) \quad = \quad \sup \left\{ k < \hat{k} \ : \ \sup_{\mathbb{Q} \in \mathcal{P}} \{ \mathbb{E}_{\mathbb{Q}}[f - \tau(k)] - \tilde{\alpha}_k(\mathbb{Q}) \} \geq 0 \right\},$$

where $\tau(k)$ is nondecreasing in k and $\tilde{\alpha}_k$ is nonnegative. Here, since $f \in F_{--}$, the decision maker is risk seeking and cannot hope to be robust: they are simply looking for *some* probability measure such that the act beats the target in (penalized) expectation. Probability measures are not chosen by an adversary, but rather an ally who has to pay penalty $\tilde{\alpha}_k(\mathbb{Q})$ for choosing probability $\mathbb{Q}$.

As k grows, the target gets larger and the overall effect of target and penalty grow, so the ally is weaker. In this setting, ρ then represents a smallest level of "assistance" the decision maker needs to provide to the ally, such that at that level, there exists a probability measure $\mathbb{Q}$ such that

$$\mathbb{E}_{\mathbb{Q}}\left[f - \tau(k)\right] - \tilde{\alpha}_k(\mathbb{Q}) \quad \geq \quad 0$$

holds. Thus, for acts $f \in F_{--}$, we can interpret in this way $\rho(f)$ to be a measure of *vulnerability.*

2.4 Stochastic dominance properties

In this section, we briefly characterize some stochastic dominance properties for aspiration measures when the decision maker has an underlying probability measure $\mathbb{P}$ to which stochastic orders can be defined. We show that aspiration measures share the stochastic dominance properties of their underlying risk family. Moreover, we show that under the mild assumption that the aspiration measure is indifferent to all acts with the same distribution under $\mathbb{P}$, then the aspiration measure preserves first-order stochastic dominance (FSD) for all acts, second-order stochastic dominance (SSD) for all acts in the diversification favoring set, and risk-seeking stochastic dominance (RSSD) for all acts in the concentration favoring set.

We first recall the definition of the stochastic orders just mentioned. Note that in this section, if not specified explicitly, expectations are taken with respect to the probability measure $\mathbb{P}$. We say that f dominates g by FSD if and only if $\mathbb{E}\left[u(g)\right] \geq \mathbb{E}\left[u(g)\right]$ for all nondecreasing functions u; in this case we write $f \geq_{(1)} g$. Similarly, f dominates g by SSD (respectively RSSD) if and only if $\mathbb{E}\left[u(f)\right] \geq \mathbb{E}\left[u(g)\right]$ for all u nondecreasing and concave (respectively convex); in this case we write $f \geq_{(2)} g$ (respectively $f \geq_{(-2)} g$). Equivalent definitions of first order, second order and risk-seeking stochastic dominance can be found in Levy [32].

We first note the following.

Proposition 3. *Let μ be a risk measure and suppose that μ preserves FSD, i.e., if $f \geq_{(1)} g$ then $\mu(f) \leq \mu(g)$. Then the risk measure $\bar{\mu}(f) = -\mu(-f)$ also preserves FSD. Moreover, if μ preserves SSD, then $\bar{\mu}$ preserves RSSD.*

Since we can always express a concave risk measure $\bar{\mu}(f)$ equivalently as $-\mu(-f)$, this proposition shows that the property of FSD carries over from convex risk measures to concave risk measures, and SSD of a convex risk measure implies RSSD of its concave counterpart. Given that convex risk measures are well-studied, Proposition 3 will allow us to use known results on stochastic dominance of convex risk measures to establish analogous such results for aspiration measures.

For this step, we first need to show that stochastic dominance properties for aspiration measures are implied by those of the associated family of risk measures. We now show this.

Proposition 4. *Let $\{\mu_k : k \in \mathbb{R}\}$ be a nondecreasing family of risk measures and let ρ be the associated aspiration measure. Suppose that μ_k preserves FSD for all k. Then ρ preserves FSD, i.e.,*

$$f \geq_{(1)} g \quad \Rightarrow \quad \rho(f) \geq \rho(g).$$

Moreover, if μ_k preserves SSD for $k > \hat{k}$ and RSSD for $k < \hat{k}$, then

$$\forall f \in F, g \in F_{++} \text{ such that } f \geq_{(2)} g \quad \Rightarrow \quad \rho(f) \geq \rho(g)$$
$$\forall f \in F, g \in F_{--} \text{ such that } f \geq_{(-2)} g \quad \Rightarrow \quad \rho(f) \geq \rho(g).$$

In general, convex risk measures do not preserve FSD or SSD, as shown by De Giorgi [16] for the case of coherent risk measures. Therefore, Proposition 4 is of little help if we do not specify conditions on the family of risk measures μ_k such that stochastic dominance is preserved. It is well known that stochastic dominance orders are fully characterized by an act's cumulative distribution function under the specified probability measure $\mathbb{P}$ (see Levy [32]). When a risk measure μ does not only depend on the distribution function of the act, we can find two acts f and g that only differ on zero-probability events, but possess different values for the risk measure, e.g., $\mu(f) > \mu(g)$. In this case, we can define a third act $h = f + \epsilon$, $0 < \epsilon < \mu(f) - \mu(g)$, which obviously dominates f by FSD (and thus also dominates g by FSD), but $\mu(f) > \mu(f) - \epsilon = \mu(h)$ and $\mu(h) = \mu(f) - \epsilon > \mu(g)$. This shows that a necessary property on risk measures in order to have preservation of stochastic dominance orders is that they only depend on the probability distribution of the act. We thus introduce the following.

Definition 4. *Let $\mathbb{P}$ be a probability measure on (S, Σ). A function $r : F \to \mathbb{R}$ is called* law-invariant *(with respect to $\mathbb{P}$) if and only if $r(f) = r(g)$ whenever f and g have the same cumulative distribution function under $\mathbb{P}$, i.e., $\mathbb{P}(\{s \in S : f(s) \leq x\}) = \mathbb{P}(\{s \in S : g(s) \leq x\})$ for all $x \in \mathbb{R}$.*

Law-invariance[8] means the underlying mapping between the event space and the consequence space is irrelevant; all that matters is the distribution of the acts under $\mathbb{P}$. It also means that zero-probability events do not matter, i.e., it might be that two acts differ on events $A \in \Sigma$, but as long as $\mathbb{P}(A) = 0$, this does not have any impact on the function r. This seems like an eminently reasonable property, common to many models of decision making under uncertainty.

In our context, law-invariance is useful because it has strong implications for stochastic dominance.

Proposition 5. *Let $(S, \Sigma, \mathbb{P})$ be an atomless probability space[9]. If ρ is a law-invariant aspiration measure, then ρ preserves FSD on F, SSD on F_{++} and RSSD on F_{--}.*

3 Strongly aspirational preferences

The representation of Theorem 1 provides an interpretation of aspirational preferences in terms of risk of beating a target function. Motivated by the strong empirical support on the importance of aspiration levels in risky choice, we now consider a special case of these preferences when the decision maker is especially fixated on the target. In particular, we consider a bounded target function. The bounds can be interpreted as decision maker aspiration levels: a minimal (reservation) aspiration level and a maximal (satiation) level.

[8] Law-invariance is also referred to as "probabilistic sophistication"; see, for instance, Machina and Schmeidler [35].

[9] Note that the assumption of an atomless probability space is quite weak; even random variables with discrete outcomes may be generated (as piecewise constant functions) by atomless probability spaces.

To place this in the context of the general framework, note that an aspiration measure $\rho : F \to \mathbb{R} \cup \{-\infty, \infty\}$ naturally defines two target acts.[10] Let $\tau_u = \inf\{a \in \mathbb{R} : \rho(a) = \infty\}$ and $\tau_l = \inf\{a \in \mathbb{R} : \rho(a) > -\infty\}$, with $\inf \emptyset = \infty$, and consider the case when both are finite. We denote by τ_u and τ_l the constant acts corresponding to individual consequences with these values. Since ρ is nondecreasing, $\tau_l \leq \tau_u$. Moreover, for all $f \in F$ with $f \geq \tau_u$, we have $\rho(f) = \infty$, and for all $f \in F$ with $f < \tau_l$, we have $\rho(f) = -\infty$. Consequently, all acts in $\{f \in F : f \geq \tau_u\}$ are "fully satisfactory," while all acts in $\{f \in F : f < \tau_l\}$ are "fully unsatisfactory." When $\tau_l = \tau_u = \tau$ for some finite τ, the target function corresponds to a single aspiration level.

This leads to the following definition.

Definition 5. *An aspiration measure $\rho : F \to \mathbb{R} \cup \{-\infty, \infty\}$ is called a* strong aspiration measure *(SAM) if $\tau_u = \inf\{a \in \mathbb{R} : \rho(a) = \infty\}$ and $\tau_l = \inf\{a \in \mathbb{R} : \rho(a) > -\infty\}$ are finite. In this case, we say the decision maker has* strongly aspirational preferences*.*

Analogously, we say when τ_l and τ_u are not both finite, as in some of the examples discussed earlier, that the decision maker has *weakly aspirational preferences* and the choice function is a *weak aspiration measure (WAM)*. We will primarily focus on SAM for the remainder of the paper.

First, note that the acts τ_l and τ_u are fully characterized by the underlying family of risk measures.

Lemma 1. *Let $\rho : F \to \mathbb{R} \cup \{-\infty, \infty\}$ be an aspiration measure and $\{\mu_k\}$ be the corresponding family of risk measures as defined in Equation (4). Then $\tau_u = \sup_{k \in \mathbb{R}} \mu_k(0)$ and $\tau_l = \inf_{k \in \mathbb{R}} \mu_k(0)$.*

Lemma 1 also implies that for the target function $\tau : k \to \tau(k) = \mu_k(0)$ we defined in the previous section, we have $\tau_l \leq \tau(k) \leq \tau_u$.

Strongly aspirational preferences in this way capture the focus of target-driven decision-making: namely, achieving aspiration levels is a central goal, and acts that always attain the satiation level (never attain the reservation level) do so should be most (least) highly valued.

Since the aim of this section is also to characterize F_{++} and F_{--} in case of SAM, we assume that $F_{++} \neq \emptyset$ and $F_{--} \neq \emptyset$. Then we can, without loss of generality, assume $\hat{k} = 0$ in Theorem 1 and make the following identifications:[11]

$$\begin{aligned} F_0 &= \{f \in F \;:\; \rho(f) = 0\} \\ F_{++} &= \{f \in F \;:\; \rho(f) > 0\} \\ F_{--} &= \{f \in F \;:\; \rho(f) < 0\}. \end{aligned} \tag{7}$$

Therefore, in what follows, the sign of $\rho(f)$ denotes whether f is in the diversification favoring set or concentration favoring set.

[10] In what follows, we assume without loss of generality that $\sup_{f \in F} \rho(f) = \infty$ and $\inf_{f \in F} \rho(f) = -\infty$. Indeed, if $\sup_{f \in F} \rho(f) = \rho_u$ and $\inf_{f \in F} \rho(f) = \rho_l$, where $\rho_u, \rho_l \in \mathbb{R}$, then one can easily define a strictly increasing and continuous transformation T such that $T \circ \rho$ satisfy the above conditions. Since T is strictly increasing and continuous, $T \circ \rho$ describes the same preference relation as ρ and also maintains all its properties as given in Theorem 1.

[11] Since $F_{++} \neq \emptyset$ and $F_{--} \neq \emptyset$, then $\hat{k}$ in Theorem 1 is finite and we can shift ρ by the constant $\hat{k}$.

3.1 Symmetric SAM

Though convex risk measures are well-studied, concave risk measures are not. We can, however, easily construct families of concave risk measures from convex families. We now discuss this.

Proposition 6. *Consider a family of convex risk measures* $\{\mu_k : k \in (0, \infty)\}$ *that is nondecreasing on* $k \in (0, \infty)$*, with* $\inf_{k>0} \mu_k(0) \geq 0$ *and* $\sup_{k>0} \mu_k(0) < \infty$*. Let* $\bar{\mu}_k(f) = -\mu_{-k}(-f) + \inf_{k>0} \mu_k(0)$ *for all* $f \in F$ *and* $k \in (-\infty, 0)$*. Then, the family of risk measures* $\{\bar{\mu}_k : k \in (-\infty, 0)\}$ *is concave and nondecreasing on* $k \in (-\infty, 0)$*. Moreover,*

$$\bar{\mu}_s(f) \leq \mu_t(f) \qquad \forall s < 0, t > 0.$$

Proposition 6 implies that a family of convex risk measures $\{\mu_k\}$ on $k > 0$ with $\inf_{k>0} \mu_k(0) \geq 0$ and $\sup_{k>0} \mu_k(0) < \infty$ can fully generate a SAM by reflecting the risk measures to generate the required concave family on $k < 0$. We additionally define $\mu_0(f) = \inf_{k>0} \mu_k(f)$ $(\geq \sup_{k<0} \mu_k(f))$ and call the family of risk measures $\{\mu_k : k \in \mathbb{R}\}$ a *symmetric* family. The corresponding SAM is also said to be *symmetric*. One can easily show that for a symmetric SAM, we have $\tau_l = \inf_{k>0} \mu_k(0) - \tau_u$.

3.2 Examples of strong aspiration measures

We now provide some examples of SAMs. In Examples 4-6, we focus on the case of symmetric SAM with $\tau_u = \tau_l = \tau$, which will be also considered in our applications of SAM in Section 4. In these examples we set $\tau = 0$ which is without loss simply by shifting acts by a constant; f then represents value in excess of the target τ and $\mu_k(0) = 0$ for all k. Finally, in Examples 4-6 we assume that (S, Σ) is endowed with a probability measure $\mathbb{P}$ and expectations are taken with respect to $\mathbb{P}$.

Example 4. Entropic SAM (ESAM) The family

$$\mu_k(f) = \frac{1}{k} \ln \mathbb{E}\left[\exp(-kf)\right] \qquad k \neq 0,$$

is a symmetric family of nondecreasing risk measures that are convex for $k > 0$. The associated symmetric SAM is given by

$$\rho(f) \quad = \quad \sup\left\{k \ : \ \frac{1}{k} \ln \mathbb{E}\left[\exp(-kf)\right] \leq 0\right\},$$

which we call the *entropic strong aspiration measure* (ESAM). If f is normally distributed under $\mathbb{P}$, then we have $\bar{\mu}_k(f) = -\mathbb{E}[f] + k\,\sigma^2(f)/2$, where $\sigma^2(f) = \mathbb{E}\left[f^2\right] - \mathbb{E}[f]^2$ is the variance of f. Therefore, μ_k rewards (i.e., has less "risk") greater variance. In this case, we have $\rho(f) = 2\,\mathbb{E}[f] / \sigma^2(f)$. Here, the diversification favoring set are those acts with positive expected value, and the concentration favoring set are those with negative expected value. For a fixed, positive expected value, one prefers smaller variance (risk aversion). For acts with a fixed, negative expected value, however, one prefers larger variance: the intuition is that larger variance gives one better hopes of attaining the target.

The positive part of the above representation yields the entropic satisficing measure of Brown and Sim [8], which is also the reciprocal of the riskiness index of Aumann and Serrano [4].

Example 5. Conditional value-at-risk (CVaR) SAM

The family

$$\mu_k(f) = \begin{cases} \mathrm{CVaR}_{e^{-k}}(f) & \text{if } k > 0 \\ -\mathrm{CVaR}_{e^{k}}(-f) & \text{if } k < 0 \end{cases}$$

where

$$\mathrm{CVaR}_{\epsilon}(f) = \inf_{\nu\in\mathbb{R}} \left\{ \nu + \frac{1}{\epsilon}\mathbb{E}\left[(-f-\nu)^{+}\right] \right\}$$

is a symmetric family of nondecreasing risk measures that are coherent (convex and positive homogeneous; see Artzner et al. [3]) on $k > 0$. The SAM given by this symmetric family is

$$\rho(f) \quad = \quad \begin{cases} \sup\left\{k > 0 \ : \ \mathrm{CVaR}_{e^{-k}}(f) \leq 0\right\} & \text{if } \mathbb{E}[f] \geq 0, \\ \sup\left\{k < 0 \ : \ \mathrm{CVaR}_{e^{k}}(-f) \geq 0\right\} & \text{otherwise,} \end{cases}$$

which we call the *CVaR SAM*. A variant of this measure (without the risk seeking part and scaled to be on $(0,1]$) is defined in Brown and Sim [8]. When f is normally distributed under $\mathbb{P}$, we have

$$\rho(f) \quad = \quad \begin{cases} \sup\left\{k > 0 \ : \ \frac{\phi(\Phi^{-1}(e^{-k}))}{e^{-k}}\sigma(f) \leq \mathbb{E}[f]\right\} & \text{if } \mathbb{E}[f] \geq 0, \\ \sup\left\{k < 0 \ : \ \frac{\phi(\Phi^{-1}(e^{k}))}{e^{k}}\sigma(f) \leq -\mathbb{E}[f]\right\} & \text{otherwise,} \end{cases}$$

where ϕ and Φ are the standard normal density and cumulative distribution functions, respectively. The SAM in this case is a monotonic transformation of the ratio $\mathbb{E}[f]/\sigma(f)$, called the Sharpe ratio.

Example 6. Homogenized entropic SAM (HESAM)

The family

$$\mu_k(f) = \begin{cases} \inf\limits_{a>0}\left\{a\ln\left(\mathbb{E}\left[\exp(-f/a)\right]\right) + ak\right\} & \text{if } k > 0 \\ \sup\limits_{a<0}\left\{a\ln\left(\mathbb{E}\left[\exp(-f/a)\right]\right) - ak\right\} & \text{if } k < 0 \end{cases}$$

is a symmetric family of nondecreasing, coherent risk measures on $k > 0$. The associated SAM is

$$\rho(f) \quad = \quad \begin{cases} \sup\left\{k > 0 \ : \ \inf\limits_{a>0}\left\{a\ln\mathbb{E}\left[\exp(-f/a)\right] + ak\right\} \leq 0\right\} & \text{if } \mathbb{E}[f] \geq 0, \\ \sup\left\{k < 0 \ : \ \sup\limits_{a<0}\left\{a\ln\mathbb{E}\left[\exp(-f/a)\right] - ak\right\} \leq 0\right\} & \text{otherwise.} \end{cases}$$

If f is normally distributed under $\mathbb{P}$, then

$$\rho(f) \quad = \quad \frac{\mathrm{sign}(\mathbb{E}[f])}{2}\left(\frac{\mathbb{E}[f]}{\sigma(f)}\right)^{2},$$

which is again a monotonic transformation of the Sharpe ratio.

Example 7. General symmetric family

Let $\{\alpha_k\}_{k\in\mathbb{R}}$ be a family of functions on the space $\mathcal{P}$ of probability measures on (S,Σ), such that α_k is convex for $k>0$, $\alpha_k(\mathbb{Q})$ is non-increasing in k, and $\alpha_k(\mathbb{Q})=-\alpha_{-k}(\mathbb{Q})$ for all $\mathbb{Q}\in\mathcal{P}$. Additionally, $\sup_{k>0}\inf_{\mathbb{Q}\in\mathcal{P}}\alpha_k(\mathbb{Q})=0$ and $\tau_l=-\tau_u=\inf_{k>0}\inf_{\mathbb{Q}\in\mathcal{P}}\alpha_k(\mathbb{Q})\in\mathbb{R}$. As discussed, the family of risk measures defined by

$$\mu_k(f) \;=\; \sup_{\mathbb{Q}\in\mathcal{P}}\{-\mathbb{E}_{\mathbb{Q}}[f]-\alpha_k(\mathbb{Q})\}=-\inf_{\mathbb{Q}\in\mathcal{P}}\{\mathbb{E}_{\mathbb{Q}}[f]+\alpha_k(\mathbb{Q})\}, \tag{8}$$

for $k>0$ is a family of convex risk measures. This family in turn generates the concave family

$$\begin{aligned}\bar{\mu}_k(f) &= -\sup_{\mathbb{Q}\in\mathcal{P}}\{\mathbb{E}_{\mathbb{Q}}[f]-\alpha_{-k}(\mathbb{Q})\}\\ &= -\sup_{\mathbb{Q}\in\mathcal{P}}\{\mathbb{E}_{\mathbb{Q}}[f]+\alpha_k(\mathbb{Q})\}\end{aligned}$$

for $k<0$, which can be interpreted as the (negative of the) best case penalized expected value over all probability measures. The corresponding symmetric SAM can be expressed as

$$\rho(f) \;=\; \max\left\{\sup\left\{k>0\;:\;\inf_{\mathbb{Q}\in\mathcal{P}}\{\mathbb{E}_{\mathbb{Q}}[f]+\alpha_k(\mathbb{Q})\}\geq 0\right\},\sup\left\{k<0\;:\;\sup_{\mathbb{Q}\in\mathcal{P}}\{\mathbb{E}_{\mathbb{Q}}[f]+\alpha_k(\mathbb{Q})\}\geq 0\right\}\right\}.$$

3.3 Properties and interpretation of the partition

In this section we again consider the case where (S,Σ) is endowed with a probability measure $\mathbb{P}$. Stochastic dominance properties are with respect to $\mathbb{P}$ and expectations are taken with respect to $\mathbb{P}$, when not stated explicitly.

As choice with SAM is a special case of choice under an aspiration measure, it follows that all results on stochastic dominance discussed in Section 2.4 apply to choice under SAM: namely, SAM will inherit the stochastic dominance properties of the underlying family of risk measures. For symmetric SAM, if $\{\mu_k\}$ on $k>0$ satisfies FSD, then so will the SAM, and, if the family satisfies SSD, then the SAM satisfies SSD on the diversification favoring set and RSSD on the concentration favoring set. Finally, choice under a law-invariant SAM automatically satisfies FSD everywhere, SSD on the diversification favoring set, and RSSD on the concentration favoring set when the probability space is atomless.

On atomless probability spaces, law-invariant risk measures also display important boundedness properties. We say that a convex (concave) risk measure $\tilde{\mu}$ is bounded from below (above) by the expectation when $\tilde{\mu}(f)\geq\mathbb{E}[-f]$ ($\tilde{\mu}(f)\leq\mathbb{E}[-f]$) for all $f\in F$. Föllmer and Schied [22] show that law-invariant convex risk measures are bounded from below by the expectation on atomless probability spaces. This implies that concave risk measures are bounded from above by the expectation. Namely, if $\tilde{\mu}$ is law-invariant and concave then $\overline{\tilde{\mu}}(f)=-\tilde{\mu}(-f)$ is law-invariant and convex, thus $\overline{\tilde{\mu}}(f)\geq\mathbb{E}[-f]$, or equivalently, $\tilde{\mu}(f)\leq\mathbb{E}[-f]$. We now show that these boundedness properties have important implications for the structure of the diversification favoring and concentration favoring sets.

Theorem 2. *Let $\{\mu_k\}$ be a nondecreasing family of risk measures inducing SAM ρ. If, for $k > 0$, $\tilde{\mu}_k = \mu_k - \mu_k(0)$ is bounded from below by the expectation and for $k < 0$, $\tilde{\mu}_k = \mu_k - \mu_k(0)$ is bounded from above by the expectation, then*

$$\begin{aligned} \mathbb{E}[f] < \tau_l &\Rightarrow \rho(f) \leq 0 \\ \mathbb{E}[f] \geq \tau_u &\Rightarrow \rho(f) \geq 0. \end{aligned}$$

If the probability space is not atomless, then it is generally not true that a convex (concave) risk measure is bounded from below (above).[12] In many cases, however, convex (concave) risk measures are bounded from below (above) even if the probability space is not atomless. This is the case for the convex risk measures of Examples 4-6, as shown in the following proposition. Recall that in these examples, we had a single target $\tau_l = \tau_u = \tau$, and took, without loss, $\tau = \mu_k(0) = 0$ for all k.

Proposition 7. *The underlying families of risk measures in ESAM, CVaR SAM and HESAM are bounded by the expectation, i.e., $\mu_k(f) \geq \mathbb{E}[-f]$ for $k > 0$ and $\mu_k(f) \leq \mathbb{E}[-f]$ for $k < 0$.*

We emphasize the generality of Theorem 2, which applies not only to any law-invariant SAM on an atomless probability space, but also to several SAMs on non-atomless spaces as seen in Proposition 7.

The result has a number of noteworthy implications. First, this provides a characterization of the "partition" describing where diversification and concentration are preferred in terms of expected values. Acts that fail to attain τ_l on average cannot be in the diversification favoring set, whereas acts that attain at least τ_u on average cannot be in the concentration favoring set. Conversely, acts that are in the diversification favoring set must satisfy $\mathbb{E}[f] \geq \tau_l$, and acts in the concentration favoring set must satisfy $\mathbb{E}[f] < \tau_u$. Note that this structure relating risk attitudes to expected values is purely a consequence of the SAM model.

Second, Theorem 2 also has an implication for choice under SAM. In particular, consider $f, g \in F$ with $\mathbb{E}[f] > \tau_u \geq \tau_l > \mathbb{E}[g]$. Note that in order to compute these expected values nothing about the structure of the underlying risk measures is required. Theorem 2 implies that any decision maker using SAM (with the underlying risk family satisfying the boundedness properties) will either prefer f to g or be indifferent between the two. Thus, f can be taken as the (weakly) preferred act in all cases. For expected utility maximizers, by contrast, all rankings are possible: the ranking will depend on the specific structure of the decision maker's utility function, which must therefore first be specified. In such settings, therefore, the decision maker using a SAM can simplify their decision problem by disregarding acts with expected payoffs lower than τ_l.

Finally, this insight into the partition also provides some prescriptive grounds for favoring concentration in the case of a target-oriented model, as we now explain. Indeed, consider a similar model of choice that favors diversification everywhere, i.e., represented by the function $\tilde{\rho}$, with

$$\tilde{\rho}(f) = \sup\{k : \mu_k(f) \leq 0\},$$

where $\{\mu_k\}$ is a family of convex risk measures bounded below by the expectation such that $\tau_u = \tau_l = 0$ (the single-target assumption is not needed but made to simplify the discussion). It is not hard to see that

[12] Rockafellar et al. [44] add boundedness as an additional property on convex risk measures for their class of *deviation measures*.

$\mathbb{E}[f] < 0$ implies that $f \preceq g$ for all $g \in F$. Indeed, since $\mu_k(f) \geq -\mathbb{E}[f] > 0$ for all k, and $\tilde{\rho}(f) = -\infty$ by convention, such acts are minimally preferred over F.

In particular, target-oriented choice under a model that favors diversification everywhere cannot distinguish between acts expected to fall short of the target: the preference relation is indifferent to all such acts. Thus, in order to be useful for decision makers with aggressive targets, it is crucial to allow for some concentration-favoring preferences. It does not seem unreasonable for decision makers to "roll the dice" and concentrate resources, rather than diversify them, when their ambitions are high relative to available choices.

4 Strongly aspirational preferences and some paradoxes

In this section, we apply choice under strong aspirational preferences to several paradoxes of decision theory. We first look at some problems that fall into the domain of Allais [1] and Ellsberg [19], then move on to a discussion of gain-loss separability, which is an issue in prospect theory.

In what follows, we exclusively consider the case of symmetric SAM with a single target ($\tau_l = \tau_u = \tau$). Recall that this is a special case of strongly aspirational preferences; application of the model with $\tau_l < \tau_u$ may further enhance the model's descriptive relevance.

4.1 Application to Allais

Consider the following two pairs of gambles:

Gamble A: Wins \$500,000 for sure.

Gamble B: 1% chance of 0, 10% chance of winning \$2,500,000 and 89% chance of winning \$500,000,

along with

Gamble C: 90% chance of 0, 10% chance of winning \$2,500,000.

Gamble D: 89% chance of 0, 11% chance of winning \$500,000.

The most typical pattern of preferences observed among actual decision makers is to choose A over B and C over D. It is not hard to see that this is inconsistent with traditional expected utility theory with any utility function.

In contrast, these choice pairs can in fact be consistent with choice under SAM over a specific range of a fixed target. For instance, let ρ be any law-invariant SAM and let τ be the target. We further assume that the corresponding family of risk measures satisfies the boundedness properties of Theorem 2. Denoting gamble A by f_A, we have, for $\tau \leq \$500{,}000$, $\rho(f_A - \tau) = \infty$, so $f_A - \tau \succeq f_B - \tau$. On the other hand, the expected value of gamble C is \$250,000 and the expected value of gamble D is \$55,000; therefore, using Theorem 2, for $\tau > \$55{,}000$, $\tau < \$250{,}000$, we have $\rho(f_C - \tau) \geq 0 \geq \rho(f_D - \tau)$, so the observed pattern above is (weakly) resolved over $\tau \in (\$55,000, \$250,000)$.

In fact, for several SAM, this pattern of preferences will be observed over an even larger range of targets. This is shown in Table 2.

In all three cases, gamble A is strictly preferred to gamble B for $\tau \leq$ \$500,000, and gamble C is strictly preferred to gamble D for $\tau \in (\underline{\tau}, \$2,000,000)$, for some $0 < \underline{\tau} < \$55,000$. The intuition in the first pair is that gamble A is guaranteed to hit the \$500,000 target; for the second case, as long as the target is not very small, the extra "upside" of \$2,500,000 versus \$500,000 outweighs the small difference in probabilities of zero payoffs. It seems plausible that this type of intuition is being used by the decision makers who make such choices.[13]

4.1.1 Common consequence effect

We now briefly generalize the above pattern over a pair of choices. The effect above, first pointed out by Allais [1], is typically called the *common consequence effect*.

Formally, consider two positive payoffs $x > y > 0$ and two probabilities $q \in (0, 1)$, $p \in (0, 1)$, with $q > p$. As before, we denote the first pair of gambles A and B. Gamble A is a sure payoff of y; B, on the other hand, pays off x with probability p, y with probability $1 - q$, and 0 with probability $q - p$.

The second pair is a pair of all-or-nothing gambles, which we denote C and D. Gamble C pays off x with probability p and 0 otherwise; D pays off y with probability q and 0 otherwise. We will assume gamble C beats gamble D in expectation, i.e., $px > qy$, though we could remove this assumption in what follows.

In observed choices, particularly when x is considerably larger than y and $q-p$ is small, real-world decision makers often prefer the "safer" choice among the first two gambles (i.e., the sure payoff of A over the risky payoff B) and the "riskier" choice among the second two gambles (i.e., C over D). The rationale, presumably, is along the lines that A offers a sure payoff, whereas B can result in a zero payoff; for the second pair, though C has a slightly higher chance of paying off nothing, this extra risk may well be worth bearing if the difference $x - y$ is large.

This is easily seen to be inconsistent with expected utility theory. Let u be any utility function, normalized to $u(0) = 0$. Then strict preference of A over B implies $u(y) > pu(x) + (1 - q)u(y) \Rightarrow qu(y) > pu(x)$; on the other hand, strict preference of C over D implies $pu(x) > qu(y)$, a contradiction. This fundamentally occurs because of the independence axiom, which imposes the requirement that common components of any two gambles be irrelevant to the direction of the preference.

The common consequence effect can be explained by SAM over an explicit range of targets; we show a formal result for entropic SAM.

Proposition 8. *Consider the two pairs of gambles, (A,B) and (C,D), as described above with $px > qy$ and let ρ denote the entropic strong aspiration measure and μ_k denote the entropic risk measure at level k. Then for every (x, y, p, q) as above, there exists a target $\tau(x, y, p, q) = \tau^\star < qy$ such that for all $\tau \in (\tau^\star, y]$,*

[13] It is worth mentioning that having some risk-seeking behavior is unnecessary to address the example of Allais; the resolution would still hold without this, but over a smaller range of targets.

$\rho(f_A - \tau) > \rho(f_B - \tau)$ and $\rho(f_C - \tau) > \rho(f_D - \tau)$. Moreover, we have

$$\tau^\star = -\mu_{\rho^\star}(f_D),$$

where $\rho^\star$ is the unique $\rho > 0$ such that $\mu_\rho(f_C) = \mu_\rho(f_D)$.

Notice that the entropic SAM is linked to an expected utility representation with an exponential function (CARA utility). Despite this connection, however, the implications for choice may be drastically different than those implied by the expected utility function.

4.1.2 Common ratio effect

The *common ratio effect* is a related well-known pattern of many observed preferences, again famously pointed out by Allais [1], that cannot be captured by EUT. This phenomenon is again found in the preferences typically observed over two pairs of gambles. Consider two positive real numbers $x > y > 0$ and two probabilities $q \in (0, 1)$, $p \in (0, 1)$.

We denote the first pair of gambles by A and B. The first, A, involves a sure bet of y. The second, B, pays off x with probability p and 0 with probability $1 - p$.

The second pair of gambles, C and D, involves two risky bets: C pays off x with probability $p\,(1 - q)$ and 0 otherwise; D pays off $y < x$ with probability $1 - q$ and 0 otherwise. Note that we can view both of these as a composition of two biased coin flips; first, a coin with probability $1 - q$ of getting a head (i.e., a positive payoff), then a coin with probability p of getting a head. C pays off x if both coins get heads; D pays off y if the first coin gets a head regardless of the outcome of the second coin. Note that, conditional on the first coin getting a head, C and D offer the exact same gambles as A and B.

In many settings, it is well-known that real-world decision makers prefer A over B while also preferring C over D. For instance, consider $p = .8$, $q = .75$, $x = 16$, and $y = 10$. Many subjects prefer the sure payoff of $y = 10$ over the 80% chance of $x = 16$ in the first case; in the second case, however, even though C offers a lower chance of a positive payoff (20% vs. 25%), the extra upside of $x = 16$ vs. $y = 10$ is well worth the extra 5% chance of zero payoff. Again, it is easy to see that such a pattern of preferences is inconsistent with expected utility theory.

We now show that this effect can in general be captured by SAM over an explicit range of targets. As with the common consequence effect, we prove the result for entropic SAM.

Proposition 9. *Consider the two pairs of gambles, (A,B) and (C,D), as described above and let ρ denote the entropic strong aspiration measure and μ_k denote the entropic risk measure at level k. Then for every (x, y, p, q) as above, there exists a target $\tau(x, y, p, q) = \tau^\star < y$ such that for all $\tau \in (\tau^\star, y)$, $\rho(f_A - \tau) > \rho(f_B - \tau)$ and $\rho(f_C - \tau) > \rho(f_D - \tau)$. Moreover, we have*

$$\begin{aligned} \tau^\star &= -\mu_{\rho^\star}(f_D) \\ \rho^\star &= \rho(f_B - y). \end{aligned}$$

Notice that when $p\,x = y$, i.e., both pairs of gambles are equal in expectation, we obtain $\rho^\star = 0$ and thus $\tau^\star = \mathbb{E}\left[f_D\right] = (1-q)y$. When $p\,x > y$, the decision maker will prefer C over D for some targets strictly less than $(1-q)\,y$. When $p\,x < y$, there is still a range of targets for which C is preferred, but $\tau^\star > (1-q)\,y$; in this case, the risk-seeking part of the aspiration measure is at work, and decision makers must have an appreciably high target such that the extra upside provided by C is worth it.

4.2 Application to Ellsberg

In this section, we will show that strongly aspirational preferences can be consistent with behavioral choices when the probability distributions of uncertain payoffs are unknown. Ellsberg's [19] famous experiments provide interesting insights that decisions made under ambiguity can be inconsistent with expected utility theory. Again, we will show that the SAMs can be extended to handle ambiguity and illustrate that we can resolve Ellsberg's paradoxes across a fairly wide range of targets.

To encompass ambiguity in SAM, we now confine the probability measure to a family of distributions, $\mathcal{Q}$. Intuitively speaking, the greater the size of the family $\mathcal{Q}$, the greater the level of ambiguity. In particular, if the family is a singleton, i.e., $\mathcal{Q} = \{\mathbb{P}\}$, then the underlying probability measure is unambiguously specified.

Ambiguity has already been studied in convex risk measures (see Föllmer and Schied, [22]), which are the building blocks of SAMs. Given a law-invariant family of risk measures, $\mu_{\mathbb{P},k}(f)$, evaluated under the probability measure $\mathbb{P}$, we can extend this to family of risk measures to encompass ambiguity. For $k > 0$, we consider an ambiguity averse risk measure,

$$\mu_k(f) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mu_{\mathbb{Q},k}(f),$$

which retains the convexity of the risk measure. For $k < 0$, the concave counterpart is given by

$$\bar{\mu}_k(f) = \inf_{\mathbb{Q}\in\mathcal{Q}} \bar{\mu}_{\mathbb{Q},k}(f),$$

which corresponds to an ambiguity favoring risk measure. For example, we can extend versions of the CVaR, entropic, and homogenized entropic risk measures in this way to handle ambiguity.

We note the following about the structure of the partition with the corresponding SAM that explicitly incorporates ambiguity.

Theorem 3. *Given a nondecreasing family of risk measures, $\{\mu_{\mathbb{Q},k}\}$ (convex for $k > 0$, concave for $k < 0$) with $\mu_{\mathbb{Q},k}(f) - \mu_{\mathbb{Q},k}(0) \geq \mathbb{E}_{\mathbb{Q}}\left[-f\right]$ if $k > 0$ and $\mu_{\mathbb{Q},k}(f) - \mu_{\mathbb{Q},k}(0) \leq \mathbb{E}_{\mathbb{Q}}\left[-f\right]$ if $k < 0$, and $-\infty < \tau_l \leq \mu_{\mathbb{Q},k}(0) \leq \tau_u < \infty$. Consider the SAM*

$$\rho(f) = \sup\left\{k: \ \mu_k(f) \leq 0\right\}$$

with

$$\mu_k(f) = \begin{cases} \sup\limits_{\mathbb{Q}\in\mathcal{Q}} \mu_{\mathbb{Q},k}(f) & \textit{if } k > 0 \\ \inf\limits_{\mathbb{Q}\in\mathcal{Q}} \mu_{\mathbb{Q},k}(f) & \textit{if } k < 0 \end{cases}$$

for some $\mathcal{Q} \subseteq \mathcal{P}$*. Then the following implications hold:*

$$\exists \mathbb{Q} \in \mathcal{Q} \ : \ \mathbb{E}_{\mathbb{Q}}[f] < \tau_l \quad \Rightarrow \quad \rho(f) \leq 0$$
$$\exists \mathbb{Q} \in \mathcal{Q} \ : \ \mathbb{E}_{\mathbb{Q}}[f] \geq \tau_u \quad \Rightarrow \quad \rho(f) \geq 0.$$

Observe that if there exist $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$ such that $\mathbb{E}_{\mathbb{Q}_1}[f] < \tau_l$ and $\mathbb{E}_{\mathbb{Q}_2}[f] \geq \tau_u$, then f is in the neutral set, i.e., $\rho(f) = 0$.

4.2.1 Ellsberg's two-color experiment

The setup for Ellsberg's two-color experiment is as follows. Box 1 contains 50 red balls and 50 blue balls. Box 2 contains red and blue balls in unknown proportions. In the first test, subjects are given the following two choices:

Gamble A: Win \$100 if ball drawn from Box 1 is red.

Gamble B: Win \$100 if ball drawn from Box 2 is red.

In the second test, subjects have to decide between the two choices:

Gamble C: Win \$100 if ball drawn from Box 1 is blue.

Gamble D: Win \$100 if ball drawn from Box 2 is blue.

In the experimental findings, the majority of subjects are ambiguity averse and strictly prefer gamble A over gamble B and gamble C over gamble D, while a smaller portion are actually ambiguity favoring and strictly prefer gamble B over gamble A and gamble D over gamble C. Ellsberg argues the experimental findings are inconsistent with the subjective expected utility theory. The reasoning is as follows: individuals who strictly prefer gamble A over gamble B may perceive that in Box 2, red balls are fewer in number than blue ones. In doing so, they would prefer gamble D over gamble C, which is inconsistent with the experimental findings.

Under Theorem 2, if the corresponding risk measures satisfy the boundedness properties, a SAM on gambles A and C yields non-negative or non-positive values when the target is below or above \$50, respectively. Specific SAMs, such as those based on CVaR, entropic and homogenized entropic risk measures, are strictly positive or negative when the target is below or above \$50, respectively. In contrast, Theorem 3 implies that for any target between \$0 and \$100, these SAMs on gambles B and D, which have unknown distributions, are neutral and thus have a value of zero. Therefore, the preference induced by these SAMs are consistent with the experimental observations.

Clearly, Ellsberg's paradox can also be resolved by several models of weakly aspirational preferences (convex or concave risk measures or by worst-case or best-case expected utility under ambiguity depending on whether the individuals are ambiguity averse or favoring (see, e.g., Föllmer and Schied [22] or Gilboa and Schmeidler [27])). The difference here, however, is that SAMs suggest that the ambiguity preferences depend heavily on the aspiration levels of the subjects. This allows the model to be consistent with some other plausible choice patterns for variations of the Ellsberg experiment. For instance, if the number of red balls in Box 1 were known to be much smaller, we expect decision makers would largely prefer gambles B and C. This remains consistent with SAM.

4.2.2 Ellsberg's three-color experiment

In the three color experiment, a box contains 30 red balls and 60 black and yellow balls with unknown proportions. In the first test, subjects choose between the following gambles:

Gamble A: Win \$300 if ball drawn from the box is black or yellow.

Gamble B: Win \$300 if ball drawn from the box is red or yellow.

In the second test, they have to decide between the two choices:

Gamble C: Win \$300 if ball drawn from the box is black.

Gamble D: Win \$300 if ball drawn from the box is red.

In gamble A, the probability of winning the \$300 prize is 2/3 and the expected payoff is \$200. In contrast, the probability of winning the same prize in gamble B ranges from 1/3 to 1. In gamble C, the probability of winning the prize ranges from 0 to 2/3. On the other hand, the probability of winning in gamble D is exactly 1/3 and the expected payoff is \$100.

Subjective expected utility theory postulates that individuals who prefer gamble A over gamble B should also prefer gamble C over gamble D. Ellsberg's experiment reveals, however, that individuals who prefer gamble A over gamble B also tended to prefer gamble D over gamble C; likewise, Ellsberg found that individuals who preferred gamble B over gamble A also tended to prefer gamble C over gamble D.

We present in Table 3 the SAM values for all the gambles evaluated using the ambiguity version of the SAMs based on entropic, homogenized entropic and CVaR risk measures. The preferences induced on these gambles by these SAMs are the same. Gamble A is preferred over gamble B if the target is less than \$200 and a reversal of preference occurs when the target exceeds \$200. On the other hand, gamble D is preferred over gamble C if the target falls below \$100 and a reversal of preference occurs when the target exceeds \$100. Thus, choice under these SAMs is consistent with the experimental findings of Ellsberg over the target ranges $\tau \leq \$100$ and $\tau \geq \$200$.

4.2.3 Another Ellsberg-like example

Machina [34] recently pointed out the following Ellsberg-like example, shown in Table 1. In this example, there are are 101 balls in an urn, and it is known that 50 balls are labeled 1 or 2, and 51 balls are labeled 3 or 4. The exact number of balls with each label is, however, unknown. The decision maker is then offered the two separate bets listed in Table 1.

Although Choquet expected utility can successfully address the classical Ellsberg examples from above, Machina [34] shows that with CEU, f_1 is preferred to f_2 if and only if f_3 is preferred to f_4, then argues why

	First pair of bets			
	50 balls		51 balls	
	E1	E2	E3	E4
f_1	\$8k	\$8k	\$4k	\$4k
f_2	\$8k	\$4k	\$8k	\$4k
	Second pair of bets			
	50 balls		51 balls	
	E1	E2	E3	E4
f_3	\$12k	\$8k	\$4k	\$0
f_4	\$12k	\$4k	\$8k	\$0

Table 1: Example from Machina [34].

such a pattern may not be observed with decision makers who have some ambiguity aversion. The reason CEU requires this is due to a comonotonicity principle that says that order-preserving shifts of acts by common amounts on identical events cannot reverse rankings. Here, one can see that (f_3, f_4) are obtained by a common tail shift of (f_1, f_2) (adding \$4k in E1 and subtracting \$4k in E4), which means that a CEU maximizer must satisfy $f_1 \succ f_2$ if and only if $f_3 \succ f_4$. Baillon et al. [5] show that Machina's example is also problematic for maxmin expected utility and variational preferences.

We apply choice under ESAM over the range of targets $\tau \in (0, \$12\text{k}]$ to this example, and the results are shown in Table 4. Across a wide range of τ we see (nonstrict) violations of the comonotonicity principle just mentioned. In fact, the only ranges for which it is not violated are quite small: $\tau \in [\inf_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[f_3], \$4\text{k}] \approx [\$3.96\text{k}, \$4\text{k}]$, $\tau \in [\sup_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[f_3], \$8\text{k}] \approx [\$7.96\text{k}, \$8\text{k}]$, as well as precisely at the single target $\tau = \mathbb{E}[f_1] \approx \$5.98\text{k}$. Though no studies were performed, Machina [34] conjectured many decision makers may prefer f_1 and f_4, and this pattern is (nonstrictly) satisfied by choice with ESAM over the middle target range $\tau \in [\$3.96\text{k}, \$5.98\text{k}]$ and also for $\tau > \$8\text{k}$.

4.3 Gain-loss separability

It is known that both prospect theory (Kahneman and Tversky [30]) and cumulative prospect theory (Tversky and Kahneman [49]) require a strong condition of *gain-loss separability*: namely, if both the gain and the loss portion of one gamble are favored over another one, then the same direction of preference must hold for the full ("mixed") gambles themselves. Wu and Markle [50] have shown systematic violations of gain-loss separability in experimental studies. As choice under SAM has both risk aversion and risk seeking, it is interesting to examine implications for gain-loss separability under SAM.

Specifically, Wu and Markle [50] consider two gambles, High and Low, each with some probability of a positive payoff and some probability of a negative payoff. For gamble High (Low), the payoffs are G or

L with probability p and $1 - p$ (G' or L' with probability p' and $1 - p'$). In all trials, it is assumed that $G > G' > 0 > L > L'$.

In "act" notation, we denote the two gambles by f_{High} and f_{Low}. For act f, the notation f^+ denotes the act $f^+(s) = \max(f(s), 0)$ for all $s \in S$, i.e., the gain part of the act, and the notation f^- denotes the act $f^-(s) = \min(f(s), 0)$ for all $s \in S$, i.e., the loss part of the act.

Wu and Markle [50] show violations of gain-loss separability by finding experimental violations of *double matching*. Double matching is the requirement

$$f^+_{\text{High}} \sim f^+_{\text{Low}} \text{ and } f^-_{\text{High}} \sim f^-_{\text{Low}} \implies f_{\text{High}} \sim f_{\text{Low}},$$

and is a necessary requirement for gain-loss separability (thus violations of double matching are even stronger than violations of gain-loss separability).

It is not hard to see that choice under SAM need not satisfy double matching. As one example, consider ESAM with a target of zero. Since $f^+_{\text{High}} \geq 0$ and $f^+_{\text{Low}} \geq 0$, then $\rho(f^+_{\text{High}}) = \rho(f^+_{\text{Low}}) = \infty$, so $f^+_{\text{High}} \sim f^+_{\text{Low}}$. In addition, for the entropic risk measure, $f \leq 0$ with $f(s) < 0$ for some state with nonzero probability implies $\bar{\mu}_k(f) > 0$ for all $k < 0$. This in turn implies that $\rho(f^-_{\text{High}}) = \rho(f^-_{\text{Low}}) = -\infty$, so $f^-_{\text{High}} \sim f^-_{\text{Low}}$. On the other hand, the gambles High and Low are different, so, in general, it will not be the case that $\rho(f_{\text{High}}) = \rho(f_{\text{Low}})$, and therefore double matching is violated.

Table 5 shows application of ESAM with zero target to a set of experiments from Wu and Markle [50]. ESAM violates double matching in every trial. What is perhaps more interesting is that the ESAM seems to match well the preferences of the subject majority over the mixed gambles: namely, the High gamble is preferred in 29 of the 34 trials, whereas a majority of the subjects preferred the High gamble in 27 of the 34 trials. Moreover, of the 5 cases in which we found $\rho(f_{\text{Low}}) > \rho(f_{\text{High}})$, 3 corresponded to cases in which a strong majority preferred Low over High (trials 1, 2, and 3; for trial 6, the subjects were nearly evenly split, and ESAM slightly favored Low over High). All told, simple application of ESAM with zero target matched the majority of subjects in 28 of the 34 cases.

While choice under ESAM seems to match the subject behavior well here, more experimentation and study is required. The important feature that we want to emphasize is that SAM does not treat decompositions of gambles into losses and gains at all the same way that prospect theory does.

5 Optimization of aspiration measures

In this section we discuss the issue of optimization of the AM choice function. The model is amenable to large-scale optimization, which is important for use in applications with many decision variables. An immediate one that comes to mind is portfolio optimization. We will provide an example of this here. While this example is by no means intended to be a completely realistic model of portfolio choice, it serves the purpose of illustrating the relevant computational issues at hand.

Specifically, given a AM ρ, we consider the problem

$$z^* = \sup\{\rho(f) \ : f \in F\}, \tag{9}$$

where F is the convex hull $\{\sum_{i=1}^n w_i\, f_i : \sum_{i=1}^n w_i = 1, w_i \geq 0,\ \forall i = 1, \ldots, n\}$ of n available assets. Here, the "act" f_i corresponds to the uncertain return, in excess of a desired target return τ, for asset i.

From a computational perspective, finding a feasible solution in a convex set is relatively easy compared to finding a feasible solution in a non-convex one. Observe that for $k > \hat{k}$, the acceptance set

$$\mathcal{A}_{\mu_k} = \{f \in F \ :\ \rho(f) \geq k\},$$

which can be empty, is convex. If the diversification favoring set is nonempty, i.e., $F_{++} \neq \emptyset$, we can efficiently obtain the optimal solution to Problem (9) using the binary search procedure of Brown and Sim [8]. Otherwise, if this set is empty (which only happens if the target τ is sufficiently large), we have $z^* \leq \hat{k}$. In this case, each of the extreme points f_i must either be in F_0 or F_{--}. If one of them, say f_j, is in F_0, then if the diversification favoring set is empty, $\rho(f_j) = \hat{k}$ attains the highest AM value over F.

Otherwise, all n available assets are in the concentration favoring set. Then, by quasi-convexity of ρ in this set, there exists an extreme point that is optimal. Hence,

$$z^* = \max_{i=1\ldots,n} \{\rho(f_i)\},$$

and we can simply enumerate the AM values for the n assets and choose the largest one in this case.

We now demonstrate this concretely on an asset allocation problem in which the underlying marginal distributions of assets' returns are not known exactly, while assets' returns are assumed to be independent. Here, independence is assumed for sake of simplicity, but we can easily extend the results to also incorporate dependence; see for instance Hall et al. [29]. We thus consider n assets with independent returns f_i, $i = 1, \ldots, n$. The exact marginal distribution of f_i is unknown but can be characterized by its support $[\underline{f}_i, \overline{f}_i]$, i.e., the probability that f_i belongs to $[\underline{f}_i, \overline{f}_i]$ is one. Also, the mean of f_i is unknown and lies in $[\underline{\mu}_i, \overline{\mu}_i] \subseteq [\underline{f}_i, \overline{f}_i]$. We then consider the problem

$$\sup \left\{ \rho\left(\sum_{i=1}^n w_i\, f_i - \tau \right) \ :\ \sum_{i=1}^n w_i = 1,\ w_i \geq 0,\ i = 1, \ldots, n \right\},$$

where τ is a given target return. The decision variables are the n weights w_i for each available asset.

We consider the case of ρ as ESAM. Since the returns are independently distributed, the underlying risk measure is given by

$$\begin{aligned}
\mu_k\left(\sum_{i=1}^n w_i\, f_i - \tau \right) &= \frac{1}{k} \ln \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{P}} \left[\exp\left(-k \left(\sum_{i=1}^n w_i\, f_i - \tau \right) \right) \right] \\
&= \sum_{i=1}^n \frac{1}{k} \ln \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{P}} \left[\exp\left(-k w_i\, f_i \right) \right] + \tau
\end{aligned}$$

for $k > 0$ and $k < 0$, and $\mathcal{Q}$ is the set of probability measures such that for each asset i f_i possesses a feasible distribution (with the given support $[\underline{f}_i, \overline{f}_i]$ and mean in the corresponding interval $[\underline{\mu}_i, \overline{\mu}_i]$) and returns are independent.

Proposition 10. *Let f be an act (random variable) and $\mathcal{Q}$ be the set of all probability measures such that f has distribution with support $[\underline{f}, \overline{f}]$ and its mean lies in $[\underline{\mu}, \overline{\mu}] \subseteq [\underline{f}, \overline{f}]$. Then*

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \left[\exp(-af)\right] = \begin{cases} \underline{p} \exp(a\underline{f}) + \underline{q} \exp(a\overline{f}) & \text{if } a \geq 0 \\ \overline{p} \exp(a\underline{f}) + \overline{q} \exp(a\overline{f}) & \text{otherwise,} \end{cases}$$

where $\underline{p} = (\overline{f} - \underline{\mu})/(\overline{f} - \underline{f})$, $\underline{q} = 1 - \underline{p}$, $\overline{p} = (\overline{f} - \overline{\mu})/(\overline{f} - \underline{f})$ and $\overline{q} = 1 - \overline{p}$.

Given a target τ, Proposition 10 enables us to compute the ESAM for this problem. Here, there is ambiguity in the return distribution. Observe that

$$\rho\left(\sum_{i=1}^{n} w_i\, f_i - \tau\right) = \rho\left(\sum_{i=1}^{n} w_i\,(f_i - \tau)\right).$$

Hence, if $\rho(f_i - \tau) < 0$ for all i, then all assets are in the concentration favoring set, and it is optimal to invest in the asset with the highest value of $\rho(f_i - \tau)$. Otherwise, we solve the following optimization problem

$$\sup\left\{k \ : \ \sum_{i=1}^{n} \frac{1}{k} \ln(\underline{p}_i \exp(-w_i\, k\, \underline{f}_i) + \underline{q}_i \, \exp(-w_i\, k\, \overline{f}_i)) + \tau \leq 0, \ \sum_{i=1}^{n} w_i = 1, \ k > 0, \ w_i \geq 0, \ i = 1, \ldots, n\right\},$$

where $\underline{p}_i = (\overline{f}_i - \underline{\mu}_i)/(\overline{f}_i - \underline{f}_i)$, $\underline{q}_i = 1 - \underline{p}_i$. The decision variables are the ESAM level k and weights w_i. By replacing k with its reciprocal, we can essentially obtain the optimal portfolio allocation by solving the following tractable convex optimization problem

$$\inf\left\{a \ : \ \sum_{i=1}^{n} a \ln(\underline{p}_i \exp(-w_i\, \underline{f}_i/a) + \underline{q}_i \, \exp(-w_i\, \overline{f}_i/a)) + \tau \leq 0, \ \sum_{i=1}^{n} w_i = 1, \ a > 0, \ w_i \geq 0, \ i = 1, \ldots, n\right\},$$

which can be solved, in high dimension, efficiently using interior point methods.[14]

We now present a numerical example based on the information presented in Table 6. Note that the asset returns are defined such that asset 1 is a risk-free asset that pays 2% in all states. Assets 2 to 6 are risky assets, with asset 2 being the one with lowest downside and upside and asset 6 being the one the highest downside and upside. We solve the optimal asset allocation for various targets as shown in Table 7. The lowest target corresponds to the risk-free rate. Here, the investor can reach the target for sure by investing in asset 1. As the target increases, the risk-free asset becomes less attractive, as it fails to attain the target with certainty. The investor puts some wealth into the risky assets. If the target becomes very high, i.e., the investor is ambitious, then they only hold asset 6, the asset with the highest upside potential and a positive probability of beating the target.

This example demonstrates the intuitive idea that if the investor possesses a high target return, then they will be willing to take more risk. This pattern is similar to that observed in mutual fund managers during the technology bubble of the 1990's (Dass et al. [14]). Managers with high contractual incentives to rank at the top (i.e., those with a high target) adopted the aggressive and risky strategy to not invest in bubble stocks, as avoiding the herd was the only way to highly outperform the market.

[14] For this example, it is convenient to use a solver that can explicitly handle the "exponential cone"; here we use the software package ROME [28] to solve our example problem.

We are not aware of other models that can accommodate differing attitudes towards both risk and ambiguity in a computationally tractable way. Maximizing the probability of beating a target is a highly difficult optimization problem in general (Nemirovksi and Shaprio [39], for instance, show that even computing the distribution of a sum of uniform random variables is NP-hard[15]). Prospect theory is also quite difficult to use - Chen et al. [12] show that optimization of the expected value of an "S-shaped" value function over box constraints is NP-hard. The α-maxmin model (Ghirardato et al. [25]) allows for ambiguity seeking and aversion but results in a choice function that is neither convex nor concave.

While all these models have important implications both theoretically and descriptively, they are difficult to use in optimization settings, and computing globally optimal solutions in general can only be done with enumeration across grid-based approximations. On a grid with 1% resolution, this approach on this six-asset example would require computation and comparison of 10^{12} values. By contrast, on a standard desktop machine, optimization of ESAM here takes about one second.

6 Discussion

We have considered the problem of risky choice over monetary acts (random variables) and examined the case of a fairly generic preference structure over such acts. In addition to the usual properties of a weak order and a mild continuity property, the preference relation obeys state-wise monotonicity and convex preferences, except perhaps on a set of unfavorable acts for which concentration is preferred. We have shown a dual representation of these *aspirational preferences*. This states that we can express aspirational preferences in terms of a maximum index level at which a measure of risk of beating a target function is acceptable.

This result provides a dual interpretation of a number of models in this context, including expected utility theory and several generalizations, and perhaps opens the door to new models of choice. One that we then considered here is the special case when the target function is bounded. These *strongly aspirational preferences* are partly motivated from the perspective of bounded rationality and, though more research is required, seem to have empirical potential. This corroborates a body of work that suggests that aspiration levels play a key role in individual decision-making.

An application like portfolio choice, where performance is often adjusted relative to a benchmark, may be a natural fit for the SAM model. One important feature of this model is that it is relatively amenable to large-scale optimization. In addition to considering new classes of choice models in this framework and investigating in more depth the empirical implications of strongly aspirational preferences, exploring use of the model in applications like this is of interest.

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Appendix

Proofs

Proof of Proposition 1

First, suppose that $F_{++} \neq G_{++}$. Then either $F_{++} \cap (G_0 \cup G_{--}) \neq \emptyset$ or $G_{++} \cap (F_0 \cup F_{--}) \neq \emptyset$. Without loss of generality we assume that $F_{++} \cap (G_0 \cup G_{--}) \neq \emptyset$. Let $f \in F_{++} \cap (G_0 \cup G_{--})$.

If $G_{++} \cup G_0 = \emptyset$, then $G_{--} = F$. Obviously,

$$G_{++} \cup G_0 = \emptyset \subseteq F_{++}$$

and

$$F_{--} \cup F_0 \subseteq F = G_{--}.$$

If $G_{++} \cup G_0 \neq \emptyset$, then take $g \in G_{++} \cup G_0$. Since $f \in G_0 \cup G_{--}$ then $g \succeq f$. Since $f \in F_{++}$, then $g \in F_{++}$. Therefore,

$$G_{++} \cup G_0 \subseteq F_{++}$$

and

$$F_{--} \cup F_0 = \overline{F_{++}} \subseteq G_{--}.$$

Since $G_0 \subseteq F_{++}$ and $F_{++} \cap F_0 = \emptyset$, then $F_0 \cap G_0 = \emptyset$. It is clear that from $G_0 \subseteq F_{++}$ ($F_0 \subseteq G_{--}$) it follows that for all $f, g \in G_0$ (resp. $f, g \in F_0$)

$$\lambda f + (1-\lambda)\, g \succeq g \sim f \quad (\text{resp.} \quad g \sim f \succeq \lambda f + (1-\lambda)\, g)$$

for all $\lambda \in [0,1]$.

Clearly for $f, g \in F_{++} \cap G_{--}$ with $f \succeq g$,

$$f \succeq \lambda f + (1-\lambda)\, g \succeq g$$

for all $\lambda \in [0,1]$.

We now suppose that $F_{++} = G_{++}$, but $F_0 \neq G_0$. Since $F_0 \cup F_{--} = G_0 \cup G_{--}$, then also $F_{--} \neq G_{--}$. Moreover, either $F_0 \cap G_{--} \neq \emptyset$ or $G_0 \cap F_{--} \neq \emptyset$

Note that if $F_0 \cap G_{--} \neq \emptyset$ *and* $G_0 \cap F_{--} \neq \emptyset$, then for $f \in F_0 \cap G_{--}$ and $g \in G_0 \cap F_{--}$ it follows $f \succ g$ and $g \succ f$. A contradiction.

Assume without loss of generality that $F_0 \cap G_{--} \neq \emptyset$ and $G_0 \cap F_{--} = \emptyset$. Let $f \in F_0 \cap G_{--}$. Let $g \in G_0$, then $g \succ f$ since $f \in G_{--}$. Moreover, since $f \in F_0$, the $g \in F_{++}$. It follows $G_0 \subseteq F_{++}$. A contradiction, since $F_{++} = G_{++}$ and $G_{++} \cap G_0 = \emptyset$. Therefore, either $G_0 = \emptyset$ or $F_0 \cap G_{--} = \emptyset$. In the former case, $F_0 \cup F_{--} = G_{--}$ and we obtain the same result as for the case $F_{++} \neq G_{++}$. In the latter case $F_0 = G_0$, i.e., the two partitions must be identical. □

Proof of Proposition 2

Clearly, all convex functions are also quasi-convex. It suffices to show that a quasi-convex function that satisfies translation invariance is always convex. For all $f, g \in F$ we have:

$$\begin{aligned}
& \mu(\lambda f + (1-\lambda) g) - (\lambda \mu(f) + (1-\lambda)\mu(g)) \\
= \; & \mu(\lambda(f + \mu(f)) + (1-\lambda)(g + \mu(g))) \\
\leq \; & \max\{\mu(f + \mu(f)), \mu(g + \mu(g))\} \\
= \; & \max\{0, 0\} = 0.
\end{aligned}$$

Hence,

$$\mu(\lambda f + (1-\lambda) g) \leq \lambda \mu(f) + (1-\lambda)\mu(g).$$

That a quasi-concave function that satisfies translation invariance is always concave follows by a an analogous argument. □

Proof of Theorem 1

First, we show that an aspirational preference relation has functional representation ρ and that ρ satisfies the properties listed in the theorem.

Let $\succeq$ be a aspiration preference relation with partition F_{++}, F_{--}, F_0. Property 1 implies the existence of an upper semi-continuous function $\rho : F \to \mathbb{R}$ such that $f \succeq g$ if and only if $\rho(f) \geq \rho(g)$ (Theorem 4, Bosi and Mehta [7]). That ρ is nondecreasing follows directly from Property 2 (that is, monotonicity of $\succeq$). Property 3(i) implies quasi-concavity for ρ on F_{++}. Indeed, let $f, g \in F_{++}$ and assume without loss of generality that $f \succeq g$, then $\lambda f + (1-\lambda)g \succeq g$, so $\rho(\lambda g + (1-\lambda)g) \geq \rho(g) = \min(\rho(f), \rho(g))$. An analogous argument follows for quasi-convexity on F_{--}. Moreover, strict ordering of the diversification favoring, neutral, and concentration favoring sets implies $\rho(f) > \rho(g) > \rho(h)$ for all $f \in F_{++}$, $g \in F_0$, and $h \in F_{--}$. Finally, let $g \in F_0$ and $\hat{k} = \rho(g)$. Since $\succeq$ is indifferent among acts in F_0, $\hat{k}$ does not depend on the choice of $g \in F_0$.

On the other hand, let $\rho : F \to \mathbb{R} \cap \{-\infty, \infty\}$ be upper semi-continuous and increasing function and F_{++}, F_{--}, F_0 be a partition of F. Assume ρ is quasi-concave on F_{++}, quasi-convex on F_{--}, and $\rho(g) = \hat{k} \in \mathbb{R} \cup \{-\infty, \infty\}$ for all $g \in F_0$, $\rho(f) > \hat{k} > \rho(h)$ for all $f \in F_{++}$, $h \in F_{--}$. Define a preference relation $\succeq$ with $f \succeq g$ if and only if $\rho(f) \geq \rho(g)$. Since ρ is upper semi-continuous, then it follows from Bosi and Mehta [7] that $\succeq$ satisfies (i) and (ii) in Property 1. It is straightforward to show that $\succeq$ also satisfies Properties 2 and 3 with the partition F_{++}, F_{--}, F_0. Thus $\succeq$ is an aspirational preference relation.

Now suppose ρ takes the form (3), where $\{\mu_k\}$ is the family of risk measures as described. Let $F_{++} = \{f \in F : \rho(f) > \hat{k}\}$, $F_{--} = \{f \in F : \rho(f) < \hat{k}\}$ and $F_0 = \{f \in F : \rho(f) = \hat{k}\}$. We show that ρ defines an aspirational preference relation with partition F_{++}, F_{--}, F_0.

1. Upper semi-continuity of ρ:

Upper semi-continuity for ρ is equivalent to $\{f \in F : \rho(f) \geq k\}$ being closed for all k. Let $k \in \mathbb{R}$ and take a sequence $(f_n)_n$ in $\{f \in F : \rho(f) \geq k\}$ such that $f_n \to f$ as $n \to \infty$. Since $\rho(f_n) \geq k$, then $\mu_k(f_n) \leq 0$. Therefore, the sequence $(f_n)_n$ belongs to the acceptance set $\mathcal{A}_{\mu_k}$. Since $\mathcal{A}_{\mu_k}$ is closed, than $f \in \mathcal{A}_{\mu_k}$, i.e., $\mu_k(f) \leq 0$. This implies $\rho(f) \geq k$, i.e., $f \in \{f \in F : \rho(f) \geq k\}$. This proves that $\{f \in F : \rho(f) \geq k\}$ is closed for all k, and thus ρ is upper semi-continuous.

2. ρ *nondecreasing*: Follows clearly from monotonicity of the underlying risk measures.

3. Mixing: First, by definition of F_{++}, F_{--} and F_0 given above, acts in F_{++} are strictly preferred to acts in F_0, which are strictly preferred to acts in F_{--}.

a. Quasi-concavity over diversification favoring acts: Let $f, g \in F_{++}$ and $k^* = \min\{\rho(f), \rho(g)\} > \hat{k}$. Note that $\mu_k(f) \leq 0$ and $\mu_k(f) \leq 0$ for all $k \in (\hat{k}, k^*)$. Then, using convexity of μ_k on $k > \hat{k}$, we have

$$
\begin{aligned}
\rho(\lambda f + (1-\lambda)g) &= \sup\left\{k \in \mathbb{R} \;:\; \mu_k(\lambda f + (1-\lambda)g) \leq 0\right\} \\
&\geq \sup\left\{k \in (\hat{k}, \infty) \;:\; \mu_k(\lambda f + (1-\lambda)g) \leq 0\right\} \\
&\geq \sup\left\{k \in (\hat{k}, \infty) \;:\; \lambda\mu_k(f) + (1-\lambda)\mu_k(g) \leq 0\right\} \\
&\geq k^* \\
&= \min\left\{\rho(X), \rho(Y)\right\}.
\end{aligned}
$$

b. Quasi-convexity over concentration favoring acts: Let $f, g \in F_{--}$ and $k^* = \max\{\rho(f), \rho(g)\} < \hat{k}$. Note that $\mu_k(f) > 0$ and $\mu_k(g) > 0$ for all $k > k^*$. Hence, for all $k \in (k^*, \hat{k})$,

$$\mu_k(\lambda f + (1-\lambda)g) \geq \lambda\mu_k(f) + (1-\lambda)\mu_k(g) > 0.$$

Since μ_k is nondecreasing in k, the above inequality also holds for $k > k^*$. Therefore, we have

$$
\begin{aligned}
\rho(\lambda f + (1-\lambda)g) &= \sup\left\{k \in \mathbb{R} \;:\; \mu_k(\lambda f + (1-\lambda)g) \leq 0\right\} \\
&= \sup\left\{k \in (-\infty, k^*] \;:\; \mu_k(\lambda f + (1-\lambda)g) \leq 0\right\} \\
&\leq k^* \\
&= \max\left\{\rho(f), \rho(g)\right\}.
\end{aligned}
$$

We now show that the functional representation ρ of an aspirational preference relation takes the form (3) where the family of risk measures $\{\mu_k\}$ is defined in Equation (4). Since ρ is nondecreasing, μ_k is nondecreasing in k. To verify that μ_k is a risk measure with a closed acceptance set, we note the following:

1. Closed acceptance set:

We show that $\mu_k(f) \leq 0$ is equivalent to $\rho(f) \geq k$. One direction is trivial, i.e., when $\rho(f) \geq k$ then $\mu_k(f) \leq 0$. For the other direction, we note that upper semi-continuity for ρ implies upper semi-continuity for $a \to \rho(f + a)$, for all $f \in F$. Moreover, since $a \to \rho(a + f)$ is also increasing due to ρ being increasing, then it is also right-continuous and the limit of Problem (4) is achievable. It follows that when $\mu_k(f) \leq 0$, then there exists an $a \leq 0$ such that $\rho(a + f) \geq k$. Due ρ being increasing we also have $\rho(f) \geq k$. We have thus showed:

$$\mathcal{A}_{\mu_k} = \{f \in F : \mu_k(f) \leq 0\} = \{f \in F : \rho(f) \geq k\}.$$

Since ρ is upper semi-continuous, $\{f \in F : \rho(f) \geq k\}$ is closed and thus so is $\mathcal{A}_{\mu_k}$.

2. Monotonicity:

Clear.

3. Translation invariance:

For all constant acts $x \in F$,

$$\begin{aligned} \mu_k(f+x) &= \inf\{a \ : \ \rho(f+x+a) \geq k\} \\ &= \inf\{a-x \ : \ \rho(f+a) \geq k\} \\ &= \mu_k(f) - x. \end{aligned}$$

4. Convexity on $k > \hat{k}$:

Given $f, \ g \in F$, since ρ is increasing and the definition of μ_k, we have for all $\epsilon > 0$,

$$\rho(f + \mu_k(f) + \epsilon) \geq k$$

and

$$\rho(g + \mu_k(g) + \epsilon) \geq k.$$

Since $k > \hat{k}$, we have $f + \mu_k(f) + \epsilon, g + \mu_k(g) + \epsilon \in F_{++}$. For every $\lambda \in [0,1]$, define

$$a_\lambda \triangleq \lambda \mu_k(f) + (1-\lambda)\mu_k(g).$$

Then, for all $\epsilon > 0$,

$$\begin{aligned} \rho(\lambda f + (1-\lambda) g + a_\lambda + \epsilon) &= \rho(\lambda(f + \mu_k(f) + \epsilon) + (1-\lambda)(g + \mu_k(g) + \epsilon)) \\ &\geq \min\{\rho(f + \mu_k(f) + \epsilon), \rho(g + \mu_k(g) + \epsilon)\} \\ &\geq k > \hat{k}. \end{aligned}$$

Then

$$\begin{aligned} \mu_k(\lambda f + (1-\lambda) g) &= \inf\{a \ : \ \rho(\lambda f + (1-\lambda) g + a) \geq k\} \\ &\leq a_\lambda \\ &= \lambda \mu_k(f) + (1-\lambda)\mu_k(g). \end{aligned}$$

5. Concavity on $k < \hat{k}$:

Since $\mu_k(f) = \inf\{a \ : \ \rho(f+a) \geq k\}$, it follows that $\rho(f + \mu_k(f) + a) < k < \hat{k}$ and $\rho(g + \mu_k(g) + a) < k < \hat{k}$ for all $a < 0$. Therefore, for all $a < 0$, $f + \mu_k(f) + a \in F_{--}$, and $g + \mu_k(g) + a \in F_{--}$; hence,

$$\rho(\lambda(f + \mu_k(f)) + (1-\lambda)(g + \mu_k(g)) + a) \leq \max\{\rho(f + \mu_k(f) + a), \rho(g + \mu_k(g) + a)\} < k$$

for all $\lambda \in [0,1]$. Therefore,

$$\begin{aligned}
& \mu_k(\lambda(f+\mu_k(f))+(1-\lambda)(g+\mu_k(g))) \\
= \;& \inf\{a \;:\; \rho(\lambda(f+\mu_k(f)+(1-\lambda)(g+\mu_k(g))+a) \geq k\} \\
= \;& \inf\{a \;:\; \rho(\lambda(f+\mu_k(f))+(1-\lambda)(g+\mu_k(g))+a) \geq k, a \geq 0\} \\
\geq \;& 0.
\end{aligned}$$

Concavity then follows from the translation invariance property of μ_k.

Finally, we need to show that

$$\rho(f) = \sup\{k \in \mathbb{R} \;:\; \mu_k(f) \leq 0\}.$$

We have seen in (i) above that the limit of Problem (4) is achievable. Therefore,

$$\begin{aligned}
\sup\{k \in \mathbb{R} \;:\; \mu_k(f) \leq 0\} &= \sup\{k \in \mathbb{R} \;:\; \exists a \leq 0 \text{ s.t. } \rho(f+a) \geq k\} \\
&= \sup\{\rho(f+a) \;: a \leq 0\} \\
&= \rho(f),
\end{aligned}$$

which completes the proof. □

Proof of Proposition 3

Suppose that $f \geq_{(1)} g$. Then $\mathbb{E}[u(f)] \geq \mathbb{E}[u(g)]$ for all u nondecreasing. Since $u(x)$ is nondecreasing if and only if $-u(-x)$ is also nondecreasing, we have $-\mathbb{E}[u(-f)] \geq -\mathbb{E}[u(-g)]$, or, equivalently, $\mathbb{E}[u(-f)] \leq \mathbb{E}[u(-g)]$ for u nondecreasing. This implies that $-g \geq_{(1)} -f$. Therefore,

$$\bar{\mu}(f) \;=\; -\mu(-f) \;\leq\; -\mu(-g) \;=\; \bar{\mu}(g).$$

For SSD, we observe that a function $u(x)$ is nondecreasing and concave if and only if $-u(-x)$ is nondecreasing and convex. Hence, $f \geq_{(2)} g$ if and only if $-g \geq_{(-2)} -g$. Similarly to above, the result follows. □

Proof of Proposition 4

Note that if $f \geq_{(1)} g$, then $\mu_k(f) \leq \mu_k(g)$ for all $k \in \mathbb{R}$ since μ_k preserves FSD. By the definition of ρ, it follows immediately that $\rho(f) \geq \rho(g)$, i.e., ρ also preserves FSD.

For the next claim, note that $g \in F_{++}$ implies that $\rho(g) > \hat{k}$. Since $f \geq_{(2)} g$ and $\mu_{\rho(g)}$ preserves SSD, we have

$$\mu_{\rho(g)}(f) \;\leq\; \mu_{\rho(g)}(g) \;\leq\; 0.$$

Therefore, $\rho(f) \geq \rho(g)$.

Likewise, $g \in F_{--}$ implies that $\rho(g) < \hat{k}$. Since $f \geq_{(-2)} g$ and $\mu_{\rho(g)}$ preserves RSSD, we have

$$\mu_{\rho(g)}(f) \;\leq\; \mu_{\rho(g)}(g) \;\leq\; 0.$$

Therefore, $\rho(f) \geq \rho(g)$. □

Proof of Proposition 5

First, it is easy to see that law-invariance of the aspiration measure implies law-invariance of the underlying family of risk measures (see Equation 4). Föllmer and Schied [22] show that on atomless probability spaces any law-invariant risk measure preserves FSD, and any convex, law-invariant risk measure preserves SSD; the claim now follows by Propositions 3 and 4. □

Proof of Lemma 1

We have:

$$\begin{aligned}\tau_u &= \inf\{a \in \mathbb{R} : \rho(a) = \infty\} = \inf\{a \in \mathbb{R} : \mu_k(a) \leq 0, \text{ for all } k \in \mathbb{R}\} \\ &= \inf\{a \in \mathbb{R} : \mu_k(0) \leq a, \text{ for all } k \in \mathbb{R}\} \\ &= \sup_{k \in \mathbb{R}} \mu_k(0).\end{aligned}$$

Similarly,

$$\begin{aligned}\tau_l &= \inf\{a \in \mathbb{R} : \rho(a) > -\infty\} = \inf\{a \in \mathbb{R} : \exists k \in \mathbb{R} \text{ with } \mu_k(a) \leq 0\} \\ &= \inf\{a \in \mathbb{R} : \exists k \in \mathbb{R} \text{ with } \mu_k(0) \leq a\} \\ &= \inf_{k \in \mathbb{R}} \mu_k(0).\end{aligned}$$

□

Proof of Proposition 6

It is straightforward to verify the concavity and nondecreasing properties of $\bar{\mu}_k$. Moreover, for all $k > 0$,

$$\begin{aligned}\mu_k(f) - \bar{\mu}_{-k}(f) &= \mu_k(f) - \left(-\mu_k(-f) + \inf_{k>0} \mu_k(0)\right) = \mu_k(f) + \mu_k(-f) - \inf_{k>0} \mu_k(0) \\ &= 2\left(\frac{1}{2}\mu_k(f) + \frac{1}{2}\mu_k(-f)\right) - \inf_{k>0} \mu_k(0) \geq 2\,\mu_k\left(\frac{1}{2} f + \frac{1}{2}(-f)\right) - \inf_{k>0} \mu_k(0) \\ &= 2\,\mu_k(0) - \inf_{k>0} \mu_k(0) \geq 2 \inf_{k>0} \mu_k(0) - \inf_{k>0} \mu_k(0) = \inf_{k>0} \mu_k(0) \geq 0.\end{aligned}$$

Hence, $\bar{\mu}_{-k}(f) \leq \mu_k(f)$ for all $k > 0$. Therefore, for all $s < 0, t > 0$,

$$\bar{\mu}_s(f) \leq \lim_{k \uparrow 0} \bar{\mu}_k(f) \leq \lim_{k \downarrow 0} \mu_k(f) \leq \mu_t(f).$$

□

Proof of Theorem 2

The boundedness properties for the family $\{\tilde{\mu}_k\}$ imply $\mu_k(f) - \mu_k(0) \geq \mathbb{E}[-f]$ for $k > 0$ and $\mu_k(f) - \mu_k(0) \leq \mathbb{E}[-f]$ for $k < 0$. It follows that when $\mathbb{E}[f] < \tau_l$, then for all $k > 0$

$$\mu_k(f) \geq \mathbb{E}[-f] + \mu_k(0) \geq \mathbb{E}[-f] + \tau_l > 0.$$

The second inequality follows from Lemma 1, since $\tau_l = \inf_{k\in\mathbb{R}} \mu_k(0)$. Since $\mu_k(f) > 0$ for all $k > 0$, it follows from Theorem 1 that $\rho(f) \leq 0$. Likewise, if $\mathbb{E}[f] \geq \tau_u$, then for all $k < 0$,

$$\mu_k(f) \leq \mathbb{E}[-f] + \mu_k(0) \leq \mathbb{E}[-f] + \tau_u \leq 0.$$

Again the second inequality follows from Lemma 1, since $\tau_u = \sup_{k\in\mathbb{R}} \mu_k(0)$. Since $\mu_k(f) \leq 0$ for all $k < 0$, from Theorem 1, we have $\rho(f) \geq 0$. □

Proof of Proposition 7

Since these are symmetric families of risk measures, it suffices to show that $\mu_k(f) \geq \mathbb{E}[-f]$ for $k > 0$, which implies that for $k < 0$

$$\mu_k(f) = -\mu_{-k}(-f) \leq \mathbb{E}[-f].$$

Henceforth, we assume $k > 0$. For the entropic risk measure, we have, by Jensen's inequality

$$\mu_k(f) = \frac{1}{k} \ln \mathbb{E}[\exp(-k\, f)] \geq \frac{1}{k} \ln \exp(\mathbb{E}[-kf]) = -\mathbb{E}[f].$$

For the CVaR risk measure, we have

$$\begin{aligned} \mu_k(f) &= \inf_{\nu\in\mathbb{R}} \left\{\nu + e^k\, \mathbb{E}\left[(-f-\nu)^+\right]\right\} \\ &\geq \inf_{\nu\in\mathbb{R}} \left\{\nu + \mathbb{E}\left[(-f-\nu)^+\right]\right\} \\ &\geq \inf_{\nu\in\mathbb{R}} \left\{\nu + \mathbb{E}[-f-\nu]\right\} \\ &= \mathbb{E}[-f]. \end{aligned}$$

Finally, for the homogenized entropic risk measure, we have

$$\begin{aligned} \mu_k(f) &= \inf_{a>0} \left\{a \ln\left(\mathbb{E}[\exp(-f/a)]\right) + ak\right\} \\ &\geq \inf_{a>0} \left\{a \ln\left(\mathbb{E}[\exp(-f/a)]\right)\right\} \\ &\geq \inf_{a>0} \left\{a \ln\left(\exp(-\mathbb{E}[f]/a)\right)\right\} \\ &= \mathbb{E}[-f]. \end{aligned}$$

□

Proof of Proposition 8

First, it is clear that $\rho(f_A - \tau) = \infty$ and $\rho(f_B - \tau) < \infty$ for any $\tau \in (0, y]$. We thus focus on comparing C to D over the range $\tau \in (0, y]$.

There is a one-to-one mapping between target levels and SAM levels. In particular, for a particular SAM level ρ, let $\tau(f_C, \rho)$ and $\tau(f_D, \rho)$ be the corresponding target levels that induce the SAM level ρ for gambles C and D, respectively. We have

$$\begin{aligned} \tau(f_C, \rho) &= -\frac{1}{\rho} \log\left[1 + p(e^{-x\rho} - 1)\right] \\ \tau(f_D, \rho) &= -\frac{1}{\rho} \log\left[1 + q(e^{-y\rho} - 1)\right]. \end{aligned}$$

Note that $\tau(f_C, \rho)$ and $\tau(f_D, \rho)$ are both decreasing and continuous in ρ. We will compare these target functions as ρ varies and will show that there exists a unique $\rho^\star > 0$ such that $\tau(f_C, \rho^\star) = \tau(f_D, \rho^\star)$, and that $\tau(f_D, \rho) > \tau(f_C, \rho)$ for all $\rho > \rho^\star$, and $\tau(f_D, \rho) < \tau(f_C, \rho)$ for all $\rho < \rho^\star$. This shows that $\rho(f_C - \tau) > \rho(f_D - \tau)$ if and only if $\tau > \tau(f_C, \rho^\star) = \tau(f_D, \rho^\star)$.

First, consider $\rho < 0$. Over this range, we have

$$\begin{aligned} \tau(f_C, \rho) > \tau(f_D, \rho) \quad &\Leftrightarrow \quad -\frac{1}{\rho} \log\left[1 + p(e^{-x\rho} - 1)\right] > -\frac{1}{\rho} \log\left[1 + q(e^{-y\rho} - 1)\right] \\ &\Leftrightarrow \quad p(e^{-x\rho} - 1) - q(e^{-y\rho} - 1) > 0. \end{aligned}$$

Let $v(\rho) = p\,(e^{-x\rho} - 1) - q(e^{-y\rho} - 1)$, the left hand side of the latter inequality. Over $\rho < 0$, we have

$$\begin{aligned} v'(\rho) &= -pxe^{-x\rho} + qye^{-y\rho} \\ &< qy(e^{-y\rho} - e^{-x\rho}) \\ &< 0, \end{aligned}$$

where in the first line we use the fact that $px > qy$ and in the second line we use $\rho < 0$ and $y > x$. In addition,

$$\begin{aligned} \lim_{\rho \uparrow 0} v(\rho) &= 0 \\ \lim_{\rho \to -\infty} v(\rho) &= +\infty. \end{aligned}$$

In sum, $v(\rho)$ is a strictly decreasing function from $+\infty$ to 0 as $\rho \uparrow 0$ and therefore must be strictly positive over the range, which implies that $v(\rho) > 0$ over $\rho < 0$, and thus $\tau(f_C, \rho) > \tau(f_D, \rho)$ over this range.

For $\rho = 0$, the target levels reduce to the expected values; thus, $\tau(f_C, 0) = px > qy = \tau(f_D, 0)$.

Finally, consider $\rho > 0$. Similar to the first case, we have over this range

$$\tau(f_C, \rho) > \tau(f_D, \rho) \quad \Leftrightarrow \quad p\,(e^{-x\rho} - 1) - q(e^{-y\rho} - 1) < 0.$$

Let $w(\rho) = p\,(e^{-x\rho} - 1) - q(e^{-y\rho} - 1)$, the left hand side of the latter inequality. We have $\lim_{\rho \downarrow 0} w(\rho) = 0$ and $\lim_{\rho \to \infty} w(\rho) = q - p > 0$. Moreover, $w'(\rho) = -pxe^{-x\rho} + qye^{-y\rho}$, so

$$w'(\rho) \le 0 \quad \Leftrightarrow \quad \rho \le \left(\frac{1}{x - y}\right) \log\left[\frac{px}{qy}\right] = \bar{\rho} > 0.$$

Thus, $w(\rho)$ over $\rho \ge 0$ has a left limit of zero, a right limit of the positive value $q - p$, and is nonincreasing for $\rho \le \bar{\rho}$ and increasing otherwise. This implies that there exists a unique $\rho^\star \ge \bar{\rho} > 0$ when $w(\rho)$ crosses zero. Note furthermore that $w(\rho^\star) = 0$ is equivalent to $\tau(f_C, \rho^\star) = \tau(f_D, \rho^\star)$, i.e., $\mu_{\rho^\star}(f_C) = \mu_{\rho^\star}(f_D)$. Also, since $\rho^\star > 0$, we must have $\tau(f_C, \rho^\star) = \tau(f_D, \rho^\star) < \mathbb{E}\left[f_D\right] = qy$ as claimed.

In summary, we have shown that there is a single target level $\tau^\star$ with the desired construction such that the SAM levels of C and D coincide at $\tau^\star$; below $\tau^\star$, D is preferred to C and vice versa for above $\tau^\star$. This completes the proof. $\square$

Proof of Proposition 9

First, notice that for any $\tau < y$, $\rho(f_A - \tau) = \infty$ and $\rho(f_B - \tau) < \infty$, so $\rho(f_A - \tau) > \rho(f_B - \tau)$. Now consider C and D. For D, any target $\tau < y$ induces a corresponding value $\hat{\rho}(\tau) = \rho(f_D - \tau)$, where using the representation theorem for ρ, we find

$$\frac{1}{\hat{\rho}(\tau)} \log\left[(1-q)e^{-\hat{\rho}(\tau)y} + q\right] \;=\; -\tau \quad \Leftrightarrow \quad \tau \;=\; -\mu_{\hat{\rho}(\tau)}(f_D)$$

must hold. Notice that $\hat{\rho}(\tau)$ is monotonically decreasing on $\tau \in (0, y)$. In order to have $\rho(f_C - \tau) > \rho(f_D - \tau)$, we must have $\mu_{\hat{\rho}(\tau)}(f_C - \tau) < 0$. Considering separately the two cases whether $\hat{\rho}(\tau) > 0$ or $\hat{\rho}(\tau) \leq 0$, we find that in either case, this is equivalent to the condition

$$\frac{1}{\hat{\rho}(\tau)} \log\left[pe^{-\hat{\rho}(\tau)x} + (1-p)\right] \;<\; -y \quad \Leftrightarrow \quad \mu_{\hat{\rho}(\tau)}(f_B - y) \;<\; 0.$$

Therefore, we can choose τ small enough such that $\hat{\rho}(\tau) \leq \rho(f_B - y)$ holds, which leads to the threshold value $\tau^\star$ in the result. Notice that $x > y > 0$ and $p > 0$ imply that $\rho^\star = \rho(f_B - y) > 0$; this in conjunction with $q < 1$ implies that $\tau^\star = -\mu_{\rho^\star}(f_D) < y$.

For $y > \tau > \tau^\star$, we have $\hat{\rho}(\tau) < \rho^\star$, so $\rho(f_C - \tau) > \rho(f_D - \tau)$ and $\rho(f_A - \tau) > \rho(f_B - \tau)$ over the range $(\tau^\star, y)$, as required. □

Proof of Theorem 3

Suppose there exists a $\mathbb{Q}^* \in \mathfrak{Q}$ such that $\mathbb{E}_{\mathbb{Q}^*}[f] < \tau_l$. From Theorem 2, we have, for $k > 0$,

$$\mu_k(f) = \sup_{\mathbb{Q} \in \mathfrak{Q}} \mu_{\mathbb{Q},k}(f) \geq \mu_{\mathbb{Q}^*,k}(f) \geq \mathbb{E}_{\mathbb{Q}^*}[-f] + \mu_{\mathbb{Q}^*,k}(0) > 0,$$

where the strict inequality follows by $\mathbb{E}_{\mathbb{Q}^*}[f] < \tau_l$ and Lemma 1. Hence, $\rho(f) \leq 0$. Likewise, suppose there exists a $\mathbb{Q}^* \in \mathfrak{Q}$ such that $\mathbb{E}_{\mathbb{Q}^*}[f] \geq \tau_u$; we have, for $k < 0$,

$$\mu_k(f) = \inf_{\mathbb{Q} \in \mathfrak{Q}} \mu_{\mathbb{Q},k}(f) \leq \mu_{\mathbb{Q}^*,k}(f) \leq \mathbb{E}_{\mathbb{Q}^*}[-f] + \mu_{\mathbb{Q}^*,k}(0) \leq 0,$$

and again invoking the representation theorem for SAM, it must be that $\rho(f) \geq 0$. □

Proof of Proposition 10

The worst case expectation can be obtained by solving the following optimization problem:

$$\begin{aligned}
\sup_{q} \quad & \mathbb{E}_q[\exp(-af)] \\
\text{s.t.} \quad & \underline{\mu} \leq \mathbb{E}_q[f] \leq \overline{\mu}, \\
& \mathbb{E}_q[1] = 1 \\
& q(y) \geq 0, \quad \forall y \in [\underline{f}, \overline{f}].
\end{aligned}$$

We can consider q to be an infinite dimensional vector indexed by $y \in [\underline{f}, \overline{f}]$. By weak duality, the upper bound to the above problem can be obtain by

$$\begin{aligned}
& \inf_{r,s,t} \left\{ r + \overline{\mu} s - \underline{\mu} t \ : \ r + ys - yt \geq \exp(-ay) \ \forall y \in [\underline{f}, \overline{f}], s, t \geq 0 \right\} \\
= & \inf_{r,s,t} \left\{ r + \overline{\mu} s - \underline{\mu} t \ : \ r \geq \max_{y \in [\underline{f}, \overline{f}]} \{ \exp(ay) - ys + yt \}, s, t \geq 0 \right\} \\
= & \inf_{r,s,t} \left\{ r + \overline{\mu} s - \underline{\mu} t \ : \ r \geq \max\{ \exp(-a\overline{f}) - \overline{f} s + \overline{f} t, \exp(-a\underline{f}) - \underline{f} s + \underline{f} t \}, s, t \geq 0 \right\} \\
= & \inf_{s,t} \left\{ \max\{ \exp(-a\overline{f}) - \overline{f} s + \overline{f} t, \exp(-a\underline{f}) - \underline{f} s + \underline{f} t \} + \overline{\mu} s - \underline{\mu} t \ : s, t \geq 0 \right\}.
\end{aligned}$$

By inspection, when $a \geq 0$, strong duality is obtained by a two point distribution $\mathbb{P}$ with $\mathbb{P}(f = \underline{f}) = \underline{p}$ and $\mathbb{P}(f = \overline{f}) = \underline{q}$ and dual variables $s = 0$, $t = (\exp(-a\underline{f}) - \exp(-a\overline{f}))/(\overline{f} - \underline{f}) \geq 0$. Likewise, when $a < 0$, strong duality is achieved by a two point distribution with $\mathbb{P}(f = \underline{f}) = \overline{p}$ and $\mathbb{P}(f = \overline{f}) = \overline{q}$ and dual variables $s = (\exp(-a\overline{f}) - \exp(-a\underline{x}))/(\overline{f} - \underline{f}) \geq 0$, $t = 0$. $\square$

Tables

ESAM				
τ	Gamble A	Gamble B	Gamble C	Gamble D
55,000	∞	83.7×10^{-6}	$\mathbf{1.90 \times 10^{-6}}$	0
250,000	∞	18.4×10^{-6}	$\mathbf{0}$	-8.36×10^{-6}
500,000	∞	4.80×10^{-6}	$\mathbf{-0.60 \times 10^{-6}}$	$-\infty$
695,000	$-\infty$	$\mathbf{0}$	$\mathbf{-0.92 \times 10^{-6}}$	$-\infty$
2,000,000	$-\infty$	$\mathbf{-4.60 \times 10^{-6}}$	$\mathbf{-4.60 \times 10^{-6}}$	$-\infty$
HESAM				
τ	Gamble A	Gamble B	Gamble C	Gamble D
55,000	∞	3.76490	**0.04798**	0.00000
250,000	∞	1.66770	**0.00000**	-0.46876
500,000	∞	0.08759	**−0.04440**	$-\infty$
695,000	$-\infty$	**0.00000**	**−0.12513**	$-\infty$
2,000,000	$-\infty$	**−1.19251**	**−1.36274**	$-\infty$
CVaR SAM				
τ	Gamble A	Gamble B	Gamble C	Gamble D
55,000	∞	4.48864	**0.08311**	0.00000
250,000	∞	3.91202	**0.00000**	-1.51413
500,000	∞	0.10259	**−0.69315**	$-\infty$
695,000	$-\infty$	**0.00000**	**−1.02245**	$-\infty$
2,000,000	$-\infty$	**−2.01490**	**−2.07944**	$-\infty$

Table 2: Values attributed to gambles A, B, C, and D described in the main text, by various strong aspiration measures for different values of the target τ. In bold are the preferred gambles in each pair for each target.

	ESAM		HESAM		CVaR SAM	
τ	Gamble A	Gamble B	Gamble A	Gamble B	Gamble A	Gamble B
10	**0.10986**	0.04055	**0.92936**	0.28243	**1.06471**	0.37156
30	**0.03662**	0.01320	**0.70421**	0.14970	**0.99325**	0.30010
50	**0.02192**	0.00688	**0.53253**	0.07043	**0.91629**	0.22314
70	**0.01542**	0.00347	**0.39361**	0.02393	**0.83291**	0.13976
90	**0.01153**	0.00104	**0.27980**	0.00254	**0.74194**	0.04879
110	**0.00876**	0.00000	**0.18730**	0.00000	**0.64185**	0.00000
130	**0.00655**	0.00000	**0.11402**	0.00000	**0.53063**	0.00000
150	**0.00462**	0.00000	**0.05889**	0.00000	**0.40547**	0.00000
170	**0.00281**	0.00000	**0.02160**	0.00000	**0.26236**	0.00000
190	**0.00097**	0.00000	**0.00246**	0.00000	**0.09531**	0.00000
210	-0.00104	**0.00000**	-0.00254	**0.00000**	-0.04879	**0.00000**
230	-0.00347	**0.00000**	-0.02393	**0.00000**	-0.13976	**0.00000**
250	-0.00688	**0.00000**	-0.07043	**0.00000**	-0.22314	**0.00000**
270	-0.01320	**0.00000**	-0.14970	**0.00000**	-0.30010	**0.00000**
290	-0.04055	**0.00000**	-0.28243	**0.00000**	-0.37156	**0.00000**

τ	Gamble C	Gamble D	Gamble C	Gamble D	Gamble C	Gamble D
10	0.00000	**0.04055**	0.00000	**0.28243**	0.00000	**0.37156**
30	0.00000	**0.01320**	0.00000	**0.14970**	0.00000	**0.30010**
50	0.00000	**0.00688**	0.00000	**0.07043**	0.00000	**0.22314**
70	0.00000	**0.00347**	0.00000	**0.02393**	0.00000	**0.13976**
90	0.00000	**0.00104**	0.00000	**0.00254**	0.00000	**0.04879**
110	**0.00000**	-0.00097	**0.00000**	-0.00246	**0.00000**	-0.09531
130	**0.00000**	-0.00281	**0.00000**	-0.02160	**0.00000**	-0.26236
150	**0.00000**	-0.00462	**0.00000**	-0.05889	**0.00000**	-0.40547
170	**0.00000**	-0.00655	**0.00000**	-0.11402	**0.00000**	-0.53063
190	**0.00000**	-0.00876	**0.00000**	-0.18730	**0.00000**	-0.64185
210	**−0.00104**	-0.01153	**−0.00254**	-0.27980	**−0.04879**	-0.74194
230	**−0.00347**	-0.01542	**−0.02393**	-0.39361	**−0.13976**	-0.83291
250	**−0.00688**	-0.02192	**−0.07043**	-0.53253	**−0.22314**	-0.91629
270	**−0.01320**	-0.03662	**−0.14970**	-0.70421	**−0.30010**	-0.99325
290	**−0.04055**	-0.10986	**−0.28243**	-0.92936	**−0.37156**	-1.06471

Table 3: Values attributed to gambles A, B, C and D given in the main text by entropic (columns 1-2), homogenous entropic (columns 3-4) and CVaR SAM (columns 5-6), as function of the target τ. In bold are the preferred gambles in each pair for each target.

	τ (\$k)						
	(0,3.96)	[3.96,4]	(4,5.98)	[5.98]	(5.98,7.96)	[7.96,8]	(8,12]
f_1 vs. f_2	=	=	f_1	=	f_2	f_2	=
f_3 vs. f_4	f_3	=	=	=	=	f_4	f_4

Table 4: Preferences for Machina [34] example using ESAM (target ranges computed from various expected values and are rounded to two digits).

	High gamble			Low gamble					ESAM results		
Trial	G	p	L	G'	p'	L'	% High	n	$\rho(f_{\text{High}})$	$\rho(f_{\text{Low}})$	ESAM prefers?
1	150	0.3	-25	75	0.8	-60	22.20%	81	0.0123	0.0246	Low
2	1800	0.05	-200	600	0.3	-250	21.00%	81	-0.0007	0.0001	Low
3	1000	0.25	-500	600	0.5	-700	28.30%	60	-0.0005	-0.0002	Low
4	200	0.3	-25	75	0.8	-100	33.30%	72	0.0134	0.0122	High
5	1200	0.25	-500	600	0.5	-800	43.10%	72	-0.0003	-0.0004	High
6	750	0.4	-1000	500	0.6	-1500	51.40%	72	-0.0008	-0.0007	Low
7	4200	0.5	-3000	3000	0.75	-6000	51.90%	81	0.0001	0.0001	High
8	4500	0.5	-1500	3000	0.75	-3000	48.30%	60	0.0004	0.0004	High
9	4500	0.5	-3000	3000	0.75	-6000	58.30%	60	0.0001	0.0001	High
10	1000	0.3	-200	400	0.7	-500	51.30%	80	0.0014	0.0014	High
12	3000	0.01	-490	2000	0.02	-500	59.30%	81	-0.0013	-0.0017	High
11	4800	0.5	-1500	3000	0.75	-3000	54.20%	72	0.0004	0.0004	High
13	2200	0.4	-600	850	0.75	-1700	51.70%	60	0.0007	0.0003	High
14	2000	0.2	-1000	1700	0.25	-1100	57.60%	59	-0.0004	-0.0005	High
15	1500	0.25	-500	600	0.5	-900	51.30%	80	0	-0.0005	High
16	5000	0.5	-3000	3000	0.75	-6000	65.00%	80	0.0001	0.0001	High
17	1500	0.4	-1000	600	0.8	-3500	58.80%	80	0	-0.0002	High
18	2025	0.5	-875	1800	0.6	-1000	71.70%	60	0.0006	0.0007	Low
19	600	0.25	-100	125	0.75	-500	57.50%	80	0.0022	-0.0009	High
20	5000	0.1	-900	1400	0.3	-1700	40.00%	60	-0.0002	-0.0007	High
21	700	0.25	-100	125	0.75	-600	71.30%	80	0.0024	-0.0014	High
22	700	0.5	-150	350	0.75	-400	63.30%	60	0.0045	0.0025	High
23	1200	0.3	-200	400	0.7	-800	70.00%	80	0.0015	0.0003	High
24	5000	0.5	-2500	2500	0.75	-6000	78.80%	80	0.0002	0.0001	High
25	800	0.4	-1000	500	0.6	-1600	57.50%	80	-0.0007	-0.0008	High
26	5000	0.5	-3000	2500	0.75	-6500	71.30%	80	0.0001	0	High
27	700	0.25	-100	100	0.75	-800	72.50%	80	0.0024	-0.0025	High
28	1500	0.3	-200	400	0.7	-1000	75.00%	80	0.0017	-0.0001	High
29	1600	0.25	-500	600	0.5	-1100	72.50%	80	0.0001	-0.0007	High
30	2000	0.4	-800	600	0.8	-3500	65.00%	80	0.0004	-0.0002	High
31	2000	0.25	-400	600	0.5	-1100	80.00%	80	0.0005	-0.0007	High
32	1500	0.4	-700	300	0.8	-3500	77.50%	80	0.0003	-0.0007	High
33	900	0.4	-1000	500	0.6	-1800	70.00%	80	-0.0005	-0.0008	High
34	1000	0.4	-1000	500	0.6	-2000	77.50%	80	-0.0004	-0.0009	High

Table 5: ESAM applied to gambles High and Low in experiments from Wu and Markle [50] (ESAM values rounded to four decimal points).

Asset	$\underline{v}_i$	$\overline{v}_i$	$\underline{\nu}_i$	$\overline{\nu}_i$
1	2.0	2.0	2.0	2.0
2	-30.0	6.0	4.0	5.0
3	-40.0	8.0	5.0	6.0
4	-50.0	10.0	8.0	9.0
5	-60.0	15.0	11.0	12.0
6	-100.0	20.0	15.0	16.0

Table 6: Supports $[\underline{v}_i, \overline{v}_i]$ of the distributions of assets' percentage returns V_i and the corresponding ranges $[\underline{\nu}_i, \overline{\nu}_i]$ for the expected returns for the portfolio choice example in Section 5.

	Asset						
τ	1	2	3	4	5	6	ρ
2.0	1.000	0.000	0.000	0.000	0.000	0.000	∞
3.0	0.718	0.065	0.049	0.077	0.054	0.036	0.3220
4.0	0.435	0.130	0.099	0.155	0.109	0.073	0.1610
5.0	0.153	0.195	0.148	0.232	0.163	0.109	0.1073
6.0	0.000	0.192	0.164	0.292	0.210	0.142	0.0795
7.0	0.000	0.069	0.138	0.348	0.261	0.183	0.0585
9.0	0.000	0.000	0.000	0.350	0.361	0.289	0.0315
11.0	0.000	0.000	0.000	0.000	0.488	0.512	0.0146
14.0	0.000	0.000	0.000	0.000	0.000	1.000	0.0031
18.0	0.000	0.000	0.000	0.000	0.000	1.000	-0.0136

Table 7: Optimal asset allocation under the ESAM for various values of the target τ for the portfolio choice example in Section 5.