Liesenfeld, Roman; Richard, Jean-François

Working Paper

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Economics working paper / Christian-Albrechts-Universität Kiel, Department of Economics, No. 2004,12

Provided in cooperation with:
Christian-Albrechts-Universität Kiel (CAU)

Suggested citation: Liesenfeld, Roman; Richard, Jean-François (2004) : Classical and Bayesian Analysis of Univariate and Multivariate Stochastic Volatility Models, Economics working paper / Christian-Albrechts-Universität Kiel, Department of Economics, No. 2004,12, http://hdl.handle.net/10419/21987

Classical and Bayesian Analysis of Univariate and Multivariate Stochastic Volatility Models

by Roman Liesenfeld and Jean-François Richard

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Classical and Bayesian Analysis of Univariate and Multivariate Stochastic Volatility Models

Roman Liesenfeld*
Department of Economics, Christian-Albrecht-Universität, Ohlshausenstr. 40-60, 24118 Kiel, Germany

Jean-François Richard
Department of Economics, University of Pittsburgh, 4S01 Wesley W. Posvar Hall, Pittsburgh, PA 15260, USA

(December 18, 2004)

Abstract

In this paper Efficient Importance Sampling (EIS) is used to perform a classical and Bayesian analysis of univariate and multivariate Stochastic Volatility (SV) models for financial return series. EIS provides a highly generic and very accurate procedure for the Monte Carlo (MC) evaluation of high-dimensional interdependent integrals. It can be used to carry out ML-estimation of SV models as well as simulation smoothing where the latent volatilities are sampled at once. Based on this EIS simulation smoother a Bayesian Markov Chain Monte Carlo (MCMC) posterior analysis of the parameters of SV models can be performed.

JEL classification: C15, C22, C52

Keywords: Dynamic Latent Variables; Markov Chain Monte Carlo; Maximum likelihood; Simulation Smoother

*Corresponding author. Tel.: +49-431-8803810; fax: +49-431-8807605. *E-mail address:* liesenfeld@stat-econ.uni-kiel.de (R. Liesenfeld)

1. Introduction

This paper shows how to use Efficient Importance Sampling (EIS) (Richard and Zhang, 2004) to perform a classical and Bayesian analysis of univariate and multivariate dynamic Stochastic Volatility (SV) models.

The standard univariate SV model due to Taylor (1982, 1986) can be represented by

$$r_t = \beta e^{\lambda_t/2}\epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0,1) \tag{1}$$

$$\lambda_t = \delta\lambda_{t-1} + \nu\eta_t, \quad \eta_t \sim \mathrm{N}(0,1), \tag{2}$$

where r_t is the asset return in period $t : 1 \to T$, λ_t is the latent log volatility of r_t, and $\{\epsilon_t, \eta_t\}$ are serially and mutually independent Gaussian random variables. The parameters to be estimated are given by $\omega = (\beta, \delta, \nu)'$. In order to ensure stationarity of r_t, it is assumed that $|\delta| < 1$. This SV model is used as an alternative to the class of AutoRegressive Conditionally Heteroscedastic (ARCH) models in accounting for the time-varying and persistent volatility. A complete description of the properties of the SV model is provided, e.g., by Ghysels et al. (1996).

A natural extension of the standard SV model is the multivariate factor SV specification introduced by Shephard (1996) and Jacquier et al. (1999). According to this model a set of asset returns are driven by latent factors which are specified as SV processes. Such multivariate volatility models are important for portfolio allocation and asset pricing which have to be discussed within a multivariate framework. Furthermore, multivariate volatility models might provide information about the factors driving volatility processes. Consider n assets with returns $r_t = (r_{1,t}, ..., r_{n,t})'$. The simplest version of a multivariate factor SV specification for r_t is the following one-factor model:

$$r_t = Dx_t + e_t \tag{3}$$

$$x_t = \beta \exp(\lambda_t/2)\epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0,1) \tag{4}$$

$$\lambda_t = \delta\lambda_{t-1} + \nu\eta_t, \quad \eta_t \sim \mathrm{N}(0,1), \tag{5}$$

where $D = (d_1, ..., d_n)'$ denotes the factor loadings, x_t a latent factor following a SV process, ϵ_t and η_t serially independent Gaussian random variables, and $e_t = (e_{1,t}, ..., e_{n,t})'$ a vector of serially independent idiosyncratic errors with $e_t \sim \mathrm{N}(0, \Sigma_e)$ and $\Sigma_e = \mathrm{diag}(\sigma^2_{e,1}, ..., \sigma^2_{e,n})$. The errors ϵ_t, η_t and $e_{j,t}$ are assumed to be mutually independent. In order to achieve identification, we impose the restriction $d_1 = 1$. This multivariate specification accounts not only for the volatility dynamics of the individual assets but also, due to the common factor, for time varying correlations between the the assets. Extensions of this basic multivariate framework recently analyzed by Pitt and Shephard

(1999) and Aguilar and West (2000) allow for additional factors and for idiosyncratic errors which also follow SV processes.

Liesenfeld and Richard (2003) used EIS to obtain Maximum Likelihood (ML) parameter estimates and filtered volatility estimates for diagnostic tests of several extensions of the standard SV model, including univariate specifications with semi-nonparametric error distributions, SV models with two independent dynamic volatility processes and the multivariate model (3)-(5). Here, we extend the application of EIS to a Markov Chain Monte Carlo (MCMC) posterior analysis of the parameters of univariate and multivariate SV models. In particular, within a Gibbs sampling algorithm for the conditional posterior distribution of the parameters and latent volatilities, EIS is used to sample the volatilities. For this purpose the EIS procedure, which is designed to produce an accurate approximation to the unknown highly multivariate conditional density of the volatilities given the observed returns, is combined with the Acceptance-Rejection Metropolis Hastings (AR-MH) algorithm of Tierney (1994).

An attractive feature of this EIS approach is that it allows for sampling the vector of latent variables as one block. This eliminates the slow convergence due to high correlation between the latent variables which typically arises in procedures where each element of the latent process is sampled individually. Furthermore, the EIS procedure is highly generic. Hence, changes in the model being analyzed can easily be accommodated. Moreover, the proposed approach can easily be adapted to other unobserved component time series models with arbitrary conditional distributions of the observed variable given the unobserved components. Examples are the dynamic parameter-driven models involving counts (see, Chan and Ledolter, 1995 and Jung and Liesenfeld, 2001), the stochastic autoregressive duration models (see, Bauwens and Hautsch, 2003) or the dynamic discrete choice panel models analyzed by Liesenfeld and Richard (2004).

Alternative block-sampling procedures for sampling a latent process in a MCMC estimation framework are the multi-move sampler of Shephard and Pitt (1997) and the 'mixture sampler' proposed by Kim et al. (1998). Using a Taylor expansion, the multi-move sampler is based on local approximations to the conditional distribution of the latent process. In contrast, EIS relies upon corresponding global approximations. The mixture sampler is based on mixture of normal approximations to the conditional distribution for a linearized version of the SV model. However, this approach does not seem able to easily deal with generalizations of the SV model (see, e.g., the discussion to Pitt and Shephard, 1999).

The outline of this paper is as follows. Section 2 provides a description of the EIS procedure and

its application to obtain ML estimations for univariate and multivariate SV models. In Section 3 we discuss how to combine EIS with MCMC to perform a Bayesian analysis of those models. In section 4 we summarize our results and conclude.

2. Classical Estimation Based on EIS

2.1. EIS

The likelihood for the basic univariate SV model (1) and (2) and its multivariate extension (3)-(5) is given by a T-fold integral of the form $L(\omega;R) = \int f(R,\Lambda;\omega)d\Lambda$, where $R = \{r_t\}_{t=1}^T$ denotes the vector of observable returns and $\Lambda = \{\lambda_t\}_{t=1}^T$ the vector of latent volatilities. The integrand represents the joint density of R and Λ, which can be factorized into the sequence of conditional densities f_t for (r_t,λ_t) given $R_{t-1} = \{r_\tau\}_{\tau=1}^{t-1}$ and $\Lambda_{t-1} = \{\lambda_\tau\}_{\tau=1}^{t-1}$. Based on this factorization the likelihood can be written as

$$L(\omega;R) = \int \prod_{t=1}^{T} f_t(r_t,\lambda_t|R_{t-1},\Lambda_{t-1};\omega)d\Lambda \tag{6}$$

and

$$f_t(r_t,\lambda_t|R_{t-1},\Lambda_{t-1};\omega) = g_t(r_t|R_{t-1},\Lambda_t;\omega)p_t(\lambda_t|R_{t-1},\Lambda_{t-1};\omega), \tag{7}$$

where g_t represents the conditional density of r_t given (R_{t-1},Λ_t) and p_t the conditional density of λ_t given (R_{t-1},Λ_{t-1}). Under the basic univariate SV model these densities are proportional to:

$$g_t(r_t|\lambda_t;\omega) \propto \exp\{-\frac{1}{2}[(r_t/\beta)^2\exp(-\lambda_t)+\lambda_t]\} \tag{8}$$

$$p_t(\lambda_t|\lambda_{t-1};\omega) \propto \exp\{-\frac{1}{2\nu^2}(\lambda_t-\delta\lambda_{t-1})\}, \tag{9}$$

where, for convenience, the initial value λ_0 is assumed to be a known constant. Under the multivariate SV model g_t is replaced by

$$g_t(r_t|\lambda_t;\omega) \propto \det(DD'\beta^2\exp(\lambda_t)+\Sigma_e)^{-1/2}\exp\{-\frac{1}{2}r_t'[DD'\exp(\lambda_t)+\Sigma_e]^{-1}r_t\}. \tag{10}$$

It is well-known that direct MC estimation of $L(\omega;R)$ based on the natural sampler for the λ_t process directly obtained from the statistical formulation of the model and given by the sequence of p_t densities is highly inefficient. This follows from the fact, that the simulated λ_t trajectories from the 'natural' (or 'initial') sampling densities p_t typically do not bear any resemblance to the actual unobserved λ_t sequence under which the observed process r_t is obtained. In other words,

the implicit 'posterior' density of Λ is much tighter than its 'prior' (the natural sampler). Hence, potential efficiency gains are enormous.

To improve the efficiency of the MC estimate of $L(\omega; R)$, EIS replaces the natural sampler by importance sampling densities which provide close approximations to the implicit posterior of Λ. In particular, let $\{m_t(\lambda_t|\Lambda_{t-1}, a_t)\}_{t=1}^T$ be a sequence of auxiliary importance samplers indexed by the auxiliary parameters $a = \{a_t\}_{t=1}^T$. Then for any choice of the auxiliary parameters, the integral in (6) can be rewritten as

$$L(\omega; R) = \int \prod_{t=1}^{T} \frac{f_t(r_t, \lambda_t|R_{t-1}, \Lambda_{t-1}; \omega)}{m_t(\lambda_t|\Lambda_{t-1}; a_t)} \prod_{t=1}^{T} m_t(\lambda_t|\Lambda_{t-1}; a_t) d\Lambda \tag{11}$$

and the corresponding importance sampling MC estimate is given by

$$\tilde{L}_N(\omega; R, a) = \frac{1}{N} \sum_{i=1}^{N} \left[\prod_{t=1}^{T} \frac{f_t(r_t, \tilde{\lambda}_t^{(i)}|R_{t-1}, \tilde{\Lambda}_{t-1}^{(i)}; \omega)}{m_t(\tilde{\lambda}_t^{(i)}|\tilde{\Lambda}_{t-1}^{(i)}; a_t)} \right], \tag{12}$$

where $\{(\tilde{\lambda}_1^{(i)}, ..., \tilde{\lambda}_T^{(i)}), i : 1 \to N\}$ are N independent trajectories drawn from the sequence of importance sampling densities m_t.

As a natural choice, the importance sampler $\{m_t\}$ is specified as a parametric extension of the natural sampler $\{p_t\}$. EIS aims at selecting a_t's that minimize the MC sampling variance of the MC likelihood estimate (12). This requires that $\prod_t m_t$ be as close as possible to being proportional to $\prod_t f_t$. Feasibility necessitates that this high-dimensional optimization problem be decomposed into a sequence of low-dimensional subproblems. However, due to the recursive structure of the λ_t process it is not possible to secure a good approximation of f_t by m_t period by period independently from one another which would essentially amount to factorizing the high dimensional integral in (6) into a product of independent univariate integrals. Accordingly, EIS approximations involve density kernels instead of densities. In particular, let $k_t(\lambda_t, \lambda_{t-1}, a_t)$ denote the density kernel of m_t satisfying

$$m_t(\lambda_t|\lambda_{t-1}, a_t) = \frac{k_t(\lambda_t, \lambda_{t-1}, a_t)}{\chi_t(\lambda_{t-1}, a_t)}, \quad \text{where} \quad \chi_t(\lambda_{t-1}, a_t) = \int k_t(\lambda_t, \lambda_{t-1}, a_t) d\lambda_t. \tag{13}$$

Note that the integrating constant χ_t does not depend on λ_t. Hence, an approximation of f_t by k_t would leave χ_t unaccounted for. But χ_t can be transferred back into the subproblem of period $t-1$ involving λ_{t-1}. Therefore, the sequential implementation of EIS requires solving a back-recursive sequence of low-dimensional least-squares problems of the form

$$(\hat{c}_t, \hat{a}_t) = \arg\min_{c_t, a_t} \sum_{i=1}^{N} \left\{ \ln \left[f_t\big(r_t, \breve{\lambda}_t^{(i)}|\breve{\Lambda}_{t-1}^{(i)}, R_{t-1}, \omega\big) \cdot \chi_{t+1}\big(\breve{\lambda}_t^{(i)}, \hat{a}_{t+1}\big) \right] - c_t - \ln k_t\big(\breve{\lambda}_t^{(i)}, \breve{\lambda}_{t-1}^{(i)}, a_t\big) \right\}^2 \tag{14}$$

for $t : T \to 1$, with $\chi_{T+1}(\lambda_T, \cdot) \equiv 1$. The N independent trajectories $\{(\breve{\lambda}_1^{(i)}, ..., \breve{\lambda}_T^{(i)}), i : 1 \to N\}$ are drawn from the sequence of p_t densities, and the c_t's are constants to be estimated jointly with the a_t's. In order to obtain maximally efficient EIS samplers, a small number of iterations of the EIS algorithm are required, where the initial sampling densities p_t are replaced by the previous stage importance samplers until a fixed point solution $\{\hat{a}_t\}$ is obtained. The final MC EIS estimate of the likelihood is obtained by substituting $\hat{a} = \{\hat{a}_t\}_{t=1}^T$ for a in (12) and ML-EIS estimates of ω are obtained by maximizing $\tilde{L}_N(\omega; R, \hat{a})$ with respect to ω using a standard numerical optimizer. The convergence of such an optimizer requires that $\tilde{L}_N$ be continuous and/or differentiable in ω. This is achieved by computing $\tilde{L}_N$ for different values of ω under a set of Common Random Numbers (CRNs). This means that all $\{\breve{\lambda}_t^{(i)}\}$ and $\{\tilde{\lambda}_t^{(i)}\}$ draws for different values of ω are obtained by transformation of a common set of canonical random numbers, here standardized normals. (For a description of the implementation of EIS for SV models, see the Appendix.) Once the parameters have been estimated, EIS also allows to compute filtered estimates of functions of λ_t. Diagnostics on the model specification are then obtained as byproducts (for details, see Liesenfeld and Richard, 2003).

As mentioned above, once EIS is implemented for a baseline model, modifications in that model only require minor adjustments of the program, essentially adjusting the regressand and regressors in the auxiliary regressions (14). For example, transforming the EIS algorithm for the baseline SV model (1) and (2) into an EIS algorithm for the multivariate factor model (3)-(5) only requires replacing the conditional density (8) by the conditional density (10).

We conclude this description of EIS with a comment on its reliability. It has long been recognized that if the importance sampling density $\prod_t m_t$ has thinner tails than the integrand $\prod_t f_t$ the variance of $\prod_t f_t/m_t$ might not exist in which case the consistency of the likelihood estimate $\tilde{L}_N(\omega; R, a)$ is not longer guaranteed (see, e.g., Geweke, 1989 and Koopman and Shephard, 2002). In order to check the existence of the variance of importance samplers for the estimation of SV models, Lee and Koopmann (2004) used diagnostics based upon extreme-value distributions. Their results indicate that the EIS approach produces reliable importance sampler for univariate SV models. Another highly sensitive test to assess the existence of the variance for the EIS sampler can be found in Richard and Zhang (2004).

2.2. Application

To illustrate the EIS procedure we analyze the daily observations on four exchange rates: Deutsche Mark (DM), British Pound (BP), Swiss Franc (SF) and Japanese Yen (YEN), all against the US-Dollar. These are weekday closing prices from October 1, 1981 to June 28, 1985. The prices $s_{j,t}, (j = \text{DM}, \text{BP}, \text{SF}, \text{YEN})$ are transformed into continuously compounded returns centered around their sample mean: $r_{j,t} = 100 \cdot [\ln(s_{j,t}/s_{j,t-1}) - (1/T)\sum_{t=1}^{T} \ln(s_{j,t}/s_{j,t-1})]$. The sample size for the returns is $T = 945$. This data set is also analyzed by Kim et al. (1998) and the British Pound exchange rate by Shephard and Pitt (1997).

The ML-EIS estimation results for the univariate SV model (1) and (2) for each of the exchange rates are given in Table 1. The ML-EIS estimates are based upon a simulated sample size of only $N = 30$ trajectories. They are numerically very accurate, as indicated by the small MC (numerical) standard deviations reported between brackets which were computed from 20 ML-EIS estimations conducted under different sets of CRNs. The estimates of the parameters for all four return series are very similar to the MCMC estimates reported by Kim et al. (1998).

The ML-EIS results for the multivariate factor SV model (3)-(5) based on $N = 50$ trajectories are summarized in Table 2. All parameter estimates are reasonable and numerically very accurate as indicated by the small MC standard deviations. The estimates of the factor loadings d_j indicate that the European currencies load more heavily on the common factor than the YEN. Moreover, the estimated volatility parameters of the factor are similar in magnitude to the those obtained under univariate SV models, and the estimate of δ, which is close to one, implies that the common factor exhibits a strongly persistent volatility process. The log-likelihood of the multivariate model is $-2,590$, which is substantially larger than the sum of the likelihood values obtained under the four independent SV models which equals $-3,699$. This significant difference reflects the fact that, in contrast to the univariate specifications, the multivariate model can account for the correlation between the returns.

3. A Bayesian MCMC Approach Based on EIS

So far we have discussed the application of EIS to evaluate the likelihood, which allows to perform a classical analysis of the univariate and multivariate SV model. We now discuss how EIS can be merged with MCMC simulation methods to perform a Bayesian posterior analysis.

Bayesian MCMC simulation methods such as Gibbs sampling construct a Markov Chain whose

equilibrium distribution is the joint posterior distribution of the parameters given the data. For the problem of simulating from the joint posterior of a vector with d blocks, say $\psi = (\psi_1, ..., \psi_d)'$, the Gibbs sampler draws the rth ψ_i from the conditional distribution $f(\psi_i^{(r)}|\tilde{\psi}_1^{(r)}, ..., \tilde{\psi}_{i-1}^{(r)}, \tilde{\psi}_{i+1}^{(r-1)}, ..., \tilde{\psi}_d^{(r-1)})$, $i : 1 \to d$. Under weak regularity conditions, the Gibbs draws $(\tilde{\psi}_1^{(r)}, ..., \tilde{\psi}_d^{(r)})$ converge to draws from the joint posterior as the number of cycles r increases (see, e.g., Tierney, 1994). For the basic SV model Jacquier et al. (1994) proposed to augment the parameter vector ω to include the vector of latent variables Λ, and to use the conditional posterior distributions $f(\omega|\Lambda, R)$ for ω given (Λ, R) and $f(\Lambda|\omega, R)$ for Λ given (ω, R) as two Gibbs blocks to simulate from the joint posterior distribution $f(\omega, \Lambda|R)$. The parameter vector ω is then estimated by reporting appropriate statistics for the simulations from $f(\omega, \Lambda|R)$.

The main difficulty with this MCMC approach for estimating SV models is that of efficiently sampling from Λ: The multivariate posterior distribution $f(\Lambda|\omega, R)$ is high dimensional and has no closed-form solution. One solution adopted, e.g., by Jacquier et al. (1994) consists of constructing a Gibbs sampler based on the T univariate conditional posteriors for $\lambda_t|\Lambda_{\setminus t}, \omega, R$, where $\Lambda_{\setminus t}$ denotes Λ without the tth element. Then, in order to sample each element in Λ *individually*, they use Tierney's (1994) AR-MH algorithm based on a proposal density, which is obtained from an inverse-gamma approximation to the density of $\exp\{\lambda_t/2\}|\Lambda_{\setminus t}, \omega, R$. (For a detailed description of the AR-MH procedure, see, Chib and Greenberg, 1995.) An attractive feature of such a 'single-move' algorithm is that it is easy to obtain a fairly good approximation to the univariate conditional posterior for λ_t. On the other hand, as illustrated by Pitt and Shephard (1997), high correlation between the elements in Λ leads to a slow convergence of the corresponding MCMC-algorithm. In order to alleviate this problem, one can consider factorizing Λ into a smaller number of multivariate blocks but doing so requires being able to construct good approximations to the higher dimensional conditional posterior densities of these blocks.

Here, we propose to use a combination of the EIS-sampler with Tierney's (1994) AR-MH algorithm to simulate $\Lambda|\omega, R$ as a single block. The basis of such a procedure is the fact that the EIS density for Λ provides a very close approximation to $f(\Lambda|\omega, R)$. In particular, our experience is that the integrand in equation (6) and hence $f(\Lambda|\omega, R)$ is a well behaved function in Λ given R which can be very accurately approximated by the EIS sampling density as indicated by the fact that the R^2 associated by the EIS least squares problems (14) are typically greater than 0.999. Hence, one can expect that the EIS density provides an efficient proposal density for the target density $f(\Lambda|\omega, R)$ in the AR-MH algorithm. In the following subsection, we first discuss the MCMC-EIS implementation for the univariate SV models. The extension to the multivariate model presented in the subsequent

subsection, is then largely straightforward.

3.1 A MCMC algorithm for the univariate SV model

The implementation of the AR-MH algorithm based on EIS for the simulation from the conditional posterior $f(\Lambda|\omega, R)$ involves the following elements. First, note that this conditional posterior density has the form

$$f(\Lambda|\omega, R) \propto f(R, \Lambda|\omega) = \prod_{t=1}^{T} f_t(r_t, \lambda_t|R_{t-1}, \Lambda_{t-1}, \omega), \tag{15}$$

where f_t is given by equations (7)-(9). As mentioned above, the EIS sampling density represents an approximation to $f(R, \Lambda|\omega)$ and hence, apart from a proportionality constant, to $f(\Lambda|\omega, R)$. The corresponding functional approximation is of the form

$$f(R, \Lambda|\omega) \simeq M(\Lambda, R, \omega) := \prod_{t=1}^{T} m_t(\lambda_t|\lambda_{t-1}, \hat{a}_t)e^{\hat{c}_t}, \tag{16}$$

where $\hat{a}_t$ and $\hat{c}_t$ are the estimated coefficients of the EIS regression (14) and are implicit functions of ω.

Standard acceptance-rejection sampling techniques, as described, e.g., by Robert and Casella (2004) require finding a constant κ (as small as possible) such that $f(R, \Lambda|\omega) \leq \kappa M(\Lambda, R, \omega)$ for all Λ. In the absence of such a constant we follow Shephard and Pitt (1997) in applying instead Tierney's (1994) accept-reject method which does not require that $\kappa M(\Lambda, R, \omega)$ is dominating. This approach is based on an additional Metropolis–Hastings step applied to the Λ–trajectories that come through the acceptance-rejection step in order to ensure that the target density is adequately sampled in the absence of a dominating function.

In particular, in the accept-reject part of the AR-MH algorithm, candidate trajectories $\tilde{Z}$ are drawn from the EIS sampler until acceptance with probability $\min\{f(R, \tilde{Z}|\omega)/M(\tilde{Z}, R, \omega), 1\}$. Then, in the Metropolis-Hastings step, given $\tilde{\Lambda}^{(k)}$ (the previously sampled trajectory) and $\tilde{\Lambda}$ (the candidate trajectory from the acceptance-rejection step), the next trajectory $\tilde{\Lambda}^{(k+1)}$ is obtained by the following scheme:

1. If $f(R, \tilde{\Lambda}^{(k)}|\omega) < M(\tilde{\Lambda}^{(k)}, R, \omega)$, set $\alpha = 1$;
   If $f(R, \tilde{\Lambda}^{(k)}|\omega) \geq M(\tilde{\Lambda}^{(k)}, R, \omega)$ and $f(R, \tilde{\Lambda}|\omega) < M(\tilde{\Lambda}, R, \omega)$, set

$$\alpha = \frac{M(\tilde{\Lambda}^{(k)}, R, \omega)}{f(R, \tilde{\Lambda}^{(k)}|\omega)};$$

If $f(R, \tilde{\Lambda}^{(k)}|\omega) \geq M(\tilde{\Lambda}^{(k)}, R, \omega)$ and $f(R, \tilde{\Lambda}|\omega) \geq M(\tilde{\Lambda}, R, \omega)$, set

$$\alpha = \min \left\{ \frac{f(R, \tilde{\Lambda}|\omega) M(\tilde{\Lambda}^{(k)}, R, \omega)}{f(R, \tilde{\Lambda}^{(k)}|\omega) M(\tilde{\Lambda}, R, \omega)}, 1 \right\}.$$

2. Generate u from a uniform $U_{[0,1]}$;

   If $u \leq \alpha$

   – return $\tilde{\Lambda}$ (as $\tilde{\Lambda}^{(k+1)}$);

   Else

   – return $\tilde{\Lambda}^{(k)}$ (as $\tilde{\Lambda}^{(k+1)}$).

As in any MCMC method, after a sufficient long 'burn-in' of say ℓ iterations, the draws $\{\tilde{\Lambda}^{(k)}, k : (\ell+1) \rightarrow N\}$ are regarded as a dependent sample from the target density. In fact, this convergence occurs under mild regularity conditions (see, e.g., Chib and Greenberg, 1995). In the application below, the AR-MH step for Λ is repeated 10 times before the parameters are updated in the Gibbs sequence. This is very cheap since draws of Λ from the EIS proposal density for fixed ω can be produced very fast and easily.

As mentioned above, alternative block sampling strategies to simulate $\Lambda|R, \omega$ are Shephard and Pitt's (1997) multi-move sampler and the mixture sampler of Kim et al. (1998). The multi-move sampling procedure divides the vector of volatilities Λ into blocks and applies a Taylor expansion in order to obtain local approximations to the conditional posterior densities of the corresponding blocks of volatility shocks. These approximations lead to multivariate Gaussian densities which are used as proposal densities to draw volatility blocks within an AR-MH procedure. In contrast, the mixture sampler linearizes the return equation of the SV model and approximates the distribution of the corresponding error term $\ln \epsilon_t^2$ by a discrete mixture of normals. Thus, within the Gibbs sequence ω and Λ are augmented to include a vector of auxiliary variables S which indicate from which of the normal distributions a particular $\ln \epsilon_t^2$ is drawn. Due to the linear form of the return equation it is then possible to sample $\lambda|\omega, R, S$ as one block. In order to correct for the approximation error, the Gibbs draws of the parameters are reweighted providing draws from the exact posterior densities.

To pursue a Bayesian analysis of the parameters ω, we need to specify the prior densities. For $\ln \beta$ we assume a flat prior. The resulting conditional posterior for β^2 is an inverted chi-squared distribution with $\beta^2|\Lambda, R, \delta, \nu \sim (\sum_{t=1}^{T} r_t^2 e^{-\lambda_t})/\chi^2_{(T)}$. Furthermore, following Kim et al. (1998), we employ for $(\delta+1)/2$ a Beta prior with parameters $\delta^{(1)} > 1/2$ and $\delta^{(2)} > 1/2$, which leads to a prior

for δ given by

$$\pi(\delta) \propto \Big(\frac{1+\delta}{2}\Big)^{\delta^{(1)}-1}\Big(\frac{1-\delta}{2}\Big)^{\delta^{(2)}-1}, \qquad \delta \in (-1,1). \tag{17}$$

In our application we set $\delta^{(1)} = 20$ and $\delta^{(2)} = 1.5$, implying a prior mean of 0.86 and a prior standard deviation of 0.11. The resulting conditional posterior is non-conjugate. To sample from this posterior for δ we use an independent MH sampler based on a Gaussian proposal density (for details, see Kim et al., 1998). Finally, for ν^2 we assume an inverted chi-squared prior with $p_0 s_0/\chi^2_{(p_0)}$. Then the conditional posterior is also an inverted chi-squared distribution with $\nu^2|\Lambda, R, \delta, \beta \sim [\sum_{t=1}^{T}(\lambda_t - \delta\lambda_{t-1})^2 + p_0 s_0]/\chi^2_{(T+p_0)}$. In the application we set $p_0 = 10$ and $s_0 = 0.01$.

The estimation results of the univariate SV model for each of the four exchange rates based on the above MCMC-EIS sampling scheme are summarized in Table 3. The results are based on 12,000 Gibbs iterations on the parameters, where the first 2000 are discarded. The table shows the posterior means, posterior standard deviations and MC standard errors. Following Shephard and Pitt (1997), the MC standard errors are computed using a Parzen based spectral estimator for the variance of a sample mean, which takes the serial correlation of the parameter draws into account. In particular, for M draws of the parameter vector $\{\omega^{(k)}\}_{k=1}^{M}$ the MC standard errors are the square root of the diagonal elements of

$$J_M = \frac{1}{M}\Big[\Gamma_0 + \frac{2M}{M-1}\sum_{\ell=1}^{L_M} K\Big(\frac{\ell}{L_M}\Big)\Gamma_\ell\Big], \qquad \text{where} \quad \Gamma_\ell = \frac{1}{M}\sum_{k=\ell+1}^{M}(\omega^{(k)} - \bar{\omega})(\omega^{(k-\ell)} - \bar{\omega})', \tag{18}$$

L_M is the bandwidth, and $K(\cdot)$ represents the Parzen kernel.

The MC standard errors for the British Pound exchange rate series are nearly the same as those reported by Shephard and Pitt (1997) for their multimove sampler applied to a posterior analysis for the same series. Hence, the MCMC-EIS implementation meets the standard of efficiency applied in the literature. This is also confirmed by the autocorrelation functions of the Gibbs draws of the parameters for the British Pound. These are shown in Figure 1 and indicate comparably fast diminishing correlations of the Gibbs draws. Figure 1 also shows the plot of the parameter draws against iteration and the corresponding the histograms. Finally, note that the MCMC-EIS estimation results for all four exchange rates given in Table 3 are very similar to the ML-EIS results shown in Table 1.

An alternative to the above MCMC-EIS implementation is one in which the AR-MH step for the Λ draws from the EIS sampler is omitted and replaced by a reweighting step applied to the corresponding parameter and Λ draws (see, e.g. Kim et al. 1998). The reweighting step corrects the error associated with the approximation of the true posterior for Λ by the EIS sampling density.

In particular, let $g(\omega)$ denote the function to be estimated by the posterior mean. Then such an estimate is obtained by reweighting the Gibbs parameter draws according to

$$\hat{\mathrm{E}}[g(\omega)|R] = \sum_{k=1}^{M} g(\omega^{(k)}) \cdot c^{(k)}, \tag{19}$$

where the weights are given by

$$c^{(k)} = \frac{f(R, \Lambda^{(k)}|\omega^{(k)})}{M(\Lambda^{(k)}, R, \omega^{(k)})} \Big/ \sum_{k=1}^{M} \frac{f(R, \Lambda^{(k)}|\omega^{(k)})}{M(\Lambda^{(k)}, R, \omega^{(k)})}. \tag{20}$$

It turned out that the MCMC-EIS results based on such a reweighting scheme (not presented here) are nearly the same as those obtained from an AR-MH correction step.

3.2 A MCMC algorithm for the multivariate factor SV model

In this section we consider the extension of the above MCMC-EIS procedure to the multivariate factor SV model (3)-(5). Following Jacquier et al. (1999), for such an extension we use the augmented joint posterior $f(X, D, \Sigma_e, \Lambda, \theta|R)$, where $X = \{x_t\}_{t=1}^T$ is the vector of the latent factors and $\theta = (\beta, \delta, \nu)'$ contains the parameters of the SV process for x_t. Accordingly, the Gibbs algorithm cycles through the conditional posterior distributions of X, D, Σ_e, Λ and θ, where $(\Lambda, \theta)|X, D, \Sigma_e, R$ can be sampled by applying directly the MCMC-EIS implementation for the univariate SV model as described in Section 3.1. The conditional posteriors of the remaining Gibbs blocks are obtained as follows.

The conditional posterior of the factors X is given by

$$\begin{aligned} f(X|D, \Sigma_e, \Lambda, \theta, R) &\propto f(R|X, D, \Sigma_e) f(X|\Lambda, \theta) && (21) \\ &\propto |\Sigma_e|^{-T/2} \exp\Big\{ -\frac{1}{2} \sum_{t=1}^{T} (r_t - Dx_t)' \Sigma_e^{-1} (r_t - Dx_t) \Big\} && (22) \\ &\quad \times \exp\{ -\frac{1}{2} \sum_{t=1}^{T} \frac{x_t^2}{\beta^2 e^{\lambda_t}} \}, \end{aligned}$$

whose r.h.s. represents the kernel of a multivariate Normal distribution. It follows that the conditional posterior of X is given by $\prod_{t=1}^{T} \mathrm{N}(\mu_{x_t}, \sigma_{x_t}^2)$ with $\mu_{x_t} = r_t' \Sigma_e^{-1} D \sigma_{x_t}^2$ and $\sigma_{x_t}^2 = 1/(D' \Sigma_e^{-1} D + \beta^{-2} e^{-\lambda_t})$.

Let $\bar{D}$ denote the sub-vector with the unrestricted elements of D. Assuming that the prior for the vector $\bar{D}$ is a multivariate Normal distribution with $\bar{D} \sim \mathrm{N}(\mu_{D_0}, \Sigma_{D_0})$ and that the priors for the elements of $\Sigma_e = \mathrm{diag}(\sigma_{e,1}^2, ..., \sigma_{e,n}^2)$ are independent inverted chi-squared distributions with $\sigma_{e,j}^2 \sim p_{j_0} s_{j_0} / \chi_{(p_{j_0})}^2$, then one obtains the corresponding conditional posteriors from standard conjugate

multivariate analysis. In particular, the conditional posterior for $\bar{D}$ is a $\mathrm{N}(\mu_D, \Sigma_D)$-distribution with $\mu_D = \Sigma_D[\sum_{t=1}^{T}(\Sigma_e^{-1} r_t x_t) + \Sigma_{D_0}^{-1}\mu_{D_0}]$ and $\Sigma_D = [(\sum_{t=1}^{T} x_t^2)\Sigma_e^{-1} + \Sigma_{D_0}^{-1}]^{-1}$. Furthermore, the conditional posterior of $\sigma_{e,j}^2$ is an inverted chi-squared distribution with $\sigma_{e,j}^2|(\cdot) \sim [\sum_{t=1}^{T}(r_{j,t} - d_j x_t)^2 + p_{j_0} s_{j_0}]/\chi^2_{(T+p_{j_0})}$.

In the application we set the hyper-parameters for $\bar{D}$, following Pitt and Shephard (1999), to $\mu_{D_0} = (1,1,1)'$ and $\Sigma_{D_0} = \mathrm{diag}(25,25,25)$, reflecting a large prior uncertainty. The hyper-parameters for $\sigma_{e,j}^2$ are set to $p_{j_0} = 10$ and $s_{j_0} = 0.01$ for all $j : 1 \to 4$. Finally, for the θ parameters of the SV factor process x_t, we use the same prior specification as in the MCMC-EIS implementation for the univariate SV model.

The estimation results of the multivariate factor model based on the above sampling scheme are summarized in Table 4; the corresponding time series, histograms and autocorrelation functions of the θ parameters are plotted in Figure 2 and the autocorrelation functions of the remaining parameters are shown in Figure 3. The results are based on 22,000 Gibbs iterations on the parameters, where the first 2000 are discarded. Note from the MC standard errors given in Table 4 and the autocorrelation functions in the Figures 2 and 3 that the MCMC-EIS procedure applied to the multivariate factor model is reasonably efficient. In particular, the correlations of the θ parameters become negligible within 500 lags and those of the D and Σ_e parameters within 50 and 150 lags, respectively. Furthermore, note that the MCMC-EIS results of the multivariate model are very similar to those obtained from the ML-EIS estimation and given in Table 2. Especially, for the D and Σ_e parameters the two estimation procedures deliver quasi-identical results which indicates that the likelihood is very informative about the factor loadings and the idiosyncratic variances.

As mentioned above, the performance of the multivariate SV model (3)-(5) might be improved further by considering extensions of the model, for example, one that allows for a second latent factor and/or idiosyncratic errors also following SV processes. The application of EIS to such extensions, which are analyzed by Pitt and Shephard (1999) and Aguilar and West (2000) using MCMC methods, is left to future research. However note that once a Gibbs algorithm for the basic one-factor SV model is implemented, the augmentation to include additional latent SV processes is straightforward and just requires to add additional steps in the Gibbs sequence of the same form as described above (for further details, see also, Pitt and Shephard, 1999). Since the Gibbs step for a particular SV process is conditional on the Gibbs draws for the remaining SV processes, the MCMC-EIS implementation for such a generalization seems to be more convenient than the corresponding ML-EIS approach, which would require an approximation of the joint 'posterior' of *all* latent SV

processes simultaneously.

4. Conclusion

This paper uses Efficient Importance Sampling (EIS) to provide the basis for ML-estimation and a Bayesian Markov Chain Monte Carlo (MCMC) analysis for univariate and multivariate SV-models. EIS is a Monte Carlo procedure for the evaluation of high-dimensional interdependent integrals which can be used to accurately compute the likelihood of dynamic latent variable models. It is based upon a global approximation to the implicit 'posterior' of the vector of latent variables given all observable variables and the parameters. The resulting importance sampling density can also be used to construct within a Gibbs approach a simulation smoother for the generation of draws from the full conditional posterior of the volatility vector.

An attractive feature of the EIS procedure is that it is highly generic. Hence, changes in the model being analyzed can be easily accommodated. Here we have focused on the ML-EIS and MCMC-EIS analysis of the standard univariate SV model and a simple multivariate factor SV specification but, in principle, extensions to appropriate generalizations of these baseline specifications are straightforward. For the ML-EIS approach this flexibility has already been illustrated by Liesenfeld and Richard (2003). A further advantage of the EIS procedure is that it allows within a MCMC analysis to sample the vector of latent volatilities as one block. This avoids the negative impacts of highly correlated elements in the volatility vector on the convergence of algorithms where the volatility elements are sampled individually.

Acknowledgements

We are grateful to participants of the workshop "Computational Finance" at the University of Kiel.

Appendix: The implementation of the EIS-algorithm

The EIS implementation for the likelihood evaluation of the univariate SV model (1) and (2) starts with the selection of the class of auxiliary samplers. Using a parametric extension of the initial sampler p_t given by Equation (9), the corresponding density kernel of the Gaussian auxiliary sampler m_t can be parameterized as

$$k_t(\lambda_t, \lambda_{t-1}, a_t) = p_t(\lambda_t|\lambda_{t-1};\omega)\zeta_t(\lambda_t, a_t), \tag{A.1}$$

with

$$\zeta_t(\lambda_t, a_t) = \exp\{a_{1t}\lambda_t + a_{2t}\lambda_t^2\}, \qquad a_t = (a_{1t}, a_{2t})', \tag{A.2}$$

where ζ_t is an auxiliary function. Under this parametrization the initial sampler p_t cancels out in the EIS auxiliary regression (14). Inserting the functional forms of p_t and ζ_t into Equation (A.1), leads to

$$k_t(\lambda_t, \lambda_{t-1}, a_t) \propto \exp\Big\{ -\frac{1}{2}\Big[\Big(\frac{\delta\lambda_{t-1}}{\nu}\Big)^2 - 2\Big(\frac{\delta\lambda_{t-1}}{\nu^2} + a_{1t}\Big)\lambda_t + \Big(\frac{1}{\nu^2} - 2a_{2t}\Big)\lambda_t^2\Big]\Big\}. \tag{A.3}$$

Hence, the auxiliary sampler is a Gaussian density $\mathrm{N}(\mu_t, \sigma_t^2)$, where

$$\mu_t = \sigma_t^2\Big(\frac{\delta\lambda_{t-1}}{\nu^2} + a_{1t}\Big), \qquad \sigma_t^2 = \frac{\nu^2}{1 - 2\nu^2 a_{2t}}. \tag{A.4}$$

Integrating k_t with respect to λ_t leads to the following expression for the integrating constant

$$\chi_t(\lambda_{t-1}; a_t) \propto \exp\left\{\frac{\mu_t^2}{2\sigma_t^2} - \frac{(\delta\lambda_{t-1})^2}{2\nu^2}\right\}. \tag{A.5}$$

Based on these functional forms the computation of an EIS MC estimate of the likelihood requires the following steps:

Step (1): Generate N independent trajectories $\{(\breve{\lambda}_1^{(i)}, ..., \breve{\lambda}_T^{(i)}), i : 1 \to N\}$ from the p_t densities.

Step (2): Use these trajectories to solve for each period $t : T \to 1$ the least squares problem defined in Equation (14). The step t regression is

$$\ln g_t(r_t|\breve{\lambda}_t^{(i)}, \omega) + \ln \chi_{t+1}(\breve{\lambda}_t^{(i)}; \hat{a}_{t+1}) = c_t + a_{1t}\breve{\lambda}_t^{(i)} + a_{2t}[\breve{\lambda}_t^{(i)}]^2 + u_t^{(i)}, \qquad i : 1 \to N, \tag{A.6}$$

where $u_t^{(i)}$ denotes the regression error term and g_t is the Gaussian density for r_t given λ_t according to Equation (8). The initial condition for the integrating constant is given by $\chi_{T+1}(\cdot) \equiv 1$.

Step (3): Use the estimates of the regression coefficients a_{1t}, a_{2t} to obtain the Gaussian EIS sampler $\{m(\lambda_t|\lambda_{t-1}, \hat{a}_t)\}_{t=1}^T$ characterized by the means and variances given in Equation (A.4). Then generate N trajectories from this EIS sampler from which the EIS MC estimate of the likelihood is calculated according to Equation (12).

The implementation for the multivariate extension of the standard SV model (3)-(5) only requires the replacement of the density (8) by density (10) in the EIS regression (A.6) and in the final EIS MC estimate (12).

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Table 1. ML-EIS Estimation Results for the Univariate SV Model

	DM	BP	SF	YEN
β	.686	.675	.724	.567
	(.068)	(.088)	(.053)	(.072)
	[.0009]	[.0021]	[.0009]	[.0015]
δ	.962	.977	.940	.984
	(.019)	(.013)	(.026)	(.015)
	[.0004]	[.0004]	[.0008]	[.0005]
ν	.170	.168	.236	.117
	(.037)	(.037)	(.048)	(.050)
	[.0010]	[.0014]	[.0019]	[.0020]
Log-likel.	-949.7	-919.0	$-1,045.1$	-785.6
	[.097]	[.104]	[.134]	[.060]

NOTE: Asymptotic (statistical) standard errors obtained from a numerical approximation to the Hessian are in parentheses and MC (numerical) standard errors are in brackets. The ML-EIS estimates are based on a MC sample size $N = 30$ and three EIS iterations.

Table 2. ML-EIS Estimation Results for the Factor SV Model

	DM	BP	SF	YEN
d_j	1.000	$-.843$	1.050	.644
		(.023)	(.024)	(.021)
		[<.0001]	[<.0001]	[<.0001]
$\sigma_{e,j}$	.169	.431	.393	.406
	(.017)	(.011)	(.013)	(.010)
	[.0001]	[<.0001]	[<.0001]	[<.0001]
	β	δ	ν	Log-Likel.
	.679	.971	.152	$-2,590.2$
	(.076)	(.017)	(.032)	
	[.0004]	[.0001]	[.0004]	[.0494]

NOTE: Asymptotic (statistical) standard errors obtained from a numerical approximation to the Hessian are in parentheses and MC (numerical) standard errors are in brackets. The ML-EIS estimates are based on a MC sample size $N = 50$ and three EIS iterations.

Table 3. MCMC-EIS Posterior Analysis of the Univariate SV Model

	DM	BP	SF	YEN
β	.787	.739	.801	.584
	(.135)	(.120)	(.122)	(.070)
	[.0121]	[.0106]	[.0121]	[.0037]
δ	.977	.983	.964	.984
	(.015)	(.009)	(.017)	(.008)
	[.0014]	[.0005]	[.0015]	[.0004]
ν	.139	.140	.185	.111
	(.026)	(.025)	(.029)	(.020)
	[.0029]	[.0022]	[.0028]	[.0021]

NOTE: The first number is the posterior mean based on 12,000 Gibbs iterations (discarding the first 2000 draws). The posterior standard deviation is in parentheses and the MC standard error in brackets. The MC standard errors are computed using a Parzen window with bandwidth $L_M = 1000$. The EIS simulation smoother is based on a MC sample size $N = 30$ and three EIS iterations.

Table 4. MCMC-EIS Posterior Analysis of the Factor SV Model

	DM	BP	SF	YEN
d_j	1.000	−.839	1.045	.641
		(.024)	(.025)	(.022)
		[.0005]	[.0008]	[.0004]
$\sigma_{e,j}$	.159	.431	.395	.405
	(.019)	(.011)	(.013)	(.010)
	[.0011]	[.0002]	[.0004]	[.0001]
	β	δ	ν	
	.757	.978	.134	
	(.137)	(.013)	(.024)	
	[.0110]	[.0009]	[.0019]	

NOTE: The first number is the posterior mean based on 22,000 Gibbs iterations (discarding the first 2000 draws). The posterior standard deviation is in parentheses and the MC standard error in brackets. The MC standard errors are computed using a Parzen window with bandwidth $L_M = 1000$. The EIS simulation smoother is based on a MC sample size $N = 30$ and three EIS iterations.

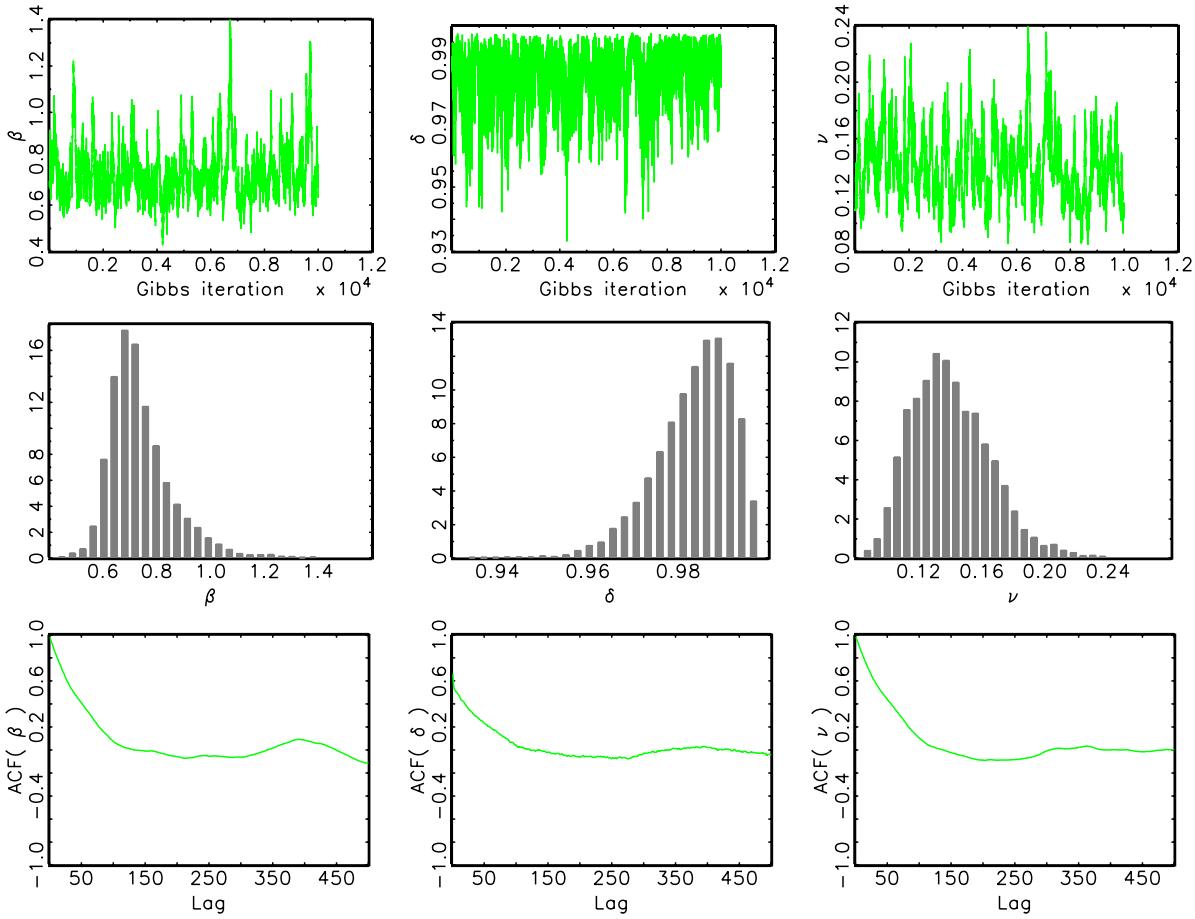

Fig. 1. Parameter draws for the univariate SV model for the British Pound/US Dollar returns from 10,000 Gibbs iterations: Draws against the Gibbs iteration (top row), histograms of the draws (middle row), autocorrelation functions of the draws (bottom row).

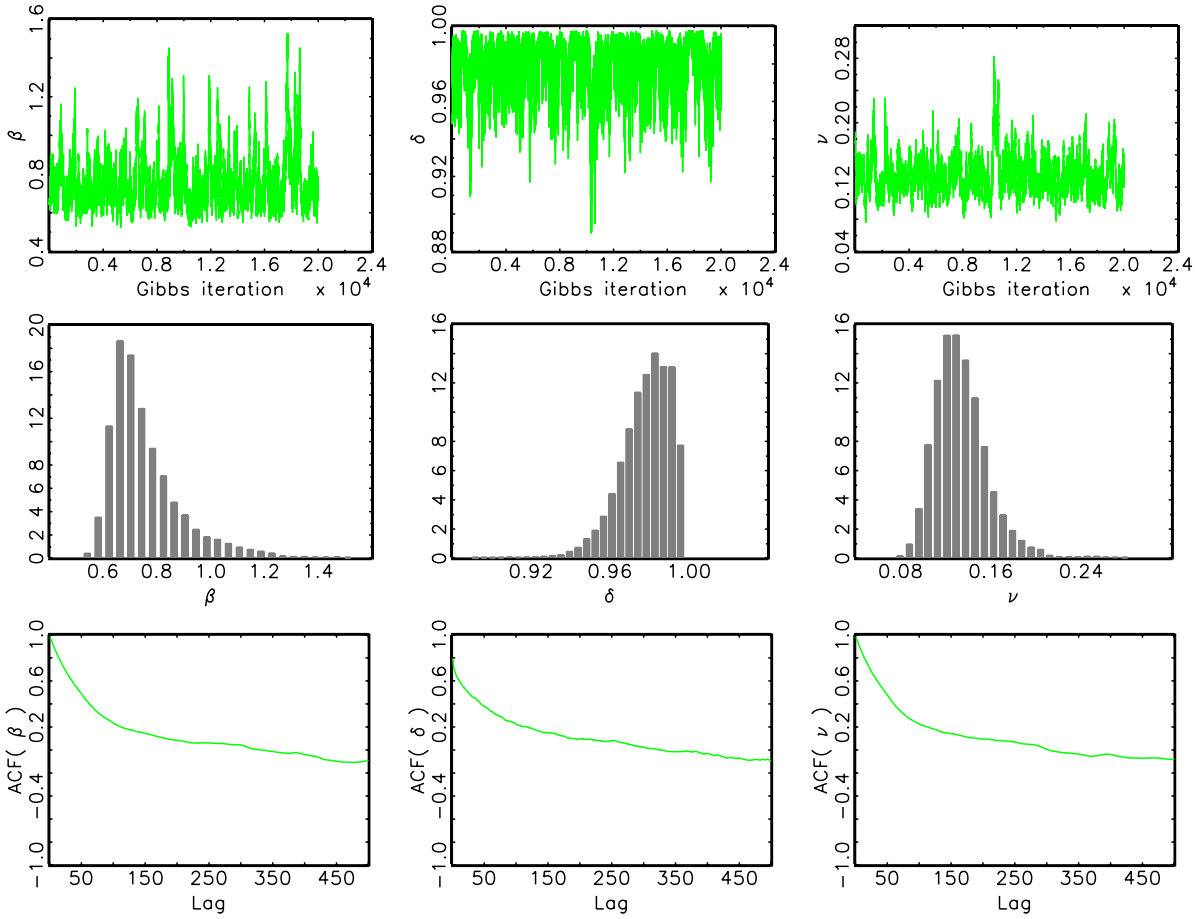

Fig. 2. Draws of the parameters β, δ, ν for the factor process in the multivariate SV model from 20,000 Gibbs iterations: Draws against the Gibbs iteration (top row), histograms of the draws (middle row), autocorrelation functions of the draws (bottom row).

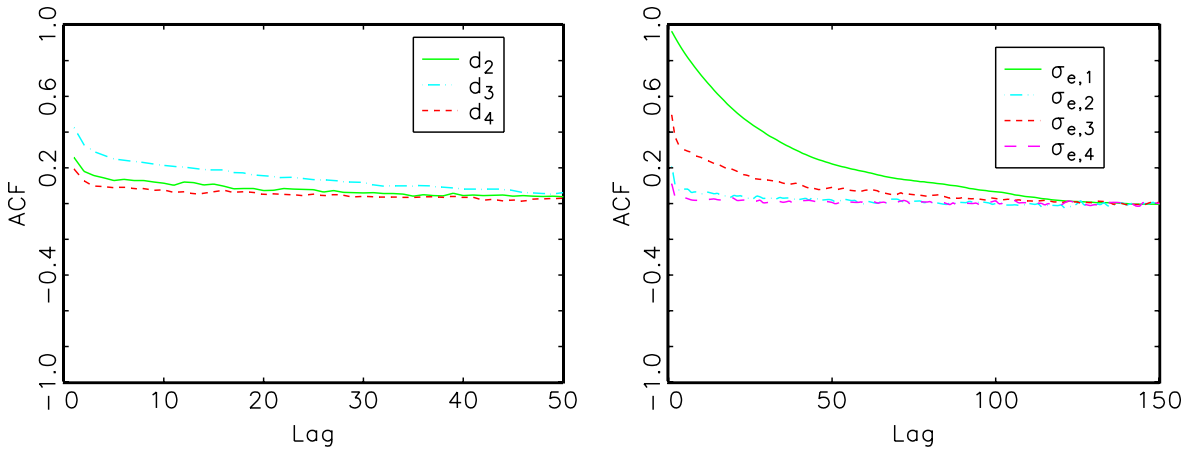

Fig. 3. Autocorrelation functions of the draws of the D and Σ_e parameters in the multivariate SV model from 20,000 Gibbs iterations. The functions for d_2, d_3, d_4 are in the left panel and for $\sigma_{e,1}$, $\sigma_{e,2}$, $\sigma_{e,3}$, $\sigma_{e,4}$ in the right panel.