On Bubbling Dynamics Generated by a Stochastic Model of Herd Behavior

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Abstract

This paper suggests a class of stochastic collective learning processes exhibiting very irregular behavior. In particular, there are multimodal long run distributions. Some of these modes may vanish as the population size increases. This may be thought of as "bubbles" persistent for a finite range of population sizes but disappearing in the limit. The limit distribution proves to be a discontinuous function of parameters determining the learning process. This gives rise to another type of "bubbles": limit outcomes corresponding to small perturbations of parameters are different.

Since an agent's decision rule involves imitation of the majority choice in a random sample of other members of the population, the resulting collective dynamics exhibit "herding" or "epidemic" features.

JEL Classification: C72, C73, D83.
Keywords: increasing returns, birth and death Markov chain, bubbles.

1 Introduction

We address here the issue of collective learning. The model studied in Kirman (1993) seems to be the closest to ours both conceptually and technically. In fact, these papers deal with a finite pool of agents whose choice among two competing products is affected by "natural" random factors (like sampling variability, mutation and imitation). Both works look at the steady state distribution of market shares and they study the asympotics of such distribution as the population grows without bound. Moreover, our paper suggests an approach to constructing a class of collective learning models accounting for variability of an agent' decision rules and a machinery to study the asympototic behavior of them (when first time goes to infinity and then the size of the population goes to infinity). In formal terms, this class of models comprises birth and death Markov chains. The tool, Laplace's method for nonstationary potentials, used to pass to the limit as the number of states increases, is new.

Our numerical simulations illustrate the extreme irregularity of random processes generated by the model. In fact, we have found parametrizations where the limit market share of either product may take three distinct values, symmetric with respect to 1/2: close to 0, 1/2 and close to 1. If all parameters but mutation rates are kept constant and the latter decreases from 10^{-4} to

10^{-5}, the support of the limit market share contains subsequently one, three and two points. The probability assigned to 1/2 seems to be a discontinuous function of mutation rates.

Conceptually, the processes analyzed here are similar to those studied in Bikhchandani et al. (1998), Orleán (1995), Banerjee (1992) and these papers contain other relevant references.

2 Description of the model

Assume that a population of N agents chooses among two products, A and B. The state variable i designates the number of agents using product A. Time is discrete $t = 0, 1, 2, \ldots$. At time $t = 0$ there are $i_0 > 0$ users of A and $N - i_0 > 0$ users of B. We do not know how these initial numbers came to exist and are not interested in this problem. What we are looking for is how the population evolves driven by sequential decisions of agents.

Let us restrict ourselves to the following decision rules. At time t an agent is picked at random. He is an A-user with probability i_t/N and a B-user with probability $1 - i_t/N$, where i_t is the state variable at t. If the agent is an A-user (B-user), he samples with replacement r_A (r_B) agents to find out which of the products they are using. The number $r_A(r_B) \geq 1$ reflects how people "get around" making decisions. Larger samples may correspond to more risk averse agents. These numbers are parameters of the model, they are fixed through the evolution of the population.

Depending upon the content of his sample, the agent may change the product in use. This change is driven by two motives: imitation and search for diversity (or mutation). In fact, if the majority in the sample is using a product different from his, being driven by conformism, the agent may choose this one. This is imitation of the majority choice. If the majority in the population is using the same product as his, or equivalently, if the minority in the population is using the concurrent one, the agent may change his product. This is mutation.

The division of population on imitators and diversity seekers is common in socio-economic theory of consumer. There it is customary to distinguish between elitarian consumers and people who prefer to follow the behavior of others (see, for example, Veblen (1965) or Duesenberry (1952)).

Let the state variable be i. Consider the probability that a sample with replacement of r agents from a pool of N agents contains at least s users of product A. This may be thought of as the probability of at least s successes in r Bernoulli trials whose probability of success equals i/N. That is,

$$\sum_{j=s}^{r} C_r^j \left(\frac{i}{N}\right)^j \left(1-\frac{i}{N}\right)^{r-j}.$$

Here C_r^j denotes the number of combinations formed from r in j,

$$C_r^j = \frac{r!}{j!\,(r-j)!}.$$

For $\mu \in (0, 1/2],\ \gamma \in [1/2, 1)$ and $x \in [0,1]$ set

$$f_r^{\gamma}(x) = \sum_{j>\gamma r} C_r^j x^j (1-x)^{r-j},$$

$$g_r^{\mu}(x) = \sum_{j=0}^{[\mu r]} C_r^j x^j (1-x)^{r-j}.$$

Here $[a]$ denotes the integer part of a real number a. Remark that $f_r^{\gamma}(x)$ equals the probability of more than γr successes and $g_r^{\mu}(x)$ equals the probability of not more than $[\mu r]$ successes in r Bernoulli trials whose probability of success is x.

The probability that the number of A-users in the population increases by 1, that is, the transition $i \to i+1$, reads

$$\left(1-\frac{i}{N}\right)\left[f_{r_B}^{1/2}\left(\frac{i}{N-1}\right)\beta_B + g_{r_B}^{1/2}\left(\frac{i}{N-1}\right)\alpha_B\right]. \tag{1}$$

The first term means that the agent picked at random is a B-user. If the majority in the sample of r_B other agents are A-users, he changes to A product with probability β_B imitating the choice of the majority. If the minority in the sample are A-users, he "mutates" to product A with probability α_B. Remark that $N-1$ comes to exist here because the agent making his choice may not be sampled.

Similary, the probability that the number of A-users in the population decreases by 1, that is, the transition $i \to i-1$, reads

$$\left(\frac{i}{N}\right)\left[f_{r_A}^{1/2}\left(1-\frac{i-1}{N-1}\right)\beta_A + g_{r_A}^{1/2}\left(1-\frac{i-1}{N-1}\right)\alpha_A\right]. \tag{2}$$

Here the sample sizes must be odd numbers (otherwise the majorities and the minorities are not well defined) and should not exceed $N-1$ (otherwise the samples are not feasible). The mutation and imitation probabilities belong to $[0,1]$ provided that $\alpha_I + \beta_I > 0,\ I = A, B$ (to avoid the trivial cases when transitions are unidirected or no transition takes place).

Now let us assume that agents' threshold values differ from 1/2. In fact, an agent, due to risk averseness, may regard 1/2 as an insufficiently reliable threshold for determining the dominant product in his sample. He may therefore use an upper $\gamma \geq 1/2$ and a lower $\mu \leq 1/2$ threshold level. Thus, A-users represent the γ-majority (μ-minority) in a sample of r agents, if their number exceeds γr (does not exceed $[\mu r]$). Then expressions (1) and (2) read,

$$\left(1-\frac{i}{N}\right)\left[f_{r_B}^{\gamma_B}\left(\frac{i}{N-1}\right)\beta_B + g_{r_B}^{\mu_B}\left(\frac{i}{N-1}\right)\alpha_B\right] \tag{3}$$

and

$$\left(\frac{i}{N}\right)\left[f_{r_A}^{\gamma_A}\left(1-\frac{i-1}{N-1}\right)\beta_A + g_{r_A}^{\mu_A}\left(1-\frac{i-1}{N-1}\right)\alpha_A\right]. \tag{4}$$

Here $\gamma_I \in (1/2, 1]$ and $\mu_I \in [0, 1/2),\ I = A, B$, designate the threshold levels for A- and B-users. If $\gamma_I = \mu_I = 1/2$ and r_I are odd numbers, expressions (3) and (4) reduce to (1) and (2).

If α_I are small compared with β_I, imitation of majority choice is the dominating behavior in the population. The corresponding dynamics may be characterized as herd behavior. In this situation one may intuitively expect the same limit patterns as in models of stochastic increasing returns based on urn schemes (see Arthur et al. (1987) and Dosi et al. (1994)). Our numerical simulations show that this expectation is, in general, correct, but the variety of outcomes is much richer than in models based on urn schemes.

3 Analysis of the Model

The random process defined above is a birth and death Markov chain $\xi^t, t = 0, 1, \dots,$ whose state space is $0, 1, \dots, N$. The transition probabilities $p_{ij} = P\{\xi^{t+1} = j \mid \xi^t = i\}$ are as follows

$$p_{ij} = \begin{cases} 0, & \text{if } |j-i| > 1, \\ (1-\frac{i}{N})p(\frac{i}{N-1}), & \text{if } j = i+1, \\ \frac{i}{N}q\left(\frac{i-1}{N-1}\right), & \text{if } j = i-1, \\ 1-\left[\left(1-\frac{i}{N}\right)p\left(\frac{i}{N-1}\right)+\frac{i}{N}q\left(\frac{i-1}{N-1}\right)\right], & \text{if } j = i. \end{cases}$$

Here $p(\cdot)$ and $q(\cdot)$ are given by the expressions in square brackets in (3) and (4).

We are interested in the long run configuration of the population as a consequence of the behavioral assumptions employed. Thus, we want to eliminate any effect of the initial state of the population on this configuration. Consequently, we have to look at the ergodic properties of the Markov chain $\xi^t,\ t \geq 0$.

Theorem 1 *If the imitation and mutation rates are positive, then the Markov chain is irreducible.*

Proof. It suffices to show that

$$p_{ii+1} > 0,\ 0 \leq i \leq N-1,\ \text{and}\ p_{jj-1} > 0,\ 1 \leq j \leq N. \tag{5}$$

Note that continuous functions $f_r^\gamma(\cdot)$ and $g_r^\mu(\cdot)$ are positive on $(0,1]$ and $[0,1)$ correspondingly. Consequently, there are positive numbers p, P, q and Q such that

$$0 < p \leq p(x) \leq P \quad \text{and} \quad 0 < q \leq q(x) \leq Q \tag{6}$$

where $0 \leq x \leq 1$,

$$\begin{aligned} p(x) &= f_{r_B}^{\gamma_B}(x)\beta_B + g_{r_B}^{\mu_B}(x)\alpha_B, \\ q(x) &= f_{r_A}^{\gamma_A}(1-x)\beta_A + g_{r_B}^{\mu_A}(1-x)\alpha_A. \end{aligned}$$

Then

$$p_{ii+1} \geq \left(1 - \frac{i}{N}\right)p,\ 0 \leq i \leq N-1, \quad \text{and} \quad p_{jj-1} \geq \frac{i}{N}q,\ 1 \leq j \leq N.$$

Thus, inequalities (5) hold true.

The theorem is proved.

An irreducible Markov chain has a unique stationary distribution. For a birth and death chain this distribution can be expressed via transition probabilities (see, for example, Hoel et al. (1972), p.51).

Since henceforth we shall be dealing with the stationary distribution, let us assume that the mutation and imitation rates are positive. Theorem 1 asserts the existence and uniqueness of this distribution.

We are interested in the steady state configuration of the population as a consequence of the behavioral assumptions formulated above. We thus want to eliminate any effect of a particular population size, concentrating on the properties invariant with respect to population size. Accordingly, we should turn to the space of shares $\frac{i}{N+1},\ i = 0, 1, \ldots N$, and look in this space at the limit of the stationary distribution as $N \to \infty$.

To analyze the asymptotic behavior of the stationary distribution as $N \to \infty$, let us apply the approach developed in Kaniovski and Pflug (1999).

For $x \in [0,1]$ set

$$\mathcal{F}_{N+1}(x) = \begin{cases} \dfrac{p_{i-1i}}{p_{ii-1}} & \text{if } [(N+1)x] = i-1,\ 1 \le i \le N, \\ \dfrac{p_{N-1N}}{p_{NN-1}} & \text{if } x = 1, \end{cases}$$

and

$$\Phi_{N+1}(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{N+1}, \\ -\dfrac{1}{N+1}\displaystyle\sum_{j=1}^{i} \ln \dfrac{p_{j-1j}}{p_{jj-1}} & \text{for } \frac{i}{N+1} \le x < \frac{i+1}{N+1},\ 1 \le i \le N-1, \\ -\dfrac{1}{N+1}\displaystyle\sum_{j=1}^{N} \ln \dfrac{p_{j-1j}}{p_{jj-1}} & \text{for } 1 - \frac{1}{N+1} \le x \le 1. \end{cases}$$

The function $\Phi_{N+1}(\cdot)$, which may be regarded as a Riemann integral sum of $\mathcal{F}_{N+1}(\cdot)$, is called the Gibbs potential of the stationary distribution (see, for example, Aoki (1996), p.57). This distribution assigns the highest probabilities to the states i^*/N where $\Phi_N(\cdot)$ attains its global minimum Φ_N^* on $[0,1]$.

Let

$$\mathcal{F}(x) = \frac{(1-x)p(x)}{xq(x)},\ 0 < x < 1.$$

By Lipschitz continuity of $\mathcal{F}(\cdot)$, for every $\epsilon \in (0, 1/2)$ there is a constant $c(\epsilon)$ such that

$$\sup_{\epsilon \le x \le 1-\epsilon} |\mathcal{F}_{N+1}(x) - \mathcal{F}(x)| \le c(\epsilon)/N \tag{7}$$

Consider

$$\Phi(x) = -\int_0^x \mathcal{F}(y)dy,\ 0 \le x \le 1.$$

Note that

$$\lim_{x\to 0} x\mathcal{F}(x) = \frac{\alpha_A}{\beta_A} \qquad \text{and} \qquad \lim_{x \to 1} \frac{\mathcal{F}(x)}{1-x} = \frac{\beta_B}{\alpha_A}.$$

Since $\mathcal{F}(\cdot)$ is positive and continuous on $(0,1)$, these relations imply Riemann integrability of $\mathcal{F}(\cdot)$ on $[0,1]$. Thus, the function $\Phi(\cdot)$ exists.

If the set of global minima X_N^* of $\Phi_N(\cdot)$ approaches as $N \to \infty$ the set X^* of global minima of $\Phi_N(\cdot)$, every weak limit point of stationary distributions belong to X^* (see, for example, Proposition 2.2 in Kaniovski and

Pflug (1999)). This convergence of sets X_N^* takes place, for example, when functions $\Phi_N(\cdot)$ converge uniformly to $\Phi(\cdot)$. We must therefore show (see the Appendix for the proof) that

$$\sup_{0\le x\le 1} |\Phi_N(x) - \Phi(x)| \to 0 \text{ as } N \to \infty. \tag{8}$$

Since $\Phi^{'}(\cdot) = -\ln \mathcal{F}(\cdot)$ is positive near 0 and negative near 1, we conclude that $X^* \subset (0,1)$. Then relations (7) and (8) imply that Theorem 3.1 by Kaniovski and Pflug (1999) may be applied here yielding the following result.

Theorem 2 *Let the imitation and mutation rates be positive. As the population size increases without bound, the steady state distribution of the market share of product A concentrates on the set of global minima of the limit Gibbs potential $\Phi(\cdot)$. If the second derivative $\Phi^{''}(\cdot)$ is positive at all points of global minima $a_i, i = 1, 2, \ldots, K$, this limit share takes the value a_i with probability*

$$\frac{1}{\sqrt{\Phi^{''}(a_i)}} \Big/ \sum_{j=1}^{K} \frac{1}{\sqrt{\Phi^{''}(a_j)}}.$$

Note that the zeros of the derivative $\Phi^{'}(\cdot)$ satisfy the following equation

$$\mathcal{F}(x) = 1 \quad \text{or} \quad xq(x) = (1-x)p(x).$$

Since $p(\cdot)$ and $p(\cdot)$ are polynoms, all connected components of the set of global minima of $\Phi(\cdot)$ are singletons.

Let us show that for all large enough sample sizes and sufficiently small mutation rates the limit Gibbs potential attains minima near to 0 and 1. Those minima are not necessarily global, as we shall demonstrate on numerical examples.

If $r_B \to \infty$ and $r_A \to \infty$, the functions $p(\cdot)$ and $q(\cdot)$ are approaching the following piesewise constant functions

$$p_\infty(x) = \begin{cases} \alpha_B, & 0 \le x < \mu_B, \\ 0, & \mu_B < x < \gamma_B, \\ \beta_B, & \gamma_B < x \le 1; \end{cases}$$

and

$$q_\infty(x) = \begin{cases} \beta_A, & 0 \le x < 1-\gamma_A, \\ 0, & 1-\gamma_A < x < 1-\mu_A, \\ \alpha_A, & 1-\mu_A < x \le 1. \end{cases}$$

This convergence is uniform on any closed interval of continuity of the limit (see, for example, Proposition 1 in Kaniovski et al. (1997)). In particular, for every $\epsilon \in (0, a)$ and every $\delta \in (0, 1-b)$

$$\sup_{x\in[0,a-\epsilon]}\left|\frac{q(x)}{p(x)}-\frac{\beta_A}{\alpha_B}\right| = \tilde{\sigma}_\epsilon(r_A, r_B) \to 0 \quad \text{as } \min(r_A, r_B) \to 0$$

and

$$\sup_{x\in[b+\delta,1]}\left|\frac{q(x)}{p(x)}-\frac{\alpha_A}{\beta_B}\right| = \check{\sigma}_\delta(r_A, r_B) \to 0 \quad \text{as } \min(r_A, r_B) \to 0,$$

where $a = \min(\mu_B, 1-\gamma_A)$ and $b = \max(\gamma_B, 1-\mu_A)$. Since the equation for the zeros of $\Phi^{'}(\cdot)$ can be rewritten as

$$\frac{q(x)}{p(x)} = \frac{1-x}{x}, \tag{9}$$

these estimates imply the following result.

Proposition *If $\beta_A/\alpha_B > (1-a)/a$, then for all sufficiently large r_A and r_B there is a local minimum approaching $\alpha_B/(\alpha_B+\beta_A)$ as* $\min(r_A, r_B) \to \infty$. *If $\beta_A/\alpha_B < (1-b)/b$, then for all sufficiently large r_A and r_B there is a local minimum converging to $\beta_B/(\alpha_A+\beta_B)$ as* $\min(r_A, r_B) \to \infty$.

Note that, if μ_B, γ_A and β_A are fixed, then, by decreasing α_B one can always satisfy the first inequality in Proposition. If μ_A, γ_B and β_B are fixed, then, by decreasing α_A one can always satisfy the second inequality in Proposition.

Another extreme case, $r_A = r_B = 1$, implies a single minimum of the limit Gibbs potential. Taking into account that

$$p(x) = (\beta_B - \alpha_B)x + \beta_B \text{ and } q(x) = (\alpha_A - \beta_A)x + \beta_A,$$

this point may be detemined by equation (9).

We now turn to the numerical simulations demonstrating the variety of limit patterns that our model can exhibit.

4 Numerical Simulations

Let all parameters for both A- and B-users be identical. Then $\ln[\mathcal{F}(x)] = -\ln[\mathcal{F}(1-x)]$. Consequently, the points of minima of $\Phi(\cdot)$ are located symmetrically with respect to $1/2$. Let us vary the mutation rate in the range $[10^{-5}, 10^{-4}]$ keeping all other parameters fixed: $r_I = 101, \beta_I = 0.05, \gamma_I = 0.65, \mu_I = 0.15$.

Figure 1 gives the limit Gibbs potential for three different mutation rates. If $\alpha = 10^{-4}$, the global minimum is attained at a single point $1/2$. By Theorem 2, the limit market shares of both products are deterministic. Thus the market is evenly divided between A and B. If $\alpha = 10^{-5}$, the global minimum is attained at two points symmetric with respect to $1/2$. By simmetricity and Theorem 2, we conclude that the limit market shares of both products take values $x_{10^{-5}} = 1.9996 \cdot 10^{-4}$ and $1 - x_{10^{-5}}$ with probabilities $1/2$.

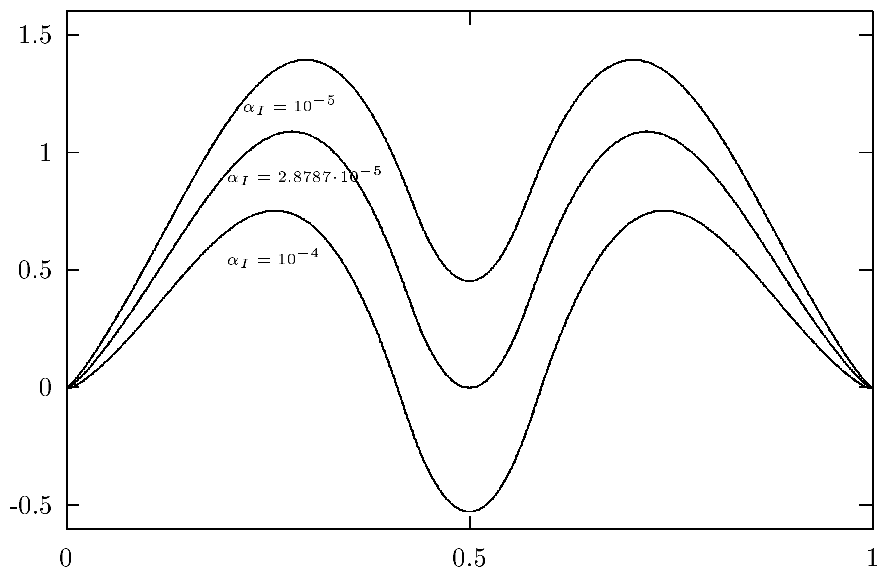
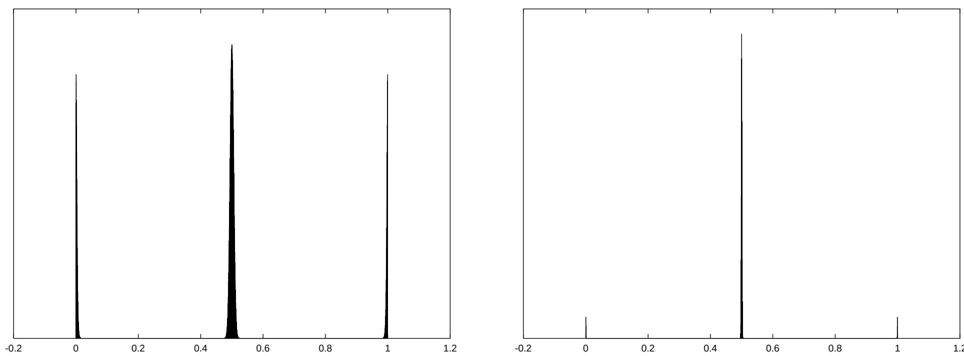


Figure 1: Limit Gibbs potential for different values of mutation rate.

Let $G(\alpha_I) = \Phi(1/2) - \Phi(\bar{x}_{\alpha_I})$, where $\bar{x}_{\alpha_I} = \min x : \Phi^{'}(x) = 0$. In our case this is a continuous function of the mutation rate α_I. Since $G(10^{-4}) < 0$ and $G(10^{-5}) > 0$, we conclude that there is an $\alpha_I^* \in (10^{-5}, 10^{-4})$ such that $G(\alpha_I^*) = 0$. Thus, there is a value for the mutation rates for which the global minimum is attained at three points: $\bar{x}_{\alpha_I^*}, 1/2$ and $1 - \bar{x}_{\alpha_I^*}$. Using an argument exploiting implicit functions, we can prove that such a value is unique. We have found that $\alpha_I^* = 2.8787 \cdot 10^{-5}$. The corresponding Gibbs potential is given in Figure 1. In this case $\bar{x}_{\alpha_I^*} \simeq 5.7541 \cdot 10^{-4}$, $\Phi(\bar{x}_{\alpha_I^*}) \simeq 5.7558 \cdot 10^{-4}$, $\Phi(1/2) \simeq -5.7751 \cdot 10^{-4}$, $\Phi(1 - \bar{x}_{\alpha_I^*}) \simeq -5.7311 \cdot 10^{-4}$, $\Phi^{''}(\bar{x}_{\alpha_I^*}) \simeq 1738.8$, $\Phi^{''}(1/2) \simeq 135.51$, $\Phi^{''}(1 - \bar{x}_{\alpha_I^*}) \simeq 1738.8$. By Theorem 2 the limit market share of A takes each of the values $\bar{x}_{\alpha_I^*}$ and $1 - \bar{x}_{\alpha_I^*}$, with probability approximately equal 0.17914. The value $1/2$ is taken with probability approximately 0.64171 . Let $p(\alpha_I)$ be the probability that the limit market share of A takes the value $1/2$. Then $p(\cdot)$ is a discontinuous function of the mutation rates in a neighborhood of α_I^*. Indeed, $p(\alpha_I) = 0$ for $\alpha_I < \alpha_I^*$, $p(\alpha_I^*) \simeq 0.64171$ and $p(\alpha_I) = 1$ for $\alpha_I > \alpha_I^*$.

Figure 2 demonstrates the "bubbles": close to 0 and 1. Here $\alpha_I = 2.88 \cdot 10^{-5}$. The limit Gibbs potential attains the global minimum at 1/2. There are two equal local minima. Since the difference between the global minimum and the local one is small, the modes of the stationary distribution near 0 and 1 persist even for $N = 5000$.



(a) N=500

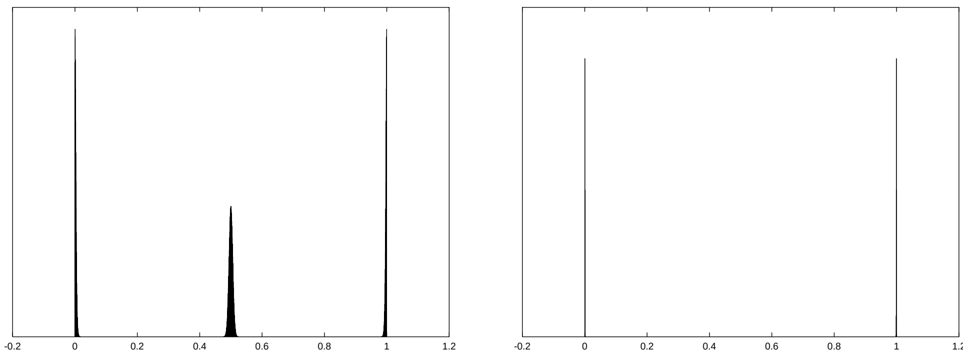
(b) N=5000

Figure 2: Stationary distribution with two "bubbles", $\alpha_I^* = 2.88 \cdot 10^{-5}$

Figure 3 shows a single "bubble" at 1/2. Here $\alpha_I = 2.875 \cdot 10^{-5}$. The limit Gibbs potential attains the global minimum at two points: close to 0 and 1. There is a local minimum at 1/2. Since the difference between the global minimum and the local one is larger than in the case depicted in Figure 2, this "bubble" disappears faster.

Thus, as the rate of mutation decreases, the collective outcome evolves from deterministic market even sharing to the outcome when either product monopolizes the market with probability 1/2. But there is an interesting intermediate configuration that has no analog in models of increasing returns based on urn schemes. In fact, for an intermediate value of mutation rates there is an outcome when monopolies and market sharing take place with positive probabilities.

An even richer variety of limit patterns is generated by the generalization of the basic setting following next.



(a) N=500 (b) N=5000

Figure 3: Stationary distribution with one "bubble", $\alpha_I^* = 2.875 \cdot 10^{-5}$

5 A Generalization of the Model

We may assume that an agent's risk aversion is not constant over time. In fact, as the same agent makes his decision several times, he may use different sample sizes and threshold levels. Also, he may imitate or mutate with different rates. In particular, on some occasions the agent may not imitate or may not mutate on others.

Since the total numer of parameters governing an agent's choice is ten (five for A-users and five for B-users), let us consider a ten-dimensional vector $\vec{R} = (R_1, R_2, \ldots, R_{10})$ taking a finite number of values $\vec{R}^{(j)},\ j = 1, 2, \ldots, K$, such that

$$P\left\{\vec{R} = \vec{R}^{(j)}\right\} = p_j > 0, \quad \sum_{j=1}^{K} p_j = 1.$$

The first five coordinates of $\vec{R}$ are allocated to the parameters governing the decisions of A-users, the other five coordinates are allocated for the parameters governing the decisions of B-users. In fact, R_1 and R_6 give the majority threshold levels, so $R_1^{(j)}$ and $R_6^{(j)}$ are reals belonging to $[1/2, 1)$; R_2 and R_7 correspond to the minority threshold levels, so $R_2^{(j)}$ and $R_7^{(j)}$ are reals from $(0, 1/2]$; R_3 and R_8 give sample sizes, so $R_3^{(j)}$ and $R_8^{(j)}$ are sufficiently large integers (but not exceeding N), if $R_1^{(j)}$ and $R_2^{(j)}$ ($R_6^{(j)}$ and $R_7^{(j)}$) equal $1/2$, then $R_3^{(j)}$ ($R_8^{(j)}$) is an odd number; R_4 and R_9 (R_5 and R_{10}) denote the imitation (mutation) rates, so these are real numbers, $R_i^{(j)} \in [0, 1],\ i = 4, 5, 9, 10$, such that $R_4^{(j)} + R_5^{(j)} > 0$ and $R_9^{(j)} + R_{10}^{(j)} > 0$.

Each time instant $t \geq 0$ an independent realization $R(t)$ of the random vector $\vec{R}$ is observed. This realization determines the parameters for the agent making his decision at t. Let us use the term "scenarios" for the possible realizations $\vec{R}^{(j)}$, $j = 1, 2, \ldots, K$, of the random vector $\vec{R}$. Then $\vec{R}(t)$ determines the scenario of the decision making process at t.

For this setting formulae (3) and (4) become as follows

$$\left(1 - \frac{i}{N}\right) \sum_{j=1}^{K} p_j \left[f_{R_8^{(j)}}^{R_6^{(j)}} \left(\frac{i}{N-1}\right) R_9^{(j)} + g_{R_8^{(j)}}^{R_7^{(j)}} \left(\frac{i}{N-1}\right) R_{10}^{(j)} \right] \tag{10}$$

and

$$\frac{i}{N} \sum_{j=1}^{K} p_j \left[f_{R_3^{(j)}}^{R_1^{(j)}} \left(1 - \frac{i-1}{N-1}\right) R_4^{(j)} + g_{R_3^{(j)}}^{R_2^{(j)}} \left(1 - \frac{i-1}{N-1}\right) R_5^{(j)} \right]. \tag{11}$$

If $K = 1$, then (10) and (11) reduce to (3) and (4). Consequently we henceforth assume that $K > 1$.

The next result is a counterpart of Theorem 1 for $K > 1$.

Theorem 3 *If for A-users as well as for B-users there are scenarios with positive imitation rate and a scenario with positive mutation rate, then the Markov chain is irreducible for all sufficiently large populations.*

Proof. Let $R_4^{j_1} > 0$, $R_5^{j_2} > 0$, $R_9^{j_3} > 0$, and $R_{10}^{j_4} > 0$ for some, not necessarily distinct, $j_i \in \{1, 2, \ldots, K\}$. That is, for A-users (B-users) the scenario $\vec{R}^{j_1}$ ($\vec{R}^{j_3}$) prescribes a positive imitation rate, while the scenario $\vec{R}^{j_2}$ ($\vec{R}^{j_4}$) prescribes a positive mutation rate. Expressions (10) and (11) are minorated by the following

$$\left(1 - \frac{i}{N}\right) \left[p_{j_1} f_{R_8^{(j_1)}}^{R_6^{(j_1)}} \left(\frac{i}{N-1}\right) R_9^{(j_1)} + p_{j_2} g_{R_8^{(j_2)}}^{R_7^{(j_2)}} \left(\frac{i}{N-1}\right) R_{10}^{(j_2)} \right]$$

and

$$\frac{i}{N} \left[p_{j_3} f_{R_3^{(j_3)}}^{R_1^{(j_3)}} \left(1 - \frac{i-1}{N-1}\right) R_4^{(j_3)} + p_{j_4} g_{R_3^{(j_4)}}^{R_2^{(j_4)}} \left(1 - \frac{i-1}{N-1}\right) R_5^{(j_4)} \right].$$

These estimates for the probabilities of transitions $i \to i-1$ and $i \to i+1$ allow application of the argument used in the proof of Theorem 1.

The theorem is proved.

To derive an analog of Theorem 2, set

$$p_G(x) = (1-x) \sum_{j=1}^{K} p_j \left[f_{R_8^{(j)}}^{R_6^{(j)}}(x) R_9^{(j)} + g_{R_8^{(j)}}^{R_7^{(j)}}(x) R_{10}^{(j)} \right],$$

$$g_G(x) = x \sum_{j=1}^{K} p_j \left[f_{R_3^{(j)}}^{R_1^{(j)}} (1-x) R_4^{(j)} + g_{R_3^{(j)}}^{R_2^{(j)}} (1-x) R_5^{(j)} \right],$$

$$\Phi_G(x) = -\int_0^x \ln \frac{(1-y)p_G(y)}{y q_G(y)} dy,$$

where $0 \le x \le 1$. A minor modification to the argument lead to Theorem 2, yields the following statement.

Theorem 4 *Let for A-users as well as for B-users there be a scenario with positive imitation rate and a scenario with positive mutation rate. As the population size increases without bound, the steady state distribution of the market share of product A concentrates on the set of global minima of the limit Gibbs potential $\Phi_G(\cdot)$. If the second derivative $\Phi_G^{''}(\cdot)$ of this potential is positive at all points of global minima $a_j,\ i = 1, 2, \ldots, l$, then the limit share of product A takes the value a_i with probability*

$$\frac{1}{\sqrt{\Phi_G^{''}(a_i)}} \Big/ \sum_{j=1}^{l} \frac{1}{\sqrt{\Phi_G^{''}(a_j)}}.$$

6 Conclusions

We have considered a class of stochastic collective learning models generating "herd" or "epidemic" dynamics of a finite pool of agents. Unlike conceptually similar models exploiting the machinery of urn schemes, these dynamics do not exhibit path dependence. In fact, they always generate ergodic Markov chains and, consequently, possess unique stationary distributions.

The most interesting phenomena occur when the population grows without bound. While the stationary distribution converges to a limit, the dynamics generate "bubbles": that is, some states that are more likely than the neighboring ones for a wide range of population sizes vanish in the limit. This limit exhibits extremely irregular behavior as a function of parameters determining the learning process.

Unlike in models of stochastic increasing returns based on urn schemes, a finite population of agents dominated by imitators may end up with a monopoly of either product as well as with sharing the market. In one of our simulations, where the imitation rates are more than 1700 times higher than the mutation rates, these outcomes coexist and the probability of sharing the market is 3.6 times higher than the probability of monopoly of either product. In another simulation the market is shared with probability one.

If samples are without replacement, then small terms have to be added in the expresions for $p(\cdot)$ and $q(\cdot)$. The correction comes to exist because the probability of success changes during sampling. These terms vanish as const$/N$ for $N \to \infty$, so they do not affect our results. Thus, all conceptual conclusions remain valid for the case of samples without replacement.

Appendix

Let us prove relation (8). By (7) it would hold if

$$\lim_{\delta\to 0}\left[|\Phi(\delta)| + |\Phi(1) - \Phi(1-\delta)|\right] = 0$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\left[|\Phi_N(\delta)| + |\Phi_N(1) - \Phi_N(1-\delta)|\right] = 0.$$

The first equality holds by continuity of Riemann integral. Let us prove the second equality.

Since both terms here are analyzed by the same argument, we shall deal only with $\Phi_N(\delta)$. One has

$$\begin{aligned}
|\Phi_N(\delta)| &\le \frac{1}{N+1}\sum_{j=1}^{[(N+1)\delta]}\left[|\ln p_{jj-1}| + |\ln p_{j-1j}|\right], \\
\ln p_{jj-1} &= \ln\frac{j}{N} + \ln q\left(\frac{j-1}{N-1}\right), \\
\ln p_{j-1j} &= \ln\left(1 - \frac{j-1}{N}\right) + \ln p\left(\frac{j-1}{N-1}\right).
\end{aligned}$$

Since $\frac{\ln(1+x)}{x} \to 1$ as $x \to 0$, $\left|\ln(1-\frac{j-1}{N})\right|$ is uniformly bounded for $1 \le j \le [(N+1)\delta]$. Consequently, by (6) there is a constant C (independent of δ) such that

$$|\Phi_N(\delta)| \le \frac{1}{N+1}\sum_{j=1}^{[(N+1)\delta]}\left(C - \ln\frac{j}{N}\right).$$

Since $\ln x$ is an increasing function, by regarding

$$\frac{1}{N}\sum_{j=2}^{[(N+1)\delta]}\ln\frac{j}{N}$$

as an integral sum, one gets

$$-\frac{1}{N}\sum_{j=2}^{[(N+1)\delta]}\ln\frac{j}{N} \le -\int_{\frac{1}{N}}^{\frac{[(N+1)\delta]-1}{N}}\ln x\,dx = -x(\ln x-1)\Big|_{\frac{1}{N}}^{\frac{[(N+1)\delta-1]}{N}}.$$

Taking into account that $\lim_{x\to 0} x\ln x = 0$, one obtains

$$\limsup_{N\to\infty}\left(-\frac{1}{N+1}\sum_{j=1}^{[(N+1)\delta]}\ln\frac{j}{N}\right) \le \delta(1-\ln\delta)$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}|\Phi_N(\delta)| \le \lim_{\delta\to 0}\delta(C+1-\ln\delta) = 0.$$

In sum, relation(8) holds true.

References

Aoki, M., 1996. New Approaches to Macroeconomic Modeling. Cambridge University Press, Cambridge.

Arthur, B., Ermoliev, Y., Kaniovski, Y., 1987. Path-dependent processes and the emergence of macrostructure. Europ. J. Oper. Res. 30, 294–303.

Banerjee, A., 1992. A simple model of herd behavior. Q. J. Economics 107, 797–817.

Bikhchandani, S., Hirschleifer, D., Welch, I., 1998. A theory of fads, fashion, custom and cultural change as informational cascades. J. Pol. Econ. 100, 992–1026.

Dosi, G., Ermoliev, Y., Kaniovski, Y., 1994. Generalized urn schemes and technological dynamics. J. Math. Econ. 23, 1–19.

Duesenberry, J., 1952. Income, Saving and the Theory of Consumer Behavior. Harvard University Press, Cambridge, Mass.

Hoel, P., Port, S., Stone, C. J., 1972. Introduction to Stochastic Processes. Houghton-Mifflin, Boston.

Kaniovski, Y., Kryazhimskii, A., Young, H., 1997. Learning equilibria in games played by heterogeneous populations. Interim Report IR-97-017, International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria.

Kaniovski, Y., Pflug, C. G., 1999. Limit theorems for stationary distributions of birth-and-death processes. Stochastic Models 15, 103–123.

Kirman, A., 1993. Ants, rationality and recruitments. Q. J. Economics 108, 137–154.

Orleán, A., 1995. Bayesian interactions and collective dynamics of opinion: Herd behavior and mimetic contagion. J. Econ. Behav. Organ. 28, 257–254.

Veblen, T., 1965. The Theory of Leisure Class. Kelley, Fairfield, N.J.

The Rock Group

Rock Group (Research on Organizations, Coordination and Knowledge) is strongly committed to develop theoretical and empirical analyses of organizational key issues such as coordination among agents, decision making processes, coalition formation, replication and diffusion of knowledge, routines, and competencies within the organizational environment.

Rock Group is rooted in the Department of Management and Computer Science (DISA) of the University of Trento, which is characterized by a strong cooperative culture among scholars from different fields, from Management to Mathematics, from Statistics to Computer Science. Rock Group's activities, both research and education benefit by the whole range of competencies of DISA.

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