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A Proof of Determinacy in the New-Keynesian Sticky Wages and Prices Model

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A Proof of Determinacy in the New-Keynesian Sticky Wages and Prices Model

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Abstract

The paper is concerned with determinacy in a version of the New-Keynesian model that integrates imperfect competition and nominal price and wage setting on goods and labour markets. The model is reformulated with an explicit period of arbitrary length and shown to remain well-defined as the period shrinks to zero. The 4×4 constituent matrix of the model's continuous-time counterpart is mathematically tractable and its determinacy results carry over to the period model at least if the period is sufficiently short. This being understood, it is proved that determinacy is (essentially) ensured if an extended Taylor principle requirement is met.

JEL classification: E31; E32; E52.

Keywords: Determinacy; New-Keynesian wage and price Phillips curves; variable period length; continuous-time limit; Taylor principle.

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1. Introduction

A basic extension of the standard New-Keynesian model with its forward-looking price Phillips curve, dynamic IS equation and a monetary policy rule is concerned with an integration of labour markets. Introducing imperfect competition and staggered nominal

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[1] We wish to thank Toichiro Asada for his mathematical hints.

wage setting in these markets, they can be treated in an analogous way to the goods markets; see Erceg et al. (2000), Woodford (2003, chapter 6), or Galí (2007, chapter 6). In its reduced form, the model now contains four dynamic variables: output gap, price inflation and wage inflation on the one hand, which are non-predetermined variables, and the real wage gap on the other hand, which is a predetermined variable.

As the model is formulated in discrete time, uniqueness of a stationary equilibrium requires that three eigen-values of a suitable 4×4 matrix lie outside, and one inside, the unit circle. From the numerical investigations to be found in the literature one can infer that this determinacy causes no problems once a modified Taylor principle is satisfied, which says that the central bank adjusts the nominal interest rate more than one-for-one in response to variations in any arbitrarily weighted average of price and wage inflation (Galí, 2007, p. 128). A mathematical proof supporting this numerical knowledge is, however, not available. It also seems hard to achieve in general, given that already the conditions for all four eigen-values to lie on either side of the unit circle are fairly complicated.

Prospects of analytical tractability appear to improve if the model were conceived in continuous time, so that three eigen-values of suitable matrix would have to lie in the right half of the complex plane and the fourth one in the left half. An *a priori* preference for continuous time on the basis of mathematical reasons fits in with a methodological precept that was put forward by Duncan Foley several decades ago: *"No substantive prediction or explanation of a well-defined macroeconomic period model should depend on the real time length of the period"* (Foley, 1975, p. 310; his emphasis). Accordingly, the length of the period should be retained as an explicit variable in the mathematical formulation of a period model, and it is to be made sure that it is possible to find meaningful limiting forms of the equations as the period goes to zero. And Foley goes on to state, "In my view, this procedure should be routinely applied as a test that any period model is consistent and well formed where no particular calendar time is specified as the natural period" (*ibid.*, p. 311).

In this paper we follow Foley's maxim and reformulate the sticky wages and prices model with an explicit period of arbitrary length. We will thus confirm that the model indeed passes the test of remaining well-defined as the period shrinks to zero. It will subsequently be possible to study the four eigen-values of the matrix that constitutes the model's continuous-time counterpart. Determinacy in this case carries over to the discrete-time framework at least if the period length is sufficiently short, and it is in this sense that we can derive conditions for determinacy in the period model. As a matter of fact, they essentially amount to the modified Taylor principle mentioned above.

The remainder of the paper is organized as followed. The next section reiterates the key equations of the New-Keynesian period model. Section 3 provides the details for the eigen-value relationships between the discrete-time and continuous-time framework. Section 4 reformulates the original model with a variable period, and Section 5 contains the determinacy proposition together with the proof. Section 6 gives some numerical evidence of how variations of the period length may affect the determinacy threshold of the sum of the two policy coefficients on wage and price inflation. Section 7 concludes with pointing out that the methodological interest of the paper may well reach beyond the scope of the specific model studied here.

2. The period model

The presentation of the key equations of the New-Keynesian model follows Galí (2007, chapter 6). In the main we also adopt his notation, except that we avoid using a tilde. Thus, let y_t be the output gap in period t, i.e. the percentage deviation of output from its natural level, and ω_t the real wage gap, which is the difference between the (log of the) real wage rate and the (log of the) natural real wage. Price and wage inflation are denoted by π_t^p and π_t^w, respectively, the nominal rate of interest is i_t and the natural interest rate r_t^n (they are explicitly supposed to be quarterly rates; cf. Galí, p. 52, fn 6). If in addition u_t and v_t designate the model's two exogenous components, Galí's equations (15), (17) – (20) on pp. 126f can be reproduced as follows:

$$\pi_t^w \;=\; \beta\, E_t[\pi_{t+1}^w] \;+\; \kappa_w\, y_t \;-\; \lambda_w\, \omega_t \tag{1}$$

$$\pi_t^p \;=\; \beta\, E_t[\pi_{t+1}^p] \;+\; \kappa_p\, y_t \;+\; \lambda_p\, \omega_t \tag{2}$$

$$y_t \;=\; E_t[y_{t+1}] \;-\; \frac{1}{\sigma}\,(i_t - E_t[\pi_{t+1}^p] - r_t^n) \tag{3}$$

$$i_t \;=\; \rho \;+\; \phi_p\, \pi_t^p \;+\; \phi_w\, \pi_t^w \;+\; \phi_y\, y_t \;+\; v_t \tag{4}$$

$$\omega_t \;=\; \omega_{t-1} \;+\; \pi_t^w \;-\; \pi_t^p \;-\; u_t \tag{5}$$

All coefficients are constant and positive, apart from the policy coefficients in (4), some of which may also attain zero values. The coefficient ρ can be interpreted as the household's discount rate, from which β derives as $\ln \beta = -\rho$ (Galí, 2007, p. 18).

Effectively, the two exogenous variables in the dynamics are $z_{1,t} := v_t - (r_t^n - \rho)$ and $z_{2,t} = u_t$. If the system is to have a solution satisfying $y_t = \pi_t^p = \pi_t^w = 0$ for all t, then $z_{1,t}$ and $z_{2,t}$ must vanish (Galí, pp. 127f). Assuming this for the rest of the paper, we are left with the four dynamic variables $x_t := (\pi_t^w, \pi_t^p, y_t, \omega_{t-1})'$. Plugging (4) into (3) and solving these four equations for the expected values of x_{t+1}, the system can be transformed into the representation,

$$E_t[x_{t+1}] \;=\; A\, x_t \tag{6}$$

where A is a suitable 4×4 matrix. The non-predetermined variables of the model are π_t^w, π_t^p and y_t, while the real wage gap ω_{t-1} of the previous quarter is a predetermined variable. Hence determinacy requires that the matrix A has three eigen-values outside, and one inside, the unit circle. [2] As a result of his numerical analysis, Galí (p. 128) asserts

[2] Which, in particular, implies that there is no scope for endogenous cyclical behaviour, or an overshooting wage-price spiral.

that a sufficient (albeit not necessary) condition for this to prevail is the inequality,

$$\phi_w \; + \; \phi_p \;\; > \;\; 1 \tag{7}$$

Accordingly, determinacy is guaranteed if the central bank adjusts the nominal interest rate more than one-for-one in response to variations in any arbitrarily weighted average of price and wage inflation. Equation (7) is thus an extended version of the famous Taylor principle.

3. Determinacy and the concept of a variable period length

Let us now consider Foley's axiom mentioned in the introduction, though still at a general level. Given a fixed a time unit, we will refer to a dynamic system as an h-economy if its period has length h. To make the results comparable across different values of h, the variables in x_t have to be expressed in terms of the time unit. In the present case this means that the inflation rates have to be 'quarterized', if the underlying time unit continues to be a quarter: $\pi^w_t = (w_t - w_{t-h})/h$ and $\pi^p_t = (p_t - p_{t-h})/h$ for the log wages w_t and prices p_t (the output gap as a ratio of two flow magnitudes and the real wage gap have no time dimension).

Transforming a quarterly model into an h-economy would be straightforward if $E_t[x_{t+1}] = x_{t+1}$ holds true in (6) and the right-hand side represents a linear partial adjustment mechanism for each variable. The matrix A can then be decomposed into $A = I + J$ (I the identity matrix) and the adjustments in the h-economy become $x_{t+h} = x_t + h\,Jx_t$. In the limit $h \to 0$ a differential equations system is obtained, $\dot{x} = Jx$, whose basic dynamic properties are characterized by the eigen-values of the matrix J. Provided no eigen-value is zero or lies on the imaginary axis, these properties will carry over to the discrete-time h-economy, at least if h is sufficiently small.

Things are a bit more involved for the present New-Keynesian model. Here the influence of h will be of a nonlinear nature and the matrix $J = (A - I)/h$ from (6) is itself dependent on h. This gives us $J = J(h)$ and

$$E_t[x_{t+h}] \;\; = \;\; [\, I + h\, J(h)\,]\; x_t \tag{8}$$

It will furthermore be established that the matrices $J(h)$ converge to some finite matrix J^o as h tends to zero. Under $E_t[x_{t+h}] = x_{t+h}$, the limit would be well-defined if Ix_t is brought to the left-hand side in (8) and the resulting equation divided by h. Hence the 'continuous-time matrix' J^o should contain all the relevant information about the qualitative behaviour of the discrete-time system (8) if h is small enough. In the present context we are interested in the number of stable and unstable eigen-values in J^o and $[I + hJ(h)]$, respectively. The precise relationship between the two is stated in the following lemma.

Lemma

Let $h \mapsto J(h)$ be a continuous function of $n{\times}n$ matrices defined on an interval $[0,\varepsilon]$ for some $\varepsilon > 0$. Suppose k eigen-values of $J^o := J(0)$ have positive, and $n{-}k$ eigen-values have negative real parts. Then there is a positive number $\overline{h}$ such that for all $0 < h < \overline{h}$ the matrix $[I + hJ(h)]$ has k eigen-values of J^o inside, and $n{-}k$ eigen-values outside, the unit circle.

Proof: It is immediate that if $\mu(h)$ is an eigen-value of $J(h)$, then $1{+}h\mu(h)$ is an eigen-value of $[I{+}hJ(h)]$. Let $\mu_o = -a_o {\pm} ib_o$ be an eigen-value of J^o with $a_o > 0, b_o \geq 0$, and let $\mu(h) = -a(h) {\pm} ib(h)$ be the eigen-values of $J(h)$ that converge to μ_o as $h \to 0$. Then for h sufficiently small we have $|1{+}h\mu(h)|^2 = [1{-}ha(h)]^2 + h^2 b^2(h) < (1{-}ha_o/2)^2 + 2h^2 b_o^2 = 1 - h\,[a_o - h(a_o^2/4{+}2b_o^2)]$.

Clearly, there exists some $\overline{h} > 0$ such that the last term in square brackets is positive for $h < \overline{h}$, which says that for all these h the eigen-values $[1 + h\mu(h)]$ are inside the unit circle. On the other hand, it is obvious that if an eigen-value $\mu_o = a_o \pm ib_o$ of J^o has a positive real part a_o, then $|1 + h\mu(h)| > 1$ for all $\mu(h)$ close to μ_o.

q.e.d

Taking $J(h) \to J^o$ for granted as h approaches zero, the main significance of the Lemma lies in the fact that it is much easier to derive the number of eigen-values of the matrix J^o that are in the left and right half of the complex plane, respectively, than the number of eigen-values of $[I + hJ(h)]$ that are inside and outside the unit circle, whether h is small or $h = 1$ as in the original economy.

4. Reformulation of the model with a variable period length

After presenting the general idea of the h-economies and expressing our hopes for its benefits in the determinacy analysis, we have now to come to terms with our specific model and introduce a period of arbitrary length h into it. We begin with the equation for the real wage gap. With the assumption $u_t = z_{2,t} = 0$ mentioned above, which means that the natural real wage is constant, eq. (5) becomes $\omega_t = \omega_{t-h} + (w_t {-} p_t) - (w_{t-h} {-} p_{t-h}) = \omega_{t-h} + h\,[(w_t - w_{t-h})/h - (p_t - p_{t-h})/h]$, or

$$\omega_t \;=\; \omega_{t-h} \;+\; h\,(\pi_t^w - \pi_t^p) \tag{9}$$

Consider next the Taylor rule in (4). Given that in the quarterly model the interest rate i_t corresponds to the log of the gross yield on bonds purchased in t and maturing in $t{+}1$ (Galí, pp. 16, 18), hi_t corresponds to the log of the gross yield when these bonds are maturing in $t{+}h$. The household's rate for discounting periods of length h is $h\rho$, and similarly so for the component v_t (see the specification of $z_{1,t}$ in Section 2). In the h-economy, the Taylor rule thus reads $hi_t = h\rho + \phi_p(p_t {-} p_{t-h}) + \phi_w(w_t {-} w_{t-h}) + h\phi_y\, y_t + hv_t$. Retranslated into quarterly magnitudes we obtain

$$i_t \;=\; \rho \;+\; \phi_p\,\pi_t^p \;+\; \phi_w\,\pi_t^w \;+\; \phi_y\,y_t \;+\; v_t \tag{10}$$

Before we turn to the counterparts of (1) – (3) in the h-economy, we have to have a look at the structural parameters entering these equations or their composed coefficients κ_w, λ_w, κ_p, λ_p, respectively. The latter are given by

$$\begin{array}{llll} \kappa_w = a_w\,\lambda_w & & \kappa_p = a_p\,\lambda_p & \\ a_w = \sigma + \phi/(1-\alpha) & & a_p = \alpha/(1-\alpha) & \\ \lambda_w = \dfrac{(1-\theta_w)\,(1-\beta\theta_w)}{\theta_w} & \dfrac{1}{1+\phi\,\varepsilon_w} & \lambda_p = \dfrac{(1-\theta_p)\,(1-\beta\theta_p)}{\theta_p} & \dfrac{1-\alpha}{1-\alpha+\alpha\,\varepsilon_p} \end{array} \tag{11}$$

(cf. Galí, 2007, pp. 121, 125f). All of these parameters are specified as positive numbers, where the following ones are independent of the length of the period: α is the exponent on labour in the production function ($\alpha < 1$, p. 18 in Galí); σ and ϕ are the elasticity coefficients in the household's utility function that refer to consumption and labour (p. 17);[3] ε_p is the household's elasticity of substitution among the differentiated consumption goods (pp. 41f, 122); and ε_w is the firms' elasticity of substitution among the varieties of labour inputs (p. 120).

The parameter β serves to discount the household's intertemporal utility and so changes with the length of the period. When instead of a quarter this parameter applies to a period of length h, it may be denoted as $\beta(h)$. Since the discount rate for a period of length h is $h\rho$ and the quarterly coefficient $\beta = \beta(1)$ was already said to be related to the quarterly discount rate by $\ln\beta = -\rho$, or equivalently $\beta = 1/(1+\rho)$,[4] the coefficient $\beta(h)$ is determined by

$$\beta(h) \;=\; 1\,/\,(1+h\rho) \tag{12}$$

The two remaining parameters θ_w and θ_p have a time dimension, too. $(1-\theta_w)$ is the fraction of households/unions that reoptimize their posted nominal wage within a given quarter, while the rest θ_w of them post the wage of the previous quarter (Galí, p. 122). Likewise, $(1-\theta_p)$ is the fraction of firms that in this period reset their price, and the rest θ_p does not (pp. 43, 47, 121).

The parameter $\theta_w(h)$ appropriate for the h-economy is obtained from the observation that in a period of length h the fraction of reoptimizing households will be $h(1-\theta_w)$. This gives us $\theta_w(h) = 1 - h(1-\theta_w)$. Using (12), the term $(1-\beta\theta_w)$ in (11) now reads $1 - \beta(h)\theta_w(h) = 1 - [1 - h(1-\theta_w)]/(1+h\rho) = h(1+\rho-\theta_w)/(1+h\rho)$. In this way the first fraction in the definition of λ_w in (11) becomes $[1-\theta_w(h)]\,[1-\beta(h)\theta_w(h)]\,/\,\theta_w(h) = h(1-\theta_w)\,h(1+\rho-\theta_w)\,/\,[1-h(1-\theta_w)](1+h\rho)$. The same reasoning applies to $\theta_p(h)$ and the first fraction in the definition of λ_p in (11). The coefficients $\lambda_w(h)$ and $\lambda_p(h)$ adjusted to the h-economy can thus be written as

[3] This ϕ may not be confused with the policy coefficients ϕ_w, ϕ_p, ϕ_y in the Taylor rule.

[4] Note that solving $\beta = 1/(1+\rho)$ for ρ gives $\rho = (1-\beta)/\beta \approx \ln[1+(1-\beta)/\beta] = \ln(1/\beta) = -\ln\beta$.

$$\lambda_w(h) \;=\; h^2\, b_w(h)\;, \qquad b_w(h) \;:=\; \frac{(1-\theta_w)\,(1+\rho-\theta_w)}{[1\,-\,h(1-\theta_w)]\,(1+h\rho)}\;\frac{1}{1+\phi\,\varepsilon_w}$$
$$\lambda_p(h) \;=\; h^2\, b_p(h)\;, \qquad b_p(h) \;:=\; \frac{(1-\theta_p)\,(1+\rho-\theta_p)}{[1\,-\,h(1-\theta_p)]\,(1+h\rho)}\;\frac{1-\alpha}{1-\alpha+\alpha\,\varepsilon_p} \tag{13}$$

If the period-dependent parameters β, θ_w and θ_p in the model are suitably adjusted, then all of the agents' optimization procedures go through unaltered. This means that we can directly refer to the Phillips curve and the dynamic IS equation as they are formulated in (1) – (3); we only have to replace the coefficients β, λ_w, λ_p with $\beta(h)$, $\lambda_w(h)$, $\lambda_p(h)$, and κ_w, κ_p with $\kappa_w(h) = a_w\,\lambda_w(h)$, $\kappa_p(h) = a_p\,\lambda_p(h)$, respectively.

So, to begin with, let us reconsider the wage Phillips curve (1) in the context of an h-economy. Employing (11) and (13) we here get $w_t - w_{t-h} = \beta(h)E_t[w_{t+h} - w_t] + a_w\,\lambda_w(h)\,y_t - \lambda_w(h)\,\omega_t = \beta(h)E_t[w_{t+h} - w_t] + h^2\,[a_w\,b_w(h)\,y_t - b_w(h)\,\omega_t]$. Dividing through by h to express the wage inflation rates as quarterly magnitudes, $\pi^w_t = (w_t - w_{t-h})/h$, solving for the expected values and using (12) as well as (9) yields $E_t[\pi^w_{t+h}] = (1+h\rho)\,\{\,\pi^w_t \,-\, h\,[a_w\,b_w(h)\,y_t - b_w(h)(\omega_{t-h} + h(\pi^w_t - \pi^p_t))]\,\}$. Expected price inflation can be treated in the same way. It is then convenient to define

$$\begin{aligned}
j_{ww}(h) &= \rho + h\,(1{+}h\rho)\,b_w(h) & j_{wp}(h) &= -h\,(1{+}h\rho)\,b_w(h)\\
j_{wy}(h) &= -(1{+}h\rho)\,a_w\,b_w(h) & j_{w\omega}(h) &= (1{+}h\rho)\,b_w(h)\\
j_{pw}(h) &= -h\,(1{+}h\rho)\,b_p(h) & j_{pp}(h) &= \rho + h\,(1{+}h\rho)\,b_p(h)\\
j_{py}(h) &= -(1{+}h\rho)\,a_p\,b_p(h) & j_{p\omega}(h) &= -(1{+}h\rho)\,b_p(h)
\end{aligned} \tag{14}$$

and write the reduced form of the expected inflation rates as

$$E_t[\pi^w_{t+h}] \;=\; \pi^w_t \;+\; h\,[\,j_{ww}(h)\,\pi^w_t + j_{wp}(h)\,\pi^p_t + j_{wy}(h)\,y_t + j_{w\omega}(h)\,\omega_{t-h}\,] \tag{15}$$
$$E_t[\pi^p_{t+h}] \;=\; \pi^p_t \;+\; h\,[\,j_{pw}(h)\,\pi^w_t + j_{pp}(h)\,\pi^p_t + j_{py}(h)\,y_t + j_{p\omega}(h)\,\omega_{t-h}\,] \tag{16}$$

Regarding the output gap, use eq. (10) to obtain the counterpart of the dynamic IS curve (3) in the h-economy, solved for the expectational variable, as $E_t[y_{t+h}] = y_t + (h/\sigma)\,(\rho + \phi_p\,\pi^p_t + \phi_w\,\pi^w_t + \phi_y\,y_t + v_t - E_t[\pi^p_{t+h}] - r^n_t)$. The magnitudes ρ, v_t and r^n_t cancel out if $z_{1,t} = 0$ from Section 2 is taken into account. Substituting (16) and defining

$$\begin{aligned}
j_{yw}(h) &= [\phi_w - h j_{pw}(h)]\,/\,\sigma & j_{yp}(h) &= [\phi_p - 1 - h j_{pp}(h)]\,/\,\sigma\\
j_{yy}(h) &= [\phi_y - h j_{py}(h)]\,/\,\sigma & j_{y\omega}(h) &= -h\,j_{p\omega}(h)\,/\,\sigma
\end{aligned} \tag{17}$$

the output equation can be written as

$$E_t[y_{t+h}] \;=\; y_t \;+\; h\,[\,j_{yw}(h)\,\pi^w_t + j_{yp}(h)\,\pi^p_t + j_{yy}(h)\,y_t + j_{y\omega}(h)\,\omega_{t-h}\,] \tag{18}$$

Lastly, put

$$j_{\omega w} \;=\; 1 \qquad j_{\omega p} \;=\; -1 \qquad j_{\omega y} \;=\; j_{\omega\omega} \;=\; 0 \tag{19}$$

and adjust the identity for the real wage gap (9) to the present notation (apart from the reference of these coefficients to h, which is here obsolete),

$$E_t[\omega_t] \;=\; \omega_{t-h} \;+\; h\,[\,j_{\omega w}\,\pi_t^w + j_{\omega p}\,\pi_t^p + j_{\omega y}\,y_t + j_{\omega\omega}\,\omega_{t-h}\,] \tag{20}$$

We have thus achieved our goal to write the h-economy version of the model compactly as $E_t[x_{t+h}] = [I + hJ(h)]\,x_t$, to which we can then apply the Lemma from the previous Section. It only remains to make explicit that the limit of the matrices $J(h)$ exists. In fact, from (14), (17), (19) we obtain:

$$\begin{aligned}
J^o \;=\; J(0) \;&=\; \begin{bmatrix} \rho & 0 & -a_w\,b_w & b_w \\ 0 & \rho & -a_p\,b_p & -b_p \\ \phi_w/\sigma & (\phi_p-1)/\sigma & \phi_y/\sigma & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \\
b_w \;=\; b_w(0) \;&=\; (1-\theta_w)(1+\rho-\theta_w)\,/\,(1+\phi\,\varepsilon_w) \\
b_p \;=\; b_p(0) \;&=\; (1-\theta_p)(1+\rho-\theta_p)\,(1-\alpha)\,/\,(1-\alpha+\alpha\,\varepsilon_p)
\end{aligned} \tag{21}$$

In contrast to the unwieldy general discrete-time matrices $J(1)$ or $J(h)$ in (8), the limit matrix J^o seems to offer some scope for an analytical treatment of the determinacy problem. This will be the upshot of the paper in the next section.

5. The determinacy proposition

According to the Lemma, for determinacy in the h-economies it has to be shown that the limit matrix J^o in (21) has one real and negative eigen-value and three eigen-values with positive real parts. The following two assumptions will prove sufficient to ensure this.

Assumption 1

$$\phi_w + \phi_p \;>\; 1 \;-\; \frac{\rho\,\phi_y\,(b_w+b_p)}{b_w\,b_p\,(a_w+a_p)}\,,$$

where a_w, a_p, b_w, b_p are defined in (11) and (21), respectively.

Assumption 2

Either $\phi_y = 0$ or $\rho^2 \le b_w + b_p$ (or both).

The first assumption is a relaxed version of the Taylor principle stated in (7). Just as in Galí's (2007, p. 130, Figure 6.2) illustration of the determinacy frontier, a positive policy coefficient on the output gap allows a (slight) weakening of the condition that the sum of the two inflation coefficients exceed unity. The assumption will also turn out to be a necessary condition for determinacy, at least in h-economies with a short period length h (if we disregard equality in Assumption 1, in which case the Lemma fails to apply).[5]

The inequality in Assumption 2 is a convenient condition to determine the sign of a partial derivative in the proof; see eq. (25) below. However, neither the condition for the sign nor the sign itself are necessarily needed in the mathematical argument. Nevertheless, since a typical value of the quarterly discount rate is $\rho = 0.01$ and so ρ^2 is extremely small, this inequality can be safely taken for granted and there is no need to seek for further (more tedious) refinements. In fact, with the numerical parameters that we will employ from the literature below, we get $b_w = 0.009$ and $b_p = 0.029$.

Proposition

1. *If the inequality in Assumption 1 is reversed, then an h-economy as it was developed above, and compactly summarized by eq. (8), exhibits indeterminacy for (at least) all h sufficiently small.*
2. *Let Assumptions 1 and 2 be satisfied. Then the steady state of the h-economy is determinate (at least) if the period length h is sufficiently short.*

The hard work to do is, of course, the proof of the determinacy part of the proposition, that is, a demonstration of the 3:1 structure in the four eigen-values of J^o. To give a short outline of our approach, the proof begins with assigning zero values to two selected parameters. They are easily seen to give rise to a negative and a positive eigen-value, and to two eigen-values on the imaginary axis. In a second step, the Implicit Function Theorem is employed to show that the real parts of the latter two become positive as one of these parameters slightly increases. The third step makes sure that upon further increases of the two parameters toward their originally given values, none of the eigen-values can change the sign of its real part.

Proof: For the analysis of the eigen-value structure of the matrix J^o we need the coefficients in its characteristic equation, $\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$.[6] With the notation $\eta := 1/\sigma$ to avoid fractions, they result as follows (e.g., see Murata, 1977, p. 14):

$$A_1 \;=\; -\text{trace } J^o \;=\; -2\,(\rho \;+\; \eta\,\phi_y)$$

$$A_2 \;=\; \text{sum of the principal second-order minors of } J^o$$

[5] It is quite a common feature in New-Keynesian models that, in the presence of a positive weight on output fluctuations, determinacy of the steady state is also guaranteed if the central bank raises interest rates a bit less than one-to-one in response to an increase in inflation. A detailed discussion of this issue can be found in Woodford (2003, p. 254). Another condition for this to hold may, however, not be neglected, namely, the absence of nominal taxes; cf. Edge and Rudd (2007).

[6] The eigen-values can here be denoted by the usual letter 'λ' since there will be no more risk of confusing them with the parameters λ_w, λ_p in (11) and (13).

$$= \begin{vmatrix} \rho & 0 \\ 0 & \rho \end{vmatrix} + \begin{vmatrix} \rho & -a_w\, b_w \\ \eta\,\phi_w & \eta\,\phi_y \end{vmatrix} + \begin{vmatrix} \rho & b_w \\ 1 & 0 \end{vmatrix} +$$

$$\begin{vmatrix} \rho & -a_p\, b_p \\ \eta\,(\phi_p - 1) & \eta\,\phi_y \end{vmatrix} + \begin{vmatrix} \rho & -b_p \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} \eta\,\phi_y & 0 \\ 0 & 0 \end{vmatrix}$$

$$= \rho\,(2\,\eta\,\phi_y + \rho) \;-\; [1 - \eta\,\phi_w\,a_w]\,b_w \;-\; [1 - \eta\,(\phi_p - 1)\,a_p]\,b_p$$

$$A_3 = -\ (\text{sum of the principal third-order minors of } J^o)$$

$$= -\begin{vmatrix} \rho & 0 & -a_w\,b_w \\ 0 & \rho & -a_p\,b_p \\ \eta\,\phi_w & \eta\,(\phi_p - 1) & \eta\,\phi_y \end{vmatrix} - \begin{vmatrix} \rho & 0 & b_w \\ 0 & \rho & -b_p \\ 1 & -1 & 0 \end{vmatrix}$$

$$- \begin{vmatrix} \rho & -a_w\,b_w & b_w \\ \eta\,\phi_w & \eta\,\phi_y & 0 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} \rho & -a_p\,b_p & -b_p \\ \eta\,(\phi_p - 1) & \eta\,\phi_y & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= \eta\,\phi_y\,(b_w + b_p - \rho^2) \;+\; \rho\,[\,(1 - \eta\,\phi_w\,a_w)\,b_w \;+\; (1 - \eta\,(\phi_p - 1)\,a_p)\,b_p\,]$$

$$A_4 = \det J^o \;=\; -\eta\,[\,\rho\,\phi_y\,(b_w + b_p) \;+\; (\phi_w + \phi_p - 1)\,(a_w + a_p)\,b_w\,b_p\,]$$

The first part of the proposition is easily verified by making use of the relationship $A_4 = \det J^o = \lambda_1\,\lambda_2\,\lambda_3\,\lambda_4$ for the four eigen-values of J^o. Recalling that for determinacy three eigen-values must have positive real parts and one must be negative, it suffices to note that the strict violation of Assumption 1 is equivalent to $A_4 > 0$, and that a positive sign of the determinant implies an even number of eigen-values with positive real parts.

In the proof of the second part of the proposition, the coefficients ρ and ϕ_y are treated as variable. Their given values may therefore be marked as $\rho^\star$ and $\phi_y^\star$. We also distinguish the two cases that the expression $[1 - \eta\,\phi_w\,a_w]\,b_w + [1 - \eta\,(\phi_p - 1)\,a_p]\,b_p$ in A_2 and A_3 is zero or nonzero, respectively. The proof begins with the normal nonzero case.

In a first step, put $\rho = \phi_y = 0$. Then $A_1 = A_3 = 0$, $A_2 \neq 0$ and, with Assumption 1, $A_4 < 0$. The characteristic equation thus reduces to $\lambda^4 + A_2\,\lambda^2 + A_4 = 0$. The quadratic equation that results from replacing λ^2 with μ has two real solutions $\mu_{1,2}$, one of which is positive and the other negative. Hence (from $\mu_1 > 0$) one eigen-value λ is a positive and one a negative real number, and (from $\mu_2 < 0$) the remaining two are a pair of purely complex eigen-values. In the next step we want to show that this pair moves into the right half of the complex plane when now ρ is slightly increased above zero. As the signs of the other two real eigen-values are preserved, this intermediate step will achieve the desired structure of the eigen-values.

For this purpose, write a complex eigen-value of J^o as $\lambda = \alpha + \beta\, i$ and compute $\lambda^2 = (\alpha^2{-}\beta^2) + 2\alpha\beta\, i$, $\lambda^3 = \alpha(\alpha^2{-}3\beta^2) + \beta(3\alpha^2{-}\beta^2)\, i$, $\lambda^4 = (\alpha^2{-}\beta^2)^2 - 4\alpha^2\beta^2 + 4\alpha\beta(\alpha^2{-}\beta^2)\, i$. The characteristic equation can be decomposed into its real and imaginary component as follows,

$$\begin{aligned} (\alpha^2-\beta^2)^2 - 4\alpha^2\beta^2 \,+\, A_1\,\alpha(\alpha^2-3\beta^2) \,+\, A_2\,(\alpha^2-\beta^2) \,+\, A_3\,\alpha \,+\, A_4 \;&=\; 0 \\ 4\alpha\beta(\alpha^2-\beta^2) \,+\, A_1\,\beta(3\alpha^2-\beta^2) \,+\, 2A_2\,\alpha\beta \,+\, A_3\,\beta \;&=\; 0 \end{aligned} \tag{22}$$

In the special situation where $\alpha = 0$, we obtain two relationships that will prove useful further below,

$$\begin{aligned} \beta^2 - A_2 \;&=\; -A_4\,/\,\beta^2 \;>\; 0 \\ -A_1\,\beta^2 + A_3 \;&=\; 0 \end{aligned} \tag{23}$$

Conceiving the composite terms A_j as functions of ρ and (later) ϕ_y, the two equations in (22) may be more compactly written as,

$$\begin{aligned} F_1(\alpha,\beta;\rho,\phi_y) \;&=\; 0 \\ F_2(\alpha,\beta;\rho,\phi_y) \;&=\; 0 \end{aligned} \tag{24}$$

Equation (24) is a typical example for an application of the Implicit Function Theorem in its two-dimensional version: α and β are two endogenous real variables that vary with the exogenous variables ρ and ϕ_y in order to reestablish equality in (24), which may be expressed as $\alpha = \alpha(\rho,\phi_y)$, $\beta = \beta(\rho,\phi_y)$. Furthermore we have a base solution $\alpha(0,0) = 0$, $\beta(0,0) > 0$ for $\rho = \phi_y = 0$. The theorem, then, gives us a formula to compute the partial derivative of the real part $\alpha = \alpha(\rho,\phi_y)$ at this point with respect to ρ, which should turn out to be positive.

Entering the formula will be all of the partial derivatives of the two function F_1 and F_2. Denote them as $F_{j\gamma} = \partial F_j/\partial\gamma$ for $j{=}1,2$, $\gamma{=}\alpha,\beta,\rho,\phi$, where in order to avoid stacked indices let here ϕ stand for ϕ_y. Likewise write $A_{j\rho} = \partial A_j/\partial\rho$ and $A_{j\phi} = \partial A_j/\partial\phi_y$ for $j = 1,2,3,4$. Generally at a point at which $\alpha = 0$, we can, in particular, use (23) to compute the derivatives and their signs:

$$\begin{aligned} F_{1\alpha} \;&=\; 2\beta^2\,(2\rho+\eta\phi_y) \;\geq\; 0 & F_{2\alpha} \;&=\; -2\beta\,(2\beta^2-A_2) \;<\; 0 \\ F_{1\beta} \;&=\; 2\beta\,(2\beta^2-A_2) \;>\; 0 & F_{2\beta} \;&=\; 2\beta^2\,(2\rho+\eta\phi_y) \;\geq\; 0 \\ F_{1\rho} \;&=\; -2\beta^2\,(\rho+\eta\phi_y) \;\leq\; 0 & F_{2\rho} \;&=\; \beta\,(\rho^2+2\beta^2-A_2) \;>\; 0 \\ F_{1\phi} \;&=\; -\eta\rho\,(2\beta^2+b_w+b_p) \;<\; 0 & F_{2\phi} \;&=\; \eta\beta\,(2\beta^2+b_w+b_p-\rho^2) \;>\; 0 \end{aligned} \tag{25}$$

In the computation of $F_{2\rho}$ it has also been exploited that $A_{3\rho} = -2\,\eta\,\phi_y\,\rho \,+\, [1 - \eta\,\phi_w\,a_w]\,b_w \,+\, [1-\eta\,(\phi_p{-}1)\,a_p]\,b_p$ equals $\rho^2 - A_2$. The positive sign of $F_{2\phi}$ is ensured by Assumption 2.

After these preparations we can take the real part of an eigen-value $\lambda = \lambda(\rho, \phi_y) = \alpha(\rho, \phi_y) + \beta(\rho, \phi_y)\, i$ and differentiate it with respect to ρ. The formula from the Implicit Function Theorem reads,

$$\frac{\partial \alpha(\rho, \phi_y)}{\partial \rho} \quad = \quad \frac{-F_{2\beta}\, F_{1\rho} \;+\; F_{1\beta}\, F_{2\rho}}{F_{1\alpha}\, F_{2\beta} - F_{1\beta}\, F_{2\alpha}} \tag{26}$$

Equation (25) ascertains that both the numerator and denominator are unambiguously positive. This holds for all nonnegative values of ρ and ϕ_y and thus, in particular, at the point $\rho = 0$, $\phi_y = 0$. The positive derivative in (26) proves the claim that at $\phi_y = 0$ and for ρ sufficiently small (but positive), one eigen-value of J^o is negative and the other three have positive real parts.

It next has to be shown that at a further rise of ρ up to the given value $\rho^\star$, none of the eigen-values can hit the imaginary axis or even move from one half-plane into the other. Suppose this happens at some value $\widetilde{\rho} > 0$. Owing to $A_4 = \det J^o \neq 0$ there must be again a pair of purely imaginary eigen-values, for which $\alpha(\widetilde{\rho}, 0) = 0$. Furthermore, since there is only a single eigen-value with a negative real part, the partial derivative $\partial \alpha(\widetilde{\rho}, 0)/\partial \rho$ must be negative or zero. This, however, contradicts the fact that (25) has just been found to be strictly positive at all ρ and ϕ that would entail $\alpha(\rho, 0) = 0$. Hence the desired 3:1 eigen-value structure also prevails at $\rho = \rho^\star$ and $\phi_y = 0$.

For the case of a positive coefficient on the output gap it remains to verify that the eigen-value structure is preserved if now ϕ_y rises from zero to the given value $\phi_y^\star$. The argument is completely analogous to the previous paragraph. In computing the partial derivative $\partial \alpha(\rho^\star, \phi_y)/\partial \phi_y$ we only have to replace $F_{1\rho}$ and $F_{2\rho}$ in (26) with $F_{1\phi}$ and $F_{2\phi}$, and observe with (25) that this does not change the sign of the numerator. Hence $\partial \alpha(\rho^\star, \phi_y)/\partial \phi_y > 0$ for all values of ϕ_y, which implies that the variations of ϕ_y cannot change the signs of the real parts of the four eigen-values, either.

Finally, consider the special case $C := [1 - \eta\, \phi_w\, a_w]\, b_w \,+\, [1 - \eta\, (\phi_p - 1)\, a_p]\, b_p = 0$. Here the above method of proof fails to apply since $A_2 = 0$ at the very beginning. Instead, we now treat η as a variable coefficient and mark its given value as $\eta^\star$. Since $C = C(\eta) \neq 0$ for $\eta \neq \eta^\star$, we know that for all these η the real parts of the corresponding eigen-values have the desired 3:1 structure. The rest of the proof makes sure that at $\eta = \eta^\star$ this property does not possibly get lost.

Suppose to the contrary that some eigen-value changes the sign of its real part at $\eta^\star$. Then by virtue of $\det J^o \neq 0$ there must be a pair $\lambda_{1,2} = \pm\, i\beta$ of purely complex eigen-values at this value. Since for $\eta \neq \eta^\star$ the other two eigen-values are real and of opposite sign, $\lambda_3 < 0 < \lambda_4$ (say) also holds true at $\eta = \eta^\star$. To check the consistency of this situation, we refer to the following two identities between the coefficients in the characteristic polynomial and the four eigen-values of J^o,[7]

[7] To show them, use the fact that the characteristic polynomial equals $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$ and expand out.

$$
\begin{aligned}
A_1 &= -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \\
A_3 &= -\lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 \lambda_4 - \lambda_1 \lambda_3 \lambda_4 - \lambda_2 \lambda_3 \lambda_4
\end{aligned}
$$

Since $A_1 < 0$ and $\lambda_1 + \lambda_2 = 0$, the first equation implies $\lambda_3 + \lambda_4 > 0$. The second equation yields $A_3 = -\lambda_1 \lambda_2 (\lambda_3 + \lambda_4) - (\lambda_1 + \lambda_2)\lambda_3 \lambda_4 = i^2 \beta^2 (\lambda_3 + \lambda_4)$, which says that A_3 is negative. With $C(\eta^\star) = 0$, on the other hand, A_3 is here given by $A_3 = \eta^\star \phi_y (b_w + b_p - \rho^2)$. Since according to Assumption 2 this expression is nonnegative, we have a contradiction, that is, $\lambda_{1,2} = \pm i\beta$ at $\eta = \eta^\star$ is impossible.

q.e.d

6. Determinacy under a variable period length

To study determinacy under variations of the period length h, let $\phi^\star_{wp}(h)$ denote the critical value of the sum of the wage and price policy coefficients $\phi_w + \phi_p$ at which, given h, the steady state of (8) becomes determinate as $\phi_w + \phi_p$ increases from zero. [8] Then, taking Assumption 2 for granted, the Proposition and the Lemma tell us that $\phi^\star_{wp}(h)$ converges toward the right-hand side of Assumption 1 as h tends to zero, which may be written as $\phi^\star_{wp}(0)$. Attempts to check this numerically will, however, face an intrinsic problem. While usually a procedure computing eigen-values with a precision of, say, five significant digits will be considered fully satisfactory, this error is no longer negligible if, at a given small value of h, one of the eigen-values of the matrix $[I + hJ(h)]$ in eq. (8) is, for example, computed as 0.99995 and thus said to be stable, although it is actually unstable with a true value of 1.00001. As a consequence, the numerical computations yield somewhat distorted values for the determinacy threshold $\phi^\star_{wp}(h)$.

In fact, in a battery of numerical explorations in which we let h tend to zero, $\phi^\star_{wp}(h)$ was typically found to converge to a value distinctly larger than $\phi^\star_{wp}(0)$. Nevertheless, in all of these cases the limit was still consistently below unity, even for very small values of the policy coefficient ϕ_y (recall that $\phi^\star_{wp}(0)$ tends to unity from below as ϕ_y approaches zero). Hence for small values of h, the pure Taylor principle $\phi_w + \phi_p > 1$ was always sufficient to ensure numerical determinacy.

ρ	α	ϕ	σ	ε_p	ε_w	θ_p	θ_w	ϕ_y
0.01	1/3	1	1	6	6	2/3	3/4	0.5/4 or 2/4

Table 1: Galí's numerical parameter scenario.

On the other hand, the rigorous mathematical formulation of the determinacy part of the Proposition is limited to sufficiently short period lengths h and so must leave it open

[8] Where it is understood that ϕ_w and ϕ_p vary in fixed proportions. Effectively, in all our numerical experiments $\phi^\star_{wp}(h)$ proved to be independent of this proportion.

whether for longer periods, up to $h = 1$, determinacy would require stronger or weaker conditions. This problem has to be investigated by numerical methods anyway. To this end we take up the numerical example discussed in Galí (2007, pp. 52, 129). The values of the structural parameters that will remain constant, or for which we only consider two alternative values as in the case of ϕ_y, are given in Table 1.[9] Note that the first value of ϕ_y is Taylor's original value for the policy coefficient on the output gap, which is here divided by 4 since Galí uses a quarter as his underlying time unit.

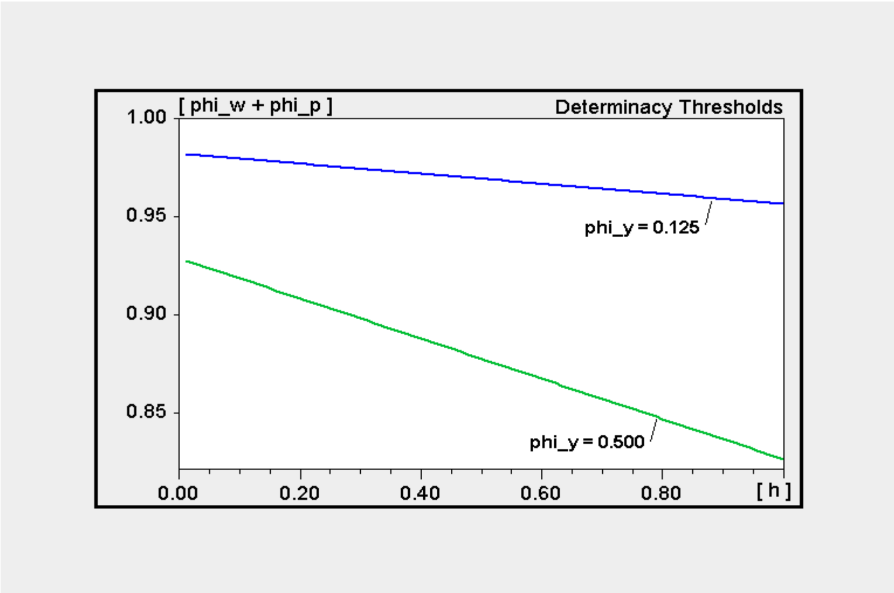

Figure 1: Determinacy thresholds $\phi^\star_{wp}(h)$ under variations of the period length h.

Given the period length h and one of the values for ϕ_y together with the other parameters in Table 1, we can compute the determinacy threshold $\phi^\star_{wp}(h)$ by way of a suitable iteration mechanism (basically a regula falsi procedure). Drawing the threshold as a function of h over the interval $[0.01, 1.00]$, the two graphs in Figure 1 are obtained; one for $\phi_y = 0.5/4 = 0.125$ and the other for $\phi_y = 2/4 = 0.500$. The diagram illustrates that the condition for determinacy is steadily relaxed as the period length increases up to a quarter, $h = 1$. It may be added that in both of the cases here depicted, the computed values of $\phi^\star_{wp}(h)$ are still persistently above the theoretical threshold $\phi^\star_{wp}(0)$, which is 0.9406 for $\phi_y = 0.125$ and 0.7623 for $\phi_y = 0.500$.

[9] Since we could not find an explicit value for ε_w, we assigned the value of $\varepsilon = \varepsilon_p$ from p. 52 to it.

7. Conclusion

The paper took up a version of the standard New-Keynesian period model from the literature that integrates goods and labour markets by designing imperfect competition and staggered price and wage setting in an analogous manner. The model can be reduced to a dynamic system in four variables, one of which is predetermined and the other three are so-called jump variables. Numerical evidence suggests that a suitably extended Taylor principle for the monetary policy rule will be sufficient to ensure determinacy of the steady state, but an analytical treatment was missing so far. We approached the determinacy problem by revitalizing a more than 30 year old methodological precept by Duncan Foley. It says that a macroeconomic model should routinely specify its period in an explicit way such that it can be of any arbitrary length h, and it should then be checked that the model remains well-defined in the limit as the period shrinks to zero. A side-effect of this procedure is that a matrix characterizing the continuous-time system will usually be much easier to analyze than the matrix from the original period model. Obviously, the significance of Foley's axiom goes well beyond the scope of the specific New-Keynesian model studied here.

In the present case of a period model it appears an extremely difficult task to locate its four eigen-values inside and outside the unit circle. In contrast, for the limiting matrix as h tends to zero it indeed turned out to be feasible to verify that, as required for determinacy, one of the eigen-values is negative and the other three have positive real parts, and that the abovementioned Taylor principle plays a key role for this. Also our method of proof can be of more general interest. The proof begins with a special set of the parameters that gives rise to one positive and one negative eigen-value, while the other two eigen-values are on the imaginary axis. Using the Implicit Function Theorem it is then shown that the latter two are moving into the positive half of the complex plane as one of the modified parameters slightly increases. A final step makes sure that upon further variations of the modified parameters toward their original values, none of the eigen-values can hit the imaginary axis again. We may thus hope that similar methods and ideas will prove fruitful for other dynamic systems of dimension three, four or perhaps even five if it comes to a mathematical analysis of their stability or determinacy.

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