

Improving the Risk Concept: A Revision of Arrow-Pratt Theory in the Context of Controlled Dynamic Stochastic Environments

Dan Protopopescu¹

Center for the Study of Organizations and Decisions in Economics

Universitat Autònoma de Barcelona

08193 Bellaterra, Barcelona, Spain

E-mail: *dan.protopopescu@uab.cat*

December 2007

Abstract

In the literature on risk, one generally assume that uncertainty is uniformly distributed over the entire working horizon, when the absolute risk-aversion index is negative and constant. From this perspective, the risk is totally exogenous, and thus independent of endogenous risks. The traditional measures of risk-aversion are generally too weak for making comparisons between risky situations. This can be highlighted in concrete problems in finance and insurance, context for which the ARROW-PRATT measures of risk-aversion give ambiguous results (ROSS 1981). We improve the ARROW-PRATT approach (1964, 1971A, 1971B), which takes into account only attitudes towards small exogenous risks, by integrating in the analysis potentially high endogenous risks that are under the control of the agent. Based on multiple theoretical and empirical arguments, this new approach offers an elegant study of the close relationship between behavior, attitude and perceived risk.

Keywords: Endogenous risk-aversion, adaptive risk management, optimal risk-aversion threshold, excessive risk-averse behavior, risk perception, changing risk behavior.

JEL Classifications: C61, D78, D81, D83.

1. Introduction

It is well-known that economic agents behave on average risk-neutral for small and repeated decisions, but the most common attitude of economic agents in all important decision-making problems is one generated by risk-aversion (AINGIER 1987). Such a behavior is characteristic for large gains as well as large losses.

Rational agents are goal oriented, they have values and reference points and base their decisions on uncertain future. In general, they are confronted with various sources of risk and uncertainty, which are generally different in different contexts.

The decision adopted is not made independently but jointly with other decisions, which place the agents in risky situations. Decisions taken to avoid, even partially, a source of risk can be affected by the presence of others.

Uncertainty always attends the risk. It must distinguish between quantifiable risks (when the objective probabilities are supposed to be known) and inherently unmeasurable uncertainties (when the objective probabilities are not given in advance). In other words, it must distinguish between decisions under risk and decisions under uncertainty.

¹Financial support from the Spanish Ministry of Education and Science through grant **PTA-2003-02-00877** and **FEDER** is acknowledged.

It is well-known that endogenous risks are generated and amplified within the system, in contrast with exogenous risks which generally refer to shocks that arrive from outside the system. Traditionally, the risk-aversion is equivalent to the concavity of the utility function (viewed as the measure upon which the agent bases his decisions) or a decreasing marginal utility. Formally speaking, it means that for any arbitrary risk, the agent will prefer the sure amount equal to the expected value of the risk rather than the risk itself. However, the condition of concavity is just a way of expressing risk-averse preferences.

In the literature on risk, two polar cases can be distinguished according to the degree of risk-aversion exhibited by the economic agent: i) the risk-neutral case; and ii) the infinitely risk-averse case. These are restrictive theoretical assumptions, which do not apply for economic studies leading to risk assessment, risk control, and risk management in a dynamic stochastic environment.

The objective of this paper is to analyze how the traditional results are modified when the agents' risk behavior depends on the system dynamics in the past and future as well as on their individual risk perception. It offers an elegant solution to the inconvenients that arise when modelling the risk-aversion as in ARROW-PRATT traditional approach.

2. Problem Statement

Consider a rational risk-averse decision-maker characterized by a consistent and efficient outcome oriented behavior (DREZE 1990; WALSH 1996) who faces uncertainty about the evolution of a dynamic stochastic environment. His objective is to drive the system as close as possible to a desired reference trajectory by employing a set of control instruments in an optimal way. The instruments are selected on the basis of a non-decreasing endogenous information set.

Facing complex decisions, the decision-maker adopts a closed-loop strategy, the information being utilized in real time. In this case, the optimal policy does not require some large periods of engagement. The control rules are sensitive to the choice of the working horizon. This may be the case when the relevant information acquisition cost is high, most likely due to permanent random shocks in the system or because of the slow inertia of the environment. The decision-maker tries to reduce the uncertainty related to the choice of his actions by acquiring information from the beginning of the control to the moment of decision. He has the possibility to learn from errors and to make a self-evaluation of his actions during the period of control.

The feedback control responds not only to the effects of random inputs, but also to the measurement errors as well. It is thus not necessary to be able to identify and measure the sources of disturbance. The feedback learning process is progressive, allowing to adjust the decision-maker's ex-ante anticipations about the future trend of the system. In this way, the difference between actual and assumed system characteristics is minimized by monitoring its random fluctuations.

The degree of information embedded in the observation of the state variable generally depends on the selected values for the control instruments, so that the extent of learning about the latent parameters can be directly influenced by the decision-maker. This is the context of a rational active learning which allows for the decision-maker to experiment. In a multiperiod setting, the decision-maker can learn about the consequences of his actions through experimentation. It helps to better anticipate the trend of the system, and thus to avoid undesirable scenarios in the future.

The decision-maker bases his actions on his state of knowledge at the point where the actions are taken. He has some influence over the rate at which information arrives. His behavior may generate information. The active learning makes the decision-maker more experienced over time. However, despite of potential benefits from active learning in stochastic optimization

problems, the potential for learning is limited if the model is noisy (EASLEY AND KIEFER 1988; KIEFER AND NYARKO 1989).

The paper is organized as follows. Section 3 discusses the general model. Section 4 deals with the choice of the performance criterion ensuring an unique optimal solution for the problem of optimization and control. Section 5 briefly summarizes the traditional approach of ARROW-PRATT (1964, 1971A, 1971B). Section 6 introduces the concept of endogenous risk-aversion by taking into account performances of the system in the past as well as anticipations of the future behavior of the system in the future. We also analyze the case of high potential shifts in fluctuating systems. Several qualitative results on adaptive risk management /perception are given in Section 7. Section 8 introduces the concept of excessive risk-averse decision-maker. A complete characterization of common /distinct types of decision-makers becomes thus possible. Section 9 refines the risk analysis by taking into account potential sensitive periods influencing the choice of the optimal risk-aversion threshold. Section 10 discusses the case of a changing risk profile, when the decision-maker becomes risk-averse, (almost) risk-neutral or risk-lover, depending on the system evolution and his individual risk perception. An interesting analysis about risk-averse preferences based on the decision-maker's type is provided. Section 11 draws several important conclusions and makes suggestions for further research.

3. The Model

The type of model we analyze corresponds to a data generating process which is dynamic, non-linear and managed by a system of discrete simultaneous equations.

Suppose that the economic agent disposes of an instrument (or control variable) with the help of which he tries to control the environment in an optimal way.

Let $x_t \in \mathbf{R}^q$ be the value of the control variable at time t . Note that x_t is not strictly exogenous, in general the actions being dependent variables on the history and current state of the system. Different contexts of decision-making generally call for different actions.

Let $y_t \in \mathbf{R}^p$ be the observable target variable in t , and let $z_t \in \mathbf{R}^r$ be an exogenous variable observed outside the system under consideration, and hence not subjected to the agent's control at the time period t . This may be forecasted but cannot be influenced by the decision-maker.

Whether or not the variable z_t is exogenous depends upon whether or not that variable can be taken as given without losing information for the purpose at hand. Specifically, the exogeneity of the variable z_t depends on the parameters of interest of the decision-maker as well as on the purpose of the model (statistical inference, forecasting, or policy analysis). Variations in the process z_t over time will generate variations in the process x_t .

Denote by $\mathbf{X}_t = \{\dots, x_{-1}, x_0, x_1, \dots, x_t\}$ the history of the process x up to time t , and similarly for \mathbf{Y}_t and \mathbf{Z}_t . We thus allow for the current state variable to depend not only on the agent's current decision but also on an arbitrarily complex history \mathbf{X}_t .

We make the following basic assumptions:

Assumption 1. The evolution of the environment is modelled by a non-linear extended-memory process generated according to the structural state equation:

$$y_t = F(\mathbf{Y}_{t-1}, \mathbf{X}_t, \mathbf{Z}_t, \boldsymbol{\beta}_t, t) + u_t, \quad t \in \mathbf{Z}$$

where $u_t \in \mathbf{R}^p$ (exogenous environmental "white noise") is the specific "risk" modelled by a normal random variable with zero mean-vector and finite variance-covariance matrix $\boldsymbol{\Psi}$.

The parameter vector $\boldsymbol{\beta}_t \in \mathbf{R}^k$ varies according to the information available at time t . It specifies the structure of the model. The shape of F and the value of the parameters of interest are generally determined from the behavior of the decision-maker over time.

In general, y_t is not a Gaussian process. Obviously, nonlinearity between y_t and \mathbf{Y}_{t-1} implies non-normality, and hence an asymmetric dynamic. The variable t plays the role of a synthesis variable in the econometric model.

Assumption 2. The agent's objective is to constrain the system to follow a feasible optimal path (or aspiration level) $\eta \stackrel{\text{not.}}{=} \{y_1^g, y_2^g, \dots, y_T^g\}$ by selecting the control variable x_t in a suitable way.

This is a pre-specified condition which cannot be changed during the period of control. We say that η characterizes the agent's preferences on the outcomes of the system.

Taking into account foreseeable movements in y as well as possible economic constraints, the agent chooses some optimal bounds l_t such that $0 < y_t^g \leq l_t < 1$, $t = 1, \dots, T$.

Since a real time control process is necessarily discrete, this cannot converge with precision to any target value, but only to a neighborhood of it. When the process of control is finished, the decision-maker will obtain a stochastic neighbouring-optimal trajectory which is expected to be close to the optimal path η . More is non-linear the model, more it will be difficult to track the targets.

Remark 1: When the targets are endogenously fixed, these are modelled sequentially, that is, simultaneously with the system dynamics. The targets must be compatible with the state of the system. Constraints may change over time.

Assumption 3. The timing of the control is as follows: At each period t , the agent implements an optimal action x_t , which is a stimulus for the dynamic environment. This is purported to contribute to the stability against deviations from the system equilibrium or to correct unavoidable deviations in the past. A shock u_t is carried out and the agent observes the output y_t (the impulse response) which allows for extracting a dynamic signal about the future trend of the system. The agent employs this signal for a strategic learning (specific to a closed-loop monitoring) in order to reduce his uncertainty about the future trend of the system. The question is: How this signal will influence the agent's risk behavior over time?

The information revealed by the output signal can increase the precision of the next control instrument, and hence to decrease the agent's risk aversion in the future. This output together with the corresponding action provide information on the data generating process.

The uncertainty is reduced only ex-post, that is, only after the informative output-signal has been received. The effect of the shock u_t on the output y_t will disappear gradually in time.

Assumption 4. The optimality of the instruments x_t ($t = 1, \dots, T$) is considered with respect to a global criterion (preference function) $W(y_1, y_2, \dots, y_T)$ which measures the system deviations $\Delta y_t \stackrel{\text{not.}}{=} y_t - y_t^g$. This is supposed twice continuously differentiable, strictly increasing and convex in the feasible area of the model.

Assumption 5. The decision problem is to find the optimal values of the control instruments which minimize the agent's loss function by taking into account the constraint relationships that exist between the controlled, partially controlled and uncontrolled variables.

The agent estimates his optimal policy $\hat{x}(\cdot) \stackrel{\text{not.}}{=} (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_T)$ before knowing the value of y_0 . He therefore obtains a random policy conditional to y_0 :

$$\hat{x}(\cdot) = \arg \min_{x(\cdot)} E_x [W(y_1, y_2, \dots, y_T) \mid y_0]$$

where E_x denotes the expectation with respect to the stochastic process induced by the decision rule x .

It is very likely that difference between ex-ante decisions and ex-post results (i.e., between ex-ante and ex-post optimality) exists.

In practice, the initial state y_0 can be fixed or random. It is crucial to treat the initial value correctly and to measure its impact. Small variations of the initial conditions can have large effects on the long-run outcomes.

This is the classical context, often employed in the literature, where the hypothesis of risk-neutrality for the decision-maker is adopted for the entire period of control.

4. Choice of the Criterion

In order to avoid several local minima, and thus to have a unique solution for the optimal control problem, a necessary condition is to use a strictly convex criterion.

Using the parsimony principle, we seek for the simplest strictly convex loss function. It is the quadratic approximation which satisfies this condition. Interpretation is simple: a quadratic objective function may be considered as a good local approximation of the true preferences, exactly as a model approaches the behavior of the environment around the observed variables. This is reasonable because it induces a high penalty for large deviations of the state from the target but a relatively weak penalty for small deviations.

Nothing impedes to consider an additive recursive criterion, on the one hand, in order to simplify the determination of the formula for the optimal instrument and, on the other hand, because it makes possible to apply the BELLMAN's (1961) optimality principle.

A limited expansion of second order of $W_{[1,T]}(y_1, \dots, y_T)$ around a given feasible point $Y^g = (y_1^g, \dots, y_T^g)$ gives us:

$$W_{[1,T]}(y_1, \dots, y_T) = \Delta Y' K \Delta Y + 2\Delta Y' d + c$$

where

$$K_{(pT \times pT)} \stackrel{not.}{=} \left[\frac{\partial^2 W}{\partial Y \partial Y'} \right]_{Y^g}, \quad d \stackrel{not.}{=} \left[\frac{\partial W}{\partial Y'} \right]_{Y^g}$$

The quadratic performance criterion W being strictly convex and twice continuously differentiable, the matrix K is symmetrical and positive semidefinite.

The decision criterion is a function that puts weight (or measure) on the possible outcomes, indicating their desirability or undesirability.

By definition,

$$W_{[1,T]}(y_1, \dots, y_T) \stackrel{def.}{=} \sum_{t=1}^T W_t(y_t)$$

(additively separable criterion)

where W_t is a quadratic asymmetric loss function given by the following analytical expression:

$$W_t(y_t) = (y_t - y_t^g)' K_t (y_t - y_t^g) + 2(y_t - y_t^g)' d_t$$

with a prime denoting transpose.

Asymmetry derives from the difference in penalty costs which the decision-maker may attach to errors, depending on whether these are errors of shortfall or overshooting about the targets.

Generally, the decision to choose certain parameters K_t and d_t reflects the decision-maker's priorities and also depend on the available quantity of information concerning the future development of the system parameters. However, it is far from probable that the decision-maker will be able to assign values to the weights which represent his preferences correctly.

If the future evolution of the system is unpredictable, then the best weighting matrix K_t which can be selected is the identity matrix, while the best value for d_t is the unity vector. If K_t is not diagonal, then the penalties also attach to the covariances of the state variable deviations from its desired level.

The weights employed are anything but objective, since the deviation of the target variable may be not of the same importance at different moments in time. The idea is to choose the parameters which yield a smoother (i.e., less fluctuating) control, and so a more stable system.

At each period t , the parameters K_t and d_t are updated and new optimal values are chosen to fulfill the requirements of the decision-maker. These requirements are based on policy values presented at each period and do not require any direct information about the actual weighting that the decision-maker may have in mind.

5. Sensitive Criterion to Risk: Static Approach

The notion of utility is fundamental in decision and risk theory. In the theory of utility maximization, the risk situations are measured by taking into consideration the agent's preferences. Thus, a decision in a risky situation (e.g., a lottery) is more risked than another, if the utility of the former is inferior to the utility of the latter, for an agent who is risk-averse.

In the static approach of expected utility (one aggregated period of time), the measure of risk-aversion is given by the ARROW-PRATT index, which requires the existence of a VON NEUMANN-MORGENSTERN utility function.

Let $W_{[1,T]}$ be the usual quadratic (non-symmetrical) criterion, and let U be the global utility of the control described by the following exponential relationship:

$$U(W_{[1,T]}) = \frac{2}{\varphi} [\exp(-\frac{\varphi}{2} W_{[1,T]}) - 1]$$

It verifies:

$$-\frac{U''(W_{[1,T]})}{U'(W_{[1,T]})} = \frac{\varphi}{2}, \quad \forall W_{[1,T]}$$

where a prime denotes the partial derivative with respect to $W_{[1,T]}$. The parameter $\frac{\varphi(W_{[1,T]})}{2}$ can be interpreted in terms of the ARROW-PRATT risk measure of risk-aversion at a particular $W_{[1,T]}$, U being a CARA exogenous utility function.

It is well-known that the non-linearity of the utility function is more commonly represented as risk-aversion. The exponential utility functions are, in this sense, the most widely utilized risk-sensitive utility functions because they can model a spectrum of risk attitudes for the players (CORNER AND CORNER 1995).

This is the case of a totally exogenous risk, when the agent's risk attitude is rigid during the entire working horizon. This type of approach is easy to handle as it allows to work with a mean-variance framework, but unfortunately, constant absolute risk-aversion is not a tenable assumption for a majority of economic /financial models.

The static approach is "myopic" with regard to potential changes in the future behavior of the agent due to inherent fluctuations of the system over time. Input effects are not thus taken into account. In dynamic stochastic environments, this procedure is not appropriate.

The optimal strategy will be φ -dependent:

$$s_{\varphi}^g(\cdot) = \arg \max_{x_1, \dots, x_T} E_0[U(W_{[1,T]})]$$

$$s_{\varphi}^g(t) \stackrel{not.}{=} \hat{x}_t(I_{t-1}) \mid y_0, \quad \forall t = 1, \dots, T$$

where I_t is the information set acquired until time t and updated each time as new observation becomes available. We define the sensitive criterion to risk by:

$$\gamma_0(\varphi) \stackrel{def.}{=} E_0[U(W_{[1,T]})]$$

It is easy to see that if the term $\varphi V_0(W_{[1,T]}(y_1, \dots, y_T))$ is small, then a limited expansion of second order justifies the following approximation:

$$\gamma_0(\varphi) \approx -E_0[W_{[1,T]}(y_1, \dots, y_T)] + \frac{\varphi}{4} V_0(W_{[1,T]}(y_1, \dots, y_T))$$

We distinguish three distinct cases:

i) If $\varphi < 0$, then the function $U(W_{[1,T]})$ is negative, strictly concave and decreasing. Note that the sign of U has no particular importance (conceptually, this is unimportant). When one maximizes $\gamma_0(\varphi)$, the above approximation shows that the variability is penalized; the agent is afraid of large accidental values of $W_{[1,T]}$.

There is a strong correlation between pessimism and risk-aversion. Indeed, if φ decreases, the agent is convinced that some large values of $W_{[1,T]}$ appear more and more frequently. Therefore, the agent will have some pessimistic expectations, being characterized by a significative loss aversion.

ii) If $\varphi > 0$, then the function $U(W_{[1,T]})$ is negative, strictly convex and decreasing. The line $W = -2/\varphi$ is an horizontal asymptote towards infinity of the curve $U(W_{[1,T]})$ and $(0, 0) \in Graph(U)$. The situation is opposed to the previous case. There is rather an interest for moderate values of $W_{[1,T]}$ than for extreme values. We say that the agent is optimistic (or risk-lover).

iii) If $\varphi = 0$, then one obtains (using the HOSPITAL's rule):

$$\lim_{\varphi \rightarrow 0} \gamma_0(\varphi) = -E_0[W_{[1,T]}(y_1, \dots, y_T)]$$

Thus:

$$\min E_0[W_{[1,T]}(y_1, \dots, y_T)] \Leftrightarrow \max \gamma_0(\varphi)$$

the term in the left-hand being the usual criterion from the risk-neutral case.

We have that $U(W_{[1,T]}) \xrightarrow{\varphi \rightarrow 0} -W_{[1,T]}$. In other words, the agent's problem is to minimize the expected loss if he is risk-neutral, or to maximize the expected utility of the loss if he is risk-averse. The shape of the utility function will determine the magnitude of the departure from the risk-neutrality case.

Although often criticized for its limitations in explaining and evaluating risky choices, the most widely used hypothesis for the analysis of economic behavior is risk-neutrality. This is a potential borderline case which must be envisaged with prudence in complex dynamic stochastic systems.

If $\varphi_1 \leq 0 \leq \varphi_2$, then one can write the following inequalities:

$$U_{\varphi_1}(W_{[1,T]}^*) \leq U_0(W_{[1,T]}^*) \leq U_{\varphi_2}(W_{[1,T]}^*)$$

We give below a graphical illustration of the three above cases.

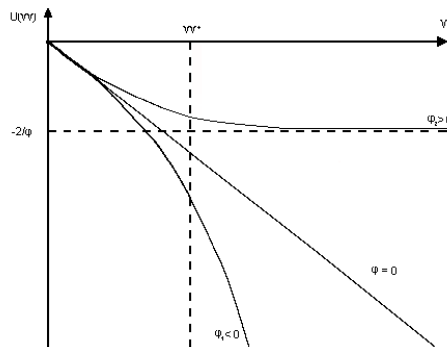


Figure 1

While in a static environment one can imagine an agent that does not learn during the process and maintains a fixed utility function, this type of behavior would be less likely to produce a positive payoff in a very dynamic environment. There is by now accumulating evidence that agents differ substantially in their risk preferences and also that CARA risk preferences are not a good description of agents' revealed risk preferences. CARA utility functions cannot capture the full behavior of agents towards risk. This is because, in this context, the risk premium is assumed constant across shifts of a lottery. Numerous empirical studies have demonstrated the need for more flexible and realistic utility functions than the CARA. It is necessary to have a more profound understanding of the way the evolution of the dynamic system can affect the risk perception, risk attitude, and risk behavior of rational decision-makers.

6. Adaptive Endogenous Risk-Aversion

6.1. General Framework

Although there is a large amount of literature on risk (FRIEDMAN AND SAVAGE 1948; BERNOULLI 1954/1738; PRATT 1964; ARROW 1951, 1971; KAHNEMAN AND TVERSKY 1979; ROSS 1981; YAARI 1987; KIMBALL 1993; RABIN 2000; RABIN AND THALER 2001; CHEVE AND CONGAR 2002; EISENHAUER 2003; EISENHAUER AND VENTURA 2003; NOVOSELOV 2003; KIRKWOOD 2004; YAO ET AL. 2004; FILBECK ET AL. 2005; NIELSEN 2005; BOMMIER 2006; CHAMBERS AND QUIGGIN 2006; EISENHAUER 2007; JERKER 2007, AMONGST OTHERS), there is no theoretical and empirical work for comparing the degree of risk-aversion of rational decision-makers in the context of controlled dynamic stochastic environments. The purpose of the present study is to extend the ARROW-PRATT traditional approach, which takes into account only attitudes towards small exogenous risks, to the context of potentially high endogenous risks that are under the control of the decision-maker.

The attitude towards risk is closely linked to the history of the process as well as to the agent's anticipations on the system behavior in the future. We formalize this point of view for a general class of models and we detail the positive implications in the context of a varying-risk environment which evolves over a finite discrete-time horizon. Because the system behavior changes continuously, the agent's risk aversion will also change, particularly when these changes are significative. Based on a dynamic adjustment process of the system deviations, the agent's objective is to diminish his risk-aversion over time. He moves from risk avoidance to risk elimination.

The agent makes decisions according to his attitude towards risk. The dynamicity of the environment might affect the design of the agent's utility in a subtle way. We consider an exponential utility function (performance criterion) U_t depending on the evolutive loss $W_{[1,t]}$ and a dynamic absolute risk-aversion index φ_t :

$$U_t(W_{[1,t]}, \varphi_t) \stackrel{def.}{=} \frac{2}{\varphi_t} [\exp(-\frac{\varphi_t}{2} W_{[1,t]}) - 1], \quad t = 1, \dots, T$$

with

$$W_{[1,t]} \stackrel{def.}{=} \sum_{s=1}^t W_s(y_s)$$

It follows that:

$$-\frac{U_t''(W_{[1,t]}, \varphi_t)}{U_t'(W_{[1,t]}, \varphi_t)} = \frac{\varphi_t}{2}$$

where a prime denotes the partial derivative with respect to $W_{[1,t]}$.

Therefore, $\frac{\varphi_t(W_{[1,t]})}{2}$ measures locally (at the point $W_{[1,t]}$) the agent's risk-aversion, U_t being a CARA endogenous time-varying utility function. This new approach extends the case of time-varying preferences embedded in the recursive utility concept introduced by EPSTEIN and ZIN (1989). The key concept here is the endogeneity of the agent's risk aversion index, which is particularly important in the policy settings. Time-varying preferences is only a secondary aspect, a direct consequence of the model specification.

We define the sensitive criterion to dynamic risk at time t by:

$$\gamma_0(\varphi_t) \stackrel{def.}{=} E_0[U_t(W_{[1,t]}, \varphi_t)]$$

We have:

$$\min E_0[W_{[1,t]}(y_1, \dots, y_t)] \Leftrightarrow \max \gamma_0(\varphi_t)$$

To illustrate why this new risk framework is informative about how attitudes towards risk of the decision-maker change in a dynamic environment, we prove several theoretical results that support the need to review and improve the traditional approach of ARROW-PRATT (1964, 1971A, 1971B).

Proposition 1. A more (less) risk-averse decision-maker is characterized by a smaller (higher) local utility level.

Proof. In the appendix. ■

This result characterizes the agent's preferences in function of his risk-aversion degree over time. The objective is not to exceed a fixed risk threshold value over which the agent becomes excessively risk-averse (see **Section 8**). He will therefore be characterized by an over-pessimism, that is, an induced deviating behavior with respect to perceived states of the system.

At each control period, the interest of the agent is to choose an optimal action with a higher ex-ante expected utility. This prevents to exceed some threshold utility levels under which the agent's ex-post preferences are suboptimal. We provide below a suggestive graphical illustration of the **Proposition 1**.

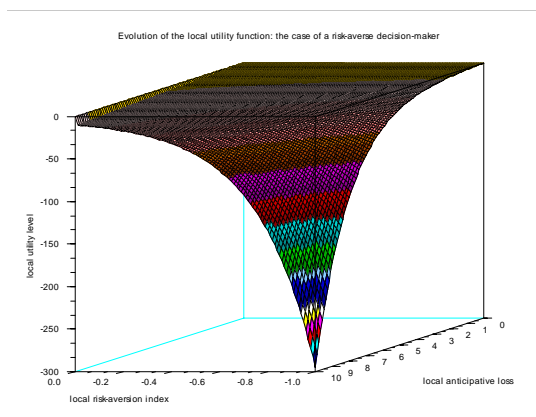


Figure 2

In an evolving environment, the utility function obviously does not remain constant over time. It changes at each stage of the control, even if not significantly (the case of smooth preferences). It does not exclude the possibility to have the same utility level for various periods. In this direction, we note a large class of decision rules which is representable by a variation of the utility between each two consecutive periods (GILBOA 1989). It comes to consider the agent at different periods as though he were different individuals (ALLAIS 1947). It is important to stress that the current decision can affect the agent's utility level in the future.

The agent can benefit from the learning of preferences. This idea was developed in first by STIGLER and BECKER (1977). However, it is far from probable that the agent exactly

maximizes his utility at each stage of the control. We rather face a nearly optimization behavior, where the control variable is continuously and optimally adjusted to maximize some objective function (VAN DE STADT ET AL. 1985; VARIAN 1990; LELAND 1990).

Generally speaking, the utility depends on the purposes for which it is developed. It does not exist but for the agent, and thus, it has a subjective character. This is derived from individual preferences. It is very rare in econometrics to be able to fully specify the utility function. No economic agent has sufficient a priori knowledge to fully specify his utility. The stochastic disturbance in the system produces random shocks in the agent's preferences over time. It is also important to note here that the decision-makers usually violate the expected utility theory when there is risk in the choices.

Before introducing new theoretical considerations on the concept of risk-aversion, a natural question arises: How the standard conclusions on risk assessment and management modifies when modelling the attitude towards risk of the decision-maker according to past and expected future dynamics of the stochastic system?

6.2. Risk-Aversion Based on a Truncated History of the Process and Rational Anticipations of the System Behavior in the Future

In a dynamic context, the agent can fully take advantage from the learning benefits. He can influence the likelihood of the system states by using a reinforcement learning strategy. The agent is not myopic in the sense of expecting. Future anticipations play an important role in how the agent will decide what strategic actions and optimal risk to take. Risk is endogenous by nature. Its estimation typically depends on forecasts of the future state of the system.

Suppose that the agent is a strategic decision-maker. He thinks about the future taking into account feedback-and-forward information. Depending on the way the agent perceives future outcomes, both risk sensitivity and optimal decisions will be affected during the process of optimization and control. An increased power of prediction is an efficient method in order to reduce the uncertainty about the future system trajectory. The forecast is updated each time as new observation becomes available. The future can thus be regarded as an extended present.

A correct evaluation of the past evolution of the system is crucial for making optimal predictions in the future. This is necessary for an optimal assessment of the agent's risk aversion over time. Fluctuations in the system target variable will generate a fluctuating risk-aversion of the agent.

We make the following useful notations:

$$S_{t, p_d} \stackrel{\text{not.}}{=} \|y_{t-1} - y_{t-1}^g\|^2 + \dots + \|y_{t-k_1} - y_{t-k_1}^g\|^2$$

(the sum of squared past deviations at time t)

$$S_{t, a_f_d} \stackrel{\text{not.}}{=} \|y_{t|I_t}^a - y_t^g\|^2 + \dots + \|y_{t+k_2|I_{t+k_2}}^a - y_{t+k_2}^g\|^2$$

(the sum of squared anticipated future deviations at time t)

$$S_{t, w_p_d} \stackrel{\text{not.}}{=} \|y_{t-1} - y_{t-1}^g\|^2 L_{t-1} + \dots + \|y_{t-k_1} - y_{t-k_1}^g\|^2 L_{t-k_1}$$

(the weighted sum of squared past deviations at time t)

$$S_{t, w_a_f_d} \stackrel{\text{not.}}{=} \|y_{t|I_t}^a - y_t^g\|^2 \bar{L}_t + \dots + \|y_{t+k_2|I_{t+k_2}}^a - y_{t+k_2}^g\|^2 \bar{L}_{t+k_2}$$

(the weighted sum of squared anticipated future deviations at time t)

where y_{t+i}^g ($i = 0, \dots, k_2$) represent fixed targets in the future (taking into account foreseeable movements in y), $y_{t+i|I_{t+i}}^a$ ($i = 0, \dots, k_2$) are expected values of the target variable at time $t+i$ based on non-decreasing endogenous information sets I_{t+i} and L_{t-j_1} ($j_1 = 1, \dots, k_1$), \bar{L}_{t+j_2} ($j_2 = 0, \dots, k_2$) are strategic weights attached to the system deviations (in the past and future) with respect to the equilibrium path η .

In a real decision-making problem, the forecast must be as accurate and efficient as possible. A necessary preliminary step for the decision-maker in order to optimally choose the target path is to make some a priori expectations on the future evolution of the system based on its past performances. A question arises: Are the current and past values of the process y_t sufficient to forecast y_{t+k} ($k = 1, \dots, k_2$)?

Ex-ante expectations refer to those which held prior to the acquisition of information and generally imply a discrete-time process of tatonnement. These must be unique and in accord with the agent's observations and generally are dependent on the initial value of the state variable. The more they are distant in time, the more they are difficult to assess (due to the extreme uncertainty of the far future). Ex-ante and ex-post forecast errors are viewed as indicators of uncertainty of the decision-making process. Because expectations differ across decision-makers, they are generally characterized by distinct degrees of risk-aversion.

We are now in a position to give a definition of the agent's risk aversion index by taking into account past performances of the system (a truncated history) and rational anticipations of the system behavior in the future.

Definition 1. Using t to denote time, the absolute risk-aversion index φ_{t,p_f}^{r-a} evolves according to:

$$\varphi_{t,p_f}^{r-a} \stackrel{def.}{=} \frac{S_{t,w_{p-d}} + S_{t,w_{a-f-d}}}{\sqrt{(S_{t,p-d} + S_{t,a-f-d})^2 + l}}, \quad t = 1, \dots, T$$

where $l \geq 1$ is an integer characterizing the agent's type, and the parameters $L_{t-j_1}, \bar{L}_{t+j_2}$ ($j_1 = 1, \dots, k_1; j_2 = 0, \dots, k_2$) verify the following inequalities:

$$-1 < L_{t-1} \leq \dots \leq L_{t-k_1} \leq 0, \quad -1 < \bar{L}_t \leq \dots \leq \bar{L}_{t+k_2} \leq 0$$

with

$$1 \leq k_1 < T; \quad k_2 \geq 0; \quad 1 \leq k_1 + k_2 \leq T - 1$$

The weights may differ across economic agents. They are updated each time as new observation becomes available. The agent gives a higher importance to the past and future deviations which are closer to the moment of implementation of a new optimal action. The smaller the weight, the higher the importance given by the agent to the system deviation from his local objective.

Taking into account the potential destabilizing role of a long memory of the process, the agent takes into account only a limited history in the risk analysis. Distant past observations might increase significantly the bias of the estimates in the econometric model. These provide an imprecise signal for the agent.

It generally exists an arbitrary element as regards the choice of the backward lag k_1 . The objective is to find the better compromise between fit and complexity. On the other hand, distant forecasts are difficult to formulate due to unpredictable external disturbances which affect the system performance. The larger the forward lag k_2 , the higher the prediction errors.

The risk can be interpreted like the agent's degree of confidence in the future. It decreases with uncertainty. It is only by taking into account the past and the expected future that the agent can optimally evaluate the risk in a dynamic system. It allows for a better risk management at each control period.

Despite the fact that risk-averse agents hate uncertainty whereas the risk-neutral are indifferent (their behavior remains unchanged), they often place a lower value on forecasting than risk-neutral agents do.

Both objectivity and subjectivity characterizes the agent's risk behavior over time. Its complexity is given by the changing environment design and the agent's typology.

Remark 2. Note the local character of the agent's risk aversion. This is defined for a neighborhood of the fixed targets $y_{t-1}^g, \dots, y_{t-k_1}^g$ and $y_t^g, \dots, y_{t+k_2}^g$, respectively. It will therefore exist some neighborhood effects of the system dynamics on the agent's risk behavior.

There is a strong relationship between the choice of the targets and the agent's attitude towards risk. Smaller targets are correlated with a smaller degree of risk-aversion, and hence a higher sensitivity to risk of the agent.

Remark 3. The higher (lower) the degree of risk-aversion at time t , the lower (higher) the absolute risk-aversion index $\varphi_{t,p-f}^{r-a}$. It is important to distinguish between local risk-aversion (at time t) and global risk-aversion (over the whole period $[1, T]$).

Remark 4. There is no loss of generality in considering that the absolute risk-aversion index $\varphi_{t,p-f}^{r-a}$ takes values in $(-1, 0]$ because one can always find an isomorphism from $(-1, 0]$ to $(a, 0]$, with $a < -1$ a fixed real number.

Remark 5. By definition, two system deviations with respect to the agent's fixed targets are comparable in magnitude if and only if their ratio is very close to 1.

Remark 6. Large deviations of the system with respect to the agent's fixed targets are expressed by the following inequalities:

$$\|y_{t-j_1} - y_{t-j_1}^g\| \gg 1, \quad j_1 = 1, \dots, k_1; \quad \|y_{t+j_2}^a|_{I_{t+j_2}} - y_{t+j_2}^g\| \gg 1, \quad j_2 = 0, \dots, k_2$$

These are generally caused by systematic random shocks within and outside the system. It is important to note here that large deviations in the past could lead to less confidence in the future.

This new approach has strong implications on the agent's adaptive behavior towards risk in a fluctuating environment. It improves the point of view of previous work in the literature on risk management /assessment /perception (COHEN 1995; EECKHOUDT AND GODFROID 2000; ISAAC AND DUNCAN 2000; COX 2001; RABIN AND THALER 2001; CHEVE AND CONGAR 2002; EISENHAEUER 2003; EISENHAEUER AND VENTURA 2003; NOVOSELOV 2003; KIRKWOOD 2004; NIELSEN 2005; CHAMBERS AND QUIGGIN 2006; EISENHAEUER 2007, AMONGST OTHERS).

Proposition 2. Risk perception changes with the behavior of the system as well as the way the agent interprets its evolution.

Proof. In the appendix. ■

In the real world, the risk is not uniformly distributed over the entire working horizon. In other words, the risk-aversion index is non-monotonous over time. The agent becomes more or less risk-averse according to the fluctuation of the system state.

Uncertainty does not necessarily diminish over time. It verifies for simple systems as well as for complex dynamic environments. We give below a suggestive graphic in this sense.

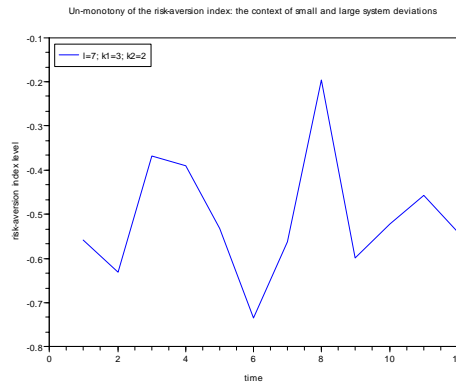


Figure 3

Note that for smaller deviations of the system with respect to the fixed targets, one obtains a smaller value of the quadratic loss function. The agent more readily accepts the risk when the loss is decreased. His behavior is thus characterized by a smaller degree of loss aversion.

Remark 7. In dynamic systems with a smooth evolution (but not only) it may be possible to obtain the same value for the risk-aversion index at distinct periods of time. Mathematically speaking, this behavior is due to the non-injectivity of the index function.

We illustrate below two scenarios in this sense, when all system deviations are small and, respectively, high.

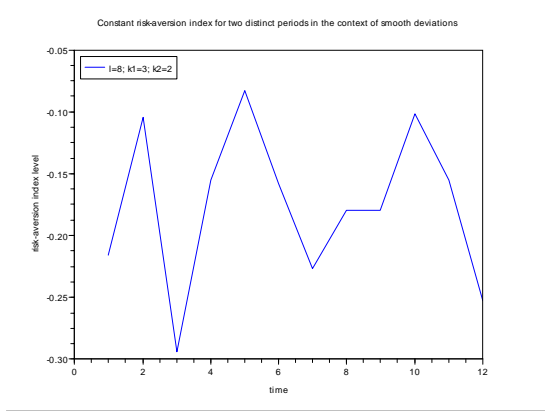


Figure 4

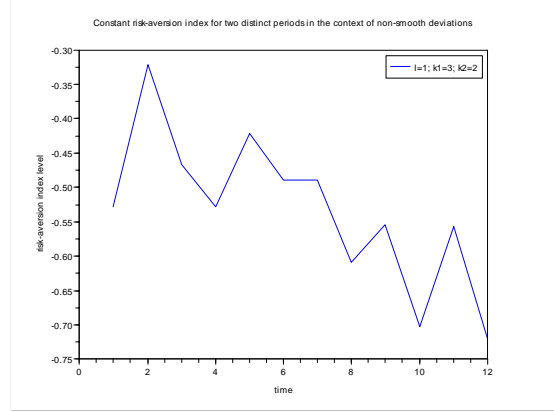


Figure 5

6.3. Risk-Aversion Based on a Truncated History of the Process

Suppose that the agent takes into account only a truncated history of the system when estimating the risk-aversion index. More exactly, only the most informative information is considered. In this particular context, it is proposed the following definition for the index:

Definition 2. Using t to denote time, the absolute risk-aversion index $\varphi_{t,p}^{r,a}$ evolves according to:

$$\varphi_{t,p}^{r,a} \stackrel{\text{def.}}{=} \frac{\|y_{t-1} - y_{t-1}^g\|^2 L_{t-1} + \dots + \|y_{t-k_1} - y_{t-k_1}^g\|^2 L_{t-k_1}}{\sqrt{(\|y_{t-1} - y_{t-1}^g\|^2 + \dots + \|y_{t-k_1} - y_{t-k_1}^g\|^2)^2 + l}}, \quad t = \overline{1, T}; \quad k_1 = \overline{1, T-1}$$

where $-1 < L_{t-1} \leq \dots \leq L_{t-k_1} \leq 0$ are strategic weights attached to the system deviations with respect to the optimal path $\{y_{t-1}^g, \dots, y_{t-k_1}^g\}$, and $l \geq 1$ is a fixed integer characterizing the agent's type.

An interesting case, very often encountered in economic /financial applications, is that of comparable deviations of the system, when these are all either high or small in magnitude. We formulate and prove a theoretical result in this direction.

Proposition 3. Suppose that all system deviations are comparable in magnitude. In this case, the agent is characterized by a higher degree of risk-aversion at the same stage of the control if a smaller value for the backward lag k_1 is considered in the estimation of the local index $\varphi_{t,p}^{r,a}$.

Proof. In the appendix. ■

This result reveals the role played by the history of the system in estimating the agent's degree of risk-aversion.

When all system deviations are small and comparable in magnitude, we say that the system is characterized by a smooth evolution (i.e., without significant shifts over time).

By contrast, a system with relevant shifts is characterized by large deviations from the fixed targets. The inertia of the system can generate either small or large fluctuations. The degree of inertia of the system is quantified by the sum of the squares of the distances of the outputs from the fixed targets.

It is important to note here that the shift amplitude is considered with respect to the optimal reference path η . We give below a graphical illustration of the **Proposition 3**.

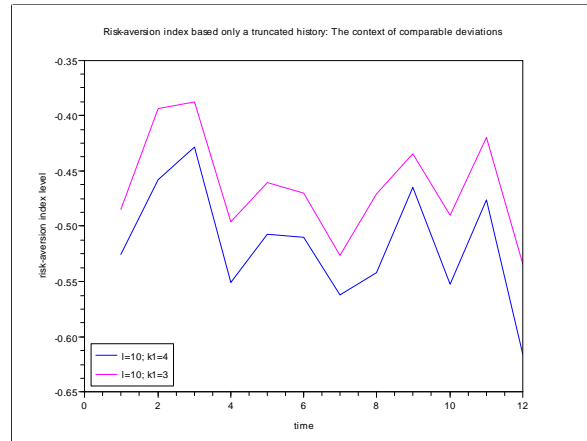


Figure 6

6.3.1 High Potential Shifts

Shift happens. The agent learns from failures. Every high deviation, seen as a failure, is analyzed in order to avoid unexpected fluctuations of the system in the future. A large deviation from the expected outcome is perceived as a shift by the agent.

We call a high positive shift, the transition of the system from consecutive small levels of performance to a high level one. In the opposite case, we call this a high negative shift. The risky shift phenomenon is one of the key issues in economics, and in the study of dynamic systems in particular.

The agent's objective during the period of control is to obtain smooth shift values with respect to the fixed targets. This contributes to the equilibrium and stability of the dynamic system.

The concept of risk-aversion is appropriate to dynamic stochastic environments whose behavior change significantly over time. This is the case of high fluctuating systems.

Proposition 4. Consider the following two opposite scenarios: the transition of the system is from consecutive small (large) deviations to a large (small) deviation. For this type of scenarios, the degree of risk-aversion is highly dependent on the weights the agent will attach to the large deviations of the system.

Proof. In the appendix. ■

Sudden significative changes in the system behavior can affect differently the agent's risk aversion. Large shifts are correlated with large deviations of the system with respect to the fixed targets. The shift amplitude generally depends on the type of transition.

We give below three distinct attitudes towards risk depending on the agent's individual perception about large fluctuations of the system.

i) the large deviation of the system at time $t - 1$ (in the context of the first scenario) is much more important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.

ii) the large deviation of the system at time $t - 1$ (in the context of the first scenario) is much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.

iii) the large deviation of the system at time $t - 1$ (in the context of the first scenario) is either much more or much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario.

The following three graphics illustrate numerically the **Proposition 4**.

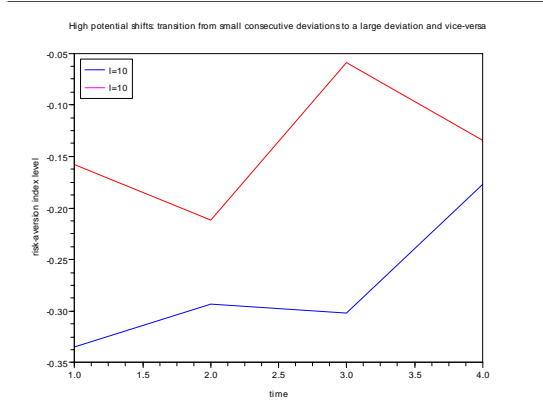


Figure 7: Case a

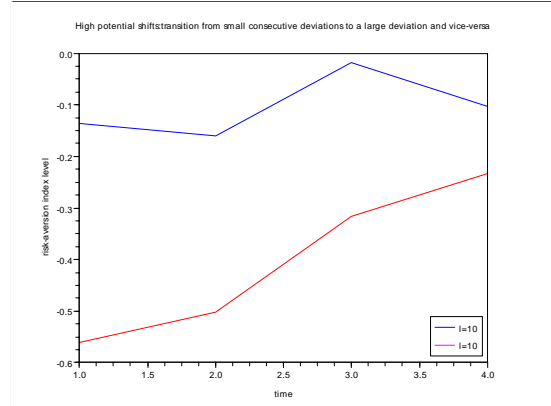


Figure 8: Case b

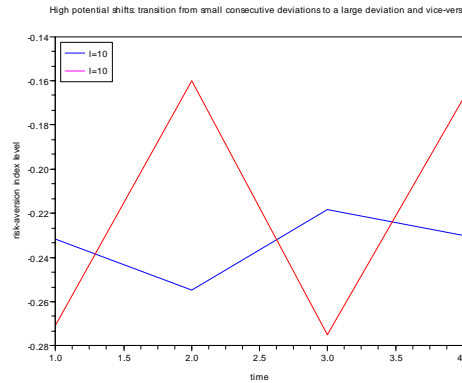


Figure 9: Case c

Note that for the three above graphics, the risk-aversion index value at time $t = 1, \dots, 4$ corresponds, respectively, to the following distinct cases:

- i) system transition from four large (small) consecutive deviations to a small (large) deviation;
- ii) system transition from three large (small) consecutive deviations to a small (large) deviation;
- iii) system transition from two large (small) consecutive deviations to a small (large) deviation;
- iv) system transition from a large (small) to a small (large) deviation.

The sense of transition is perceived differently by a risk-averse agent. The initial state of transition plays a crucial role in estimating his degree of risk-aversion at a given period of time. The agent's objective is to anticipate significant fluctuations in the target variable during the period of control.

6.3.2. Fluctuating System

In what follows, we analyze the impact of the system fluctuations on the agent's attitude towards risk during the period of control.

A natural question arises: In the real world, does more often envisage to pass from a high deviation of the system to a small one or the opposite? The answer is not obvious. When the

system inertia is high, the first scenario is not easy to be carried out. An effective control will also prevent the realization of the second scenario.

The agent must avoid to deviate from the fixed targets during the period of control. In order to reach this strategic objective, a robust control strategy (i.e., with a low sensitivity to changes in the input data) is necessary.

We give below two suggestive graphics which illustrate the agent's risk behavior in the context of a fluctuating system.

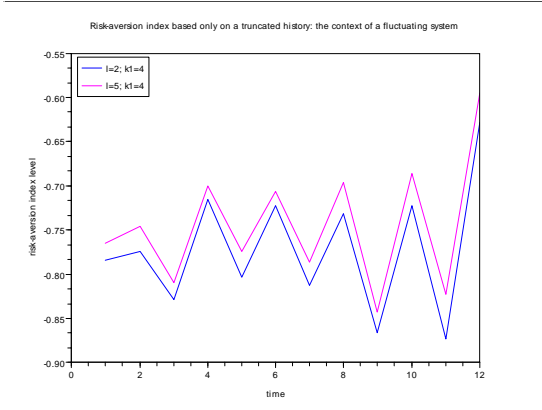


Figure 10

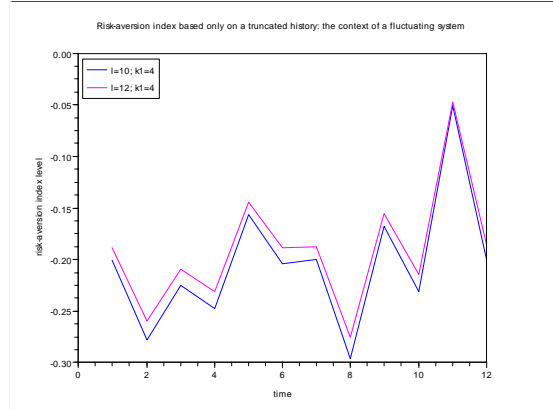


Figure 11

The first (second) graph corresponds to a highly (weakly) risk-averse agent who implements his optimal policy in the context of a dynamic environment characterized by a large inertia. Effective management of the system inertia will have a positive impact on the agent's risk attitude.

It is well-known the role of risk preferences in explaining financial system inertia. It is important to distinguish here between exogenous and endogenous inertia. These influence differently the agent's risk behavior and his optimal decisions over time. The endogenous inertia present in all dynamic processes is one of the most critical parameters affecting the evaluation of alternative policies. There is generally an inherent inertia effect of the system due to its capacity of reaction. The degree of inertia is modelled as an endogenous rational choice made by the agent.

It is interesting to analyze the correlation between the agent's risk behavior and his degree of inertia over time. This is an exciting fruitful area for future research, with strong implications in terms of policy-making.

6.3.2. Risk-Aversion Based on a Progressive History of the Process

Suppose that the agent has the interest to use a progressive history of the process for estimating the risk-aversion index. It may be the case where more information is needed in improving the risk assessment process. Past overall performances of the system are taken into account in measuring the agent's degree of risk-aversion. The index value is updated each time as new observation becomes available. In this particular context, we introduce the following definition for the risk-aversion index:

Definition 3. Using t to denote time, the absolute risk-aversion index $\varphi_{t,w}^{r-a}$ evolves according to:

$$\varphi_{t,w}^{r-a} \stackrel{def.}{=} \frac{\|y_{t-1} - y_{t-1}^g\|^2 L_{t-1} + \dots + \|y_0 - y_0^g\|^2 L_0}{\sqrt{(\|y_{t-1} - y_{t-1}^g\|^2 + \dots + \|y_0 - y_0^g\|^2)^2 + l}}, \quad t = 1, \dots, T$$

with $l \geq 1$, a fixed integer which characterizes the agent's type, and $-1 < L_{t-1} \leq L_{t-2} \leq \dots \leq L_0 \leq 0$, some strategic weights attached to the system deviations with respect to the reference path $\{y_{t-1}^g, y_{t-2}^g, \dots, y_0^g\}$.

Denote by y_0^g the target variable at time $t = 0$ employed by the agent in the previous optimization scheme. This is chosen to be a small value.

The initial state of the process influences the system trajectory and implicitly the agent's risk attitude during the period of control. Agent's sensitivity to risk (generally non-uniform over time) may be captured by the shape of the agent's index curve. An empirical analysis in the case of a free-floating initial state can illustrate the agent's risk-sensitive behavior. One can imagine two distinct scenarios in this direction:

- i) y_0 is small, in which case the deviation $\| y_0 - y_0^g \|$ is small in magnitude, and thus the risk-aversion index $\varphi_{1,w}^{r,a}$ has a high value.
- ii) y_0 is high, in which case the deviation $\| y_0 - y_0^g \|$ is high in magnitude, and thus the risk-aversion index $\varphi_{1,w}^{r,a}$ has a small value.

In the real world, the relationship between the agent's reaction to the perceived states of nature and his attitude towards risk is complex. This is a consequence of the importance the agent places on the system states. The observability of the system is dependent only on the system states and the system output.

Generally, risk-aversion makes the reaction stronger than risk-neutrality. However, when both agents are risk-averse but at very different degrees, the less cautious of the two can have a weaker reaction than in the risk-neutral case. This is an astonishing result that naturally follows from the analysis of the relationship between high deviations and perceived states of nature.

Proposition 5. A risk-averse agent who manages more and more hardly the evolution of the system is characterized by an increasing risk-aversion over time.

Proof. In the appendix. ■

As the number of consecutive failures increases, the economic agent will become less and less confident, and thus his degree of risk-aversion will be more and more raised. It is the context when it is possible for the agent to become excessively risk-averse during the period of control.

This type of scenario is possible when large deviations are correlated with a high inertia of the system. The potential for learning is limited in this particular context.

In a noisy environment, the actions implemented by the agent have a weak impact on the irregular trend of the system.

Reasoning by analogy, we conclude that in the case where the agent controls better and better the system trajectory, he will become more and more confident over time. His degree of risk-aversion will then be less and less raised. It is important to note that the two above scenarios are not symmetrical.

We give below a graphical illustration of these two distinct scenarios when the amplitude of the system deviation at time $t = 1$ is relatively small or very large.

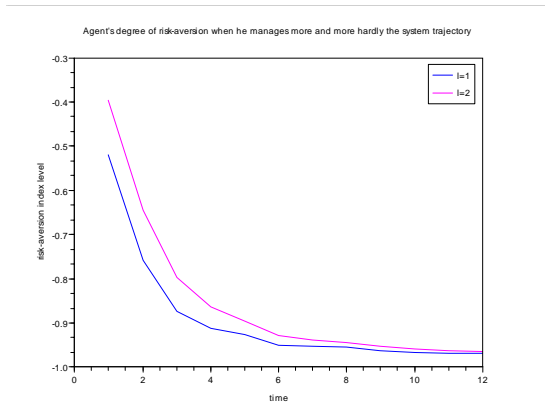


Figure 12

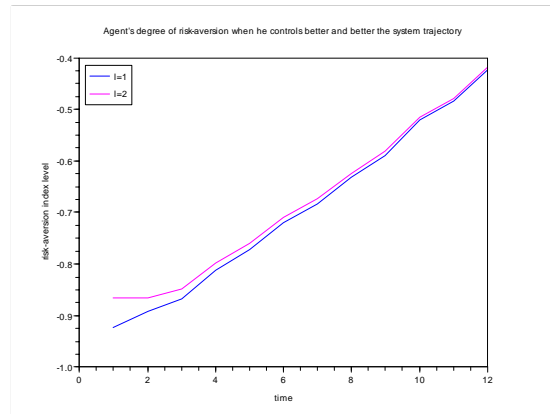


Figure 13

It is interesting to remark that in the case where the agent more and more hardly controls the system trajectory, the index value decreases more quickly, while in the opposite case, it increases slowly. In other words, it is more easy to loss the control of the system than to improve its trajectory.

For the second scenario, the shape of the index curve is almost linear, while for the first scenario, this is characterized by an important degree of nonlinearity. This is a surprising result, far from intuitive.

6.4. Risk-Aversion Based on Rational Anticipations of the System Behavior in the Future

In this section, we analyze the case where the agent takes into account only rational anticipations about the system behavior when modelling the risk estimates over time. It implies a risk-attitude adjustment during the period of control.

This may be the case where the agent looks towards the future rather than evaluating the evolution of the system in the past. Deviations from past targets are “ignored” except to the extent that they affect the future.

In this particular context, the absolute risk-aversion index is defined as follows:

Definition 4. Using t to denote time, the absolute risk-aversion index $\varphi_{t,f}^{r-a}$ evolves according to the following relationship:

$$\varphi_{t,f}^{r-a} \stackrel{def.}{=} \frac{S_{t,w-a-f-d}}{\sqrt{(S_{t,a-f-d})^2 + l}}, \quad t = 1, \dots, T.$$

6.5. Comparing the four Dynamic Risk-Aversion Definitions

Proposition 6. In a dynamic environment, the agent’s degree of risk-aversion varies according to his adopted strategy to manage endogenous uncertainty.

Proof. In the appendix. ■

Future uncertainty affects the agent’s risk attitude. Anticipation of higher (smaller) system deviations with respect to the fixed targets corresponds to a higher (smaller) degree of risk-aversion only if the agent attaches appropriate weights to these deviations.

More information helps for better forecasting the risk but it does not necessarily decrease the agent’s uncertainty. Adaptive dynamics of the system correlated with its inherent inertia can produce significative changes in the agent’s risk behavior. The shorter the planning horizon, the easier it is to be accurate in forecasting. It is therefore expected to obtain a better evaluation of the risk.

Errors from learning occur. These can be improved but not eliminated when analyzing real phenomena. The process of learning is generally non-monotonous over time.

The agent dynamically adjusts his actions in order to minimize the distance between the actual state of the system and the fixed target. Large endogenous risks are not easy to avoid due to their correlation with high fluctuations of the system.

Permanent or transitory shocks in observed movements of the system will affect the agent’s degree of risk-aversion. Contrary to what is generally believed or intuition would suggest, the sensitivity to risk can be lower when the agent does not explicitly integrate past information in the risk-aversion index formula. The observed outputs depend upon his ability to anticipate the future.

A natural question arises: How to define the optimal trade-off between past and expected future when dealing with adaptive risk perception and optimal risk assessment?

We give below four suggestive graphics (corresponding to the four proposed risk-aversion index definitions) which illustrate numerically the above theoretical result.

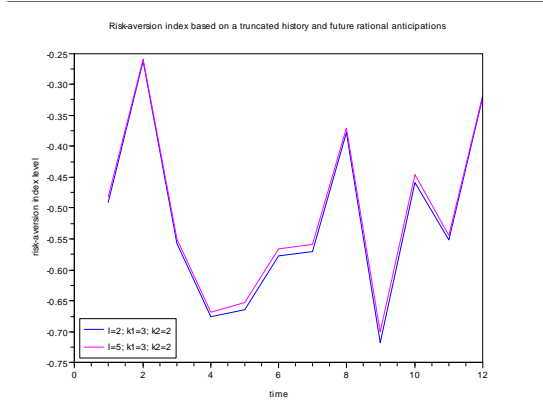


Figure 14

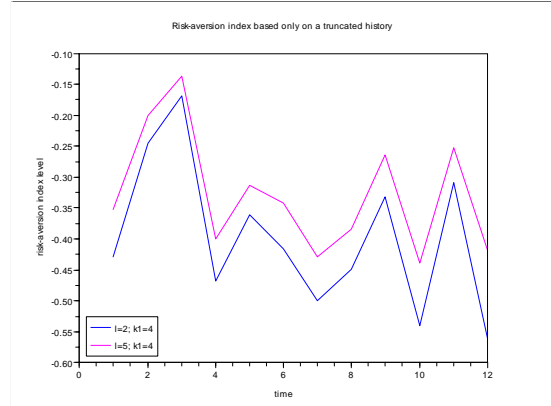


Figure 15

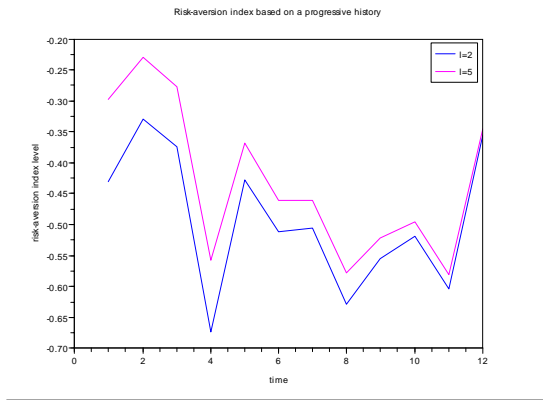


Figure 16

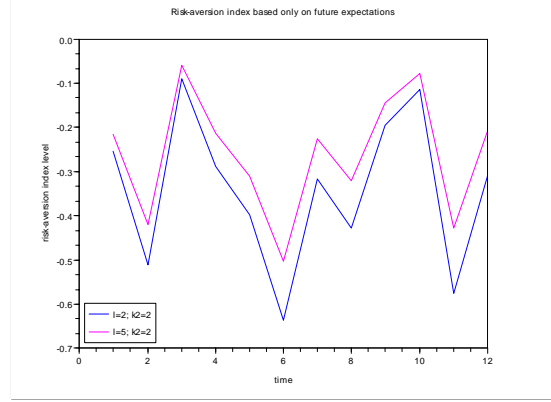


Figure 17

We have the following inequalities for the risk-aversion index:

$$\begin{aligned} \varphi_{1,p_f}^{r-a} &< \varphi_{1,w}^{r-a} < \varphi_{1,p}^{r-a} < \varphi_{1,f}^{r-a}, \text{ while } \varphi_{2,f}^{r-a} < \varphi_{2,w}^{r-a} < \varphi_{2,p_f}^{r-a} < \varphi_{2,p}^{r-a} \\ \varphi_{12,p}^{r-a} &< \varphi_{12,w}^{r-a} < \varphi_{12,p_f}^{r-a} < \varphi_{12,f}^{r-a}, \text{ while } \varphi_{6,f}^{r-a} < \varphi_{6,p_f}^{r-a} < \varphi_{6,w}^{r-a} < \varphi_{6,p}^{r-a} \\ \varphi_{3,p_f}^{r-a} &< \varphi_{3,w}^{r-a} < \varphi_{3,p}^{r-a} < \varphi_{3,f}^{r-a}, \text{ while } \varphi_{8,w}^{r-a} < \varphi_{8,p}^{r-a} < \varphi_{8,f}^{r-a} < \varphi_{8,p_f}^{r-a} \end{aligned}$$

Distinct agents, characterized by distinct risk-averse preferences, have generally distinct perceptions of endogenous risks in a given environment. It depends on how they succeed to manage the available information from the system as well as future information from outside the system.

It is important to make distinction between risk-aversion before and after starting the control process. They are not generally based on the same information set.

Before starting the control, the agent's degree of risk-aversion is measured on the basis of an exogenous information set. It defines the agent's type at a given period of time.

After starting the control, the agent's degree of risk-aversion is estimated on the basis of a non-decreasing endogenous information set. In a dynamic evolving environment, the risk-aversion profile of the agent can change.

7. Qualitative Consequences on Strategic Risk Management

The object of this section is to explore the theoretical and empirical implications of the proposed risk-aversion concept on the agent's risk behavior by developing realistic scenarios for the dynamic system.

7.1. Small System Deviations

Proposition 7. Suppose that all system deviations in the past and future are small. For this type of scenario, the agent is characterized by a small risk-aversion during the period of control. In particular, when all system deviations are very close to zero in magnitude, the agent becomes almost risk-neutral.

Proof. In the appendix. ■

It is interesting to note that for small symmetrical deviations with respect to the fixed targets, the agent will adopt the same attitude towards risk over time.

Zero risk can exist only if the agent does not attribute any importance to the system deviations. It may be the case of a system characterized by very small fluctuations.

For large deviations of the system with respect to the fixed targets, this particular scenario is not compatible with a rational behavior (DOUARD 1996).

Because at least one amongst the system deviations (in the past and future) is inherently strictly positive, it follows that the risk-aversion index value is generally non-null.

This can be explained by the presence of random shocks that are beyond the agent’s control, and inevitable forecast errors in predicting the system trajectory.

The hypothesis of risk-neutrality for the entire working horizon, very often employed in the literature, is very restrictive.

Taking into account the dynamic complexity of the system, this work-hypothesis is non-realistic. Its use in theoretical and empirical studies is associated with a “myopic” behavior of the agent, being convenient only for parsimony purposes. This restricts an endogenous behavior to a rigid exogenous one.

The apparent need for parsimony is derived by the facility brought in the construction of the models. Parsimony may seem desirable, but is in fact not because this generally introduces non-negligible errors in the model.

We give below two suggestive graphics which illustrate the adaptive behavior towards risk of the agent in the context of small (almost null) system deviations.

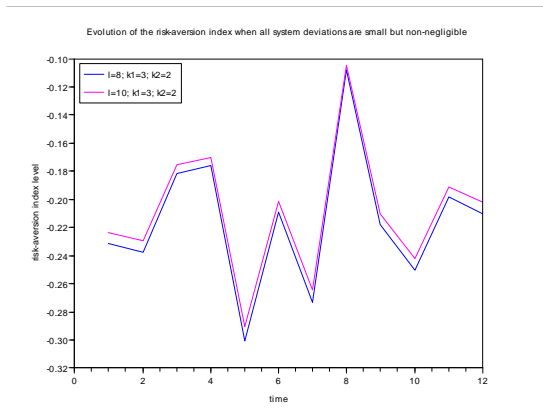


Figure 18

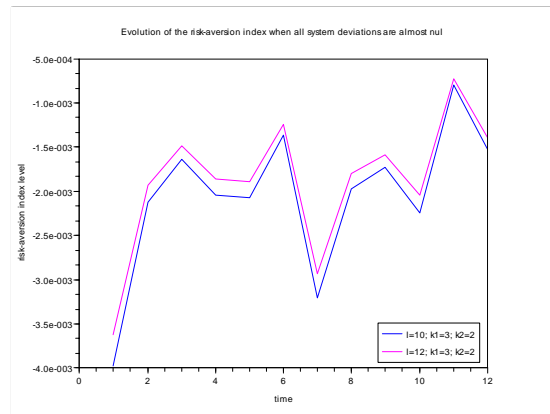


Figure 19

There is an intimate correlation between the choice of the parameter l (which characterizes the agent’s type) and the agent’s risk perception during the period of control.

7.2. Small and Large System Deviations

Proposition 8. When the system is characterized by small and large deviations, the agent is not necessarily more risk-averse for the same period of control compared to the case where all system deviations are small.

Proof. In the appendix. ■

The intuition for this result is based on two considerations:

- i) the effect of high deviations of the system in the past diminishes with time;
- ii) large expected deviations in distant future have a non significant effect on the agent's risk behavior. Economic agents are more risk tolerant on distant horizons.

The agent's degree of risk-aversion depends on two distinct factors:

- i) the moment when high deviations of the system arrive in the past or are expected to be realized in the future;
- ii) the strategic weights the agent will attach to the large deviations of the system.

We give below a realistic scenario for the system evolution which confirms this intuition.

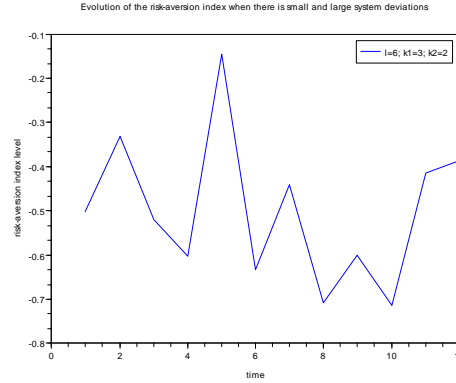


Figure 20

As is easily seen, at time $t = 5$, we ascertain a higher index value compared to that one obtained when all deviations of the system are small.

7.3. Large System Deviations

Assumption 6. The parameter l verifies the additional conditions:

$$l / \left\| y_{t-j_1} - y_{t-j_1}^g \right\|^4 \cong 0, \quad \forall j_1 = 1, \dots, k_1$$

$$l / \left\| y_{t+j_2}^a | I_{t+j_2} - y_{t+j_2}^g \right\|^4 \cong 0, \quad \forall j_2 = 0, \dots, k_2$$

Proposition 9. When all system deviations are large and comparable in magnitude, the agent's degree of risk-aversion is highly dependent on the strategic weights attached to the system deviations.

Proof. In the appendix. ■

There is a close relationship between risk-neutrality and perverted perception of the system. This is the case where the agent does not attribute any importance to the system deviations.

When the weights attached to the system deviations are negligible, the agent will be characterized by an almost risk-neutral behavior.

In contrast, for significative weights, the agent's degree of risk-aversion is non-negligible.

It is interesting to note that for this type of scenario, the boundary condition imposed on the fixed targets does not allow for a symmetrical evolution of the system with respect to the optimal path η .

We give below three suggestive graphics which show the strong correlation between the agent's risk perception and the size of the strategic weights attached to the system deviations.

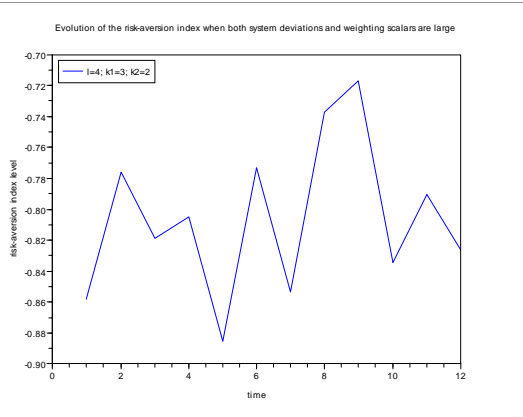


Figure 21

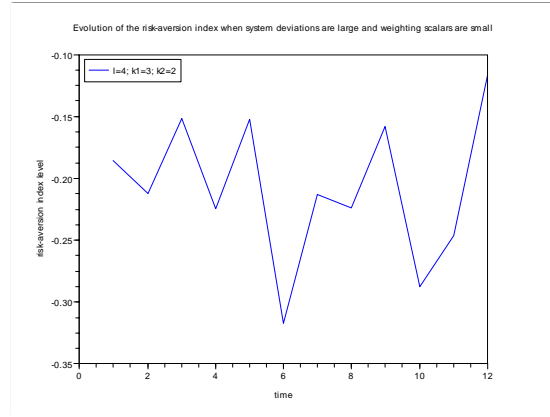


Figure 22

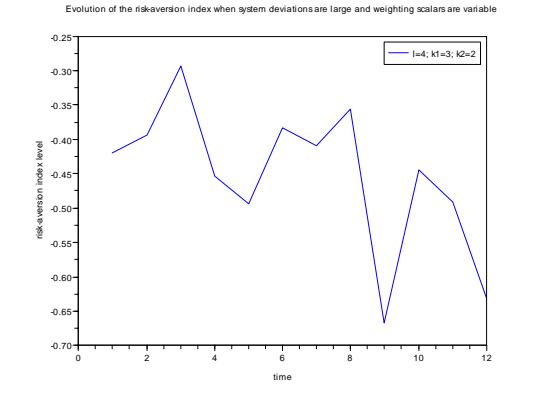


Figure 23

Proposition 10. Suppose that all system deviations are comparable in magnitude. In this case, a higher value of the sum of k_1 (feedback lag parameter) and k_2 (forward lag parameter) does not necessarily diminish the agent's risk-aversion at the same stage of control.

Proof. In the appendix. ■

This result proves the trade-off between past and future evolution of the system. Neither the past nor the future have a dominant effect on the agent's risk behavior over time. In general, this is influenced by the mixed cumulative effect of both temporal dimensions.

The past is often a backwards indicator of the future. The weight that the agent places on the future may be correlated with exogenous signals from the past.

Depending on the context, the past can provide more or less informative signals about the future trend of the system. The agent's risk behavior can be influenced by signals from the past, and thus future decisions can change.

A smooth (or almost constant) evolution of the system will not change significantly the agent's risk attitude. In this case, the past and the expected future will have almost the same impact on the agent's risk behavior.

Generally, the agent's risk sensitivity is highly correlated with a significant change in the system evolution. The more the system fluctuates, the more marked is the agent's degree of risk-aversion over time.

There is a close relationship between the system inherent inertia and the agent's risk perception. Distinct agents develop individual behaviors with regard to how they respond in similar risky situations. They are characterized by different degrees of risk-aversion. However, it may be possible that their attitudes towards risk be similar in a given particular context.

We give below two suggestive graphics illustrating the **Proposition 10**.

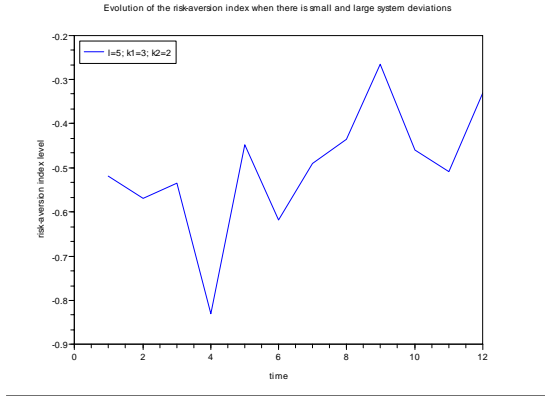


Figure 24

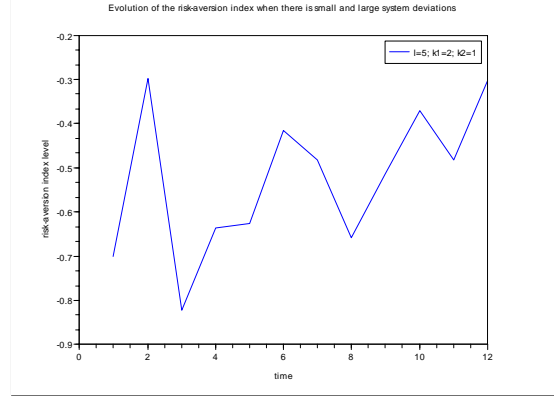


Figure 25

8. Excessive Risk-Averse Decision-Maker

Experimental evidence shows that economic agents overweight extreme events. These can modify their individual behavior towards risk. The ability to assess future risks associated with extreme events is increasingly important to economic behavior.

Let φ_{\min}^{p-f} be an optimal risk-aversion threshold fixed by the agent before starting the control and for the entire working horizon. The objective is not to exceed it. Otherwise, the agent becomes excessively risk-averse for the current period of control, and thus will be characterized by an extreme pessimism.

The exceeding of the threshold φ_{\min}^{p-f} during the period of control is correlated with large deviations of the system from the agent's fixed targets. There is a close relationship between the choice of optimal targets and the possibility to exceed the risk-aversion threshold φ_{\min}^{p-f} . Smaller targets generally imply a smaller risk-aversion index, and thus a higher possibility to exceed the optimal threshold φ_{\min}^{p-f} .

An agent with a higher (smaller) risk-aversion before starting the control will choose a smaller (higher) threshold φ_{\min}^{p-f} .

Note that $\varphi_{t,p-f}^{r-a}$ characterizes the agent's local risk-aversion (at time period t), while φ_{\min}^{p-f} characterizes his global risk-aversion (over the whole period $[1, T]$). The optimal threshold φ_{\min}^{p-f} is selected such that it offers the best characterization of the agent's type. It also depends on the particular environmental context. It must distinguish between φ_{\min}^{p-f} and $\varphi_{1,p-f}^{r-a}$. It is a strategic attitude for the agent to fix a threshold φ_{\min}^{p-f} inferior to $\varphi_{1,p-f}^{r-a}$.

The definition of the risk-aversion threshold φ_{\min}^{p-f} offers a good explanation why the risk-aversion index is non-monotonous with respect to the time variable t :

i) an increasing risk-aversion index would be in contradiction with the definition of φ_{\min}^{p-f} , in the sense that it would be never exceeded. In the real world, this limit threshold is most often exceeded by risk-averse agents.

ii) a decreasing risk-aversion index would constrain the decision-maker to exceed the threshold φ_{\min}^{p-f} over time, and hence to become excessively risk-averse for the remaining period of control. This type of scenario would not be realistic.

The agent's global utility function corresponding to the fixed risk-aversion optimal threshold φ_{\min}^{p-f} is given by:

$$U_{[1,T]}(W_{[1,T]}, \varphi_{\min}^{p-f}) \stackrel{def.}{=} \frac{2}{\varphi_{\min}^{p-f}} \left[\exp\left(-\frac{\varphi_{\min}^{p-f}}{2} W_{[1,T]}\right) - 1 \right]$$

Proposition 11. A risk-averse agent is characterized by an optimal risk-aversion threshold φ_{\min}^{p-f} fixed according to his individual type.

Proof. In the appendix. ■

Risk is not symmetric across agents. They perceive differently the system evolution, and thus give different interpretations to the system dynamics. It may be possible that they implement distinct optimal actions for the same period of time. This can be explained by arguments based on the non-injectivity of the control decision rule, regarded as a function of the risk-aversion index.

This type of modelling allows for a complete characterization of common /distinct types of risk-averse agents. We deal with a continuum of agent types and risk-averse preferences.

We illustrate below a realistic scenario concerning the choice of the agent's risk aversion threshold φ_{\min}^{p-f} .

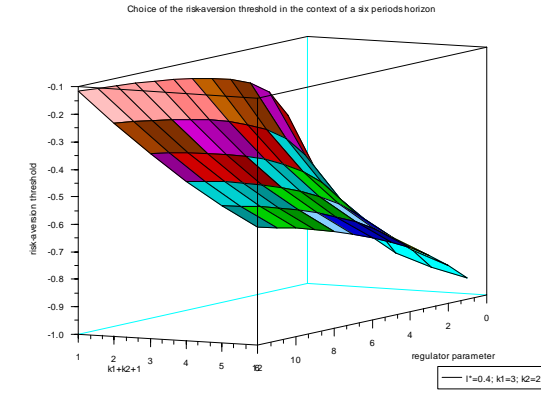


Figure 26

In the following, we give an interesting theoretical result which characterizes the boundaries of the agent's utility function during the period of control.

Proposition 12. The upper and lower bounds of the agent's local utility function vary according to the risk-aversion index level relative to the fixed optimal threshold φ_{\min}^{p-f} .

Proof. In the appendix. ■

The above result allows for characterizing the agent's risk-averse preferences in function of his individual type. The inherent disutility of the risk associated with extreme events is taken into consideration in the present model.

We give below two suggestive graphics in this sense, in the case where $\varphi_{\min}^{p-f} = -0.5$.

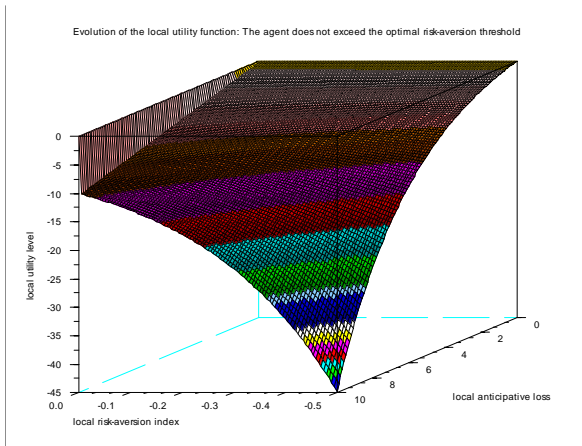


Figure 27

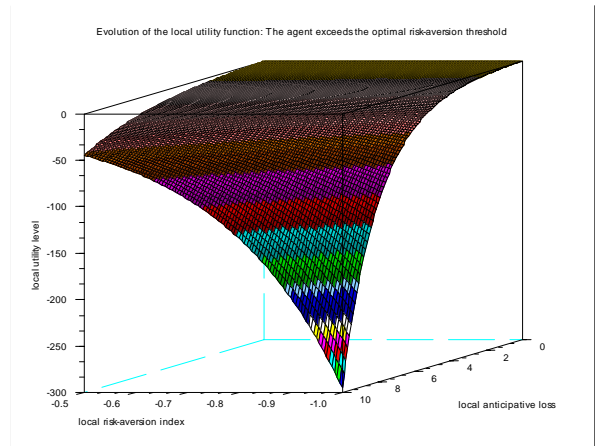


Figure 28

In order to illustrate the positive correlation between the length of working horizon and the agent's degree of risk-aversion over time, we give below an interesting result in this direction.

Proposition 13. In a dynamic stochastic environment, there is a positive correlation between the horizon length and the agent’s risk profile.

Proof. In the appendix. ■

In particular, this result allows us for characterizing the agent’s risk behavior when adopting a closed-loop strategy. Although this type of strategy is generally employed for short periods of time, the agent may be characterized by a higher risk-aversion before starting the control.

Contrary to what intuition would suggest, a shorter planning horizon does not necessarily ensure a lower risk-aversion. It generally depends on the expected behavior of the system in the future. Large expected deviations from the fixed path η induce a higher degree of uncertainty in terms of decision-making, and hence a higher risk-aversion for the agent.

We illustrate here a realistic scenario for the above theoretical result.

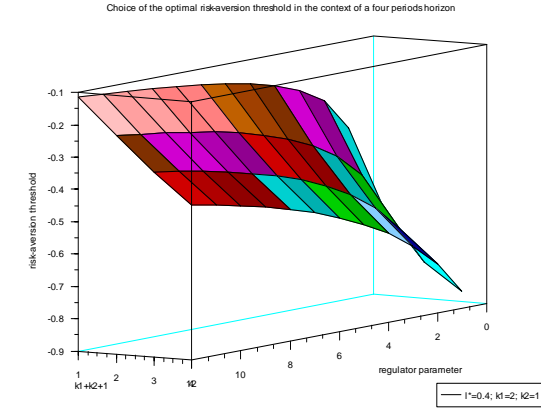


Figure 29

We remark that in the context of a four periods horizon, the decision-maker can choose an optimal risk-aversion threshold φ_{\min}^{p-f} inferior to -0.45 , while for a six periods horizon, φ_{\min}^{p-f} is inferior to -0.57 . In other words, the agent’s risk behavior is sensitive to the fixed horizon length. More exactly, risk-aversion increases with horizon length. This is in accordance with the result obtained by BOMMIER (2006).

9. Potential Sensitive Periods

It may exist some periods with an important degree of uncertainty, when the agent’s objective is not to exceed a local fixed risk-aversion index level. More exactly, suppose that for each significative period t , the agent chooses an optimal local value $\varphi_{t, p-f}^{loc.}$ such that $\varphi_{\min}^{p-f} \leq \varphi_{t, p-f}^{loc.} \leq \varphi_{t, p-f}^{r-a}$. This can be explained by the fact that a global minimum is always smaller or equal than a local one.

It must not confuse $\varphi_{t, p-f}^{loc.}$ (which is fixed before starting the control, and thus it does not depend on the system evolution path) with $\varphi_{t, p-f}^{r-a}$ (estimated after the control begins).

It is also important to distinguish between φ_{\min}^{p-f} and $\varphi_{t, p-f}^{loc.}$. The first one defines the agent’s type (for the entire period of control), while the second depends on his local objective at time t .

Let t_j ($j = 1, \dots, \bar{k}$) be all significative periods, with $\{t_1, \dots, t_{\bar{k}}\} \subseteq \{1, \dots, T\}$. Note that $\varphi_{t_j, p-f}^{loc.}$ are defined only for the periods t_j , while φ_{\min}^{p-f} is chosen by the agent according to his expectations on all $\varphi_{t_j, p-f}^{loc.}$. Consequently, these are $\varphi_{t_j, p-f}^{loc.}$ ($j = 1, \dots, \bar{k}$) that will influence φ_{\min}^{p-f} and not vice versa. One can say that $\varphi_{t_j, p-f}^{loc.}$ is the equivalent of φ_{\min}^{p-f} , but only locally. We can write the following inequality:

$$\varphi_{\min}^{p-f} \leq \min\{\varphi_{t_1, p-f}^{loc.}, \dots, \varphi_{t_{\bar{k}}, p-f}^{loc.}\}$$

The agent's utility at time t , defined for a fixed threshold $\varphi_{t, p_{-f}}^{loc.}$, is given by:

$$U_t(W_{[1,t]}, \varphi_{t, p_{-f}}^{loc.}) = \frac{2}{\varphi_{t, p_{-f}}^{loc.}} [\exp(-\frac{\varphi_{t, p_{-f}}^{loc.}}{2} W_{[1,t]}) - 1]$$

We have:

$$U_t(W_{[1,t]}, \varphi_{\min}^{p-f}) \leq U_t(W_{[1,t]}, \varphi_{t, p_{-f}}^{loc.}) \leq U_t(W_{[1,t]}, \varphi_{t, p_{-f}}^{r-a})$$

if and only if

$$\varphi_{\min}^{p-f} \leq \varphi_{t, p_{-f}}^{loc.} \leq \varphi_{t, p_{-f}}^{r-a} \quad \forall t = 1, \dots, T$$

Note that $\varphi_{t_j, p_{-f}}^{loc.}$ can be regarded as a first control-threshold imposed by the agent in order not to exceed φ_{\min}^{p-f} . At each stage t_j , it is possible to exceed $\varphi_{t_j, p_{-f}}^{loc.}$ but not φ_{\min}^{p-f} . It can be explained by the fact that the local conditions are generally more restrictive than the global ones. Since a real time control process is necessarily discrete, the risk-aversion index $\varphi_{t_j, p_{-f}}^{r-a}$ cannot converge with precision to the fixed optimal threshold $\varphi_{t_j, p_{-f}}^{loc.}$, but only to a neighborhood of it. When the process of control is finished, the agent will obtain a stochastic neighbouring-optimal trajectory of the risk-aversion index which is expected to be close to the optimal risk-aversion path $\{\varphi_{1, p_{-f}}^{loc.}, \dots, \varphi_{T, p_{-f}}^{loc.}\}$. It is very likely that difference between ex-ante and ex-post behavior towards risk exists.

10. Agent with a Changing Risk Profile

The way the risk-aversion index is parameterized depends on the problem statement. We refine the analysis by taking into account the case of a changing risk profile. In the real world, there may be periods when the agent is risk-averse and periods when he becomes (almost) risk-neutral or risk-lover. A mixture of pessimism and optimism can exist (TOULET 1982; JERKER 2007).

The definition of the risk-aversion index is similar to that one given in **Chapter 5.2**, with the difference that the strategic weights attached to the system deviations lie inside the unit circle. Let us make the following notations:

$$\tilde{S}_{t, w_{p-d}} \stackrel{not.}{=} \left\| y_{t-1} - y_{t-1}^g \right\|^2 \tilde{L}_{t-1} + \dots + \left\| y_{t-k_1} - y_{t-k_1}^g \right\|^2 \tilde{L}_{t-k_1}$$

(the weighted sum of squared past deviations at time t)

$$S'_{t, w_{a-f-d}} \stackrel{not.}{=} \left\| y_{t|I_t}^a - y_t^g \right\|^2 L'_t + \dots + \left\| y_{t+k_2|I_{t+k_2}}^a - y_{t+k_2}^g \right\|^2 L'_{t+k_2}$$

(the weighted sum of squared anticipated future deviations at time t)

where

$$-1 < \tilde{L}_{t-1} \leq \dots \leq \tilde{L}_{t-k_1} < 1, \quad -1 < L'_t \leq \dots \leq L'_{t+k_2} < 1$$

are strategic weights attached to both observed and expected deviations of the system. Their values are selected according to the importance the agent places on the past and future.

Remark 8. We distinct, in this general context, two completely opposite scenarios:

i) the pure risk-aversion case, defined according to the following boundaries conditions:

$$-1 < \tilde{L}_{t-1} \leq \dots \leq \tilde{L}_{t-k_1} < 0, \quad -1 < L'_t \leq \dots \leq L'_{t+k_2} < 0$$

ii) the pure risk-taking case, defined according to the symmetrical boundaries conditions:

$$0 < \tilde{L}_{t-1} \leq \dots \leq \tilde{L}_{t-k_1} < 1, \quad 0 < L'_t \leq \dots \leq L'_{t+k_2} < 1$$

A changing risk behavior can be thus regarded as a mixed case of risk-aversion, risk-neutrality and risk-taking. System evolution can change the agent's type.

Definition 5. Using t to denote time, the absolute risk index $\varphi_{t,p_f}^{r-a-n-l}$ evolves according to:

$$\varphi_{t,p_f}^{r-a-n-l} \stackrel{def.}{=} \frac{\tilde{S}_{t,w_p-d} + S'_{t,w_a-f-d}}{\sqrt{(S_{t,p-d} + S_{t,a-f-d})^2 + \tilde{l}}}, \quad t = 1, \dots, T$$

where $\tilde{l} \geq 1$ is a positive integer which characterizes the agent's type, and k_i ($i = 1, 2$) represent the backward (forward) lag parameters verifying the following constraints:

$$1 \leq k_1 < T, \quad k_2 \geq 0, \quad 1 \leq k_1 + k_2 \leq T - 1$$

In this case, the boundary condition for the index will change. It is possible to obtain negative values for some periods and positive (or null) values for others. This depends on several factors including, the available exogenous information, the accuracy of the agent's perception and, respectively, the inherent discrepancies between ex-ante anticipations and the corresponding ex-post quantities. The evolution of the system will affect differently the agent's risk behavior. His optimal actions are therefore different compared to those selected in a pure risk-aversion (risk-taking) context. An important distinction between temporal and timeless risks is provided by the present model. This offers a refinement of the traditional concept of temporal risk-aversion employed in the literature (MACHINA 1984; VAN DER PLOEG 1992). This also improves the standard analysis of economic behavior under temporal uncertainty (CHAVAS AND LARSON 1994). Moreover, the present study contributes to the bounded rationality literature (SIMON 1982A, 1982B; SARGENT 1993) in that it allows for a refinement of the agent's risk-averse preferences.

Proposition 14. A more (less) risk-taker decision-maker is characterized by a smaller (higher) local utility level.

Proof. In the appendix. ■

Agent's preferences level decreases with higher risk-taker index. His objective is not to exceed a fixed limit threshold of the index. Otherwise, the agent becomes excessively risk-taker, and thus will be characterized by an over-optimism. This strategic attitude prevents to exceed some threshold utility level over which the agent's ex-post preferences are suboptimal. In order to reach his objective, the agent must avoid to deviate from the fixed optimal path. Over-optimism can be defined as an induced deviating behavior with respect to perceived states of the system. We give below a suggestive graphic illustrating the evolution of the agent's utility function with respect to the anticipative loss and the risk-taker index.

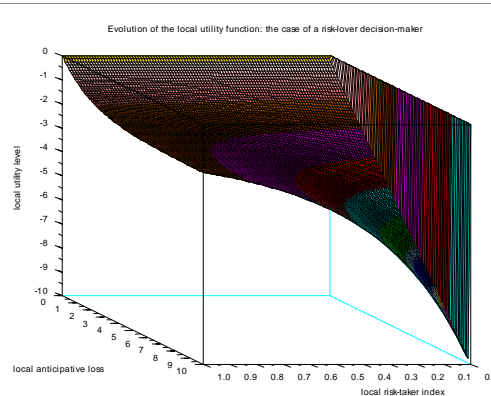


Figure 30

Proposition 15. A risk-lover decision-maker is characterized by an optimal threshold φ_{\max}^{p-f} fixed according to his risk-taker type.

Proof. In the appendix. ■

It is useful to note that risk-taking and risk-aversion are non-symmetrical behaviors. These are generally analyzed in non-symmetrical contexts.

We give below a suggestive graphic illustrating the **Proposition 15**.

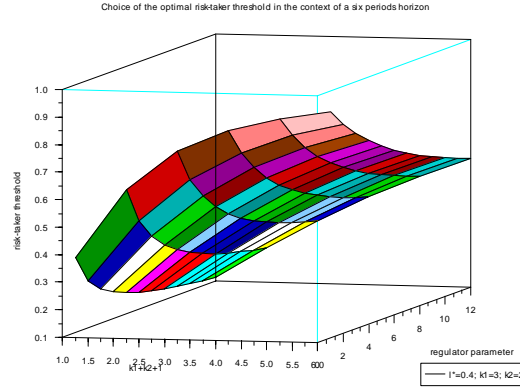


Figure 31

Proposition 16. A risk-lover decision-maker is characterized by a local utility function whose upper and lower bounds vary according to the risk-taker index level relative to the fixed optimal threshold φ_{\max}^{p-f} .

Proof. In the appendix. ■

A direct consequence of this result is the complete separability of the agent's risk-taking preferences. It depends on the index variation with respect to the fixed threshold φ_{\max}^{p-f} . It allows for characterizing the agent's preferences in function of his individual type.

We give below two suggestive graphics in this sense, in the case where $\varphi_{\min}^{p-f} = 0.5$.

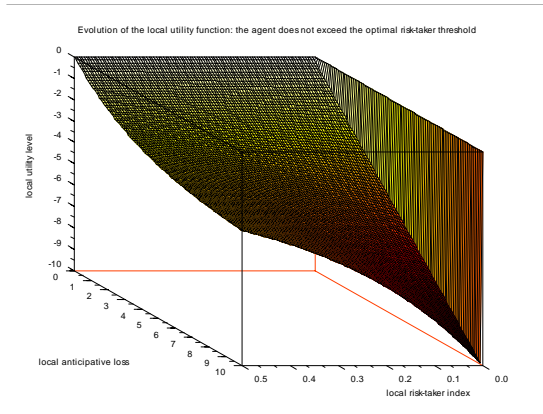


Figure 32

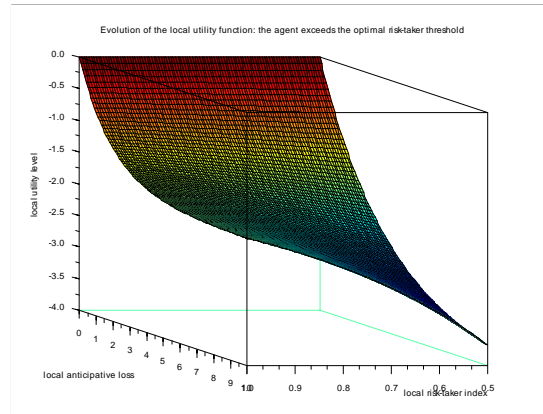


Figure 33

Proposition 17. There is a positive correlation between the horizon length and the agent's risk-taking behavior during the period of control.

Proof. In the appendix. ■

The agent's objective is to reduce the exogenous effect of the horizon length on the optimal policy decisions.

A particular environmental context can change the agent's risk-taking behavior. To fix a smaller /higher working horizon depends, on the one hand, on the strategic objective of the agent and, on the other hand, on his risk-taker type.

We give below a graphical illustration of the **Proposition 17**.

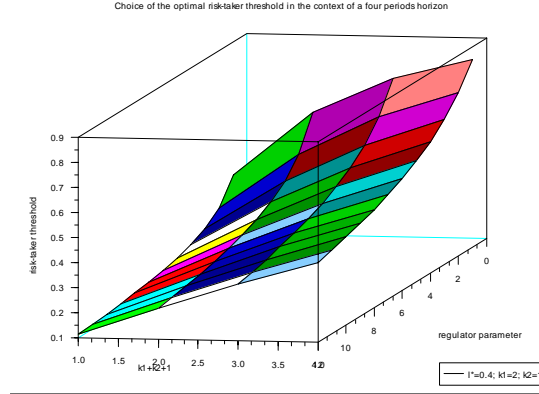


Figure 34

We remark that in the context of a four periods horizon, the decision-maker can fix a threshold φ_{\max}^{p-f} inferior to 0.4, while for a six periods horizon, φ_{\max}^{p-f} is inferior to 0.55.

This study takes a step towards a more refined understanding of the concepts of short and long horizon risk-aversion. This is a key topic for future research with implications in a wide range of economic applications intrinsically linked to the study of policy implementation and analysis.

Proposition 18. A rational agent with a changing risk profile is characterized by two distinct risk thresholds fixed according to his individual type: i) a risk-aversion threshold φ_{\min}^{p-f} ; and ii) a risk-taker threshold φ_{\max}^{p-f} .

Proof. In the appendix. ■

The objective of the agent is to maintain the level of the risk index inside the interval defined by the risk thresholds φ_{\min}^{p-f} and φ_{\max}^{p-f} . This scenario is possible if and only if the agent succeeds to constrain the dynamic system to follow the optimal path η .

When at least one of these risk thresholds are exceeded, the agent is characterized by an excessive risk behavior.

Significative changes in the system evolution are necessary such that a transition from an extreme risk behavior to the opposite one be possible.

An interesting question arises: Which of these two extreme attitudes are most probable to arrive in the real world? The answer is not obvious.

The risk-neutrality case can be regarded as a transitory state in this dynamic process. Changing risk behavior can be defined as a free floating state with respect to the agent's risk attitude over time.

For a stochastic system characterized by uniform trend towards lower values of the target variable, the agent's risk behavior is defined in a reasonably neighborhood of the risk-neutrality state.

Generally, risk-taking makes the reaction stronger than risk-neutrality. However, when both agents are risk-taker but at very different degrees, the more optimistic of the two can have a weaker reaction than in the risk-neutral case. This is an astonishing result that can be explained by the relationship between high deviations and perceived states of nature.

The evolution of the system can encourage risk-taking or induce risk-aversion behavior. Outcomes as large losses can induce risk-aversion, while outcomes as large gains can induce risk-taking. In other words, the endogenous fluctuations of the system can generate endogenous waves of optimism and pessimism.

We give below a realistic scenario illustrating the attitude towards risk of a strategic rational decision-maker characterized by a changing risk profile during the period of control.

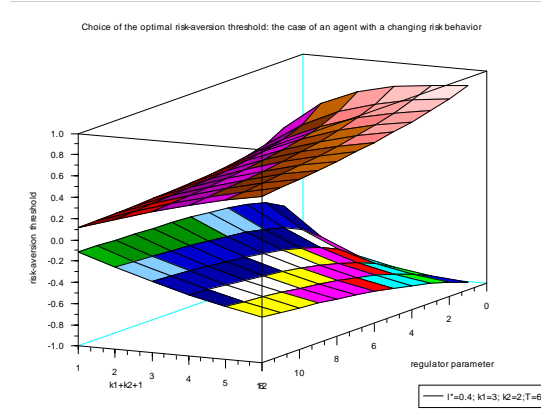


Figure 35

For this scenario, the agent's objective is not to exceed the optimal fixed thresholds $\varphi_{\min}^{p-f} = -0.6$ and $\varphi_{\max}^{p-f} = 0.4$.

If φ_{\min}^{p-f} (respectively φ_{\max}^{p-f}) is exceeded at time t , the agent will implement an excessive risk-averse (respectively risk-taking) decision for the period t . However, this does not necessarily mean that the agent's actions do not perform well at time t .

During the entire period of control, the agent's utility function evolves according to the following formula:

$$U_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a-n-l}) \stackrel{def.}{=} \begin{cases} \frac{2}{\varphi_{t,p-f}^{r-a-n-l}} [\exp(-\frac{\varphi_{t,p-f}^{r-a-n-l}}{2} W_{[1,t]}) - 1] & \text{if } -1 < \varphi_{t,p-f}^{r-a-n-l} < 0 \\ -W_{[1,t]} & \text{if } \varphi_{t,p-f}^{r-a-n-l} = 0 \\ \frac{2}{\varphi_{t,p-f}^{r-a-n-l}} [\exp(-\frac{\varphi_{t,p-f}^{r-a-n-l}}{2} W_{[1,t]}) - 1] & \text{if } 0 < \varphi_{t,p-f}^{r-a-n-l} < 1 \end{cases}$$

We illustrate below the agent's changing risk-averse preferences and his adaptive behavior to risk in a multiple-horizon dynamic environment.

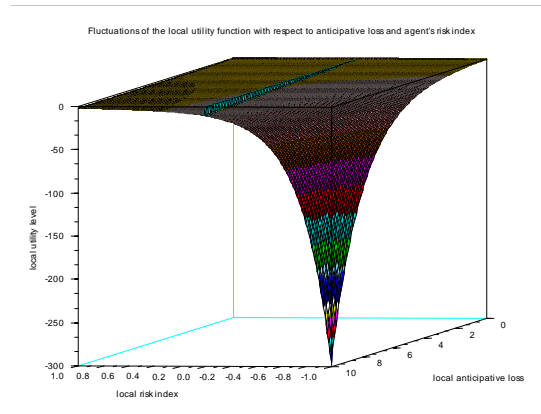


Figure 36

An interesting question arises: From what values, the optimal thresholds φ_{\min}^{p-f} and φ_{\max}^{p-f} can be regarded as small or large? This generally depends on the particular environmental context as well as on the agent's risk profile.

In order to illustrate the complexity of the agent's risk behavior during the period of control, we give below a suggestive graphic in this sense.

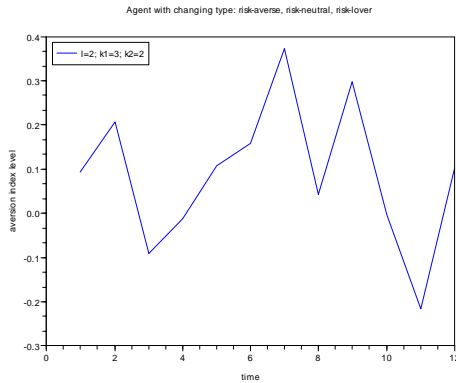


Figure 37

The agent's objective is not to exceed the optimal thresholds $\varphi_{\min}^{p-f} = -0.3$ and $\varphi_{\max}^{p-f} = 0.4$ during the entire working horizon.

The action taken at time $t = 1$ will induce a weak risk-taking for the next period. The agent is thus characterized by a small degree of confidence. A reinforcement of the active learning correlated with an efficient closed-loop strategy will progressively decrease the agent's degree of confidence in the two next periods.

At time $t = 4$, the agent becomes (almost) risk-neutral. For the following three periods, we ascertain an increasing tendency in taking higher risks. However, the agent succeeds in not overreaching the fixed optimal threshold φ_{\max}^{p-f} .

At time $t = 8$ and $t = 10$, the agent's degree of confidence is almost null. Given the inherent inertia of the system, the agent's risk aversion will increase (even if not significantly) for the next period. He succeeds in not overreaching the fixed optimal threshold φ_{\min}^{p-f} .

The strategic objective of the agent is thus reached. On the one hand, he succeeds to optimally manage his risk-aversion during the period of control, and on the other hand, the implemented optimal policy succeeds to constrain the dynamic system to follow the optimal reference path η .

11. Concluding Remarks and Possible Extensions

The present paper investigates the crucial role played by the past and future dynamics of the system in characterizing the agent's risk behavior over time. The definitions of the risk-aversion index proposed in this approach allow for a more profound understanding on how exogenous and endogenous uncertainty is captured over environmental changes. The agent interprets the evolution of the system and reveals his adaptive behavior to risk in a fluctuating environment. It is only by optimizing the past that the agent will optimally anticipate endogenous risks in the future. A better risk management is thus possible by the use of both feedback and forward information. An interesting characterization of the agent's preferences according to his degree of risk-aversion is put into evidence. A significative relationship emerges between behavior to risk and length of working horizon. The hypothesis of risk-neutrality appears in the present model as a potential borderline case. We propose a natural extension of the ARROW-PRATT theory (which takes into account only attitudes towards small exogenous risks) by including in the analysis potentially high endogenous risks that are under the control of the agent. This is necessary to overcome the limits imposed by the standard measures of risk-aversion often employed in the literature. Moreover, we refine the risk analysis by developing the case of a changing risk profile over time. This type of modelling has the potential to be a powerful tool for characterizing the agent's risk behavior, including risk-aversion, risk-neutrality and risk-taking. It can be seen as a step further in the refinement of the risk concept, providing

a better understanding on the complexity of the risk psychology in a discrete-time dynamic environment. The proposed analysis can be easily extended to the context of stochastic dynamic games (cooperative or non-cooperative), the objective here being to define and characterize the equilibrium of the game according to the optimal risk-sharing between strategic players. We can also study the identification of emerging endogenous coalitions in interactive strategic environments, in function of the players' types and their risky objectives. Exploring such possibilities appears to be a good topic for further research, providing new perspectives for theorists and empirical analysts.

Appendix

Proof of Proposition 1.

Differentiating $U_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a})$ with respect to $\varphi_{t,p-f}^{r-a}$ (for an arbitrary fixed value of $W_{[1,t]}$), we then obtain:

$$U'_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) = \frac{2\left[\left(-\frac{\varphi_{t,p-f}^{r-a}}{2}W_{[1,t]} - 1\right)\exp\left(-\frac{\varphi_{t,p-f}^{r-a}}{2}W_{[1,t]}\right) + 1\right]}{(\varphi_{t,p-f}^{r-a})^2}$$

Defining:

$$V_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) \stackrel{def.}{=} \left(-\frac{\varphi_{t,p-f}^{r-a}}{2}W_{[1,t]} - 1\right)\exp\left(-\frac{\varphi_{t,p-f}^{r-a}}{2}W_{[1,t]}\right) + 1$$

as a function of $\varphi_{t,p-f}^{r-a}$, we find for the first derivative of V_t :

$$V'_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) = \varphi_{t,p-f}^{r-a} \frac{W_{[1,t]}^2}{4} \exp\left(-\frac{\varphi_{t,p-f}^{r-a}}{2}W_{[1,t]}\right) < 0$$

In other words, V_t decreases with $\varphi_{t,p-f}^{r-a}$:

$$\varphi_{t,p-f}^{r-a} < 0 \Rightarrow V_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) > V_t(W_{[1,t]}, 0) = 0$$

It follows that $U'_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) > 0$. The local utility function $U_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a})$ is therefore increasing in $\varphi_{t,p-f}^{r-a}$. We also have that U_t is decreasing in $W_{[1,t]}$. This is in agreement with real world and strong empirical evidence.

Proof of Proposition 2.

Simple algebraic manipulations show that the sign of the difference:

$$\begin{aligned} \varphi_{t+1,p-f}^{r-a} - \varphi_{t,p-f}^{r-a} &= \frac{\xi_0 L_t + \dots + \xi_{k_1-1} L_{k_1-1} + \bar{\xi}_1 \bar{L}_{t+1} + \dots + \bar{\xi}_{k_2+1} \bar{L}_{t+1+k_2}}{\sqrt{(\xi_0 + \dots + \xi_{k_1-1} + \bar{\xi}_1 \dots + \bar{\xi}_{k_2+1})^2 + l}} \\ &\quad - \frac{\xi_1 L_{t-1} + \dots + \xi_{k_1} L_{k_1} + \bar{\xi}_0 \bar{L}_t + \dots + \bar{\xi}_{k_2} \bar{L}_{t+k_2}}{\sqrt{(\xi_1 + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{k_2})^2 + l}} \end{aligned}$$

is either positive or negative, depending on the values taken by the norm-deviations:

$$\xi_{j_1} \stackrel{not.}{=} \|y_{t-j_1} - y_{t-j_1}^g\|, \quad j_1 = 0, \dots, k_1$$

(exogenous variables)

$$\bar{\xi}_{j_2} \stackrel{not.}{=} \|y_{t+j_2|I_{t+j_2}}^a - y_{t+j_2}^g\|, \quad j_2 = 0, \dots, k_2 + 1$$

(endogenous variables)

Proof of Proposition 3.

Suppose that $k_1 < k'_1$. Let us consider that $k'_1 = k_1 + k$, with $k \geq 1$ a fixed integer. By hypothesis, we have:

$$L_{t-1} < L_{t-2} < \dots < L_{t-k_1} < L_{t-(k_1+1)} < \dots < L_{t-(k_1+k)}$$

One can write:

$$\begin{aligned} & \frac{L_{t-1} + \dots + L_{t-k_1} + \dots + L_{t-(k_1+k)}}{k_1 + k} - \frac{L_{t-1} + \dots + L_{t-k_1}}{k_1} \\ &= \frac{-k[L_{t-1} + \dots + L_{t-k_1}] + k_1[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k)}]}{k_1(k_1 + k)} \\ &= \frac{[-kL_{t-1} + L_{t-(k_1+1)} + \dots + L_{t-(k_1+k)}] + \dots + [-kL_{t-k_1} + L_{t-(k_1+1)} + \dots + L_{t-(k_1+k)}]}{k_1(k_1 + k)} \gg 0 \end{aligned}$$

In other words, we have proved that:

$$k_1 < k'_1 \Rightarrow \frac{L_{t-1} + \dots + L_{t-k'_1}}{k'_1} \gg \frac{L_{t-1} + \dots + L_{t-k_1}}{k_1}$$

(the average of k'_1 weights) (the average of k_1 weights)

Proof of Proposition 4.

Consider first the case where the transition of the system is from consecutive small deviations to a large deviation. Denote by $\varphi_{t,p}^{r-a, s-h}$ the risk-aversion index at time t for this type of scenario. We have that $\varphi_{t,p}^{r-a, s-h} \rightarrow L_{t-1}$, where L_{t-1} represents the weight attached to the system deviation at time $t-1$. Denote also by $\varphi_{t,p}^{r-a, h-s}$ the risk-aversion index at time t when the system transition is from consecutive large deviations to a small deviation. In this case, we have:

$$\varphi_{t,p}^{r-a, h-s} \rightarrow \frac{\tilde{L}_{t-2} + \dots + \tilde{L}_{t-k_1}}{k_1 - 1}$$

where $\tilde{L}_{t-2}, \dots, \tilde{L}_{t-k_1}$ are strategic weights attached to large deviations of the system (in the context of the second scenario). Depending on the magnitude /size of the parameters L_{t-1} and $\frac{\tilde{L}_{t-2} + \dots + \tilde{L}_{t-k_1}}{k_1 - 1}$, one can imagine three distinct situations:

a) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much more important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-1} \ll \tilde{L}_{t-2}, \dots, L_{t-1} \ll \tilde{L}_{t-k_1} \Rightarrow L_{t-1} \ll \frac{\tilde{L}_{t-2} + \dots + \tilde{L}_{t-k_1}}{k_1 - 1}$$

(sufficient condition)

In this case, it follows that $\varphi_{t,p}^{r-a, s-h} < \varphi_{t,p}^{r-a, h-s}$, and hence a higher degree of risk-aversion for the same period of control in the context of the first scenario.

b) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-1} \gg \tilde{L}_{t-2}, \dots, L_{t-1} \gg \tilde{L}_{t-k_1} \Rightarrow L_{t-1} \gg \frac{\tilde{L}_{t-2} + \dots + \tilde{L}_{t-k_1}}{k_1 - 1}$$

(sufficient condition)

In contrast with the previous case, one obtains that $\varphi_{t,p}^{r-a, s-h} > \varphi_{t,p}^{r-a, h-s}$, that is, a smaller degree of risk-aversion at time t in the context of the first scenario.

c) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much more or much less important for the agent than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-1} \ll (\text{or } \gg) \tilde{L}_{t-2}, \dots, L_{t-1} \ll (\text{or } \gg) \tilde{L}_{t-k_1}$$

In this case, we obtain either $\varphi_{t,p}^{r-a, s-h} < \varphi_{t,p}^{r-a, h-s}$ or $\varphi_{t,p}^{r-a, s-h} > \varphi_{t,p}^{r-a, h-s}$. This completes the proof.

Proof of Proposition 5.

The conclusion results from the following sequence of inequalities:

$$\frac{L_0}{1} \gg \frac{L_0 + L_1}{2} \gg \dots \gg \frac{L_0 + \dots + L_{t-1}}{t}$$

Each above ratio corresponds (in this order) to the estimation of the risk-aversion index at time $\tau = 1, 2, \dots, t$. For this type of scenario, it is supposed that $L_1 \ll L_0$. In other words, the impact of system deviations from $\tau = 1$ to $\tau = t$ on the agent's future uncertainty is much more important compared to the impact at $\tau = 0$.

Proof of Proposition 6.

It is easy to see that the sign of the following differences:

$$\varphi_{t,p-f}^{r-a} - \varphi_{t,f}^{r-a}, \varphi_{t,p-f}^{r-a} - \varphi_{t,p}^{r-a}, \varphi_{t,f}^{r-a} - \varphi_{t,p}^{r-a}, \varphi_{t,f}^{r-a} - \varphi_{t,w}^{r-a}$$

may be either positive or negative. In other words, the agent's degree of risk-aversion is correlated with the adopted strategy to manage endogenous uncertainty.

Proof of Proposition 7.

Denote by $\xi_{j_1} \stackrel{\text{not.}}{=} \|y_{t-j_1} - y_{t-j_1}^g\|$, $j_1 = 1, \dots, k_1$, and $\bar{\xi}_{j_2} \stackrel{\text{not.}}{=} \|y_{t+j_2|t+j_2}^a - y_{t+j_2}^g\|$, $j_2 = 0, \dots, k_2$, the system norm-deviations at time $t-1, \dots, t-k_1$, and $t, \dots, t+k_2$, respectively. They are supposed to be small but non-negligible in magnitude. The absolute risk-aversion index $\varphi_{t,p-f}$ evolves according to:

$$\varphi_{t,p-f}^{r-a} = \frac{\xi_1 L_{t-1} + \dots + \xi_{k_1} L_{t-k_1} + \bar{\xi}_0 \bar{L}_t + \dots + \bar{\xi}_{k_2} \bar{L}_{t+k_2}}{\sqrt{(\xi_1 + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{k_2})^2 + l}}, \quad t = 1, \dots, T$$

For ease of exposition, we assume that $l_t = l^*$ (a small value from $(0, 1)$) for all $t = 1, \dots, T$. It follows that:

$$\xi_{j_1} < l^*, \quad \forall j_1 = 1, \dots, k_1 \quad \text{and} \quad \bar{\xi}_{j_2} < l^*, \quad \forall j_2 = 0, \dots, k_2$$

We have:

$$\begin{aligned} |\varphi_{t,p-f}^{r-a}| &\leq \frac{\xi_1 + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{k_2}}{\sqrt{(\xi_1 + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{k_2})^2 + l}} \\ &< \frac{l^*(k_1 + k_2 + 1)}{\sqrt{(l^*(k_1 + k_2 + 1))^2 + l}} = \frac{1}{\sqrt{1 + l/(l^*(k_1 + k_2 + 1))^2}} \end{aligned}$$

Because the agent anticipates small deviations of the system with respect to the fixed targets, his degree of risk-aversion will be small before starting the control. Consequently, he

will choose a high value for the parameter l . As k_1 and k_2 have generally small values, the ratio $\frac{1}{\sqrt{1+l/(l^*(k_1+k_2+1))^2}}$ (and implicitly the risk-aversion index $\varphi_{t,p-f}^{r-a}$) will be small. We have:

$$-\frac{1}{\sqrt{1+l/(l^*(k_1+k_2+1))^2}} \leq \varphi_{t,p-f}^{r-a} < 0$$

In particular, when $\|y_{t-j_1} - y_{t-j_1}^g\| \rightarrow 0, \forall j_1 = 1, \dots, k_1$ and $\|y_{t+j_2|I_{t+j_2}}^a - y_{t+j_2}^g\| \rightarrow 0, \forall j_2 = 0, \dots, k_2$, it follows that $\varphi_{t,p-f}^{r-a} \rightarrow 0$ (a borderline case).

In the real world, this type of scenario is rare to be realized due to endogenous fluctuations of the system over time. The assumption of risk-neutrality for the entire working horizon is generally non-realistic.

We have the following implication:

$$\varphi_{t,p-f}^{r-a} \rightarrow 0 \Rightarrow U_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a}) \rightarrow -W_{[1,t]}$$

and thus

$$\max E_0[U_t(W_{[1,t]}, \varphi_{t,p-f}^{r-a})] = \min E_0[W_{[1,t]}(y_1, \dots, y_t)]$$

Proof of Proposition 8.

Suppose there is some large deviations of the system (in the past and future), say $\xi_{\alpha_1}, \dots, \xi_{\alpha_{N_1}}$ and $\bar{\xi}_{\beta_1}, \dots, \bar{\xi}_{\beta_{N_2}}$, respectively. We have:

$$\alpha_1, \dots, \alpha_{N_1} \in \{1, \dots, k_1\}, N_1 < k_1 \text{ and } \beta_1, \dots, \beta_{N_2} \in \{0, \dots, k_2\}, N_2 < k_2$$

For simplicity, we assume that all high deviations of the system are comparable in magnitude. Denote by $\varphi_{t,p-f}^{r-a,s-h}$ the risk-aversion index at time t in the case of mixed (i.e., small and large) deviations of the system. Its value is defined by the following expression:

$$\frac{\xi_1 L_{t-1} + \dots + \xi_{\alpha_{N_1}} L_{t-\alpha_{N_1}} + \dots + \xi_{k_1} L_{t-k_1} + \bar{\xi}_0 \bar{L}_t + \dots + \dots + \bar{\xi}_{\beta_{N_2}} \bar{L}_{t+\beta_{N_2}} + \dots + \bar{\xi}_{k_2} \bar{L}_{t+k_2}}{\sqrt{(\xi_1 + \dots + \xi_{\alpha_{N_1}} + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{\beta_{N_2}} + \dots + \bar{\xi}_{k_2})^2 + l}}$$

where ξ_{j_1} and $\bar{\xi}_{j_2}$ ($j_1 = 1, \dots, k_1, j_1 \neq \alpha_{N_1}^*, N_1^* = 1, \dots, N_1; j_2 = 0, \dots, k_2, j_2 \neq \alpha_{N_2}^*, N_2^* = 0, \dots, N_2$) represent small deviations of the system.

Denote by $\varphi_{t,p-f}^{r-a,s}$ the risk-aversion index at time t in the case where there is only small deviations of the system with respect to the fixed targets. One can write:

$$\varphi_{t,p-f}^{r-a,s} = \frac{\xi_1 L_{t-1} + \dots + \xi_{k_1} L_{t-k_1} + \bar{\xi}_0 \bar{L}_t + \dots + \bar{\xi}_{k_2} \bar{L}_{t+k_2}}{\sqrt{(\xi_1 + \dots + \xi_{k_1} + \bar{\xi}_0 + \dots + \bar{\xi}_{k_2})^2 + l}}$$

Without loss of generality, we assume that all small deviations ξ_{j_1} and $\bar{\xi}_{j_2}$ are comparable in magnitude. It follows that:

$$\varphi_{t,p-f}^{r-a,s-h} \rightarrow \frac{L_{t-\alpha_1} + \dots + L_{t-\alpha_{N_1}} + \bar{L}_{t+\beta_1} + \dots + \bar{L}_{t+\beta_{N_2}}}{\sqrt{(N_1 + N_2)^2 + l}}$$

and

$$\varphi_{t,p-f}^{r-a,s} \rightarrow \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{\sqrt{(k_1 + k_2 + 1)^2 + l}}$$

Depending on the moment when high deviations arrive in the past or are expected to be realized in the future, as well as on the strategic weights the agent will attach to the system deviations, one can have two distinct scenarios:

$$\frac{L_{t-\alpha_1} + \dots + L_{t-\alpha_{N_1}} + \bar{L}_{t+\beta_1} + \dots + \bar{L}_{t+\beta_{N_2}}}{\sqrt{(N_1 + N_2)^2 + l}} \ll (\text{or } \gg) \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{\sqrt{(k_1 + k_2 + 1)^2 + l}}$$

Therefore:

$$\varphi_{t, p_f}^{r-a, s-h} < (\text{or } >) \varphi_{t, p_f}^{r-a, s}$$

In other words, the agent's risk aversion at time t is not necessarily higher for mixed deviations of the system, compared to the case when these deviations are all small.

This surprising result reveals the complexity of the agent's risk behavior in a changing environment.

Proof of Proposition 9.

By hypothesis, we have:

$$\begin{aligned} & \| y_{t-j'_1} - y_{t-j'_1}^g \| / \| y_{t-j''_1} - y_{t-j''_1}^g \| \cong 1 \text{ for } j'_1, j''_1 \in \{1, \dots, k_1\}, j'_1 \neq j''_1 \\ & \| y_{t+j'_2|I_{t+j'_2}}^a - y_{t+j'_2}^g \| / \| y_{t+j''_2|I_{t+j''_2}}^a - y_{t+j''_2}^g \| \cong 1 \text{ for } j'_2, j''_2 \in \{0, \dots, k_2\}, j'_2 \neq j''_2 \\ & l / \| y_{t-j_1} - y_{t-j_1}^g \|^4 \cong 0, \forall j_1 = \overline{1, k_1}; l / \| y_{t+j_2|I_{t+j_2}}^a - y_{t+j_2}^g \|^4 \cong 0, \forall j_2 = \overline{0, k_2} \end{aligned}$$

It follows that:

$$\varphi_{t, p_f}^{r-a} \rightarrow \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{k_1 + k_2 + 1}$$

(the average of all past and future weights)

Depending on the weighting scalars values attached to the system deviations, the risk-aversion index level at time t may be more or less close to -1 or 0 . If all weighting scalars approach -1 , then the index value will be close enough to -1 . In this case, the agent will be characterized by an excessive risk-aversion. Contrary to what is generally believed or intuition would suggest, it is possible to have a small risk-aversion for large deviations of the system. It is the case where all weighting scalars approach 0 . The agent is thus almost risk-neutral, that is, there is a very small variability in his risk-aversion.

Proof of Proposition 10.

Suppose that all deviations of the system are comparable in magnitude. For ease of exposition, we consider that $k_1 < k'_1$ and $k_2 < k'_2$.

Let us make the following notations before proceeding: $k'_1 \stackrel{\text{not.}}{=} k_1 + k^*$ and $k'_2 \stackrel{\text{not.}}{=} k_2 + k^{**}$, where k^*, k^{**} are two integers superior to 1 .

We have:

$$\begin{aligned} & \frac{L_{t-1} + \dots + L_{t-k'_1} + \bar{L}_t + \dots + \bar{L}_{t+k'_2}}{k'_1 + k'_2 + 1} - \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{k_1 + k_2 + 1} = \\ & \frac{(L_{t-1} + \dots + L_{t-k_1}) + L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)} + (\bar{L}_t + \dots + \bar{L}_{t+k_2}) + \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}}{k_1 + k_2 + 1 + k^* + k^{**}} \\ & - \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{k_1 + k_2 + 1} = \end{aligned}$$

$$\frac{(k_1 + k_2 + 1)[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)} + \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}]}{(k_1 + k_2 + 1 + k^* + k^{**})(k_1 + k_2 + 1)} \\ - \frac{(k^* + k^{**})(L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2})}{(k_1 + k_2 + 1 + k^* + k^{**})(k_1 + k_2 + 1)}$$

It is easy to see that:

$$\bar{k}(L_{t-1} + \dots + L_{t-k_1}) < k_1[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}], \forall \bar{k} \in \{k^*, k^{**}\} \\ \bar{k}(\bar{L}_t + \dots + \bar{L}_{t+k_2}) < (k_2 + 1)[\bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}], \forall \bar{k} \in \{k^*, k^{**}\}$$

It follows that:

$$(k^* + k^{**})(L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}) \\ < 2k_1[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}] + 2(k_2 + 1)[\bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}]$$

Let us examine the sign of the following difference:

$$k_1[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}] + (k_2 + 1)[\bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}] \\ - \{k_1[\bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}] + (k_2 + 1)[L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}]\} \\ = (k_2 + 1 - k_1)\{[\bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})}] - [L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}]\}$$

Several distinct scenarios are possible:

i)

$$k_2 < k_1 \text{ and } \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})} \gg L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}$$

ii)

$$k_2 > k_1 \text{ and } \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})} \ll L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}$$

iii)

$$k_2 < k_1 \text{ and } \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})} \ll L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}$$

iv)

$$k_2 > k_1 \text{ and } \bar{L}_{t+(k_2+1)} + \dots + \bar{L}_{t+(k_2+k^{**})} \gg L_{t-(k_1+1)} + \dots + L_{t-(k_1+k^*)}$$

For the first two scenarios, one obtains:

$$\frac{L_{t-1} + \dots + L_{t-k'_1} + \bar{L}_t + \dots + \bar{L}_{t+k'_2}}{k'_1 + k'_2 + 1} \gg \frac{L_{t-1} + \dots + L_{t-k_1} + \bar{L}_t + \dots + \bar{L}_{t+k_2}}{k_1 + k_2 + 1}$$

and hence a smaller degree of risk-aversion (at the same stage of the control) when the agent considers a higher value of the sum of k_1 and k_2 . We underline here the trade-off between complexity and perception.

For the last two scenarios, the agent's degree of risk-aversion does not necessarily diminish (at the same stage of the control) when the sum of k_1 and k_2 is higher. It is the case where the agent gives more importance either to the past or to the future.

Proof of Proposition 11.

One can write the inequality:

$$|\varphi_{t,p-f}^{r-a}| < \frac{S_{t,p-d} + S_{t,a-f-d}}{\sqrt{(S_{t,p-d} + S_{t,a-f-d})^2 + l}}, \quad t = 1, \dots, T$$

The agent's objective is to constrain the system in such a way that:

$$\begin{aligned} \| y_{t-j_1} - y_{t-j_1}^g \| &< l^*, \quad j_1 = 1, \dots, k_1 \\ \| y_{t+j_2|I_{t+j_2}} - y_{t+j_2}^g \| &< l^*, \quad j_2 = 0, \dots, k_2 \end{aligned}$$

where l^* is a small value from $(0, 1)$.

It follows that:

$$| \varphi_{t, p-f}^{r-a} | < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l}}$$

that is,

$$-1 < -\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l}} < \varphi_{t, p-f}^{r-a} < 0$$

The agent's objective is not to exceed the fixed optimal threshold φ_{\min}^{p-f} during the entire period of control (i.e., $\varphi_{t, p-f}^{r-a} > \varphi_{\min}^{p-f} \forall t = 1, \dots, T$). Two distinct scenarios are possible:

Either

$$-1 < \varphi_{\min, more}^{p-f} < -\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l_{more}}}$$

or

$$-\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l_{less}}} < \varphi_{\min, less}^{p-f} < 0$$

with $1 \leq l_{more} < l_{less}$ two parameters which characterize the agent's type.

The first (second) scenario corresponds to a more (less) risk-averse agent by nature. The two fixed thresholds, $\varphi_{\min, more}^{p-f}$ and $\varphi_{\min, less}^{p-f}$, are not exceeded during the period of control if and only if the agent succeeds in controlling the system fluctuations.

Proof of Proposition 12.

Suppose that $W_{[1,t]} > 0, \forall t = 1, \dots, T$. We have the following inequality:

$$\exp\left(-\frac{\varphi_{t, p-f}^{r-a}}{2} W_{[1,t]}\right) - 1 > -\frac{\varphi_{t, p-f}^{r-a}}{2} W_{[1,t]} > 0$$

and hence

$$U_t(W_{[1,t]}, \varphi_{t, p-f}^{r-a}) < -W_{[1,t]} < 0$$

The agent's local utility level varies with the value of the risk-aversion index. If the agent succeeds to manage the risk at time t (i.e., $\varphi_{\min}^{p-f} < \varphi_{t, p-f}^{r-a} < 0$), then:

$$U_t(W_{[1,t]}, \varphi_{\min}^{p-f}) < U_t(W_{[1,t]}, \varphi_{t, p-f}^{r-a}) < -W_{[1,t]} = \lim_{\varphi_{t, p-f}^{r-a} \rightarrow 0} U_t(W_{[1,t]}, \varphi_{t, p-f}^{r-a})$$

In the case where the agent does not succeed to manage the risk at time t (i.e., $-1 < \varphi_{t, p-f}^{r-a} < \varphi_{\min}^{p-f}$), we have:

$$\lim_{\varphi_{t, p-f}^{r-a} \rightarrow -1} U_t(W_{[1,t]}, \varphi_{t, p-f}^{r-a}) = U_t(W_{[1,t]}, -1) < U_t(W_{[1,t]}, \varphi_{t, p-f}^{r-a}) < U_t(W_{[1,t]}, \varphi_{\min}^{p-f})$$

What is optimal for the agent in the first context becomes unoptimal in the second context. Optimality generally depends on context and adopted criteria.

Proof of Proposition 13.

One can write the sequence of inequalities:

$$\begin{aligned} \varphi_{\min, more}^{p-f} &< -\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l_{more}}}, \\ &-\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + l_{less}}} < \varphi_{\min, less}^{p-f} \end{aligned}$$

For a higher (smaller) time horizon T , the value of the parameters k_1 and k_2 can be higher (smaller), and thus the ratio $-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l_{less}}}$ (respectively $-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l_{more}}}$) can take a smaller (higher) value. In other words, a less (more) risk-averse decision-maker can choose a smaller (higher) risk-aversion threshold $\varphi_{\min, less}^{p-f}$ (respectively $\varphi_{\min, more}^{p-f}$) depending on the length of the working horizon.

We must distinguish between “nature” and “type”. The agent is considered risk-averse by nature, while his type is more or less risk-averse. The evolution of the system over time will refine the agent’s type.

Proof of Proposition 14.

Following the same reasoning as in **Proposition 1**, we obtain that $U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l})$ is a decreasing function in $\varphi_{t, p-f}^{r-l}$ and $W_{[1,t]}$, where:

$$U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l}) \stackrel{def.}{=} \frac{2}{\varphi_{t, p-f}^{r-l}} \left[\exp\left(-\frac{\varphi_{t, p-f}^{r-l}}{2} W_{[1,t]}\right) - 1 \right]$$

represents the utility function at time t of a risk-taker decision-maker.

Proof of Proposition 15.

The strategic weights verify the following conditions:

$$0 < \tilde{L}_{t-1} < \dots < \tilde{L}_{t-k_1} < 1, \quad 0 < L'_t < \dots < L'_{t+k_2} < 1$$

Following the same reasoning as in **Proposition 11**, one can write:

$$\varphi_{t, p-f}^{r-l} < \frac{\tilde{S}_{t, p-d} + S'_{t, a-f-d}}{\sqrt{(S_{t, p-d} + S_{t, a-f-d})^2 + \tilde{l}}}, \quad t = 1, \dots, T$$

It follows that:

$$\varphi_{t, p-f}^{r-l} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}}}$$

where l^* is a small value from $(0, 1)$ such that:

$$\begin{aligned} \|y_{t-j_1} - y_{t-j_1}^g\| &< l^*, \quad j_1 = 1, \dots, k_1 \\ \|y_{t+j_2|I_{t+j_2}} - y_{t+j_2}^g\| &< l^*, \quad j_2 = 0, \dots, k_2 \end{aligned}$$

The agent’s objective is not to exceed a fixed optimal threshold φ_{\max}^{p-f} during the entire period of control (i.e., $\varphi_{t, p-f}^{r-l} < \varphi_{\max}^{p-f} \forall t = 1, \dots, T$). Two distinct scenarios are possible:

Either

$$0 < \varphi_{\max, less}^{p-f} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{less}}}$$

or

$$\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{more}}} < \varphi_{\max, more}^{p-f} < 1$$

where \tilde{l}_{more} and \tilde{l}_{less} (with $1 \leq \tilde{l}_{more} < \tilde{l}_{less}$) are two strategic parameters which characterize the agent's type.

The first (second) scenario corresponds to a less (more) risk-taker agent by nature. The thresholds $\varphi_{\max, more}^{p-f}$ and $\varphi_{\max, less}^{p-f}$ are not exceeded during the period of control if and only if the agent succeeds in controlling the system fluctuations.

Proof of Proposition 16.

Following the same reasoning as in **Proposition 12**, one obtains the inequality:

$$U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l}) > -W_{[1,t]}, \quad t = 1, \dots, T$$

where by hypothesis, $W_{[1,t]} > 0$. We distinguish two asymmetric cases:

- i) the agent does not exceed the optimal threshold φ_{\max}^{p-f} at time t (i.e., $\varphi_{t, p-f}^{r-l} < \varphi_{\max}^{p-f}$).
- ii) the agent exceeds the optimal threshold φ_{\max}^{p-f} at time t (i.e., $\varphi_{t, p-f}^{r-l} > \varphi_{\max}^{p-f}$).

In the first case, we have:

$$U_t^{r-l}(W_{[1,t]}, \varphi_{\max}^{p-f}) < U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l}) < U_t^{r-l}(W_{[1,t]}, 0)$$

In the second case, one can write:

$$\lim_{\varphi_{t, p-f}^{r-l} \rightarrow 0} U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l}) = -W_{[1,t]} < U_t^{r-l}(W_{[1,t]}, \varphi_{t, p-f}^{r-l}) < U_t^{r-l}(W_{[1,t]}, \varphi_{\max}^{p-f})$$

Proof of Proposition 17.

One can write the sequence of inequalities:

$$\varphi_{\max, less}^{p-f} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{less}}},$$

$$\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{more}}} < \varphi_{\max, more}^{p-f}$$

For a higher (smaller) time horizon T , the value of the parameters k_1 and k_2 can be higher (smaller), and thus the ratio $\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\tilde{l}_{less}}}$ (respectively $\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\tilde{l}_{more}}}$) can take a (higher) smaller value. In other words, a less (more) risk-taker decision-maker can choose a higher (smaller) threshold $\varphi_{\max, less}^{p-f}$ (respectively $\varphi_{\max, more}^{p-f}$) depending on the length of the working horizon.

Proof of Proposition 18.

We have the inequality:

$$|\varphi_{t, p-f}^{r-a-n-l}| < \frac{\tilde{S}_{t, p-d} + S'_{t, a-f-d}}{\sqrt{(S_{t, p-d} + S_{t, a-f-d})^2 + \tilde{l}}}, \quad t = 1, \dots, T$$

It follows that:

$$\varphi_{t, p-f}^{r-a-n-l} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}}} \text{ if } \varphi_{t, p-f}^{r-a-n-l} > 0$$

or

$$\varphi_{t, p-f}^{r-a-n-l} > -\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}}} \text{ if } \varphi_{t, p-f}^{r-a-n-l} < 0$$

where l^* is a small value from $(0, 1)$ such that:

$$\begin{aligned} \|y_{t-j_1} - y_{t-j_1}^g\| &< l^*, \quad j_1 = 1, \dots, k_1 \\ \|y_{t+j_2|I_{t+j_2}} - y_{t+j_2}^g\| &< l^*, \quad j_2 = 0, \dots, k_2 \end{aligned}$$

Following the same reasoning as in **Proposition 11** and **Proposition 15**, one obtains the following distinct scenarios:

i) either

$$-1 < \varphi_{\min, more}^{p-f, c-b} < -\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{more}}}$$

or

$$-\frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{less}}} < \varphi_{\min, less}^{p-f, c-b} < 0$$

ii) either

$$0 < \varphi_{\max, less}^{p-f, c-b} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{less}}} < 1$$

or

$$0 < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \tilde{l}_{more}}} < \varphi_{\max, more}^{p-f, c-b} < 1$$

with $1 \leq \tilde{l}_{more} < \tilde{l}_{less}$ two strategic parameters which characterize the agent's type.

References

- [1] AIGINGER, K., 1987. Production and Decision Theory under Uncertainty.
- [2] ALLAIS, H., 1947. *Economie et Intérêt*, Paris, Imprimerie Nationale.
- [3] ARROW, K. J., 1951. "Alternative approaches to the theory of choice in risk-taking situations", *Econometrica*, **19**, 404-437.
- [4] ARROW, K. J., 1971A. *The Theory of Risk Aversion*, Amsterdam, North-Holland.
- [5] ARROW, K. J., 1971B. *Essays in the Theory of Risk Bearing*, Chicago, Markham Publishing Co.
- [6] BELLMAN, R., 1961. "Adaptive Control Processes: A Guided Tour", *Princeton University Press*.
- [7] BERNOULLI, D., 1954. "Exposition of a New Theory on the Measurement of Risk", *Econometrica*, **22**, 23-36.
- [8] BOMMIER, A., 2006. "Risk Aversion and Planning Horizons", *Journal of the European Economic Associations*, **4** (4), 708-734.
- [9] CHAMBERS, R. G., AND J. QUIGGIN, 2006. "Dual Approaches to the Analysis of Risk Aversion", *Economica*, **74**, 189-213.

- [10] CHEVE, M., AND R. CONGAR, 2002. “Managing environmental risks under scientific uncertainty and controversy”, in E. C. van Ierland, H. P. Weikard and J. Wesseler (eds).
- [11] COHEN, M. D., 1995. “Risk-aversion concepts in expected and non-expected utility models”, *The GENEVA Papers on Risk and Insurance Theory*, **20** (1), 73-91.
- [12] CORNER, J. L., AND P. D. CORNER, 1995. “Characteristics of decisions in decision analysis practice”, *The Journal of Operational Research Society*, **46**, 304-314.
- [13] COX, L. A., 2001. *Risk-Analysis: Foundations, Models, and Methods*, Boston, Kluwer.
- [14] DOUARD, J., 1996. “Is Risk Neutrality Rational?”, *Medical Decision Making*, **16** (1), 10-11.
- [15] DREZE, J. H., 1990. “Essays on Economic Decisions under Uncertainty”, *Cambridge University Press*.
- [16] EASLEY, D., AND N. KIEFER, 1988. “Controlling a stochastic process with unknown parameters”, *Econometrica*, **56**, 1045-1064.
- [17] EECKHOUDT, L., AND P. GODFROID, 2000. “Risk Aversion and the Value of Information”, *The Journal of Economic Education*.
- [18] EISENHAUER, J., G., 2003. “Approximation bias in estimating risk aversion”, *Economics Bulletin*.
- [19] EISENHAUER, J., AND L. VENTURA, 2003. “Survey measures of risk aversion and prudence”, *Applied Economics*, **35**, 1477-1484.
- [20] EISENHAUER, J. G., 2007. “Measuring Risk Aversion”, *Journal of Risk and Insurance*.
- [21] EPSTEIN, L. G., AND S. E., ZIN, 1989. “Substitution, Risk Aversion, and the Temporal Behavior of Consumption Growth and Asset Returns I: A Theoretical Framework”, *Econometrica*, **57** (4), 937-969.
- [22] FILBECK, G. P., HATFIELD, P., AND HORVATH, P., 2005. “Risk Aversion and Personality Type”, *Journal of Behavioral Finance*, **6** (4), 170-180.
- [23] FRIEDMAN, M., AND L. J. SAVAGE, 1948. “The Utility Analysis of Choices Involving Risk”, *Journal of Political Economy*, **56**, 279-304.
- [24] GILBOA I., 1989. “Expectation and Variation in Multi-Period Decisions”, *Econometrica*, **5**, 1153-1169.
- [25] ISAAC, M. R., AND J. DUNCAN, 2000. “Just Who are You Calling Risk-Averse?”, *Journal of Risk and Uncertainty*, **20** (2).
- [26] JERKER, D., 2007. “Adaptive learning and risk taking”, *Psychological Review*, **114** (1), 177-187.
- [27] KAHNEMAN, D., AND A. TVERSKY, 1979. “Prospect Theory: An Analysis of Decision under Risk”, *Econometrica*, **47**, 263-292.
- [28] KIEFER, N., AND Y. NYARKO, 1989. “Optimal control of an unknown linear process with learning”, *International Economic Review*, **30**, 571-586.
- [29] KIMBALL, M. S., 1993. “Standard Risk Aversion”, *Econometrica*, **61** (3), 589-611.
- [30] KIRKWOOD, G. W., 2004. “Approximating Risk Aversion in Decision Analysis Applications”, *Decision Analysis*, **1** (1), 51-67.
- [31] LELAND, J. W., 1990. “A Theory of Approximate Expected Utility Maximization”, Mimeo, Department of Social and Decision Sciences, Carnegie Mellon University.
- [32] MACHINA, M., 1984. “Temporal Risk and the Nature of Induced Preferences”, *Journal of Economic Theory*, **36**, 199-231.
- [33] NIELSEN, L. T., 2005. “Monotone Risk Aversion”, *Economic Theory*, **25**, 203-215.
- [34] NOVOSELOV, A. A., 2003. “Risk-Aversion: A Qualitative Approach and Quantitative Estimates”, *Automation and Remote Control*, **64** (7).

- [35] PRATT, J. W., 1964. "Risk Aversion in the small and in the large", *Econometrica*, **32**, 122-136.
- [36] RABIN, M., 2000. "Risk Aversion and Expected-Utility Theory: A Calibration Theorem", *Econometrica*, **68** (5), 1281-1292.
- [37] RABIN, M., AND THALER, R. H., 2001. "Anomalies: Risk aversion", *The Journal of Economic Perspectives*, **15**, 219-232.
- [38] ROSS, S. A., 1981. "Some stronger measures of risk aversion in the small and in the large with applications", *Econometrica*, **3**, 621-638.
- [39] SARGENT, T. J., 1993. "Bounded Rationality in Macroeconomics", Oxford, *Oxford University Press*.
- [40] SIMON, H. A., 1982A. "Models of Bounded Rationality", **1**, Cambridge, *MIT Press*.
- [41] SIMON, H. A., 1982B. "Models of Bounded Rationality", **2**, Cambridge, *MIT Press*.
- [42] STIGLER, G. S., AND BECKER, G. S., 1977. "De Gustibus non est disputandum", *American Economic Review*, **67**, 76-90.
- [43] TOULET, C., 1982. "Respect de l'axiome d'indépendance dans l'incertain total: Pessimisme, Optimisme, Indécision", *Bulletin de Mathématiques Economiques*, **17**.
- [44] VAN DER PLOEG, F., 1992. "Temporal risk aversion, intertemporal substitution and Keynesian propensities to consume", *Economics Letters*, **39**, 479-484.
- [45] VAN DE STADT, H., KAPTEYN, A., VAN DE GEER, S., 1985. "The Relativity of Utility: Evidence from Panel Data", *Review of Economics and Statistics*, **2**, 179-187.
- [46] VARIAN, H. L., 1990. "Goodness-of-fit in optimizing models", *Journal of Econometrics*, **46**, 125-140, North-Holland.
- [47] WALSH, V., 1996. "Rationality, Allocation and Reproduction", *Clarendon Press*, Oxford.
- [48] YAARI, M., 1987. "The dual theory of choice under risk", *Econometrica*, **55** (1), 95-115.
- [49] YAO, R., HANNA, S. D., AND LINDAMOOD, S., 2004. "Changes in financial risk tolerance", *Financial Services Review*, **13** (4).