Improving the Effort Concept: A Revision of the Traditional Approach in the Context of Controlled Dynamic Stochastic Environments

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Abstract

The objective of this paper is to re-evaluate the attitude to effort of a risk-averse decision-maker in an evolving environment. In the classical analysis, the space of efforts is generally discretized. More realistic, this new approach employes a continuum of effort levels. The presence of multiple possible efforts and performance levels provide a better basis for explaining real economic phenomena. In the context of a principal-agent relationship, not only the incentives of the Principal can determine the private agent to exert a good effort, but also the evolution of the dynamic system. The dynamic incentives can be ineffective when the environment does not sufficiently incite the agent to allocate effort. This possible scenario explains why some efficient strategic incentive-compatible constraints that cover the entire period of contract do not generally exist. The proposed approach offers an elegant study of the close relationship between behavior, attitude and effort allocation.

Keywords: Rational decision-maker, endogenous dynamic learning, adaptive effort management, optimal effort threshold, effort aversion, excessive effort behavior.

JEL Classifications: C91, D78, D82, D83.

1. Introduction

Consider a rational decision-maker characterized by a consistent and efficient outcome oriented behavior (DREZE 1990; WALSH 1996) who utilizes a set of control instruments in order to constrain the system to follow an optimal trajectory ensuring its equilibrium and stability.

The decision-maker adjusts to keep small the difference between the actual and assumed system characteristics by supervising and managing the system behavior. At each control period, his uncertainty level is determined by the deviations of the system from a fixed reference path. High (small) deviations from the fixed targets correspond to a high (small) level of uncertainty.

In general, a consistent behavior in problems of decision-making implies risk-aversion characterized by risky actions implemented by the decision-maker. It characterizes the majority of decision-makers, at least for high profits or important losses.

The degree of information included in the observation of the state variable generally depends on the values chosen for the control instruments, so that the extent of the learning on the

latent parameters of the system can be directly influenced by the decision-maker. He has a certain influence over the rate at which the information arrives, so that his behavior may generate information. The active learning makes the decision-maker more experienced over time. However, despite of potential benefits from active learning in stochastic optimization problems, the potential for learning is very limited if the model is noisy (EASLEY AND KIEFER 1988; KIEFER AND NYARKO 1989).

The analysis is placed in the context of a closed-loop strategy, the information being utilized by the decision-maker in real time. His objective is to reduce the uncertainty related to the choice of his actions by acquiring information from the beginning of the control to the moment of decision. The feedback control responds not only to the effects of random inputs, but also to the measurement errors as well. It is thus useless to identify and measure the sources of disturbance.

The fact that the decision-maker is risk-averse by nature does not give any obvious reason for which one should also suppose that he is effort-averse. Uncertainty generally depreciates the activity of the decision-maker and produces a temporary stability followed by a longer or shorter period of adaptation in instability which implies an additional effort allocated by the decision-maker in order to reduce the probability of a high deviation to occur.

The evolution of the dynamic system has an important impact on the decision-maker's effort behavior over time. The higher the system fluctuations, the higher the effort exerted. The effort level invested by the decision-maker at each period of control will also depend on the importance attached to the history of the process, as well as on his effort-averse type.

The objective of the present study is to explore the relationship between attitude, behavior, and perceived effort in dynamic risky environments.

The paper is organized as follows. Section 2 presents the model. Section 3 proposes a refinement of the effort concept by integrating in the analysis a truncated /progressive history of the dynamic environment. Section 4 obtains several qualitative results with regard to the decision-maker's attitude to effort during the period of control. It also provides an interesting analysis of the decision-maker's effort preferences and discusses the case of high potential shifts. Section 5 introduces the concept of (excessive) effort-aversion according to an optimal effort-threshold which characterizes the agent's type. It is analyzed here the relationship between agent's effort allocation and horizon length. Section 6 concludes and makes suggestions for future research.

2. The Model

The type of model we analyze corresponds to a data generation process which is dynamic, nonlinear and managed by a system of discrete simultaneous equations.

Let $x_t \in \mathbf{R}^q$ be the control variable at time t, let $y_t \in \mathbf{R}^p$ be the target variable in t, and let $z_t \in \mathbf{R}^r$ be an exogenous variable not subjected to the agent's control at the time period t.

Denote by $\mathbf{X}_t = \{..., x_{-1}, x_0, x_1, ..., x_t\}$ the history of the process x up to time t, and similarly for $\mathbf{Y}_t$ and $\mathbf{Z}_t$. We allow for the current state variable to depend not only on the agent's current decision but also on an arbitrarily complex history $\mathbf{X}_t$. We make the following basic assumptions:

Assumption 1. The evolution of the environment is modelled by a nonlinear extended-memory process generated according to the structural state equation:

$$y_t = F(\mathbf{Y}_{t-1}, \mathbf{X}_t, \mathbf{Z}_t, e_t, \boldsymbol{\beta}_t, t) + u_t, \qquad t \in \mathbf{Z}$$

The adjustment function F is assumed twice continuously differentiable with respect to $\boldsymbol{\beta}_t \in \mathbf{R}^k$ (the time-varying parameter to be estimated). The variable e_t represents the agent's

effort employed for the period t. The exogenous unobserved random shock $u_t \in \mathbf{R}^p$ is the specific "risk" modelled by a normal distribution with zero mean-vector and finite variance-covariance matrix $\boldsymbol{\Psi}$.

In general, y_t is not a Gaussian process. Obviously, non-linearity between y_t and $\mathbf{Y}_{t-1}$ implies non-normality and hence an asymmetric dynamic. The variable t plays the role of a synthesis variable in the econometric model.

Assumption 2. The agent's objective is to constrain the system to follow a feasible optimal path $\eta \stackrel{not.}{=} \{y_1^g, y_2^g, ..., y_T^g\}$ by selecting the control variable x_t in a suitable way. Taking into account foreseeable movements in y as well as possible economic constraints, the agent will fix some optimal bounds l_t such that $0 < y_t^g \leq l_t < 1$, $t = 1, ..., T$.

Assumption 3. The timing of the control is as follows: At each period t, the agent implements an optimal action x_t, which is a stimulus for the dynamic environment. This is purported to contribute to the stability of the evolving system. A shock u_t is carried out and the agent observes the output y_t (the impulse response) from which he extracts a dynamic signal about the future trend of the system. The information revealed by the output signal can increase the precision of the next control instrument. This signal and the corresponding action provide information on the data generating process. The agent will employ this output signal for a strategic learning (specific to a closed-loop monitoring) in order to reduce his uncertainty over time. The question is: How this signal will influence the agent's effort behavior during the period of control?

Assumption 4. The optimality of the instrument x_t is considered according to a global criterion $W(y_1, y_2, ..., y_T)$ which measures the system deviations $\triangle y_t \stackrel{not.}{=} y_t - y_t^g$, $t = 1, ..., T$. This is supposed twice continuously differentiable, strictly increasing and convex in the feasible area of the model.

Using the traditional approach (VAN DER PLOEG 1984), we consider a quadratic additive recursive criterion:

$$W_{[1,T]}(y_1, ..., y_T) \stackrel{def.}{=} \sum_{t=1}^{T} W_t(y_t)$$

where W_t is a quadratic asymmetric loss function given by:

$$W_t(y_t) \stackrel{def.}{=} (y_t - y_t^g)' K_t(y_t - y_t^g) + 2(y_t - y_t^g)' d_t$$

with a prime denoting transpose.

The choice of the parameters K_t and d_t reflects the priorities of the agent and also depends on the available amount of information concerning the future development of the system parameters. At each time t, the parameters K_t and d_t are updated and new optimal values are selected to fulfill the requirements of the agent.

Assumption 5. At each period t, the agent must compute his optimal policy $\widehat{x}_t$ before knowing the initial state y_0. He therefore obtains a random policy, conditional to y_0:

$$\widehat{x}_t = \underset{x_t}{\arg\max} E_{t-1}[U_t(W_{[1,t]}, \varphi_t) \mid y_0]$$

where $E_{t-1}(\cdot) \stackrel{not.}{=} E(\cdot \mid I_{t-1})$ is the operator of conditional expectation based on the information available in $t-1$, φ_t represents the absolute risk-aversion index in t, and U_t is the agent's local utility function defined by:

$$U_t(W_{[1,t]}, \varphi_t) \stackrel{def.}{=} \frac{2}{\varphi_t}[\exp(-\frac{\varphi_t}{2} W_{[1,t]}) - 1]$$

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with

$$W_{[1,t]} \stackrel{def.}{=} \sum_{s=1}^{t} W_s(y_s) \text{ (evolutive loss)}$$

We have:

$$-\frac{U_t''(W_{[1,t]}, \varphi_t)}{U_t'(W_{[1,t]}, \varphi_t)} = \frac{\varphi_t}{2}$$

where a prime denotes the partial derivative with respect to $W_{[1,t]}$.

Therefore, $\frac{\varphi_t(W_{[1,t]})}{2}$ measures locally (at the point $W_{[1,t]}$) the agent's risk aversion, U_t being a CARA utility. Different contexts call for different optimal actions.

3. Dynamic Effort Modelling

Although there is an extensive literature on effort (BLOCK AND HEINEKE 1973; SAP-PINGTON 1991; LAFFONT AND TIROLE 1993; SALANIE 1997; LAFFONT AND MARTIMORT 2002; LAFFONT 2003; IPPOLITO 2003; OYER 2004; BOLTON AND DEWATRIPONT 2005; EPSTEIN AND NITZAN 2006; SHAPIRO 2006; LEE AND RUPP 2007; FONG AND TOSI JR 2007; STROBL AND WALSH 2007, AMONGST OTHERS), there is no theoretical and empirical work for comparing and evaluating the degree of effort-aversion of risk-averse decision-makers in the context of controlled dynamic stochastic environments.

The objective of this section is to propose a new definition of the effort concept with important implications on the agent's adaptive behavior in an evolving environment.

In dynamic stochastic optimization problems, but not only, the agent's effort level depends on the history of the process, as well as on the importance the agent places on the past. It implies a permanent adjustment process of the effort variable over time.

We formalize this point of view for a general class of models and we detail the positive effects that it implies in the context of a dynamic stochastic system which evolves over a finite and discrete horizon.

We make the following useful notations:

$$S_{t,\,p\_d} \stackrel{not.}{=} \parallel y_{t-1} - y_{t-1}^g \parallel^2 + ... + \parallel y_{t-k_1} - y_{t-k_1}^g \parallel^2$$
$$\text{(the sum of squared past deviations at time } t)$$

$$S_{t,\,w\_p\_d} \stackrel{not.}{=} \parallel y_{t-1} - y_{t-1}^g \parallel^2 L_{t-1} + ... + \parallel y_{t-k_1} - y_{t-k_1}^g \parallel^2 L_{t-k_1}$$
$$\text{(the weighted sum of squared past deviations at time } t)$$

where L_{t-j_1} $(j_1 = 1, ..., k_1; 1 \le k_1 < T)$ are strategic weights attached to the system deviations with respect to the equilibrium path $\{y_{t-1}^g, ..., y_{t-k_1}^g\}$ verifying the sequence of inequalities:

$$-1 < L_{t-1} \le ... \le L_{t-k_1} \le 0$$

Definition 1. Using t to denote time, the absolute risk-aversion index $\varphi_{t,\,p}^{r\_a}$ evolves according to the following relationship:

$$\varphi_{t,\,p}^{r\_a} \stackrel{def.}{=} \frac{S_{t,\,w\_p\_d}}{\sqrt{S_{t,\,p\_d}^2 + l}}, \ t = 1, ..., T$$

where $l \ge 1$ is an integer characterizing the agent's type.

The weights may differ across individuals. They are updated each time as new observation becomes available. The agent gives a higher importance to the deviations which are closer to the moment of implementation of a new optimal action. The higher (smaller) the weight,

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the smaller (higher) the importance given by the agent to the system deviation from his local objective. Given the potential destabilizing role of a long memory of the process, the agent takes into account only a limited history in the risk analysis. Distant past observations might increase significantly the bias of the estimators in the econometric model. It generally exists an arbitrary element as regards the choice of the backward lag k_1. The objective is to find the better compromise between fit and complexity. For further details, see PROTOPOPESCU (2007).

In order to optimize the effort during the period of control, the agent will take into account two distinct aspects: i) the effect of the last deviation of the system (at time $t - 1$); and ii) the mixed effect of the other system deviations, from $t - 2$ to $t - k_1 - 1$. The agent's degree of risk-aversion at time $t - 1$ integrates this latter effect. It is assumed that the effort variable is multiplicative separable in these two distinct effects.

We are now in a position to give a definition of the effort variable by taking into account a truncated history of the system performances as well as the agent's degree of risk-aversion.

Definition 2: Using t to denote time, the effort variable $e_{t,\,p}^{r\,-a}$ evolves according to the following relationship:

$$e_{t,\,p}^{r\,-a} \stackrel{def.}{=} -\frac{\parallel y_{t-1} - y_{t-1}^{g} \parallel^2 L_{t-1}}{\parallel y_{t-1} - y_{t-1}^{g} \parallel^2 + s} \cdot \frac{\varphi_{t-1,\,p}^{r\,-a} - 1}{\varphi_{t-1,\,p}^{r\,-a} - d}, \quad t = 1, ..., T$$

$$0 < s \leq 1; \ 1 < d \leq 2 \text{ (fixed real numbers)}$$

It is important to note that s and d are two strategic parameters characterizing the agent's type.

The above definition expresses the idea that the risk-aversion can be viewed as effort incentive. Moreover, the planning effort can be regarded as an effective risk management tool. The effort invested depends on the way the agent exploits and interprets the system evolution. It does not always take a minimal value as the intuition would suggest.

For this type of modelling, the effort is no more seen as a pure disutility, like in the traditional approach, but rather as an efficient instrument to optimally manage the system trajectory. There may be periods when the interest of the agent is to increase the effort in order to perform his objectives.

The effort level at a given period t depends on all previous efforts invested by the agent. It allows for an addaptive effort management during the period of control. There exist a close relationship between the agent's effort allocation and his strategic objectives. The smaller the fixed targets, the higher the effort level invested. Increases in the effort cost (measured in terms of disutility) result in reduced effort levels. This is a consequence of the importance the agent places on the system deviations.

Since a real time control process is necessarily discrete, this cannot converge with precision to any target value, but only to a neighborhood of it. When the process of control is finished, the agent will obtain a stochastic neighbouring-optimal trajectory which is expected to be close to the optimal path η. The effort level will be hence strictly positive. In other words, the effort invested by the agent is always necessary (but not always sufficient) in order to reach a fixed objective. For small symmetric deviations with respect to the fixed targets, the agent will adopt the same attitude to effort at time t. By contrast, the restriction imposed on the fixed targets does not allow for a symmetric evolution of the system for large variations.

Remark 1. There is no loss of generality in considering that the effort variable e_t takes values in $[0, 1)$ because one can always find an isomorphism from $[0, 1)$ to $[0, a)$, with $a \geq 1$ a fixed real number.

In the traditional approach (SHAPIRO AND STIGLITZ 1984; CANTOR 1987; BOADWAY ET AL. 2003, AMONG OTHERS), it is often supposed that workers either provide no effort or a unit level of effort over time. In the first case, the effort is regarded to be low, while in the second case, it is considered to be large. This type of modelling is "myopic" with regard to potential changes in the agent's attitude to effort due to inherent endogenous fluctuations of the system over time. These two restrictive cases offer a poor characterization of the agent's effort behavior in an evolving environment.

Remark 2. Large system deviations with respect to the optimal path η leads to the following condition:

$$\| \, y_{t-j_1} - y^g_{t-j_1} \, \| \; \gg 1, \; j_1 = 1, ..., k_1$$

4. Qualitative Results on Effort Allocation

The traditional approach does not take into account the potential effect of the system dynamics on the agent's effort behavior over time. The objective of this section is to develop realistic scenarios for the environment in order to reveal the agent's adaptive effort behavior during the period of control.

To illustrate why the proposed definition is informative about how attitudes to effort of the agent change over time, we prove several theoretical results in this direction.

Proposition 1. The effort invested by a risk-averse agent is non-monotonous over time.

Proof. In the appendix. ∎

This type of behavior is natural in the real world. However, in particular cases, it is possible to obtain a monotonous configuration of the effort during the period of control.

We develop this idea later when analyzing the impact of a progressive transition of the system on the agent's effort behavior over time.

We give below a graphical illustration of the **Proposition 1**.
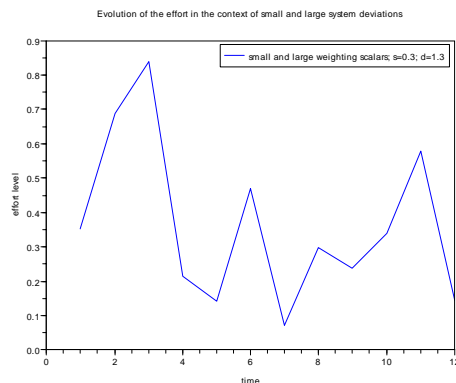

Figure 1

Different contexts do not necessarily call for different effort levels. It is possible for the agent to assign the same effort level for distinct periods of time. This may be the case of a smooth (almost constant) evolution of the system. By smooth evolution we understand either small or large comparable deviations. A slow inertia of the system contributes to the realization of this type of scenario.

We illustrate below this type of behavior in three different contexts:
i) large system deviations and large weighting scalars;
ii) small system deviations and small weighting scalars;
iii) mixed (small and large) deviations and mixed (small and large) weighting scalars.

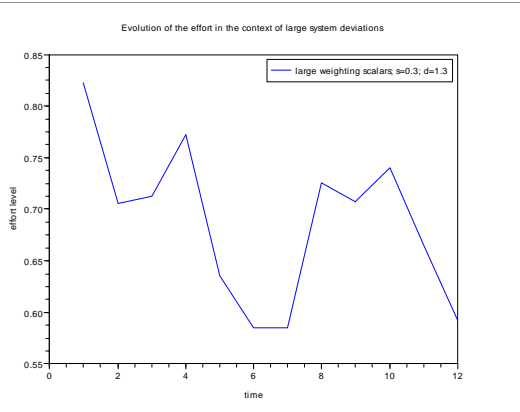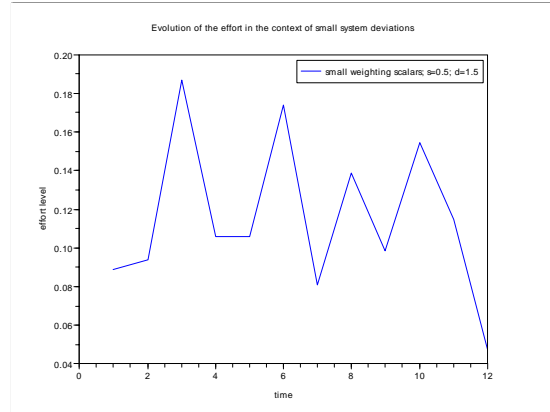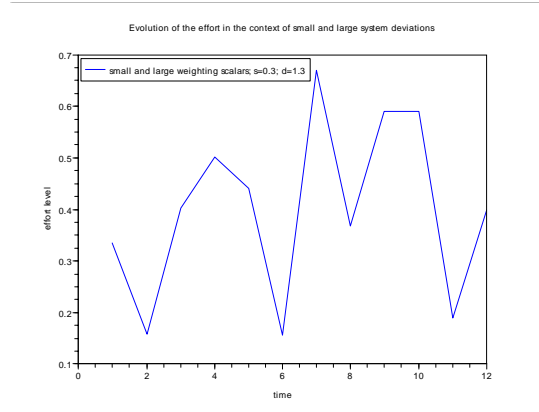Figure 2-i)

Figure 3-ii)

Figure 4-iii)

4.1. Analysis of Agent's Preferences

In a dynamic stochastic environment, the agent's preferences evolve according to the system fluctuations. These are represented at each time period by an utility function which is supposed to be multiplicative separable in the effort level and the evolutive loss:

$$U_t(W_{[1,t]}, \varphi_{t,\,p}^{r,-a}, e_{t,\,p}^{r,-a}) \stackrel{def.}{=} \frac{2D(e_{t,\,p}^{r,-a})}{\varphi_{t,\,p}^{r,-a}}[\exp(-\frac{\varphi_{t,\,p}^{r,-a}}{2}W_{[1,t]}) - 1]$$

$$W_{[1,t]}(y_1, ..., y_t) \stackrel{def.}{=} \sum_{s=1}^{t} W_s(y_s)$$

with D a twice continuously differentiable function such that:

$$D(e_{t,\,p}^{r,-a}) > 0, \quad D'(e_{t,\,p}^{r,-a}) > 0, \quad D''(e_{t,\,p}^{r,-a}) > 0, \ \forall \ e_{t,\,p}^{r,-a} \in [0,1), \forall \ t = 1, ..., T$$

The agent's preferences are refined on the basis of a non-decreasing endogenous information set. The agent can draw benefit from the learning of his preferences. The intuition for this point of view comes from the dynamic dimension of the problem.

Let $W_{[1,t]}$ and $\varphi_{t,\,p}^{r,-a}$ be arbitrarly fixed. It is easy to see that $U_t(e_{t,\,p}^{r,-a})$ is a decreasing concave function in $e_{t,\,p}^{r,-a}$. This is in agreement with economic theory and empirical evidence. For the generality sake, let us suppose for D the following quadratic parametrization:

$$D(e_{t,\,p}^{r,-a}) = c \cdot (e_{t,\,p}^{r,-a})^2, \ \forall \ e_{t,\,p}^{r,-a} \in [0,1), \ \ t = 1, ..., T$$

where $c \in (0,1)$ is a strategic parameter chosen according to the agent's effort-averse type.

We develop this idea later when introducing the concept of more /less effort-averse agent.

Note that a smaller value of the parameter c ensures to the agent a higher utility level during the entire period of control. For an illustration, we give below two superposed graphics for different values of c.
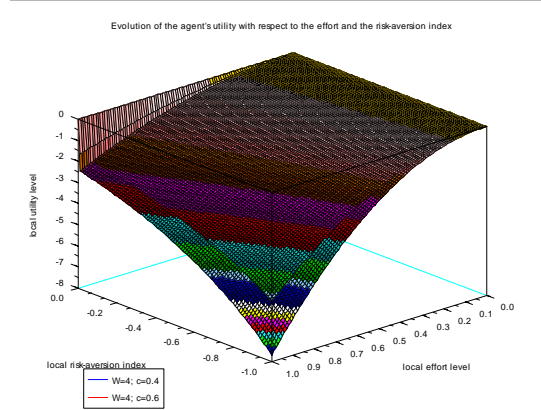

Figure 5

Proposition 2. If $\varphi_{t,\,p}^{r\,-a}$ is large (small) and $W_{[1,t]}$ and $e_{t,\,p}^{r\,-a}$ are small (large), then $U_t(W_{[1,t]},$ $\varphi_{t,\,p}^{r\,-a}, e_{t,\,p}^{r\,-a})$ is large (small).

Proof. In the appendix. ■

It is far from probable that the agent exactly maximizes his utility function at each stage of the control. We rather face a nearly optimization behavior, where the control variable is continuously and optimally adjusted over time to maximize some objective function (VAN DE STADT ET AL. 1985; VARIAN 1990).

We point out that a great class of decision rules are representable by a variation of the utility between two consecutive periods (GILBOA 1989). This comes to treat the same agent at different times as different individuals (ALLAIS 1947).

This situation is formally equivalent to that one specific to a game in which decisions are made by a sequence of heterogenous planners (PHELPS AND POLLAK 1968).

The stochastic disturbance in the system will produce random shocks in the agent's preferences over time. Uncertainty can change the agent's behavior.

We give below two suggestive graphics illustrating the evolution of the agent's utility function U_t with respect to $\varphi_{t,\,p}^{r\,-a}$, $W_{[1,t]}$, and $e_{t,\,p}^{r\,-a}$.
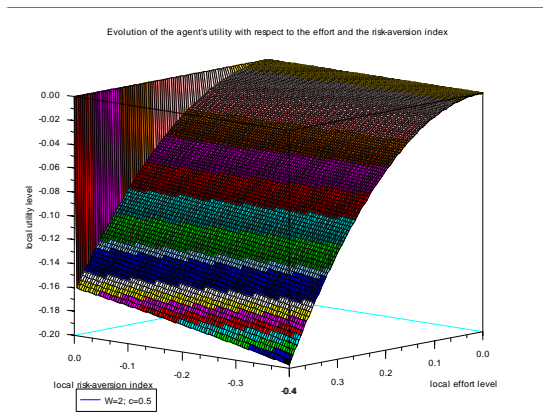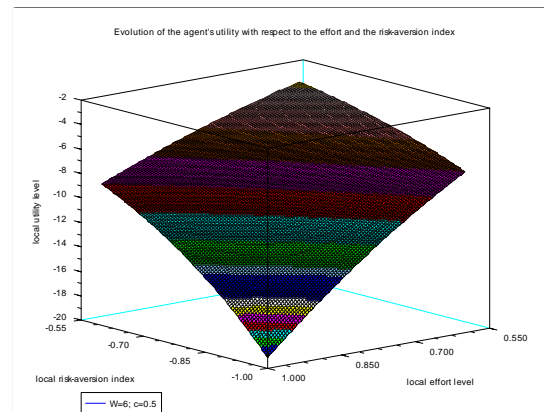

Figure 6

Figure 7

It is possible for the agent to have the same utility level for distinct periods of time. Different contexts do not necessarily call for different preferences. We illustrate this possibility by a suggestive graphic.
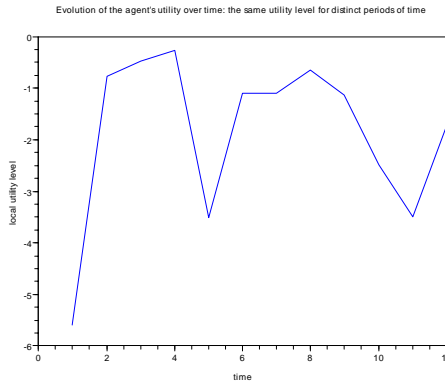
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Figure 8

Remark 3. The agent will allocate a negligible (almost null) effort level at time t when the system deviation at time $t-1$ is negligible (almost null).

The dynamic environment does not always incite the economic agent to invest a good effort level. It may be the case when all system deviations are small or the last deviation of the system is negligible (almost null). We illustrate this particular behavior in **Figure 9** below.
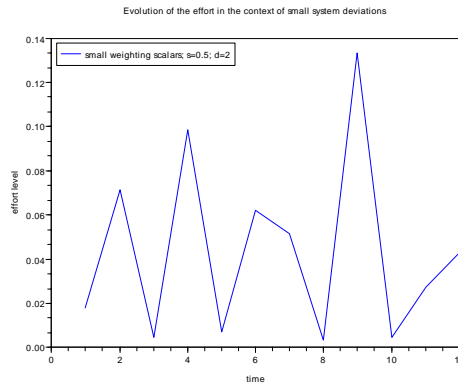

Figure 9

We observe here that the agent's effort invested at time $t = 3$, $t = 5$, $t = 8$, and $t = 10$ is almost null. This is a consequence of the importance the agent places on the system deviations at the previous periods. The other values of the effort can be considered as negligible.

This type of result has important implications for the design of incentive mechanism in the context of a dynamic principal-agent relationship. In general, the compensation paid to the agent is based on his performances because his inputs are not verifiable in court. Note here that different levels of performance can occur when agents exert similar efforts.

Given that the effort is not fully observable, it cannot be made the subject of any explicit contract. The objective of the Principal is to influence the choice of the private agent's action by conditioning his utility on the outcome, and hence offering to him a recompense which depends on its level.

For this type of contract, the output is considered to be of strategic importance, and hence it is not allowed to exceed a specific magnitude. The recompense is supposed to increase with the success of the agent in obtaining a high performance. The Principal's decision cost increases with the incentive levels.

For the above scenario, regardless of the system deviation in $t = 2$, $t = 4$, $t = 7$ and $t = 9$, the effort invested by the agent for the next periods is almost null. Thus, in the case where all system deviations would be small, the recompenses granted to the agent would be uncorrelated with the effort allocated. Equilibrium outcome will be realized with a very low effort level.

Contrary to what is generally believed or intuition would suggest, this type of scenario proves that the effort level is not always high in equilibrium. This explains why, in the context of a dynamic principal-agent relationship, it may be possible for the incentives to be ineffective when the environment does not incite the agent to allocate a good effort. In other words, some efficient strategic incentive-compatible constraints that cover the entire period of contract do not generally exist.

The strategy of the Principal to restrict attention to the class of allocations satisfying the (ex-ante) incentive-compatible constraints is not always optimal. This does not necessarily ensure that the agent will not misrepresent his private information and preferences. It makes sense to announce his characteristics if the Principal can design a mechanism that incites the agent to do this truthfully.

The revelation principle ("the truth is dominant strategy") simplifies the Principal's problem but it cannot always be restricted to an allocation that gives to the agent no incentive to misrepresent his type. In general, there is not a strict implementation because the truth is not the unique equilibrium of the game. Due to the externality in the utility functions, strict dominant strategy implementation is not generally feasible.

The information asymmetry and costly acquisition of information impose restrictions on the Principal's behavior. The incentives will typically be provided at each period of control. However, even if the process is incentive-compatible at each step, the agent's actions will not necessarily reveal the truth at any time of the contract period. Generally, the incentive mechanism is subjected to inescapable informational constraints. This is sensitive to the environment description and the size of informational asymmetries.

The strategic weights attached to the system deviations with respect to the fixed targets are correlated with the importance the agent places on the incentives implemented by the Principal. Agent's actions are not directly punished, but only his poor performances.

The present model has the potential to explain the adaptive effort behavior of the agent. It can be seen as a step further in the refinement of the effort concept, providing new perspectives of research for theorists and empirical analysts.

The possibility of a null effort of the agent at a given period t is taken into account in the present model. It is the case when the agent does not attribute any importance to the system deviation in $t-1$. We illustrate below this particular behavior by a numerical example.
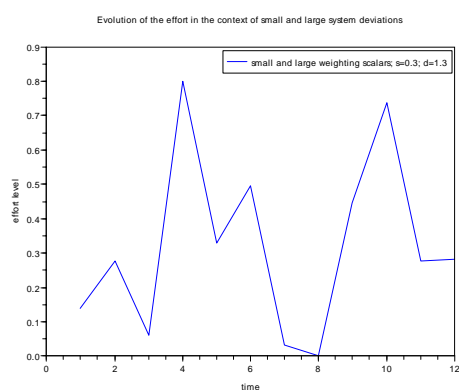

Figure 10

Remark 4. An interesting relationship emerges between risk and effort. The strategy adopted by the agent to minimize the risk will also minimize the effort. The opposite is not generally true.

Definition 3. Two deviations of the system are said to be comparable in magnitude if and only if their ratio is very close to 1.

Proposition 3. An almost risk-neutral agent can allocate a higher effort than a risk-averse one.

Proof. In the appendix.   ■

This is an astonishing result, far from intuitive. It proves that effort attitude and behavior towards risk are not always positive correlated.

The proposed model reveals surprising attitudes to effort of risk-averse agents. We give below two suggestive graphics in this sense.
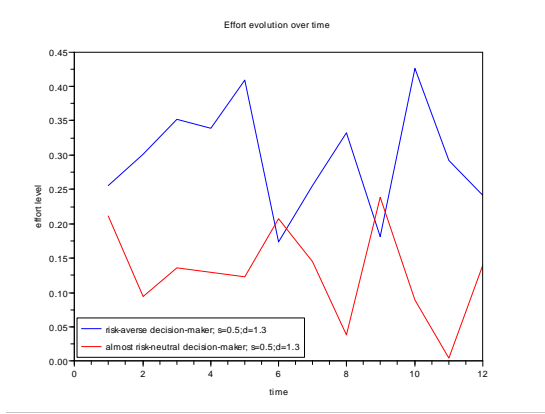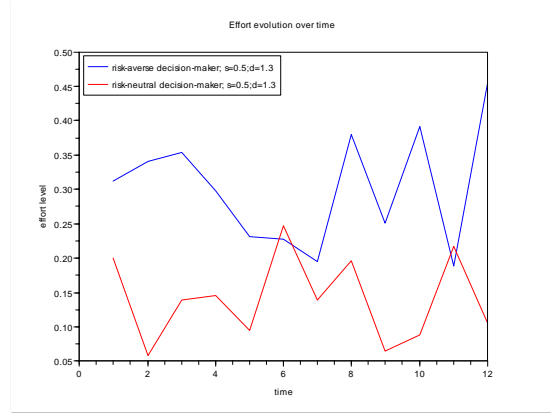

Figure 11

Figure 12

Proposition 4. If all system deviations are high and comparable in magnitude, then the effort invested by the agent is highly correlated with the importance he places on the system evolution in the past.

Proof. In the appendix.   ■

We consider two distinct scenarios: i) when the strategic weights are large; and ii) when the strategic weights are small. The first (second) scenario corresponds to the case of a less (more) effort-averse agent by nature. For an illustration, we give below two suggestive graphics in this sense.
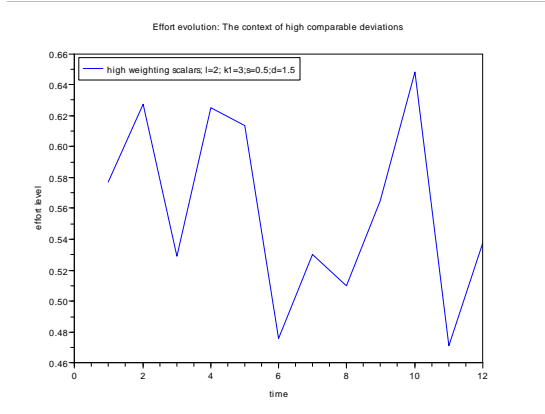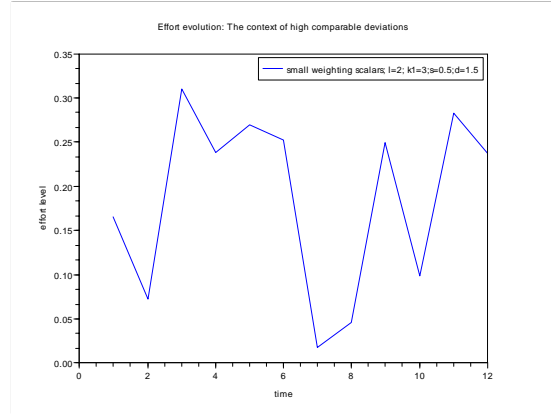

Figure 13

Figure 14

Let us now consider the context where all system deviations are high and comparable in magnitude.

Proposition 5. If a higher number of backward periods are taken into account when estimating the risk-aversion index, then a lower effort will be invested by the agent for the same period of control.

Proof. In the appendix.   ■

This result proves the importance of the history of the process in the effort assignment over time. The effort varies with the environmental context and the way the agent exploits and interprets the system dynamics.

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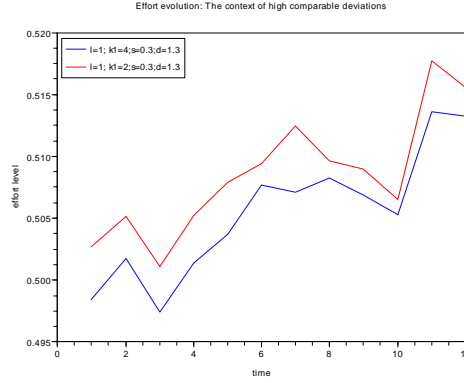We give below a graphical illustration of the **Proposition 5**.

Figure 15

Let us now consider the context where the agent has the interest to use a progressive history of the process for the estimation of the risk-aversion index. This may be the case where the horizon length is short and the economic agent needs more information useful in improving the process of risk assessment. In this particular context, the risk-aversion index is given by:

Definition 4. Using t to denote time, the absolute risk-aversion index $\varphi_{t,w}^{r-a}$ evolves according to:

$$\varphi_{t,w}^{r-a} \overset{def.}{=} \frac{\parallel y_{t-1} - y_{t-1}^g \parallel^2 L_{t-1} + ... + \parallel y_0 - y_0^g \parallel^2 L_0}{\sqrt{(\parallel y_{t-1} - y_{t-1}^g \parallel^2 + ... + \parallel y_0 - y_0^g \parallel^2)^2 + l}}, \ t = 1, ..., T$$

where $-1 < L_{t-1} \leq L_{t-2} \leq ... \leq L_0 \leq 0$ are strategic weights attached to the system deviations with respect to the optimal reference path $\{y_{t-1}^g, ..., y_0^g\}$ and $l \geq 1$ is a fixed integer characterizing the agent's type. For further details, see Protopopescu (2007).

Proposition 6. A risk-averse agent who manages more and more hardly the evolution of the system will invest more and more effort over time.

Proof. In the appendix. ∎

The environmental context can decide the agent's effort behavior during the period of control. The effort invested by the agent is necessary but not always sufficient to improve the system target variable. It may be the case of a system characterized by a slow inertia, whose tendency is to move towards a disequilibrium state. The efficiency of the control instruments plays a crucial role in the effort allocation over time. Better instruments allow for better effort management. For a numerical illustration of the **Proposition 6**, we give below a suggestive graphic.
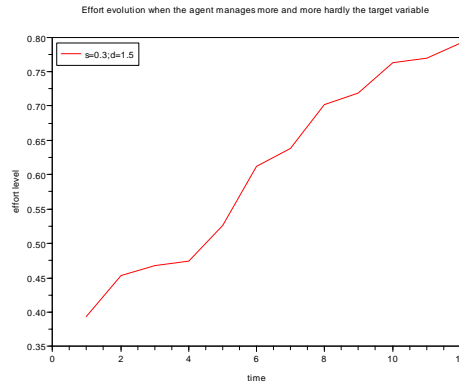

Figure 16

This result proves that in environments with a slow inertia, the incentive mechanism implemented by the Principal may not ensure an optimal equilibrium of the game. Even for a high

effort invested, the private agent may not be recompensed. Moreover, the agent can be punished by the Principal for his deviating behavior. In this particular context, a principal-agent relationship on a long-term cannot be incitative for the private agent.

Proposition 7. A risk-averse agent who manages better and better the evolution of the system will invest less and less effort over time.

Proof. In the appendix. ■

This type of scenario is specific to dynamic environments characterized by a high inertia. In this particular context, it is possible to obtain small deviations of the system by investing a small effort. For a numerical illustration, we give below a suggestive graphic.
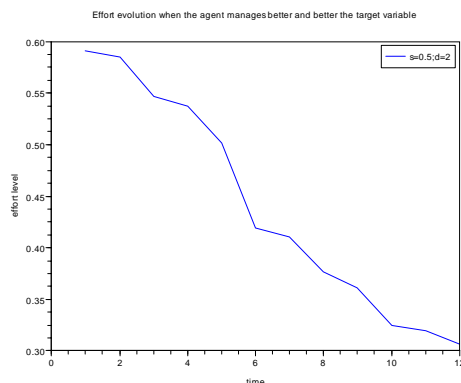

Figure 17

This type of result has the potential to explain why, in the context of a dynamic principal-agent relationship, the incentive-compatibility constraints imposed by the Principal are not always optimal. The agent can obtain a high recompense with a low effort level.

4.2. High Potential Shifts

Shift happens. The agent learns from failures. Every high deviation, seen as a failure, is analyzed in order to avoid unexpected fluctuations of the system in the future. A large deviation from the expected outcome is perceived as a shift by the agent. We call a high positive shift the transition of the system from consecutive small levels of performance to a high level one. In the opposite case, we call this type of transition a high negative shift. The agent's objective during the period of control is to obtain smooth shifts with respect to the fixed targets. This contributes to the equilibrium and stability of the system. The concept of effort-aversion is appropriate to dynamic stochastic environments whose behavior change significantly over time. It is the case of high fluctuating systems.

Proposition 8. Consider the following two opposite scenarios: the transition of the system is from consecutive small (large) deviations to a large (small) deviation in $t-2$. Assume that the variation in the target variable at time $t-1$ is high and comparable with all other previous deviations of the system. For this type of scenarios, the effort level is highly dependent on the strategic weights the agent will attach to the large deviations of the system.

Proof. In the appendix. ■

Sudden significative changes in the system behavior can affect differently the agent's strategy to effort over time. Large shifts are correlated with large deviations of the system with respect to the fixed targets. The shift magnitude generally depends on the system transition type.

We give below three distinct attitudes to effort depending on the agent's individual perception about large fluctuations of the system.

a) the large deviation of the system at time $t-2$ (in the context of the first scenario) is much more important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.
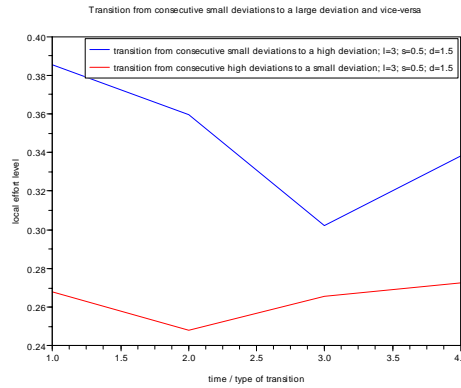

Figure 18-a)

b) the large deviation of the system at time $t-2$ (in the context of the first scenario) is much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.
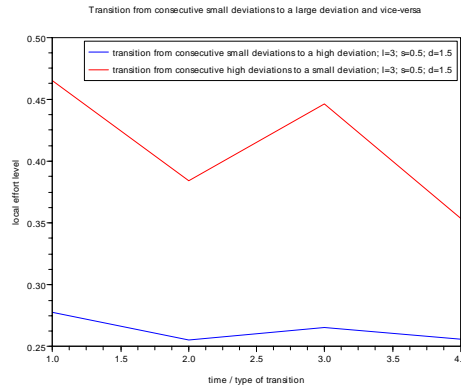

Figure 19-b)

c) the large deviation of the system at time $t-2$ (in the context of the first scenario) is either much more or much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.
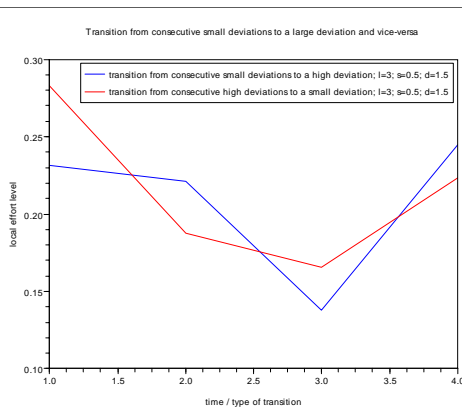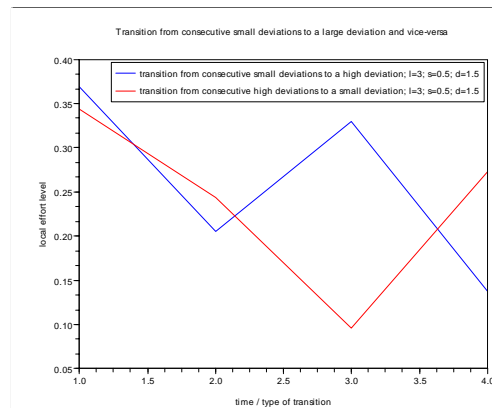

Figure 20-c)

Figure 21-c)

For the four above graphics, the effort level at time $t = 1, ..., 4$ corresponds, respectively, to the following distinct contexts:

i) system transition from four large (small) deviations to a small (large) deviation; ii) system transition from three large (small) deviations to a small (large) deviation; iii) system transition

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from two large (small) deviations to a small (large) deviation; and iv) system transition from a large (small) to a small (large) deviation.

5. Effort-Aversion

The fact that the agent is risk-averse does not give any obvious reason for which one should also suppose that he is effort-averse. In the real world, this is a common characteristic of most risk-averse agents. However, this important behavior aspect is ignored in the literature on risk and effort. Suppose that the agent will fix before starting the control an optimal effort-threshold $e_{\max}^{r\_a}$ which must not be exceeded during the entire working horizon. Otherwise, the agent becomes excessively effort-averse for the current period of control. The effort-threshold $e_{\max}^{r\_a}$ must be selected in order to offer the best characterization of the agent's type. A higher (smaller) effort-aversion is correlated with a smaller (higher) threshold $e_{\max}^{r\_a}$. Thinking strategically, the agent will fix $e_{\max}^{r\_a}$ superior to $e_{1,\,p}^{r\_a}$. A natural question arrises: From what level, the effort-threshold can be regarded as large? It will generally depend on the particular environmental context and the agent's effort-averse type. Let $\varphi_{\min}^{r\_a}$ be a suitable risk-aversion threshold fixed by the decision-maker before starting the control and for the entire period $[1, T]$. When this threshold is exceeded, the agent becomes excessively risk-averse for the current period of control, being characterized by an extreme pessimism (PROTOPOPESCU 2007). Note that the exceeding of the threshold $\varphi_{\min}^{r\_a}$ in $t-1$ does not necessarily imply an excessive effort behavior in t. This is generally correlated with the amplitude size of the system deviation in $t-1$. We plot below this interesting behavior.
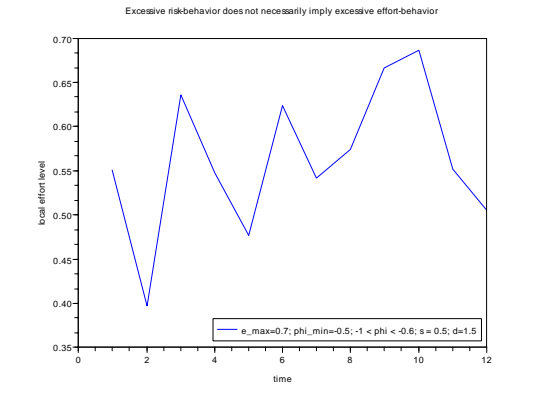

Figure 22

On the other hand, the exceeding of the threshold $e_{\max}^{r\_a}$ in t is correlated with large fluctuations in the target variable, but not necessarily implies an excessive risk behavior in $t-1$. For an illustration, we give below a suggestive graphic.
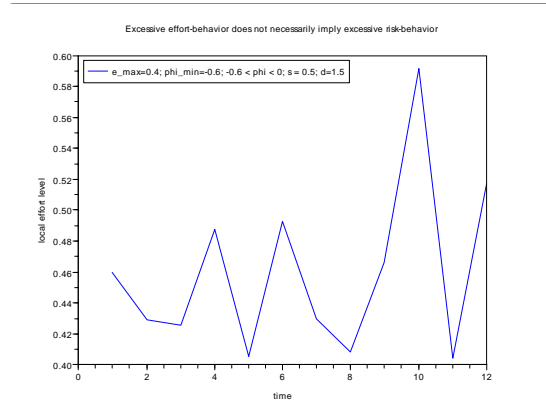

Figure 23

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Remark 5. For a low risk-aversion index $\varphi_{t-1,\,p}^{r\_a}$ and a high system deviation $\parallel y_{t-1} - y_{t-1}^g \parallel$, it is possible for the agent to exceed the optimal effort-threshold $e_{\max}^{r\_a}$ for the period t. This may be the case of a dynamic system characterized by large deviations with respect to the optimal path η. We illustrate this possibility by a numerical example.
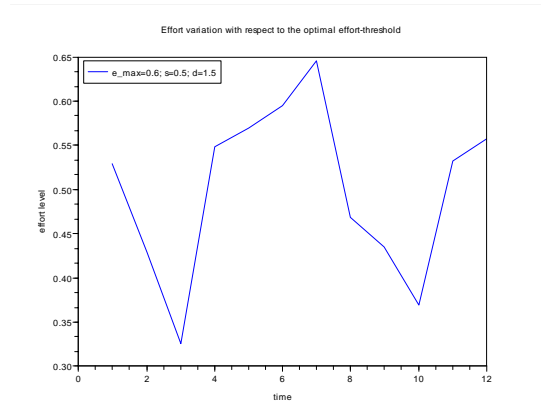

Figure 24

Remark 6. We note here the local character of the agent's effort-aversion. This is defined for a neighborhood of the fixed targets $y_{t-1}^g, ..., y_{t-k_1}^g$. It will therefore exist some neighborhood effects of the system dynamics on the effort variable.

Remark 7. The higher (lower) the degree of effort-aversion at time t, the higher (lower) the value of the effort variable $e_{t,\,p}^{r\_a}$. It is important to distinguish between local effort-aversion (at time t) and global effort-aversion (over the whole period $[1, T]$).

In the context of a principal-agent problem, the relationship between the agent's reaction to the incentive strategy adopted by the Principal and his atitude to effort is generally complex. Effort-aversion makes this reaction stronger than effort-neutrality.

Proposition 9. A risk-and-effort averse agent is characterized by a bounded utility function during the entire period of control.

Proof. In the appendix. ■

This result can be seen as a step further in the refinement of the agent's preferences in an evolving environment. An interesting characterization of the agent's preferences with respect to the optimal thresholds $\varphi_{\min}^{r\_a}$ and $e_{\max}^{r\_a}$ is thus possible. For a numerical illustration, we give below a suggestive graphic in this sense.
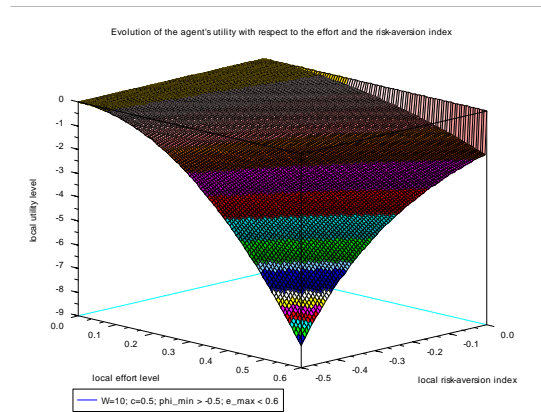

Figure 25

Note that in the case where both thresholds $\varphi_{\min}^{r\_a}$ and $e_{\max}^{r\_a}$ are exceeded, the agent's preferences become suboptimal. We illustrate below this type of behavior.
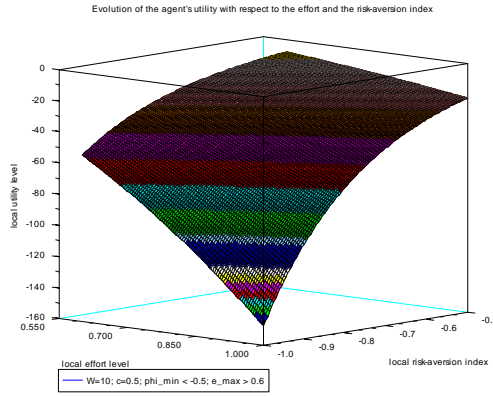

Figure 26

Remark 8. The constant c characterizes the agent's type and is correlated with the choice of the optimal threshold $e_{\max}^{r\_a}$. The smaller (higher) the threshold $e_{\max}^{r\_a}$, the smaller (higher) the value of the parameter c fixed by the agent.

This allows to better understand the nature of moral hazard and the problems it causes in the case of a principal-agent relationship stated as a dynamic Stackelberg game. We give below a suggestive graphic in this sense.
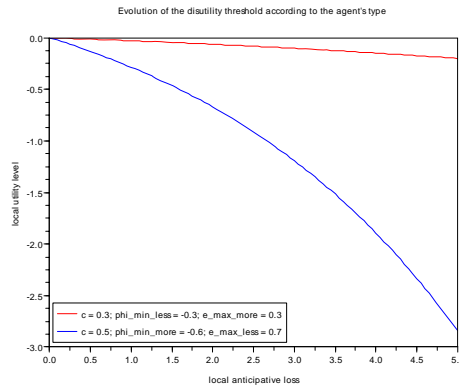
Figure 27

More exactly, if the Principal will condition the recompense that he is disposed to grant by an effort level superior to $e_{\max}^{r\_a}$, then the agent will try to shirk by misrepresenting his type. Thus the incentive-compatibility constraints imposed by the Principal may not be optimal. There is a trade-off between incentive gains and payoff losses.

5.1. Definition of the threshold $e_{\max}^{r\_a}$

During the period of control, the agent's objectives are the following:

$$e_{t,\,p}^{r\_a} \overset{def.}{=} \frac{-\parallel y_{t-1} - y_{t-1}^g \parallel^2 L_{t-1}}{\parallel y_{t-1} - y_{t-1}^g \parallel^2 + s} \cdot \frac{\varphi_{t-1,\,p}^{r\_a} - 1}{\varphi_{t-1,\,p}^{r\_a} - d} < e_{\max}^{r\_a}, \; \forall \, t = 1, ..., T$$

$$\parallel y_{t-j} - y_{t-j}^g \parallel \; < 1, \;\; t - j = 1, ..., T$$

$$\varphi_{t,\,p}^{r\_a} > \varphi_{\min}^{r\_a}, \; \forall \, t = 1, ..., T$$

It implies that:

$$e_{t,\,p}^{r\_a} < \frac{1}{s+1} \cdot \frac{\varphi_{\min}^{r\_a} - 1}{\varphi_{\min}^{r\_a} - d}$$

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We can thus define the effort-aversion threshold $e_{\max}^{r\_a}$ by the following formula:

$$e_{\max}^{r\_a} \stackrel{def.}{=} \frac{1}{s+1} \cdot \frac{\varphi_{\min}^{r\_a} - 1}{\varphi_{\min}^{r\_a} - d}$$

We give below an empirical characterization of the relationship that exists between the two optimal thresholds $\varphi_{\min}^{r\_a}$ and $e_{\max}^{r\_a}$.
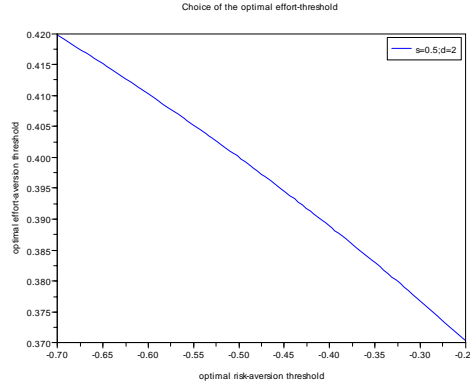
Figure 28

Following PROTOPOPESCU (2007), one can distinguish between two types of effort-averse agents:

i) less effort-averse, in which case, the effort-threshold is defined by:

$$e_{\max,\,less}^{r\_a} \stackrel{def.}{=} \frac{1}{s+1} \cdot \frac{\varphi_{\min,\,more}^{r\_a} - 1}{\varphi_{\min,\,more}^{r\_a} - d}$$

ii) more effort-averse, in which case, the effort-threshold is defined by:

$$e_{\max,\,more}^{r\_a} \stackrel{def.}{=} \frac{1}{s+1} \cdot \frac{\varphi_{\min,\,less}^{r\_a} - 1}{\varphi_{\min,\,less}^{r\_a} - d}$$

For an illustration, we plot below possible values of the effort-thresholds $e_{\max,\,less}^{r\_a}$ and $e_{\max,\,more}^{r\_a}$.

Figure 29-i)

Figure 30-ii)

Proposition 10 An effort-averse agent is characterized by an optimal effort-threshold fixed according to his individual type.

Proof. In the appendix. ∎

In this approach, we deal with a continuum of types of agents. The above result allows to distinguish between agents of common /distinct types. It has strong implications in problems of moral hasard and adverse selection, improving our understanding as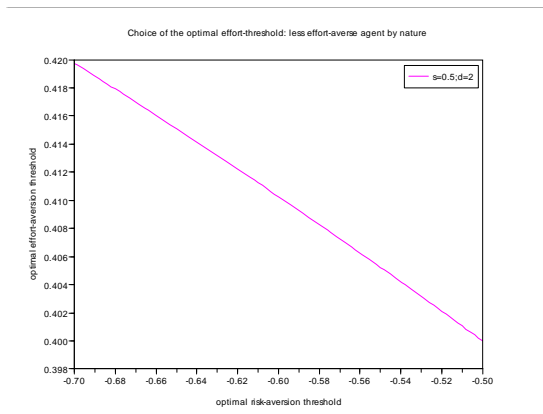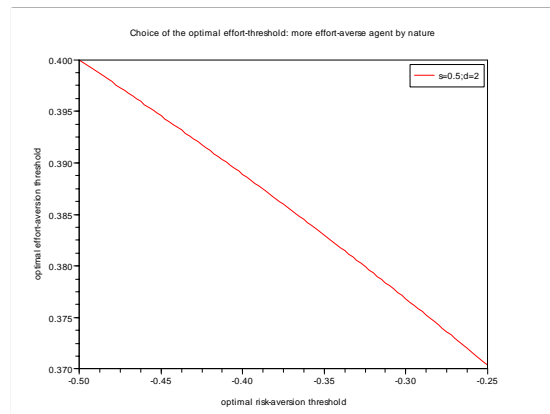 regards the agent's psychology to effort in dynamic risky environments. We give below a graphical illustration of the **Proposition 10**.
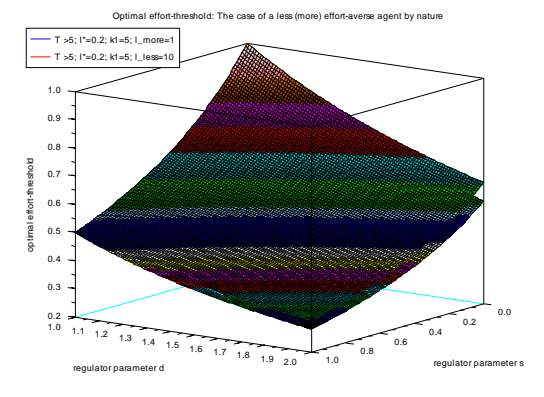
Figure 31

Proposition 11. There is a positive correlation between the agent's effort type and the planning horizon length.

Proof. In the appendix. ∎

Because risk-aversion increases with horizon length (PROTOPOPESCU 2007), the agent will choose a smaller risk-aversion threshold $\varphi_{\min}^{r\_a}$, and hence a higher effort-aversion threshold $e_{\max}^{r\_a}$. The agent is thus more receptive in terms of effort allocation with respect to the fixed objectives. The horizon length has a non-negligible impact on the agent's effort behavior over time. The higher the planning horizon, the larger its impact. The distinction between more and less effort-averse agents is more (less) pronounced for a large (small) horizon length. We illustrate below this theoretical result.
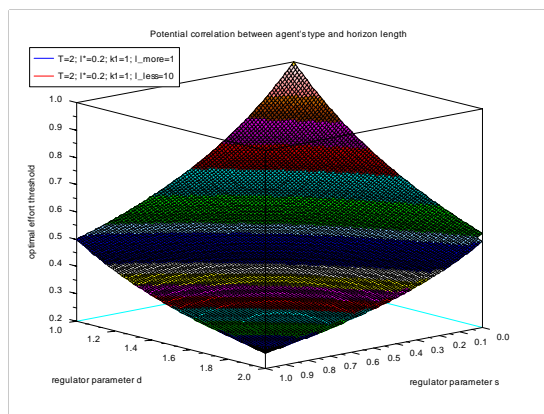
Figure 32

This type of result contributes to the theory of incentive contracts by revealing the relationship that exists between contract length and effort allocation. This non-negligible component must be taken into account when modelling a dynamic mechanism design problem. It has a positive effect on the agent's attitude to effort over time.

6. Concluding Remarks and Possible Extensions

The present study explores the impact of endogenous dynamics of an evolving system on the behavior to effort of a risk-averse agent. From this new perspective, the effort is no more seen

as a pure disutility, as in the classical approach, but rather as a potential efficient instrument in order to better manage the system evolution. The agent's type is defined according to an optimal effort-threshold fixed before starting the control and for the entire working horizon. One can thus distinguish between common and distinct types of agents according to their individual effort-averse preferences. Moreover, it allows to introduce the concept of excessive effort-aversion, which has strong implications in the context of strategic dynamic interactions, where both moral hazard and adverse selection effects coexist. This may be the case of a principal-agent relationship in which the agent's private information may not be monitorable. In particular, this type of modelling has the potential to explain why some efficient strategic incentive-compatibility constraints that cover the entire horizon of the contract do not generally exist. These do not always induce the agent to reveal his private information, and therefore do not always capture the agent's incentive to shirk. The presence of moral hazard and adverse selection makes the analysis difficult but interesting, revealing the difficulties to implement an incentive Pareto-optimal contract. The Principal cannot observe the agent's behavior directly, but only the consequences of his behavior, and those consequences are also influenced by the environment. More exactly, the Principal will only observe a variable correlated with the agent's effort (the output, an imperfect signal of the effort invested) but he cannot force the agent to choose an action which is Pareto-optimal. The asymmetric informational structure between the agent and Principal makes unable this later to solve optimally the effort prediction problem. It generates a conflict between incentive-compatibility and Pareto efficiency. The interaction of the Principal's policy with the dynamic learning process of the private agent is a reality which must be fully recognized in incentive design. The proposed study can be extended in the context of a strategic Nash game with cooperative /non-cooperative players, the objective here being to define and characterize the equilibrium path according to an optimal dynamic effort-sharing. Exploring such possibilities appears to be a good topic for further research. This provides new theoretical perspectives in understanding the complexity of the effort mechanism.

Appendix

Proof of Proposition 1.

The ratio of the efforts in $t+1$ and t is given by:

$$\frac{e_{t+1}}{e_t} = \frac{-\|y_t - y_t^g\|^2 L_t}{\|y_t - y_t^g\|^2 + s} \cdot \frac{\varphi_{t,\,p}^{r_{-a}} - 1}{\varphi_{t,\,p}^{r_{-a}} - d} \cdot \frac{\|y_{t-1} - y_{t-1}^g\|^2 + s}{-\|y_{t-1} - y_{t-1}^g\|^2 L_{t-1}} \cdot \frac{\varphi_{t-1,\,p}^{r_{-a}} - d}{\varphi_{t-1,\,p}^{r_{-a}} - 1}$$

One can distinguish three distinct scenarios:

i) $\varphi_{t,\,p}^{r_{-a}} > \varphi_{t-1,\,p}^{r_{-a}}$, $\| y_t - y_t^g \| < \| y_{t-1} - y_{t-1}^g \|$ and $L_t > L_{t-1}$. In this case, one obtains that $e_{t+1} < e_t$.

ii) $\varphi_{t,\,p}^{r_{-a}} < \varphi_{t-1,\,p}^{r_{-a}}$, $\| y_t - y_t^g \| > \| y_{t-1} - y_{t-1}^g \|$ and $L_t < L_{t-1}$. Contrary to the previous case, we have that $e_{t+1} > e_t$.

iii) $\varphi_{t,\,p}^{r_{-a}} \lessgtr \varphi_{t-1,\,p}^{r_{-a}}$, $\| y_t - y_t^g \| \lessgtr \| y_{t-1} - y_{t-1}^g \|$ and $L_t \lessgtr L_{t-1}$. In this case, one obtains either $e_{t+1} < e_t$ or $e_{t+1} > e_t$.

There is thus a trade-off between three endogenous factors: i) the risk-aversion index, ii) the last deviation of the system, and iii) the importance attached to the last deviation.

Proof of Proposition 2.

This is a direct consequence of the monotony of the local utility function U_t. This is increasing in $\varphi_{t,\,p}^{r_{-a}}$ (PROTOPOPESCU 2007) and respectively decreasing in $W_{[1,t]}$ and $e_{t,\,p}^{r_{-a}}$ (by construction).

Proof of Proposition 3.

It is supposed that a risk-averse decision-maker will place a higher importance on the system deviations compared to an almost risk-neutral one.

If the following conditions are verified:

$$\| y_{t-1}^{a\_r\_n} - y_{t-1}^g \| \quad < \quad \| y_{t-1}^{r\_a} - y_{t-1}^g \| \ \text{ and } L_{t-1}^{a\_r\_n} > L_{t-1}^{r\_a}$$

then one can write:

$$e_{t,\,p}^{a\_r\_n} \overset{def.}{=} -\frac{\| y_{t-1}^{a\_r\_n} - y_{t-1}^g \|^2 \ L_{t-1}^{a\_r\_n}}{\| y_{t-1}^{a\_r\_n} - y_{t-1}^g \|^2 + s} \cdot \frac{\varphi_{t-1,p}^{a\_r\_n} - 1}{\varphi_{t-1,p}^{a\_r\_n} - d}$$

$$< -\frac{\| y_{t-1}^{r\_a} - y_{t-1}^g \|^2 \ L_{t-1}^{r\_a}}{\| y_{t-1}^{r\_a} - y_{t-1}^g \|^2 + s} \cdot \frac{\varphi_{t-1,p}^{r\_a} - 1}{\varphi_{t-1,p}^{r\_a} - d} \overset{def.}{=} e_{t,\,p}^{r\_a}$$

In the case where $\| y_{t-1}^{a\_r\_n} - y_{t-1}^g \| > \| y_{t-1}^{r\_a} - y_{t-1}^g \|$, it is possible to obtain the opposite scenario, that is, $e_{t,\,p}^{a\_r\_n} > e_{t,\,p}^{r\_a}$.

Proof of Proposition 4.

By hypothesis:

$$\| y_{t-j_1'} - y_{t-j_1'}^g \| \,/\, \| y_{t-j_1''} - y_{t-j_1''}^g \| \ \cong 1 \text{ for } j_1',\ j_1'' \in \{1, ..., k_1\},\ j_1' \neq j_1''$$

$$l / \| y_{t-j_1} - y_{t-j_1}^g \|^4 \cong 0,\ \forall \ j_1 = 1, ..., k_1$$

It implies that:

$$e_{t,\,p}^{r\_a} \to -L_{t-1} \cdot \frac{\frac{L_{t-2}+...+L_{t-(k_1+1)}}{k_1} - 1}{\frac{L_{t-2}+...+L_{t-(k_1+1)}}{k_1} - d}$$

Depending on the weights attached to the system deviations, the risk-aversion index level at time $t-1$ may be more or less close to -1 or 0. If all weights approach -1, then the index value will be close enough to -1. The agent is characterized by an excessive risk-aversion at time $t-1$ and a significative effort level at time t.

Contrary to what is generally believed or intuition would suggest, the agent can have a small degree of risk-aversion when the deviations of the system are large. It is the case where all weights approach 0. The agent is almost risk-neutral, being characterized by a very small variability in his risk-aversion over time. For this type of scenario, the agent's effort allocated at time t is negligible (almost null).

Proof of Proposition 5.

We have the following implication:

$$k_1 < k_1' \Rightarrow \underbrace{\frac{L_{t-2} + ... + \ L_{t-(k_1'+1)}}{k_1'}}_{\text{(the average of } k_1' \text{ weights)}} \ggg \underbrace{\frac{L_{t-2} + ... + \ L_{t-(k_1+1)}}{k_1}}_{\text{(the average of } k_1 \text{ weights)}}$$

The left (right) hand of the above inequality corresponds to the agent's degree of risk-aversion at time $t-1$ in the case where a higher (smaller) number of backward periods are taken into account when defining the risk-aversion index. The agent is supposed to attach a higher importance to the system deviation at time $t-1$ for the first scenario (k_1 periods) compared to the second one (k_1' periods). It follows that:

$$-L_{t-1,\,k_1'} \cdot \frac{\frac{L_{t-2}+...+ \ L_{t-(k_1'+1)}}{k_1'} - 1}{\frac{L_{t-2}+...+ \ L_{t-(k_1'+1)}}{k_1'} - d} < -L_{t-1,\,k_1} \cdot \frac{\frac{L_{t-2}+...+ \ L_{t-(k_1+1)}}{k_1} - 1}{\frac{L_{t-2}+...+ \ L_{t-(k_1+1)}}{k_1} - d}$$

This completes the proof.

Proof of Proposition 6.

A risk-averse agent who manages more and more hardly the evolution of the system is characterized by an increasing risk-aversion over time (PROTOPOPESCU 2007).

Denote by $\widetilde{L}_\tau$ the weight attached to the system deviation at time $\tau = 0, ..., t-1$. The weights are supposed to gradually decrease during the period of control. One can write the sequence of inequalities:

$$-\widetilde{L}_0 \cdot \frac{\varphi^{m.m.h.}_{0,\,p} - 1}{\varphi^{m.m.h.}_{1,\,p} - d} < -\widetilde{L}_1 \cdot \frac{\varphi^{m.m.h.}_{1,\,p} - 1}{\varphi^{m.m.h.}_{1,\,p} - d} < ... < -\widetilde{L}_{t-1} \cdot \frac{\varphi^{m.m.h.}_{t-1,\,p} - 1}{\varphi^{m.m.h.}_{t-1,\,p} - d}$$

Each above term (in this order) corresponds to the agent's effort assignment at time $\tau = 1, 2, ..., t$. The conclusion follows.

Proof of Proposition 7.

A risk-averse agent who manages better and better the evolution of the system is characterized by a decreasing risk-aversion over time (PROTOPOPESCU 2007).

Denote by $\widehat{L}_\tau$ the weight attached to the system deviations at time $\tau = 0, ..., t-1$. In this particular context, the weights are supposed to gradually increase during the period of control. One can write the sequence of inequalities:

$$-\widehat{L}_0 \cdot \frac{\varphi^{b.b.}_{0,\,p} - 1}{\varphi^{b.b.}_{1,\,p} - d} > -\widehat{L}_1 \cdot \frac{\varphi^{b.b.}_{1,\,p} - 1}{\varphi^{b.b.}_{1,\,p} - d} > ... > -\widehat{L}_{t-1} \cdot \frac{\varphi^{b.b.}_{t-1,\,p} - 1}{\varphi^{b.b.}_{t-1,\,p} - d}$$

This completes the proof.

Proof of Proposition 8.

Denote by $\varphi^{s\_h\_k_1}_{t-1,\,p}$ the risk-aversion index at time $t-1$ in the case where the system transition is from consecutive small deviations to a large deviation. In this case, we have that $\varphi^{s\_h\_k_1}_{t-1,\,p} \to L_{t-2}$. Let us denote by $\varphi^{h\_s\_k_1}_{t-1,\,p}$ the risk-aversion index at time $t-1$ when the system transition is from consecutive large deviations to a small deviation. One can write:

$$\varphi^{h\_s\_k_1}_{t-1,\,p} \to \frac{\overline{L}_{t-3} + ... + \overline{L}_{t-(k_1+1)}}{k_1 - 1}$$

where $\overline{L}_{t-3}, ..., \overline{L}_{t-(k_1+1)}$ are strategic weights attached to large system deviations (in the context of the second scenario). One can imagine three distinct situations:

a) the large deviation of the system at time $t-2$ (in the context of the first scenario) is much more important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$\underset{\text{(sufficient condition)}}{L_{t-2} \ll \overline{L}_{t-3}, ..., \ L_{t-2} \ll \overline{L}_{t-(k_1+1)}} \Rightarrow L_{t-2} \ll \frac{\overline{L}_{t-3} + ... + \overline{L}_{t-(k_1+1)}}{k_1 - 1}$$

It follows that $\varphi^{s\_h\_k_1}_{t-1,\,p} < \varphi^{h\_s\_k_1}_{t-1,\,p}$, and hence a higher degree of risk-aversion at time $t-1$ in the context of the first scenario. We have the inequality:

$$\frac{\varphi^{s\_h\_k_1}_{t-1,\,p} - 1}{\varphi^{s\_h\_k_1}_{t-1,\,p} - d} > \frac{\varphi^{h\_s\_k_1}_{t-1,\,p} - 1}{\varphi^{h\_s\_k_1}_{t-1,\,p} - d}$$

If the variation in the target variable at time $t-1$ is high and $L_{t-1} \leq \overline{L}_{t-1}$, then one can write:

$$-L_{t-1} \cdot \frac{\varphi^{s\_h\_k_1}_{t-1,\,p} - 1}{\varphi^{s\_h\_k_1}_{t-1,\,p} - d} > -\overline{L}_{t-1} \cdot \frac{\varphi^{h\_s\_k_1}_{t-1,\,p} - 1}{\varphi^{h\_s\_k_1}_{t-1,\,p} - d}$$

In other words, the effort invested by the agent at time t will be higher for the first scenario.

b) the large deviation of the system at time $t-2$ (in the context of the first scenario) is much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-2} \gg \overline{L}_{t-3}, ..., \ L_{t-2} \gg \overline{L}_{t-(k_1+1)} \Rightarrow L_{t-2} \gg \frac{\overline{L}_{t-3} + ... + \ \overline{L}_{t-(k_1+1)}}{k_1 - 1}$$
$$\underset{\text{(sufficient condition)}}{}$$

In contrast with the previous case, one obtains that $\varphi_{t-1,\,p}^{s\_h\_k_1} > \varphi_{t-1,\,p}^{h\_s\_k_1}$, and thus a smaller degree of risk-aversion at time $t-1$ in the context of the first scenario. If, in addition, the variation in $t-1$ is high and $L_{t-1} \geq \overline{L}_{t-1}$, then one can write:

$$-L_{t-1} \cdot \frac{\varphi_{t-1,\,p}^{s\_h\_k_1} - 1}{\varphi_{t-1,\,p}^{s\_h\_k_1} - d} < -\overline{L}_{t-1} \cdot \frac{\varphi_{t-1,\,p}^{h\_s\_k_1} - 1}{\varphi_{t-1,\,p}^{h\_s\_k_1} - d}$$

In other words, the agent's effort at time t will be smaller in the context of the first scenario compared to the second one.

c) the large deviation of the system at time $t-2$ (in the context of the first scenario) is much more or much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-2} \ll (\text{or } \gg) \ \overline{L}_{t-3}, ..., \ L_{t-2} \ll (\text{or } \gg) \ \overline{L}_{t-(k_1+1)}$$

In this case, it may be possible to obtain either $\varphi_{t-1,\,p}^{s\_h\_k_1} < \varphi_{t-1,\,p}^{h\_s\_k_1}$ or $\varphi_{t-1,\,p}^{s\_h\_k_1} > \varphi_{t-1,\,p}^{h\_s\_k_1}$. It follows that:

$$-L_{t-1} \cdot \frac{\varphi_{t-1,\,p}^{s\_h\_k_1} - 1}{\varphi_{t-1,\,p}^{s\_h\_k_1} - d} \lessgtr -\overline{L}_{t-1} \cdot \frac{\varphi_{t-1,\,p}^{h\_s\_k_1} - 1}{\varphi_{t-1,\,p}^{h\_s\_k_1} - d}$$

In other words, the agent's effort at time t will be either higher or smaller for the first scenario compared to the second one. This completes the proof.

Remark 9. We have the following sequence of inequalities:

$$\frac{L_{t-2} + ... + L_{t-(k_1+1)}}{k_1} < \frac{L_{t-3} + ... + L_{t-(k_1+1)}}{k_1 - 1} < ... < \frac{L_{t-k_1} + L_{t-(k_1+1)}}{2} < \frac{L_{t-(k_1+1)}}{1}$$

Each above ratio (in this order) corresponds to the risk-aversion index at time $t-1$ in the case of a gradual transition from an ineffective control to an effective one.

Suppose that the variation in the target variable at time $t-1$ is high and comparable with all other previous large deviations of the system. Denote by $L_{t-1,\,\overline{k}}$ $(\overline{k} = k_1, ..., 1)$ the weights attached to the system deviation at time $t-1$ for each particular configuration described above. It is supposed that the following inequalities are satisfied:

$$L_{t-1,\,k_1} \leq L_{t-1,\,k_1-1} \leq ... \leq L_{t-1,\,2} \leq L_{t-1,\,1}$$

It follows that:

$$-L_{t-1,\,1} \cdot \frac{\frac{L_{t-(k_1+1)}}{1} - 1}{\frac{L_{t-(k_1+1)}}{1} - d} < -L_{t-1,\,2} \cdot \frac{\frac{L_{t-(k_1+1)} + L_{t-k_1}}{2} - 1}{\frac{L_{t-k_1} + L_{t-(k_1+1)}}{2} - d} < ... <$$

$$< -L_{t-1,\,k_1-1} \cdot \frac{\frac{L_{t-(k_1+1)} + ... + L_{t-3}}{k_1 - 1} - 1}{\frac{L_{t-(k_1+1)} + ... + L_{t-3}}{k_1 - 1} - d} < -L_{t-1,\,k_1} \cdot \frac{\frac{L_{t-(k_1+1)} + ... + L_{t-2}}{k_1} - 1}{\frac{L_{t-(k_1+1)} + ... + L_{t-2}}{k_1} - d}$$

In this particular context, the environment will incite the agent to invest a smaller effort level for the same period of control t.

Remark 10. One can write:

$$\frac{L_{t-1}}{1} < \frac{L_{t-2}+L_{t-3}}{2} < ... < \frac{L_{t-2}+...+L_{t-(k_1+1)}}{k_1}$$

Each above ratio (in this order) corresponds to the risk-aversion index at time $t-1$ in the case of a gradual transition from an effective control an ineffective one.

For this type of scenario, it is supposed that:

$$L_{t-1,\,1} \leq L_{t-1,\,2} \leq ... \leq L_{t-1,\,k_1}$$

As above, the variation in the target variable at time $t-1$ is assumed to be high and comparable with all other previous large deviations of the system.

It follows that:

$$-L_{t-1,1} \cdot \frac{\frac{L_{t-2}}{1}-1}{\frac{L_{t-2}}{1}-d} > -L_{t-1,2} \cdot \frac{\frac{L_{t-2}+L_{t-3}}{2}-1}{\frac{L_{t-2}+L_{t-3}}{2}-d} > ... > -L_{t-1,k_1} \cdot \frac{\frac{L_{t-2}+...+L_{t-(k_1+1)}}{k_1}-1}{\frac{L_{t-2}+...+L_{t-(k_1+1)}}{k_1}-d}$$

In other words, the environment will incite the agent to invest a higher effort for the same period of control t.

Proof of Proposition 9.

Denote by $U_t(W_{[1,t]}, \varphi_{\min}^{r\_a}, e_{\max}^{r\_a})$ the agent's disutility threshold at time t. We have the implication:

$$\varphi_{\min}^{r\_a} \leq \varphi_{t,\,p}^{r\_a} \text{ and } e_{t,\,p}^{r\_a} \leq e_{\max}^{r\_a} \Rightarrow 0 > U_t(W_{[1,t]}, \varphi_{t,\,p}^{r\_a}, e_{t,\,p}^{r\_a}) \geq U_t(W_{[1,t]}, \varphi_{\min}^{r\_a}, e_{\max}^{r\_a})$$

We can thus distinguish between two disutility thresholds according to the agent's individual type:

i) a smaller disutility threshold, denoted by $U_t^s(W_{[1,t]}, \varphi_{\min,\,more}^{r\_a}, e_{\max,\,less}^{r\_a})$.

ii) a higher disutility threshold, denoted by $U_t^h(W_{[1,t]}, \varphi_{\min,\,less}^{r\_a}, e_{\max,\,more}^{r\_a})$.

Proof of Proposition 10.

Proof. Following PROTOPOPESCU (2007), we have:

$$\underbrace{-1 < \varphi_{\min,\,more}^{r\_a} < -\frac{l^*k_1}{\sqrt{[l^*k_1]^2+l_{more}}}}_{\text{(more risk-averse agent by nature)}}$$

and

$$\underbrace{-\frac{l^*k_1}{\sqrt{[l^*k_1]^2+l_{less}}} < \varphi_{\min,\,less}^{r\_a} < 0}_{\text{(less risk-averse agent by nature)}}$$

where l_{more} and l_{less} (with $1 \leq l_{more} < l_{less}$) are two parameters which characterize the agent's type and $l_t = l^* \in (0,1)$ for $t = 1,...,T$.

Therefore, we have:

$$\underbrace{1 > e_{\max,\,less}^{r\_a} > \frac{1}{s+1} \cdot \frac{-\frac{l^*k_1}{\sqrt{[l^*k_1]^2+l_{more}}}-1}{-\frac{l^*k_1}{\sqrt{[l^*k_1]^2+l_{more}}}-d}}_{\text{(less effort-averse agent by nature)}}$$

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and

$$0 < e^{r\_a}_{\max,\,more} < \frac{1}{s+1} \cdot \frac{-\frac{l^*k_1}{\sqrt{[l^*k_1]^2 + l_{less}}} - 1}{-\frac{l^*k_1}{\sqrt{[l^*k_1]^2 + l_{less}}} - d}$$

(more effort-averse agent by nature)

A less (more) effort-averse agent is characterized by a higher (smaller) degree of effort-aversion. We distinguish here between "nature" and "type". The agent is considered effort-averse by nature, while his "type" is more or less effort-averse. The evolution of the system over time will refine the agent's type.

We thus obtain a complete characterization of common /distinct types of agents. The two fixed thresholds $e^{r\_a}_{\max,\,less}$ and $e^{r\_a}_{\max,\,more}$ are not exceeded during the period of control if and only if the agent succeeds in managing the system trajectory. Two common /distinct types of agents generally adopt different attitudes to effort for the same period of control. In other words, they are characterized by different degrees of effort-aversion over time.

Proof of Proposition 11.

For a higher number of periods T, the value of the parameter k_1 can be higher, and thus the ratio $-\frac{l^*k_1}{\sqrt{[l^*k_1]^2 + l_{less}}}$ (respectively $-\frac{l^*k_1}{\sqrt{[l^*k_1]^2 + l_{more}}}$) can take a smaller value.

It follows that a more /less effort-averse agent by nature can choose a higher effort-aversion threshold $e^{r\_a}_{\max,\,more}$ (respectively $e^{r\_a}_{\max,\,less}$). In other words, the agent is disposed to invest a higher effort over time.

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