Simplified preferences, voting, and the power of combination*

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Abstract

In this paper we interpreted the decision to vote for a particular party as a process of delegation to decision makers having a simplified system of preferences. Each person in a population votes for the political party that place priority on one or more issues that they consider important. Moreover, on the basis of a survey on preferences of population, we have simulated a delegation procedure which chart the selection process of a particular party. Finally, making use of noncommutative harmonic analysis, we decomposed the delegation function, and isolated the effect of a particular affinity, or a combination of either the pair of items that characterize a party. We used noncommutative harmonic analysis as an application of some results obtained by Michael E. Orrison and Brian L. Lawson in relation to spectral analysis applied in voting in political committees (see [9], [10] and [11]). *JEL* Classification number: D71. *2000 Mathematics Subject Classification*: 20C15, 62M15, 65T50.

1 Introduction

Individuals facing a choice are often not able to make a full comparison between alternatives. Even if they are able to pin down their preferences for certain characteristics of an object (for instance, a car), they would probably be able to compare only a few of them. In the case of a car, one person would take into account room and safety, while somebody else's order ranking would be based on speed and acceleration. We can interpret this evaluation imagining that our "complete" selves delegate choices to a sort of simplified self. Competition among products will be, in this way, not directed to the "real" population, but to the population of delegates that will choose products on the basis of a small subset of parameters. Car makers advertising speed and acceleration will not be considered by families who prioritize room and safety.
In public choices, political parties present themselves as decision makers committed to following a given preference order when faced with future choices. Parties collect delegations from people having similar preferences: in this way, instead of comparing all possible alternatives of the whole population, the number of alternatives is reduced to the number of parties. Traditionally this was intended in a similar way to the one used in economic location theory (see [5]). Parties have a complete system of preferences and they collect a delegation from the nearest people, i.e. from people having an order of preferences "not far" from the one expressed by the party. In this paper, instead of following this traditional path, we adopt a similar approach to the one presented in "car choice". We describe parties as simplified systems of preferences and the process of delegation as giving the power of choice to parties that correspond to this simplified preference order.
Given that parties compete to attract electors in a simplified preference space, the distribution of preferences will depend on the way preferences are simplified. If, for instance, parties simplify things proposing a couple of items to which they attach more importance, it could be that the items chosen complement themselves well, being able to attract a large share of voters, or alternatively the two items could reciprocally depress their power of attraction. When facing a simplified set of options, the right combination could be of fundamental importance.

In this paper we have a twofold goal. First, we present a general frame to formalize delegation over simplified preference orders. Second, we illustrate a way to detect the "power of mixing" in a delegation procedure; this approach is based on the so–called noncommutative harmonic analysis or generalized spectral analysis (see [2] and [3]). Our work is an application of some recent results on spectral analysis of voting in committees due to Michael E. Orrison and Brian L. Lawson (see [9], [10] and [11]); they used the machinery of spectral analysis to detect influential coalition in the voting procedures of the United State Supreme Court.

2 Individual preferences, parties and public choice

We begin by introducing some notations and terminology: a *party* will be defined as a simplified system of preferences, while the process of delegation to a party will be the power of choice corresponding to a simplified preference order.

Let X be a set of n objects. Let Λ be a set of m individuals. Λ will be called a **society** and the members of Λ **voters**. Suppose that each individual of Λ is asked to **rank objects** of X putting them in a strict order, providing a total order on X.
Let Z be the set of all possible rankings over the elements of X; each $z \in Z$ may be viewed as a permutation of the n elements of X and each individual of Λ is asked to choose an element of Z.

We may define a total order on Z according to the choices of individuals of Λ, by counting the number of individuals that prefer each ranking. Let $z \in Z$, define β_z as the number of individuals of Λ choosing z. If $z_1, z_2 \in Z$, we define "$z_1 \leq_Z z_2$" if $\beta_{z_1} \leq \beta_{z_2}$ (where this last order "$\leq$" is the usual order on the natural numbers).
We call $(Z, \leq_Z)$ a set of **population preferences** over alternative rankings, according to the choices of individuals of Λ.

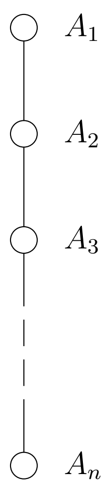
The set $(Z, \leq_Z)$ encapsulates the **individual preferences** arising from the society Λ; the theory of public choice allows us to define public choice functions which lead to a collective choice by starting from a collection of individual preferences. Let $Z^\Lambda = \{f : \Lambda \longrightarrow Z\}$ be the set of functions from Λ to Z. Then a **public choice function** is a function $G : Z^\Lambda \longrightarrow X$. In other words, a public choice function associates a single ranking to each n-tuple of rankings which defines the consent of the population, according to some specified criterion.

In many cases it is difficult to obtain a public choice directly from the set of individual preferences, due to the large variety of possible preference orders. It is worthwhile then to look at "simplified" preference sets. In some sense, people **delegate** (see Vickers [20]) choices to delegates who have "similar" preferences. In political choices this is done by voting for a *party*.

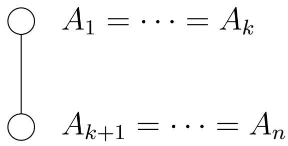
Let $X = \{A_1, \ldots, A_n\}$ be a set of n objects. Suppose that a total order $\leq_X$ is defined on X. We do not require that elements of X are strictly ordered through $\leq_X$, so obviously "=" may hold between some elements (this is different from the initial requests, according to which individuals of Λ are asked to order strictly the elements of X). Trivially $(X, \leq_X)$ is a lattice and may have a representation through lattice diagrams.

A **party** $\mathcal{P}$ may be defined by the preference order $\leq_X$, so $\mathcal{P}$ may be identified with lattice $(X, \leq_X)$.

We define $\mathcal{P}$ a **complete party** if $\mathcal{P}$ is associated to a lattice $(X, \leq_X)$ where $\leq_X$ provides a strict order on the elements of X. An example of a complete party is

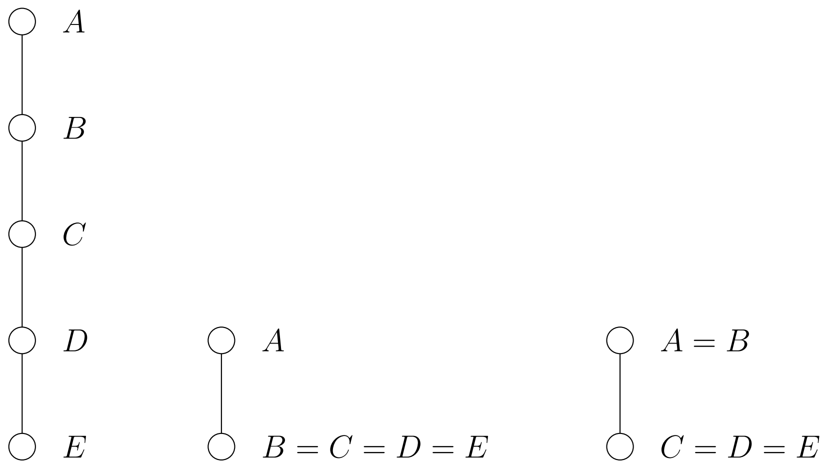


We define $\mathcal{P}$ an **incomplete party** if $\mathcal{P}$ is associated to a lattice $(X, \leq_X)$ where $\leq_X$ provides a not strict order on the elements of X. An example of an incomplete party is



with $k < n$.

For example, some preference orders over five objects A, B, C, D, E are



(a) (b) (c)

Parties like (a) are complete and parties like (b) or (c) are incomplete.

Let P be the set of all parties over X. We define a **party delegating** as a map

$$\xi : Z \longrightarrow P \tag{1}$$

from the set of population preferences Z to the set of parties P.

We are going to investigate a party delegating map ξ empirically, in order to detect particular properties. We will decompose data related to ξ into many components, each of which will have specific meanings, according to the mathematical framework illustrated in section (4.1).

We observe that ξ in general does not satisfy order preservation; we recall that ξ preserves order if

$$\text{for each } z_1, z_2 \in Z \text{ with } z_1 \leq_Z z_2, \text{ then } \xi(z_1) \leq_P \xi(z_2) \tag{2}$$

where the order on P is defined as on Z. In general it is not meaningless to have no order preservation: an individual may delegate a party even if it does not preserve his order of preferences; a distance may be defined between individual rankings arising from the choices of Λ and parties. ξ is then meaningful if it minimizes this distance, even if it does not preserve order.
As mentioned in the introduction, our approach is far from the traditional one used in economic location theory, where parties collect delegations from individuals having *similar* preferences. We describe parties as "simplified" systems of preferences and in the process of delegation an order preservation may be required. For this reason, we assume that ξ satisfies condition (2).

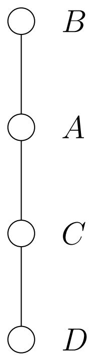
3 Voting for incomplete parties and the power of combination

Let $\mathcal{P}$ be a complete party. An individual of society Λ votes for party $\mathcal{P}$ by the selection in Z of a complete ranking of the n objects of X.

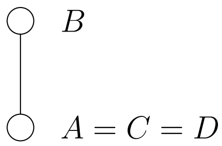
Let $X = \{A_1, \dots, A_n\}$ be a set of objects. Consider an incomplete party $\mathcal{P}_k$ of the form

$$\begin{array}{l} A_1 = \cdots = A_k \\ \\ A_{k+1} = \cdots = A_n \end{array} \tag{3}$$

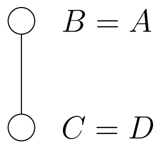
By selecting the incomplete party $\mathcal{P}_k$ the attention is focused on the first k alternatives chosen by an individual of Λ. A voter of Λ does not directly select an incomplete party, but proposes a complete ranking of preferences as described in section (2). If, for example, a person chooses the order $BACD$ over four objects, then this order corresponds directly to the complete party



But we may relate it also to the incomplete parties



or



and so on, according to which simplification we are dealing with; in this sense we talk about "simplification".

The approach through incomplete parties can be very useful if we suppose it is more straightforward and meaningful for a voter to concentrate on k alternatives, instead of n alternatives, with $k < n$.
In other words, by the delegation to an incomplete party as (3) the first k alternatives of a ranking are mixed together and act as a global single first choice, while the last $(n - k)$ items of the same ranking are also mixed and act as a global last choice. Obviously this interpretation of delegation to incomplete parties can be adapted to each type of incomplete party, not only to the example considered in (3).

4 Detecting the power of combination

In section (3) we focused our attention on simplified preference systems and their related incomplete parties. In this way a party earns consent in a reduced preference space and the distribution of preferences will depend on the way they are simplified. For example, suppose that an incomplete party is structured in such a manner that it proposes a pair of items as predominant: it could happen that these alternatives complement themselves strongly or alternatively they could weaken themselves reciprocally.

In the last years, Michael E. Orrison and Brian L. Lawson (see [9], [10] and [11]) made use of the mathematical framework of the so-called spectral analysis to locate influential coalitions in political voting procedures.
We will use a similar machinery for detecting the "power of mixing" in delegation procedures.

4.1 Noncommutative harmonic analysis

Noncommutative harmonic analysis is a generalization of classical spectral analysis. Spectral analysis is also called discrete Fourier analysis and it is basic for time–series analysis (see [1]) and other types of analysis in the computational science, engineering and natural sciences. It is a non–model based approach to data analysis and was formulated in a general group theoretic setting by Diaconis (see [2] and [3]), who extended the classical spectral analysis of time series to a non–time series subject, for the analysis of discrete data which has a noncommutative structure.
The main idea of spectral analysis is that often data has natural symmetries, which are hidden in the existence of a symmetric group (which is obviously non commutative and so the name of noncommutative harmonic analysis) for the domain of the data. The leading principle of spectral analysis is the interpretation of data through its decomposition according to these symmetries.

New efforts have been made in order to apply spectral analysis to a non–time series subject in the political sciences, above all in the analysis of voting.
Recently, Michael E. Orrison and Brian L. Lawson (see [9] and also [10], [11] with David T. Uminsky) introduced a generalization of spectral analysis as a new instrument for political scientist; they used the powerful machinery of spectral analysis to analyze political voting data. In particular, they analyzed votes of the nine judges of the United States Supreme Court (Warren Court 1958– 1962, Burger Court 1967–1981, Renquist Court 1994–1998) and detected influential coalitions.
The idea followed by Orrison and Lawson is to consider political voting data as elements of a mathematical framework; then the features of that framework can be used to work out natural interpretations of the data. The mathematical framework corresponding to voting data has many components, each of which encapsulates information on particular "coalition effects"; the decomposition of data with respect to these components provides the identification of influential coalitions.

In paragraphs (4.1.1) (and related Appendix A) we will follow Orrison and Lawson (see [9]) to explain how their setting works in relation to spectral analysis of political voting data. In section (4.2) and (5) we will apply similar machineries to the analysis of data related to preferences expression and delegation procedures.

4.1.1 Mathematical background

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $f : X \longrightarrow \mathbb{C}$ a complex–valued function on X. Let M be the vector space of all complex–valued functions on X.
Let S_n be the symmetric group of order n, that is the group of permutations of n elements. Any $\pi \in S_n$ acts on the elements of X, but also on the elements of M. Indeed, for any $\pi \in S_n$ and $f \in M$, we define $\pi(f(x)) := f(\pi^{-1}(x))$, for each $x \in X$.
Let $N \subseteq M$ be a subspace of M; N is called **invariant** with respect to S_n if $\pi(f) \in N$, for any $f \in N$ and $\pi \in S_n$.
According to the terminology of group representation theory (refer to Serre [19]), M is in particular a **representation** of S_n or a representation space of S_n. A well–known result of group representation theory (see again Serre [19]) claims that any representation space of a finite group admits a **decomposition** into a direct sum of invariant subspace of the representation space. Precisely, M may always be decomposed into a direct sum

$$M = M_0 \oplus \cdots \oplus M_h \tag{4}$$

for some positive integer h, where each M_i is an invariant subspace of M. In particular, each function $f \in M$ may be written uniquely as a sum

$$f = f_0 + \cdots + f_h \tag{5}$$

with $f_i \in M_i$ and $\pi(f_i) \in M_i$, for all $\pi \in S$. There are many ways to decompose M as the direct sum of invariant subspaces; the idea behind spectral analysis is to choose the decomposition of M that provides invariant subspaces that encapsulate important properties of the data.

Orrison and Lawson set the analysis of political voting data in the previous framework. Suppose that $X = \{X_1, \ldots, X_n\}$ is a set of n voters. Assume we have the results of N non–unanimous votes and that each person casts a ballot on each vote. They define

$$X^{(n-k,k)} = \text{ the set of } k\text{-elements subsets of the voters of } X \tag{6}$$

with $1 \leq k \leq \frac{n}{2}$ and denote with $f^{(n-k,k)}$ a function on $X^{(n-k,k)}$ defined as

$$f^{(n-k,k)}(\omega) = \text{ the number of times that } \omega \text{ is in the minority} \qquad (7)$$

for each $\omega \in X^{(n-k,k)}$. Define also $M^{(n-k,k)}$ as the vector space of all complex–valued functions on $X^{(n-k,k)}$.
We observed that the permutations of S_n act on X, but also on the subsets in $X^{(n-k,k)}$, for each k. Then, as outlined in equation (4), $M^{(n-k,k)}$ may be decomposed as a direct sum

$$M^{(n-k,k)} = M_0 \oplus M_1 \oplus \cdots \oplus M_k \qquad (8)$$

where each M_i is a subspace of $M^{(n-k,k)}$ invariant with respect to the action of S_n. The space M_0 is said to be corresponding to the **mean response**, that is the average number of times an element of $M^{(n-k,k)}$ is in the minority. M_1 corresponds to the so–called **first order effects**, whereas M_i is related to higher order effects, called **coalition effects**.
Spectral analysis focuses on the computation of the decomposition of each function $f \in M^{(n-k,k)}$ onto the components of (8), that is

$$f = f_0 + \cdots + f_k. \qquad (9)$$

In Appendix A we illustrate how this setting works on an example similar to the one suggested by Orrison and Lawson in [9].

4.2 Application to preferences combination

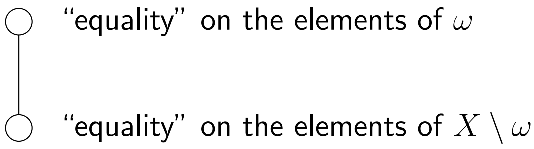
As explained in section (4.1.1) and Appendix A, harmonic analysis applied to an analysis of voting allows us to detect influential coalitions between voters of a committee. The context we are dealing with proposes a set of voters Λ and a finite set $X = \{A_1, \ldots, A_n\}$ of alternatives to rank. In this setting it seems quite meaningless to look for influential coalitions between voters, because the society Λ can be composed by a huge number of members or by a sample of a population. We are interested in seeking a sort of "influential coalition" between preferences, even if this "dual" approach seems meaningless at this point.

Let $X = \{A_1, \ldots, A_n\}$ be a set of n alternatives and suppose people of a society Λ is asked to rank $A_1, \ldots A_n$, as prescribed in section (2). We refer to a notation of paragraph (4.1.1). Define

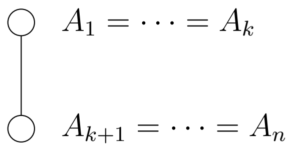
$$X^{(n-k,k)} = \text{ the set of } k\text{-elements subsets of the alternatives of } X$$

with $1 \leq k \leq \frac{n}{2}$.

Let $\omega \in X^{(n-k,k)}$, that is a set of k elements of X. Let $\mathcal{P}_{(k,\omega)}$ be the incomplete party corresponding to the lattice



For example, if $\omega = \{A_1, \ldots, A_k\}$ is the set of the first k elements of X, then $\mathcal{P}_{(k,\omega)}$ has the form



Define

$\gamma_{\mathcal{P}_{(k,\omega)}} :=$ the number of individuals of Λ who choose $A_1, \ldots, A_k$ as their first k alternatives in their rankings (independently of the order) and $A_{k+1}, \ldots, A_n$ as their last $n-k$ alternatives (independently of the order).

In other words, integer $\gamma_{\mathcal{P}_{(k,\omega)}}$ represents the number of individuals of Λ voting the party $\mathcal{P}_{(k,\omega)}$. Define a function $f^{(n-k,k)}$ on $X^{(n-k,k)}$ as

$$f^{(n-k,k)} := \gamma_{\mathcal{P}_{(k,\omega)}} \tag{10}$$

for each $\omega \in X^{(n-k,k)}$. We are interested in the spectral expansion of $f^{(n-k,k)}$, for each $1 \leq k \leq \frac{n}{2}$.

5 An application to a survey

We used the approach explained in section (4) to analyze some results of a survey on the preferences of the Trentino population. On the basis of the survey results, we simulated a delegation to hypothetical incomplete parties as defined in section (2). Moreover, making use of noncommutative harmonic analysis, we decomposed the resulting delegation function. In this way, the meaning of spectral expansion of the function defined in (10) will became clearer.

The *Indagine sulle preferenze della popolazione trentina* (see [16]) is a survey carried out on a sample of about 2000 adults resident in the province of Trento. One of the research's aims was find out about the population preferences relative to some general themes of collective well–being; the knowledge of these preferences can be advantageously used to estimate potential impacts on the population of different types of public policies. One question in particular was useful for finding out about preferences:

Question n. 5 - collective well–being

In your opinion, what is more important between:

1. [A] full employment and [B] environment preservation?
2. [A] full employment and [C] health?
3. [A] full employment and [D] local income increase?
4. [A] full employment and [E] preservation of water and air quality?
5. [B] environment preservation and [C] health?
6. [B] environment preservation and [D] local income increase?
7. [B] environment preservation and [E] preservation of water and air quality?
8. [C] health and [D] local income increase?
9. [C] health and [E] preservation of water and air quality?
10. [D] local income increase and [E] preservation of water and air quality?

Denote with

A	full employment
B	environment preservation
C	health
D	local income increase
E	preservation of water and air quality

Questions on collective well-being are structured as pairs comparisons between alternatives; such an approach may lead to preference systems that do not satisfy transitivity: for example, an interviewee may prefer A to B, B to C, but C to A. For our purpose it is meaningful to concentrate our investigation only on preference systems which satisfy transitivity. For this reason we established a simple way to detect if a preference system does satisfy transitivity or not. This method is illustrated in Appendix B and it is based on simple considerations about matrices associated to preference systems.
We refer to preference systems satisfying transitivity as **consistent preference systems**.

We examined all the preference systems arising from question n. 5 of “Indagine” and established that they was inconsistent for a percentage of 34,2 %. For our analysis we used only data arising from consistent preference systems.

5.1 Spectral analysis

Let $X = \{A, B, C, D, E\}$. According to notation of paragraph (4.1.1) we have

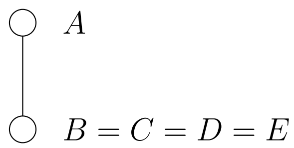
$$X^{(4,1)} = \{A, B, C, D, E\}$$
$$X^{(3,2)} = \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\}$$

where notation AB stands for the subset $\{A, B\}$. Let $f^{(n-k,k)}$ be the function defined in (10). In our context

$$f^{(4,1)}(A) = \gamma_{\mathcal{P}_{(1,A)}}$$

that is

$$f^{(4,1)}(A) = \text{the number of people choosing the incomplete party}$$



We need to count the incomplete parties with 1 alternative in the *first position*, arising from the data of “Indagine”. In table (7) of Appendix B we show how the consistent preference systems from “Indagine” have been chosen by the interviewed people. We rewrite table (7) as

A	33
B	17
C	987
D	6
E	170

Table 1: Parties with 1 predominant preference

where the number of people choosing an order with A *at the first place* or B or C and so on is pointed out . So we have

$$f^{(4,1)} = \begin{array}{c} \begin{pmatrix} 33 \\ 17 \\ 987 \\ 6 \\ 170 \end{pmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

In the same way

$$f^{(3,2)}(AB) = \gamma_{\mathcal{P}_{(2,AB)}}$$

that is

$$f^{(3,2)}(AB) = \text{the number of individuals choosing the incomplete party}$$

$$A = B$$

$$C = D = E$$

For this data, table (7) becomes

AB	1	BD	0
AC	276	BE	28
AD	3	CD	45
AE	6	CE	728
BC	117	DE	9

Table 2: Parties with 2 predominant preferences

where the number of people choosing an order with A and B *at the first two positions* **independently of the order**, or choosing A and C and so on, is pointed out. We have

$$f^{(3,2)} = \begin{pmatrix} 1 \\ 276 \\ 3 \\ 6 \\ 117 \\ 0 \\ 28 \\ 45 \\ 728 \\ 9 \end{pmatrix} \begin{array}{l} AB \\ AC \\ AD \\ AE \\ BC \\ BD \\ BE \\ CD \\ CE \\ DE \end{array}$$

5.1.1 First order effects

As outlined in the example of Appendix A, we may project the function $f^{(4,1)}$ onto the invariant subspaces of the decomposition $M^{(4,1)} = M_0^{(4,1)} \oplus M_1^{(4,1)}$ and get the Fourier expansion of $f^{(4,1)}$:

$$f^{(4,1)} = \begin{pmatrix} 33 \\ 17 \\ 987 \\ 6 \\ 170 \end{pmatrix} = \underset{f_0^{(4,1)}}{\begin{pmatrix} 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \end{pmatrix}} + \underset{f_1^{(4,1)}}{\begin{pmatrix} -88.30 \\ -104.30 \\ 865.70 \\ -115.30 \\ 48.70 \end{pmatrix}} \begin{array}{l} A \\ B \\ C \\ D \\ E \end{array}$$

The function $f_1^{(4,1)}$ shows the **first order effect**, that is the amount which each party of the form $\mathcal{P}_{(1,A)}$ differs from the mean. In this case the interpretation of the first order effects does not yield new information in relation to the initial data $f^{(4,1)}$.

Parties proposing 1 alternatives	first order effects
C = party: health	865.70
E = party: preservation of water and air quality	48.70
A = party: full employment	-88.30
B = party: environment preservation	-104.30
D = party: local income increase	-115.30

Table 3: First order effects of parties with 1 alternatives

5.1.2 Second order effects

We may project the function $f^{(3,2)}$ onto the invariant subspaces of the decomposition $M^{(3,2)} = M_0^{(3,2)} \oplus M_1^{(3,2)} \oplus M_2^{(3,2)}$ and get

$$f^{(3,2)} = \begin{pmatrix} 1 \\ 276 \\ 3 \\ 6 \\ 117 \\ 0 \\ 28 \\ 45 \\ 728 \\ 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \\ 121.30 \end{pmatrix}}_{f_0^{(3,2)}} + \underbrace{\begin{pmatrix} -179.47 \\ 160.53 \\ -209.13 \\ 28.87 \\ 113.87 \\ -255.80 \\ -17.80 \\ 84.20 \\ 322.20 \\ -47.47 \end{pmatrix}}_{f_1^{(3,2)}} + \underbrace{\begin{pmatrix} 59.17 \\ -5.83 \\ 90.83 \\ -144.17 \\ -118.17 \\ 134.50 \\ -75.50 \\ -160.50 \\ 284.50 \\ -64.83 \end{pmatrix}}_{f_2^{(3,2)}} \begin{array}{c} AB \\ AC \\ AD \\ AE \\ BC \\ BD \\ BE \\ CD \\ CE \\ DE \end{array}$$

The function $f_2^{(3,2)}$ represents the **second order effects**, that is the weight of a party which proposes two alternatives, after removing the mean effects and the first order effects.

Parties proposing 2 alternatives			second order effects
CE = party: health	+	water-air	284.50
BD = party: environment	+	income	134.50
AD = party: employment	+	income	90.83
AB = party: employment	+	environment	59.17
AC = party: employment	+	health	-5.83
DE = party: income	+	water-air	-64.83
BE = party: environment	+	water-air	-75.50
BC = party: environment	+	health	-118.17
AE = party: employment	+	water-air	-114.17
CD = party: health	+	income	-160.50

Table 4: Second order effects of parties with 2 alternatives

5.2 Interpretation

5.2.1 Mallow's method

We use Mallow's method (see Appendix A) to interpret the first and second order effects $f_1^{(3,2)}$ and $f_2^{(3,2)}$.

To interpret the first order effects we compute the inner product between $f_1^{(3,2)}$ and the "naturally interpretable" functions defined in section Appendix A. We get

	f_A	f_B	f_C	f_D	f_E
$f_1^{(3,2)}$	-199.20	-399.20	680.80	-428.20	285.80

Table 5: First order effects

Table (5) shows that the first order effect *lies mostly in the direction* of C and least in the direction of D, which confirms the results for parties proposing one alternatives of table (3).

To interpret the second order effects we compute the inner product between $f_2^{(3,2)}$ and the related "naturally interpretable" functions and get

	f_{AB}	f_{AC}	f_{AD}	f_{AE}	f_{BC}
$f_2^{(3,2)}$	59.17	-5.83	90.83	-144.17	-118.17

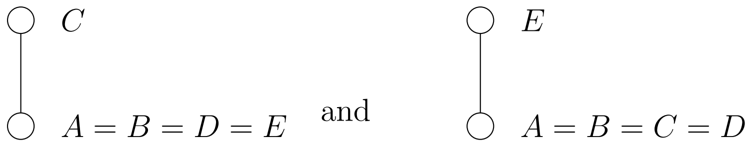
	f_{BD}	f_{BE}	f_{CD}	f_{CE}	f_{DE}
	134.50	-75.50	-160.50	284.50	-64.83

Table 6: Second order effects

which is exactly the vector $f_2^{(3,2)}$. We observe that the second order effect *lies in the direction* of pairs

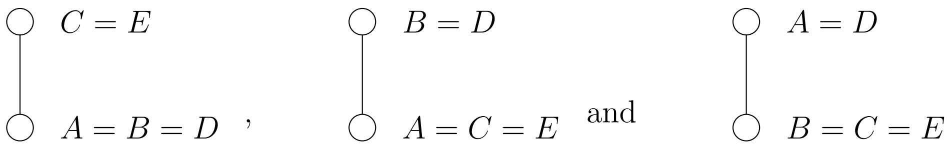
$CE =$ party: health + water/air preservation
$BD =$ party: environment + income
$AD =$ party: employment + income

This means that there is an *intrinsic affinity* between pairs for which the second order effect is high. We try to explain this concept. According to the first order effects of $f^{(4,1)}$ of table (3), two items have a particular value, C and E respectively, which correspond to the incomplete parties



(a_1) (a_2)

According to the second order effects of $f^{(3,2)}$ of table (6), the powerful pairs are CE, BD and AD respectively, that is the incomplete parties



(a_3) (a_4) (a_5)

are winning. This means that C and E are powerful items either alone (that is when a party proposes one predominant alternative) or together (that is when a party proposes two predominant alternatives). This is not the case of parties (a_4) and (a_5); for example, B and D are weak alone, but they get stronger in a pair. The same for A and D.
The analysis of second order effects allows to understand if there are "intrinsic affinities" between items and if coupling items contributes to weaken or strengthen them.

Orrison and Lawson, in their spectral analysis of voting data of the United State Supreme Court (see [10]), suggest to display the results of the second order effects analysis in a graphical way which helps to single out particular "coalition effects".
This approach can be applied in our context. Figure (1) displays information of table (6) in a way that makes it easy to identify the "coalition effects" arising from the second order effects analysis.

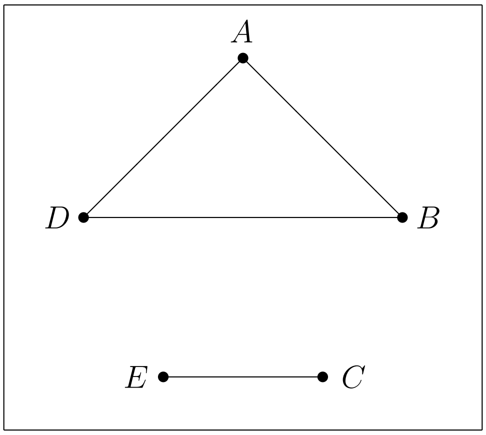


Figure 1: Second order effects

Two items are joined by a line when a positive second order effect exists between them; the absence of a line means that the second order effect of the pairs is negative.

5.2.2 Significance

One of the main idea of harmonic analysis (both commutative, that is the classical spectral analysis, or noncommutative) is to find subspaces into which data can be decomposed, while preserving the most important structure of the data, as explained in section (4.1.1), but also to see *which subspaces contain the largest amount of the data*. This is done by considering the "length" of the vectors arising from the decomposition, in order to determine which vector is significant.
The traditional method for determining **significance** is to compare the norm squared of the f_i vectors divided by the dimension of the subspaces M_i. As suggested by Diaconis (see Diaconis [3] pag. 954 and also Orrison and Lawson [9]), for the type of data we are examining this may be misleading; it is better to consider the norm squared of the vectors. In our situation we have

$$||f_1^{(3,2)}||^2 = 294420.94$$

$$||f_2^{(3,2)}||^2 = 173074.77$$

Comparing these two values suggests that **the first order effects are more significant than the second order effects**.

6 Conclusions

In this paper we have interpreted the way people vote for parties as a process of delegation to decision makers using a simplified system of preferences. Moreover, on the basis of a survey on preferences of the population, we have simulated a delegation procedure to parties. Finally, making use of noncommutative harmonic analysis, we decomposed the delegation function, and isolated the effect of affinity, or mixing, between the pair of items that characterize a party.

This approach appears to be promising both to understand how people, with limited rationality, act given a simplified set of options, and to empirically study the best way to simplify a preference set, in order to gain from complementarity among objects over which people express an order of preference.

Further studies should be devoted to enlarge the model. A first approach could be an extension of the voting model to a general choice among objects, in the line of the introductory example. As it was stated by [18], "if one replace the term 'individual i' with 'property i', social choice theory is transformed from a theory of social decision into a theory of formation of individualistic preferences" (p. 58). In this direction, we argue that our frame could be used to model a process of choice under limited rationality assumptions, where agents are unable to evaluate all the characteristics of goods, defined as in [8], and to compare them with their complete preferences.

A second approach could be directed to refine the supply side of the model. In our model we assumed parties as given; but in fact there could be a competitive formation of parties. They could in fact choose to be more or less specialized, proposing a shorter or longer list of characterizing items. An empirical analysis of the kind we described in this paper could help to ex ante define the best positioning of parties.

Appendix A - Example of spectral analysis of voting data

In [9] Orrison and Lawson suggest an example to show how generalized spectral analysis can be applied to analysis of voting data. The following is similar. Let $X = \{A, B, C, D, E\}$ be a committee of five people and suppose we have the results of 128 non–unanimous votes. Data is viewed as a function

f defined on the subsets of X; in particular we have

$$f^{(4,1)} = \text{the number of times one person of } X \text{ is in the minority.}$$
$$f^{(3,2)} = \text{the number of times two people of } X \text{ are in the minority.}$$

Suppose

$$f^{(4,1)} = \begin{array}{c c} \begin{pmatrix} 10 \\ 9 \\ 3 \\ 2 \\ 1 \end{pmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \end{array} \qquad \text{and} \qquad f^{(3,2)} = \begin{array}{c c} \begin{pmatrix} 22 \\ 21 \\ 24 \\ 11 \\ 5 \\ 2 \\ 10 \\ 2 \\ 1 \\ 5 \end{pmatrix} & \begin{array}{c} AB \\ AC \\ AD \\ AE \\ BC \\ BD \\ BE \\ CD \\ CE \\ DE \end{array} \end{array}$$

This means that, in this example, A is in the minority against the other four people for 10 times, whereas AB are in the minority against the other three for 22 times, and so on.

Let M be the vector space of the complex–valued functions on $X^{(4,1)}$ and $X^{(3,2)}$; M may be naturally decomposed as $M = M^{(4,1)} \oplus M^{(3,2)}$, where $M^{(4,1)}$ is the subspace of the functions on $X^{(4,1)}$ and $M^{(3,2)}$ on $X^{(3,2)}$. These two subspaces may be again decomposed into invariant subspaces

$$M^{(4,1)} = M_0^{(4,1)} \oplus M_1^{(4,1)} \tag{11}$$
$$M^{(3,2)} = M_0^{(3,2)} \oplus M_1^{(3,2)} \oplus M_2^{(3,2)}. \tag{12}$$

We may project the functions $f^{(4,1)}$ and $f^{(3,2)}$ onto these invariant subspaces and obtain

$$f^{(4,1)} = f_0^{(4,1)} + f_1^{(4,1)} \tag{13}$$
$$f^{(3,2)} = f_0^{(3,2)} + f_1^{(3,2)} + f_2^{(3,2)}. \tag{14}$$

One person in the minority.

Going back to the example, according to decomposition (13), we get

$$f^{(4,1)} = \begin{pmatrix} 10 \\ 9 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}}_{f_0^{(4,1)}} + \underbrace{\begin{pmatrix} 5 \\ 4 \\ -2 \\ -3 \\ -4 \end{pmatrix}}_{f_1^{(4,1)}} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix}$$

The number of votes in which one person is in the minority is 25, so the average of the individual minority is $5 = 25/5$; then $f_0^{(4,1)}$ is the mean response function. The function $f_1^{(4,1)}$ shows the **first order effect**, which counts the number of votes in which each person differs from the mean. In this case the interpretation of the first order effects doesn't yield new information in relation to the initial data; the largest value is for A, that is most often in the minority, and the smallest value is for E, that is less often in the minority.

Two people in the minority.

We can appreciate the power of spectral analysis in the analysis of higher order effects. According to decomposition (14), we obtain

$$f^{(3,2)} = \begin{pmatrix} 22 \\ 21 \\ 24 \\ 11 \\ 5 \\ 2 \\ 10 \\ 2 \\ 1 \\ 5 \end{pmatrix} = \underbrace{\begin{pmatrix} 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \\ 10.3 \end{pmatrix}}_{f_0^{(3,2)}} + \underbrace{\begin{pmatrix} 11.53 \\ 8.20 \\ 9.53 \\ 7.53 \\ -4.80 \\ -3.47 \\ -5.47 \\ -6.80 \\ -8.80 \\ -7.47 \end{pmatrix}}_{f_1^{(3,2)}} + \underbrace{\begin{pmatrix} 0.17 \\ 2.50 \\ 4.17 \\ -6.83 \\ -0.50 \\ -4.83 \\ 5.17 \\ -1.50 \\ -0.50 \\ 2.17 \end{pmatrix}}_{f_2^{(3,2)}} \begin{matrix} AB \\ AC \\ AD \\ AE \\ BC \\ BD \\ BE \\ CD \\ CE \\ DE \end{matrix}$$

The function $f_0^{(3,2)}$ is the mean response function; the number of votes in which two people are in the minority is 103, then the average of the minority of pairs is $10.3 = 103/10$. The functions $f_1^{(3,2)}$ and $f_2^{(3,2)}$ capture the **first order and second order effects**. In order to interpret these effects, Orrison and Lawson [9] suggest to use *Mallow's method* (see [12]).

To interpret the first order effects, for each subset of voters H, define a function $f_H \in M^{(3,2)}$ which identifies the elements of $f^{(3,2)}$ "containing" H with 1 and those "not containing" H with 0. In particular,

$$\begin{aligned}
f_A &= (1,1,1,1,0,0,0,0,0,0)\\
f_B &= (1,0,0,0,1,1,1,0,0,0)\\
f_C &= (0,1,0,0,1,0,0,1,1,0)\\
f_D &= (0,0,1,0,1,0,0,1,0,1)\\
f_E &= (0,0,0,1,0,0,1,0,1,1)
\end{aligned}$$

The inner product between $f_1^{(3,2)}$ and f_H describes how much $f_1^{(3,2)}$ *lies in the direction* of H. Computing the inner products we get

	f_A	f_B	f_C	f_D	f_E
$f_1^{(3,2)}$	36.79	-2.21	-12.20	-8.21	-14.21

We observe that the first order effect lies most in the direction of A, being often in the minority with other voters, but lies least in the direction of E, being only occasionally in the minority of the pairs.

To interpret the second order effects, for each pair HK of X, define functions according to the criterion already explained, that is $f_{HK} \in M^{(3,2)}$ identifies the elements of $f^{(3,2)}$ which "contain" HK with 1 and the others with 0. So

$$\begin{aligned}
f_{AB} &= (1,0,0,0,0,0,0,0,0,0)\\
f_{AC} &= (0,1,0,0,0,0,0,0,0,0)\\
f_{AD} &= (0,0,1,0,0,0,0,0,0,0) \qquad \text{etc.}\\
f_{AE} &= (0,0,0,1,0,0,0,0,0,0)\\
f_{BC} &= (0,0,0,0,1,0,0,0,0,0)
\end{aligned}$$

Computing the inner products between $f_2^{(3,2)}$ and f_{HK} we get the exact data vector $f_2^{(3,2)}$

	f_{AB}	f_{AC}	f_{AD}	f_{AE}	f_{BC}	f_{BD}	f_{BE}	f_{CD}	f_{CE}	f_{DE}
$f_2^{(3,2)}$	0.17	2.50	4.17	-6.83	-0.50	-4.83	5.17	-1.50	-0.50	2.17

These results represents the **pure second order effects**, namely the pair's weight in the minority, after removing the mean effects and the effects of the individual. The values of

$$
f_2^{(3,2)} = \begin{array}{c} \left(\begin{array}{c} 0.17 \\ 2.50 \\ 4.17 \\ -6.83 \\ -0.50 \\ -4.83 \\ 5.17 \\ -1.50 \\ -0.50 \\ 2.17 \end{array} \right) \end{array} \begin{array}{c} AB \\ AC \\ AD \\ AE \\ BC \\ BD \\ BE \\ CD \\ CE \\ DE \end{array}
$$

represent the pair's weight in the voting process. We observe that the second order effect lies most in the direction of BE and least in the direction of AE. Through this analysis we may point out *particular coalition effects* that do not arise from a direct analysis of data; for example, the pair DE has a quite high second order effect, whereas D and E have low values in the first order effects of the minority related to the individual ($f_1^{(4,1)}$). **This means that D and E are seldom in the minority alone, while they are often in the minority of the pairs.**

Appendix B - Transitivity of preferences

Let $X = \{A, B, C, D, E\}$ be the set of five alternatives investigated in question n. 5 of "Indagine". Each interviewee's answer can be realized as a table of the following type

Questions	1	2	3	4	5	6	7	8	9	10
Answers	B	C	A	A	C	D	B	C	C	D

where answer n. 1 stands for the choice between A and B, answer n. 2 for the choice between A and C and so on.

In general, the pairwise comparison adopted by the investigation of question n. 5 does not lead to a **consistent ordering** of all feasible alternatives. To make choices one needs only a choice function that allows one to select a best alternative from a set of possible alternatives. For example, an answer of type

Questions	1	2	3	4	5	6	7	8	9	10
Answers	B	C	A	A	C	D	B	C	C	D

does not lead to a total order of preferences, because a cycle between A, B and D exists, indeed B is preferred to A, A is preferred to D, but D is preferred to B. Conversely, an answer of type

Questions	1	2	3	4	5	6	7	8	9	10
Answers	A	A	A	A	C	D	E	C	D	D

satisfies transitivity and leads to the total preference order $CDEAB$.

6.1 Total orders

Let us recall some notations and terminology. Let $X = \{A_1, \ldots, A_n\}$ be a set of n elements. Define on X a relation "$\leq_X$" (if there is no misunderstanding, we will use notation $\leq$) satisfying:

(A) reflexivity: $A_i \leq A_i$, $\forall i = 1, \ldots, n$

(B) antisymmetry: if $A_i \leq A_j$ and $A_j \leq A_i$, then $A_i = A_j$, $\forall i, j = 1, \ldots, n$

(C) comparability: for any $A_i, A_j \in \Omega$, either $A_i \leq A_j$ or $A_j \leq A_i$.

If "$\leq$" satisfies also

(D) transitivity: if $A_i \leq A_j$ and $A_j \leq A_k$, then $A_i \leq A_k$ $\forall i, j, k = 1, \ldots, n$

"$\leq$" is called a **total order** on X.

We associate to $(X, \leq)$ a matrix which encapsulates the relation on X. Define $M_{(X,\leq)} := (m_{ij})$ where

$$m_{ij} = \begin{cases} 1 \text{ if } A_i \leq A_j \\ 0 \text{ otherwise} \end{cases} \qquad i, j = 1, \ldots, n \tag{15}$$

We observe that $M_{(X,\leq)}$ satisfies the properties

$$i) \sum_{i,j=1}^{n} m_{ij} = \frac{n(n+1)}{2}$$

$$ii) \; m_{ij} = \begin{cases} 1 \text{ if } m_{ji} = 0 \\ 0 \text{ if } m_{ji} = 1 \end{cases} \qquad \forall i \neq j, \quad i, j = 1, \ldots, n$$

Property *ii)* is an obvious consequence of the definition of $M_{(X,\leq)}$. Property *i)* is a consequence of counting the number of 1 in the diagonal of $M_{(X,\leq)}$ plus the number of 1 appearing in the rest of the matrix, which is the number of possible unordered pairs of n elements.

PROPOSITION 6.1 *Let $X = \{A_1, \ldots, A_n\}$ be a set of n elements with a relation "$\leq$" satisfiyng (A), (B) and (C). Let $M_{(X,\leq)}$ be the matrix defined in (15). Then "$\leq$" satisfies transitivity (and in particular is a total order) if and only if, up to re-ordering the indexes of the elements of X, the matrix $M_{(X,\leq)}$ satisfies*

$$m_{ij} = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

for all $i, j = 1, \ldots, n$, that is $M_{(X,\leq)}$ is strictly lower triangular of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}.$$

PROOF. Suppose that "$\leq$" satisfies transitivity; in particular, it is a total order on X, so we may re-order the indexes of the elements of X so that $A_n \leq \cdots \leq A_2 \leq A_1$. Consequently:

$$\begin{aligned} 1 &= m_{11} = m_{21} = m_{31} = \cdots\cdots = m_{n1} \\ 1 &= m_{22} = m_{32} = m_{42} = \cdots = m_{n2} \\ 1 &= m_{nn} \end{aligned}$$

and so $M_{(X,\leq)}$ has the desired form.
Conversely, suppose that $M_{(X,\leq)}$ satisfies (16). We want to prove that $A_n \leq \cdots \leq A_1$, so "$\leq$" is a total order on X and in particular satisfies transitivity. We proceed by induction on n. For $n = 1$ there is nothing to prove. Suppose that $A_{j-1} \leq \cdots \leq A_1$. By hypothesis $m_{ij} = 0$ for each $i < j$. Then $A_j \leq A_1$, $A_j \leq A_2$, $\ldots$, $A_j \leq A_{j-1}$, for each $j = 1, \ldots, n$; by inductive hypothesis $A_{j-1} \leq \cdots \leq A_1$, then also $A_j \leq A_{j-1} \leq \cdots \leq A_1$. By induction we get $A_n \leq \cdots \leq A_1$. $\square$

Proposition (6.1) does not provide a direct method to check if $M_{(X,\leq)}$ corresponds to a transitive relation. Nevertheless, an operative procedure can be easily found. Let $M_{(X,\leq)} = (m_{ij})$ be the matrix associated to $(X, \leq)$. Let

$M^1, \ldots, , M^n$ be the column–vectors of $M_{(X,\leq)}$; obviously $M^i \in \mathcal{M}(n \times 1, \mathbb{R})$. Define

$$\alpha_j := \sum_{i=1}^{n} m_{ij} \qquad \forall j = 1, \ldots, n, \tag{17}$$

in other words α_j is the sum of the elements of column M^j in $M_{(X,\leq)}$. Observe that $\alpha_j \in \mathbb{N}$ and $0 < \alpha_j \leq n$, for each $j = 1, \ldots, n$. The following corollary is a different interpretation of proposition (6.1).

COROLLARY 6.1 *Let $X = \{A_1, \ldots, A_n\}$ be a set of n elements with a relation "$\leq$" satisfiyng (A), (B) and (C). Let $M_{(X,\leq)}$ be the matrix associated to $(X, \leq)$. Then "$\leq$" satisfies transitivity if and only if $\alpha_1, \ldots, \alpha_n$ can be "strictly" ordered.*

PROOF. Suppose that "$\leq$" satisfies transitivity. Then according to proposition (6.1)

$$M_{(X,\leq)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix},$$

so $\alpha_1 = n, \alpha_2 = n - 1, \ldots, \alpha_n = 1$ and $\alpha_1 > \cdots > \alpha_n$.
Conversely, suppose that $\alpha_1, \ldots, \alpha_n$ can be strictly ordered. Suppose that, after re-ordering the indexes, $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Necessarily $\alpha_n = 1, \alpha_{n-1} = 2, \ldots, \alpha_2 = n - 1, \alpha_1 = n$, so

$$M_{(X,\leq)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

and by proposition (6.1) "$\leq$" is transitive. $\square$

EXAMPLE 6.1 *Let $X = \{A_1, A_2, A_3, A_4\}$ and*

$$M_{(X,\leq)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can see that $\alpha_1 = 3, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 4$. We may associate to the α_js a vector $\alpha_{(X,\leq)} := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 1, 2, 4)$ whose entries can be strictly ordered, so $\leq$ is a total order on Ω. In this case vector $\alpha_{(X,\leq)}$ provides also the order which is $A_2 \leq A_3 \leq A_1 \leq A_4$.

EXAMPLE 6.2 *Let* $X = \{A_1, A_2, A_3, A_4\}$ *and*

$$M_{(X,\leq)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

We see that $\alpha_1 = 3, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 2$. The entries of $\alpha_{(X,\leq)} = (3, 2, 3, 2)$ cannot be strictly ordered, because some values are equal. According to corollary (6.1), $\leq$ does not satisfy transitivity. Actually, there is at least the cycle $A_1 \leq A_2, A_2 \leq A_4, A_4 \leq A_1$.

6.1.1 Consistent preference systems

The preferences expressed by the answers to question n. 5 define relations "$\leq$" on $X = \{A, B, C, D, E\}$ which satisfies reflexivity, antisymmetry, comparability, but not necessarily transitivity. Let $M_{(X,\leq_i)}$ be the 5×5 matrix associated to X and relation $\leq_i$ arising from the i-interviewee's answer. $M_{(X,\leq_i)}$ corresponds to a total order of preferences if and only if it satisfies the conditions of proposition (6.1) or equivalently of corollary (6.1).

Regarding question no. 5, the "Indagine" provided answers from the 1.898 people interviewed, but 54 of them gave incomplete answers; we are omitting such set of answers considering only **1.844** interviews. We associated a matrix $M_{(X,\leq_i)}$ to each i-interviewee's answer and analyzed the results of "Indagine" using techniques of corollary (6.1), seeking **consistent preference systems**, that is, preference systems which satisfy transitivity. We found out that **1.213** preference systems are **consistent**, against **631** which are not, for a percentage of **34,2 %** of inconsistent systems. Among the consistent systems, we found the distribution of orders illustrated in table (7), in which the number of people who choose some order is associated to each chosen order.

ABEDC	1	CADBE	18	CEABD	189	EADCB	1
ACBDE	1	CADEB	35	CEADB	90	EBACD	1
ACBED	5	CAEBD	97	CEBAD	210	EBCAD	14
ACDBE	1	CAEDB	48	CEBDA	49	EBCDA	4
ACDEB	9	CBADE	4	CEDAB	28	ECABD	43
ACEBD	7	CBAED	20	CEDBA	24	ECADB	19
ACEDB	7	CBDEA	4	DABEC	1	ECBAD	49
ADCBE	1	CBEAD	64	DCABE	1	ECBDA	15
ADCEB	1	CBEDA	17	DCAEB	1	ECDAB	6
BCAED	1	CDABE	8	DCEAB	1	ECDBA	6
BCDAE	1	CDAEB	17	DEACB	1	EDABC	1
BCEAD	6	CDBAE	2	DECBA	1	EDBAC	1
BECAD	7	CDBEA	2	EABCD	2	EDBCA	1
BECDA	2	CDEAB	7	EACBD	2	EDCAB	1
CABDE	18	CDEBA	6	EACDB	1	EDCBA	3
CABED	30						

Table 7: Consistent preferences systems

We observe that not all the possible orders on the five elements A, B, C, D, E have been chosen; the possible consistent preferences systems on 5 alternatives are $120 = 5!$ whereas only 61 have been selected.

References

[1] C. Chatfield, *The analysis of time series. An introduction*, Chapman and Hall, London, 1975.

[2] P. Diaconis, *Group representations in probability and statistics*, IMS, Hayward, CA, 1988.

[3] P. Diaconis, *A generalization of spectral analysis with application to ranked data*, The 1987 Wald Memorial Lectures, The Annals of Statistics, **17** no. 3, (1989), 949–979.

[4] P. Diaconis and D. N. Rockmore, *Efficient computation of the Fourier Transform on finite groups*, J. Amer. Math. Soc., **3** no. 2, (1990), 297–332.

[5] J. J. Gabszewicz, J. F. Thisse, M. Fujita and U. Schweizer, *Location theory*, Harwood Academic Publishers, Chur, 1986.

[6] G. D. James, *The representation theory of the symmetric groups*, Springer–Verlag, Berlin, 1978.

[7] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Addison–Wesley, Reading, Massachusetts, 1981.

[8] K. J. Lancaster, *A New Approach to Consumer Theory*, Journal of Political Economy, **74** no. 2, (1966), 132–157.

[9] B. L. Lawson and M. E. Orrison, *Analyzing voting from a new perspective: Applying spectral analysis to the U.S. Supreme Court*, Presented at the annual meeting of the American Political Science Association, Boston, August 29, 2002.

[10] B. L. Lawson, M. E. Orrison and D. T. Uminsky, *Discrete Analysis of Voting in Committees*, Presented at the annual meeting of the Midwest Political Science Association, Chicago, April 3–6, 2003.

[11] B. L. Lawson, M. E. Orrison and D. T. Uminsky, *Noncommutative Harmonic Analysis of Voting in Small Committees*, July, 2003.

[12] C. Mallows, *Non–null rankings models I*, Biometrika, **44**, (1957), 114–130.

[13] D. K. Maslen, M. E. Orrison and D. N. Rockmore, *Computing the isotypic projections with the Lanczos iteration*, SIAM J. Matrix Analysis and Application, **25** no. 3, (2004), 784–803.

[14] D. C. Mueller, *Public choice II.* A revised edition of *Public choice.* Cambridge University Press, Cambridge, 1989.

[15] M. E. Orrison, *An eigenspace approach to decomposition representations of finite group*, Ph.D. Thesis, Dartmouth College, 2001.

[16] M. Pisati, *Benessere collettivo e preferenze della popolazione: un'analisi empirica*, Quaderni della programmazione PAT, *Metodi e applicazioni di ricerca valutativa per la pubblica amministrazione*, Trento, 2004.

[17] D. N. Rockmore, *Some applications of generalized FFTs*, in *Groups and computation II*, eds. L. Finkelstein and W. M. Kantor, DIMACS Series in Discrete Math. and Theoret. Comput. Sci., AMS, Providence, RI, **28**, (1997), 329–369.

[18] A. Rubinstein, *Economics and Language. Five essays*, Cambridge University Press, Cambridge, 2000.

[19] J. P. Serre, *Linear representations of finite groups*, Springer–Verlag, New York, 1977.

[20] J. Vickers, *Delegation and the theory of the firm*, Economic Journal, **95**, (1985), 138–147.