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26. June 2009

Natural Rates of Profit, Natural Prices, and the Actual Economic Systems — A theoretical framework

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June 26, 2009

Abstract The aim of the present paper is that of exploiting the notions of *vertically (hyper-)integrated sectors*, as discussed in Pasinetti (1973) and Pasinetti (1988), and of *natural system*, as defined in Pasinetti (1981), in order to build up an analytical framework in which data from national accounts can be consistently inserted and hence used to analyse actual economic systems.

Pasinetti (1981) built up an analytical framework in which the economy is divided into as many vertically (hyper-)integrated sectors as there are consumption goods. Each vertically integrated sector is summarised, as explained in Pasinetti (1973), by a composite commodity — a unit of *vertically integrated productive capacity* — made up by all the goods that enter, both directly and indirectly, the production of the corresponding consumption good, and a unit of *vertically integrated labour*, i.e. the quantity of labour that directly and indirectly enters the production of the consumption goods.

Being the model developed in Pasinetti (1981) a vertical integrated model, there is a link between it and the analytical formulations provided by Pasinetti (1973) and Pasinetti (1988). However, such an analytic link has not been singled out explicitly, thus making it difficult to use the model itself for empirical purposes.

The aim of the present paper is therefore precisely that of making this further step, in order to set up a theoretical framework which can be fruitfully used for empirical analyses of actual economic systems.

Keywords Natural system, vertical integrated sectors, functional income distribution, natural rates of profit, natural prices.

JEL classification B51, C67, D46, O33, O41.

1 Introduction

Some attempts have been made, both in the '80s[1] and more recently, [2] to apply the concept of vertical integration, developed by Pasinetti,[3] for empirical purposes. All these works are based on the original formulation of Siniscalco (1982), who computed a linear operator that, whenever applied to a vector of magnitudes expressed in terms of industries, transforms it in a vector expressed in terms of vertically integrated sectors. This approach has proven to be very useful in providing a tool for the analysis of the structural transformations of actual economic systems.

However, it does not provide a way to use the notion of *natural system*.

Such a notion has been first developed by Pasinetti in the early '60s[4] and then further characterised in the following years, especially in Pasinetti (1981). It does not simply study the structural dynamics of an economic system, but it also singles out those fundamental — *natural* — relations[5] that *must* be satisfied for the system to reproduce itself and grow at a pace consistent with the rates of change of demand in the various vertically integrated sectors.

Therefore, we can say that the *natural system* is an *ideal* configuration, and hence provides a *normative* criterion to evaluate and orientate the performances of *actual* economic systems.

There is of course a link between the notion of vertically integrated sector and that of *natural system*. Such a link consists in the fact that the model developed in Pasinetti (1981) actually is a vertical integrated model. However, the analytic link has never been singled out explicitly, thus making it difficult to use the model itself for empirical purposes.

The aim of the present paper is therefore that of making this further step.

Section 2 provides a succint presentation of the analytical framework as presented in Pasinetti (1973) and Pasinetti (1988). Then, section 3 deals with the simplest case, i.e. that of a stationary system in which all produced

[1]See for instance Siniscalco (1982) and Siniscalco and Momigliano (1986).

[2]See for instance Montresor and Vittucci Marzetti (2007a) and Montresor and Vittucci Marzetti (2008).

[3]See Pasinetti (1973) and Pasinetti (1988).

[4]See Pasinetti (1962).

[5]We are talking about *physical* requirements: were they not satisfied, growth could not take place consistently with demand and therefore with the growth potentialities of the economic system.

goods are consumed as well. Section 4 extends this last case to a situation in which there is population and demand growth. Section 5 introduces natural rates of profit and prices, i.e. the *natural system*. Finally, section 6 generalises the whole framework to the more complicated case in which not all produced goods are consumed, and therefore industries and vertically hyper-integrated sectors are not in the same number.

2 Analytical framework

The economic system can be described by:

$\mathbf{q}$	$=$	$[q_i]$:	(column) vector of total quantities;
$\mathbf{x}$	$=$	$[x_i]$:	(column) vector of final demand;
$\mathbf{A}$	$=$	$[a_{ij}]$:	(square) matrix of direct (circulating) capital requirements;
$\mathbf{a}^{\mathbf{T}}$	$=$	$[a_{Ni}]$:	(row) vector of direct labour requirements;
$\mathbf{s}$	$=$	$[s_i]$:	(column) vector of the stock of capital goods necessary for the production of quantities q_i;
$\mathbf{p}^{\mathbf{T}}$	$=$	$[p_i]$:	(row) vector of commodity prices;
		x_N:	(scalar) quantity of total labour.
			$(i = 1, \dots, m)$

We are here considering a context in which there is no growth, i.e. in which the rate of growth of population (g) and the rates of growth of sectoral demands (r_i) are all equal to zero. Hence, the physical quantity system can be written as:

$$\mathbf{q} = \mathbf{A}\mathbf{q} + \mathbf{x} \tag{2.1}$$

and therefore:

$$\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} \tag{2.2}$$

The physical quantities to be produced in the economic system as a whole are given by the direct and indirect physical requirements for the production of the goods composing the vector of final demand $\mathbf{x}$.

The stock of capital goods necessary for the production process to take place are given by:

$$\mathbf{s} = \mathbf{A}\mathbf{q} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \mathbf{H}\mathbf{x} \tag{2.3}$$

$\mathbf{H}$ is a matrix where each column represents a unit of vertically integrated productive capacity for final good , i, i.e. a composite commodity formed by the quantities of the m goods produced in the economic system that must be — directly and indirectly — available in order to produce one unit of commodity i as a final good. Symmetrically, we can express the total amount of labour required for the production of final demand $\mathbf{x}$ as:

$$x_N = \mathbf{a}^{\mathbf{T}}\mathbf{q} = \mathbf{a}^{\mathbf{T}}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \mathbf{v}^{\mathbf{T}}\mathbf{x} \tag{2.4}$$

$\mathbf{v}^{\mathbf{T}}$ is the vector of vertically integrated labour, i.e. the vector of the quantities of labour directly and indirectly employed for the production of the quantities $\mathbf{x}$.

Following Pasinetti (1973) we can now define the notion of vertically integrated sector.

Let's first of all define a new series of m vectors $\mathbf{x_i}$, whose elements are all zeros except the ith one, which is equal to the quantity of commodity i actually produced as a final good.

For each particular x_i, we can write, from equations (2.2)-(2.4):

$$\mathbf{q}^{(\mathbf{i})} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_i} \tag{2.5}$$

$$\mathbf{s}^{(\mathbf{i})} = \mathbf{A}\mathbf{q}^{(\mathbf{i})} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_i} = \mathbf{H}\mathbf{x_i} \tag{2.6}$$

$$x_N^{(i)} = \mathbf{a}^{\mathbf{T}}\mathbf{q}^{(\mathbf{i})} = \mathbf{a}^{\mathbf{T}}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_i} = \mathbf{v}^{\mathbf{T}}\mathbf{x_i} \tag{2.7}$$

As $i = 1, \ldots, m$, we have defined m vertically integrated sectors — or *sub-systems*, using Sraffa's terminology — which add up to the complete economic system, and composed by the ith element of vector $\mathbf{q}$, the ith column of matrix $\mathbf{H}$ and the ith element of vector $\mathbf{v}^{\mathbf{T}}$.

We can now go on and use the just given definitions into Pasinetti's (1981) vertical integrated model.

3 A stationary system in which all produced goods are consumed

This is the simplest case we can deal with. Apart from the fact that there is no growth, and hence the economic system is a stationary one, here the number of produced goods and of goods entering final demand is the same.

Each vertically integrated sector is composed, as we have already said at the end of section 2, by a consumption good, a unit of vertically integrated productive capacity, and a unit of vertically integrated labour. Therefore, in an economic system in which not all produced goods are consumed, the number of vertically integrated sectors would be smaller than the number of industries. This would entail some analytical complications which we will deal with below, in section 6.

For the time being, let us face the simplest case in which industries and vertically integrated sectors are in the same number.

3.1 The quantity system

To begin with, let us define a new series of vectors:

$\mathbf{a_{k_i i}}$	=	$[h_{ij}]$:	i-th column of matrix $\mathbf{H}$;
$\mathbf{x_{k_i}}$	=	$[x_{jk_i}]$:	(column) vector of the quantities of goods j, ($j = 1, \ldots, m$) directly and indirectly required for the production of comodity i as a final good.

According to the definition of vertically integrated sectors, the physical quantity system is given by:

$$\begin{cases} x_1 = a_{1N}x_N \\ \vdots \\ x_m = a_{mN}x_N \\ \mathbf{x_{k_1}} = \mathbf{a_{k_1 1}}x_1 \\ \vdots \\ \mathbf{x_{k_m}} = \mathbf{a_{k_m m}}x_m \\ x_N = \sum_{i=1}^{m} v_{ni}x_i \end{cases} \tag{3.1}$$

i.e.:

$$\begin{cases} x_1 = a_{1N}x_N \\ \vdots \\ x_m = a_{mN}x_N \\ x_{1k_1} = h_{11}x_1 \\ \vdots \\ x_{mk_1} = h_{m1}x_1 \\ x_{1k_m} = h_{1m}x_m \\ \vdots \\ x_{mk_m} = h_{mm}x_m \\ x_N = \sum_{i=1}^{m} v_{Ni}x_i \end{cases} \tag{3.2}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|ccc|c} -1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{mN} \\ \hline h_{11} & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{m1} & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & h_{1m} & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & h_{mm} & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ \hline v_{N1} & \cdots & v_{Nm} & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \\ \hline x_{1k_1} \\ \vdots \\ x_{mk_1} \\ \hline x_{1k_m} \\ \vdots \\ x_{mk_m} \\ \hline x_N \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array}\right] \tag{3.3}$$

System (3.3) is clearly a linear and homogeneous one. The mathematical condition for non-trivial solutions to exist is:

$$\sum_{i=1}^{m} v_{Ni} a_{iN} = 1 \tag{3.4}$$

The economic meaning of such a mathematical condition should be clear: it is a *macroeconomic condition* for full employment of labour and full expenditure.

3.2 The price system

As to the price system, with respect to Pasinetti (1973), we will introduce the slight modification, allowing the rate(s) of profit to be non-homogeneous through the various sectors. I.e. we have a whole series of rates of profit π_i, arranged in a diagonal matrix $\widehat{\mathbf{\Pi}}$. Therefore, the price system can be written as:

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{a}^{\mathrm{T}} + \mathbf{p}^{\mathrm{T}}\mathbf{A} + \mathbf{p}^{\mathrm{T}}\mathbf{A}\widehat{\mathbf{\Pi}} \tag{3.5}$$

System (3.5) can be rewritten as:

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{a}^{\mathrm{T}}(\mathbf{I}-\mathbf{A})^{-1} + \mathbf{p}^{\mathrm{T}}\mathbf{A}\widehat{\mathbf{\Pi}}(\mathbf{I}-\mathbf{A})^{-1} = w\mathbf{v}^{\mathrm{T}} + \mathbf{p}^{\mathrm{T}}\mathbf{A}\widetilde{\mathbf{\Pi}} \tag{3.6}$$

Let us have a look at the second addendum of expression (3.6).

The j-th element of vector $\mathbf{p^T A} = [\tilde{a}_{ij}]$ represents the *value* of *direct requirements* for the production of one unit of good j:

$$\tilde{a}_{ij} = \sum_{i=1}^{m} p_i a_{ij}$$

Such vector has to be multiplied by matrix $\widehat{\mathbf{\Pi}}(\mathbf{I} - \mathbf{A})^{-1} = \tilde{\mathbf{\Pi}} = [\tilde{\pi}_{ij}]$. Computing this product, we obtain a row vector whose k-th element is:

$$\sum_{j=1}^{m} \sum_{i=1}^{m} p_i a_{ij} \tilde{\pi}_{jk}$$

Such a summation decomposes the total profit of vertically integrated sector k into m components:

$$\sum_{j=1}^{m} (p_1 a_{1j} + p_2 a_{2j} + \ldots + p_m a_{mj}) \tilde{\pi}_{jk}$$

Each one of these m components represents that part of the unitary profit of vertically integrated sector k which is proportional to the value of capital goods directly needed to produce all commodities which enter a unit of vertically integrated productive capacity. I.e. it is the amount of the profit of *industry* j which accrues to the vertically integrated sector k.

This means that matrix $\tilde{\mathbf{\Pi}}$ provides us with a way to 're-distribute' profits from industries to vertically integrated sectors.

Moreover, we can say that matrix $\mathbf{A}\tilde{\mathbf{\Pi}}$ is analagous to matrix $\mathbf{H}$ but in 'value' rather than in physical terms. Let us therefore call $\tilde{\mathbf{H}} = [\tilde{h}_{ij}]$ such a matrix, and re-write equation (3.6) as:

$$\mathbf{p^T} = w\mathbf{v^T} + \mathbf{p^T}\tilde{\mathbf{H}} \tag{3.7}$$

However, we still lack the last step in order to be able to write the price system as in Pasinetti (1981). Actually, matrix $\tilde{\mathbf{H}}$ does not provide us with a single rate of profit for each vertically integrated sector, but rather gives us a whole series of m 'weights' that we can use to redistribute profits from the industry- to the vertically integrated sector-level.

In order to find the rates of profit of the m vertically integrated sectors we have to divide unitary profits by the value of one unit of vertically integrated productive capacity, i.e.:

$$\widehat{\boldsymbol{\pi}} = [\widehat{\pi}_i] = (\widehat{\mathbf{p_k}})^{-1}\left(\widehat{\mathbf{p^T}\widetilde{\mathbf{H}}}\right) = \left(\widehat{\mathbf{p^T}\mathbf{H}}\right)^{-1}\left(\widehat{\mathbf{p^T}\widetilde{\mathbf{H}}}\right)$$

where (row) vector $\mathbf{p_k^T}$ is the vector of the prices of the m units of vertically integrated productive capacity, defined as the value of that composite commodity which is needed to produce *one unit* of good i as a final good.

Therefore, equation (3.7) becomes:

$$\mathbf{p^T} = w\mathbf{v} + \mathbf{p_k^T}\widehat{\boldsymbol{\pi}} = w\mathbf{v} + \mathbf{pH}\widehat{\boldsymbol{\pi}} \tag{3.8}$$

and the price system is given by:

$$\begin{cases} p_1 = wv_{n1} + p_{k_1}\widehat{\pi}_1 \\ \vdots \\ p_m = wv_{nm} + p_{k_m}\widehat{\pi}_m \\ p_{k_1} = \sum_{i=1}^m h_{i1}p_i \\ \vdots \\ p_{k_m} = \sum_{i=1}^m h_{im}p_i \\ w = \sum_{i=1}^m a_{iN}p_i - \sum_{i=1}^m \widehat{\pi}_i a_{iN} p_{k_i} \end{cases} \tag{3.9}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|c} -1 & \cdots & 0 & \widehat{\pi}_1 & \cdots & 0 & v_{N1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & \widehat{\pi}_m & v_{Nm} \\ \hline h_1 1 & \cdots & h_{m1} & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1m} & \cdots & h_{mm} & 0 & \cdots & -1 & 0 \\ \hline a_{1N} & \cdots & a_{mN} & -\widehat{\pi}_1 a_{1N} & \cdots & -\widehat{\pi}_m a_{mN} & -1 \end{array}\right] \left[\begin{array}{c} p_1 \\ \vdots \\ p_m \\ \hline p_{k_1} \\ \vdots \\ p_{k_m} \\ \hline w \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array}\right] \tag{3.10}$$

System (3.10) is a linear and homogeneous one as well. The condition for non-trivial solutions to exist is precisely the same as the one we found for system (3.3), namely equation (3.4):

$$\sum_{i=1}^m v_{Ni}a_{iN} = 1$$

3.3 Solutions for physical quantities and prices

When condition (3.4) is satisfied, both system (3.3) and (3.10) give solutions for m variable, while the $(m+1)$th can be fixed arbitrarily.

For the quantity system, the most obvious variable to be fixed arbitrarily is total population: $x_N = \overline{x}_N$. The solutions for the quantity system therefore are:

$$\begin{cases} x_1 = a_{1N}\overline{x}_N \\ \vdots \\ x_m = a_{mN}\overline{x}_N \\ \mathbf{x_{k_1}} = \mathbf{a_{k_1 1}} x_1 = \mathbf{a_{k_1 1}} a_{1N}\overline{x}_N \\ \vdots \\ \mathbf{x_{k_m}} = \mathbf{a_{k_m m}} x_m = \mathbf{a_{k_m m}} a_{mN}\overline{x}_N \end{cases} \tag{3.11}$$

For the price system, on the contrary, there is no such variable. In this case, we have to arbitrarily chose one price as the *numéraire* of the system. We could chose any commodity price, but here we will choose the wage rate: $w = \overline{w}$.

The solutions for commodity prices hence are:

$$\mathbf{p} = \overline{w}\mathbf{v}(\mathbf{I} - \mathbf{H}\widehat{\pi})^{-1} \tag{3.12}$$

while the solution for the prices of the units of vertically integrated productive capacity are:

$$\mathbf{p_k^T} = \mathbf{pH} = \overline{w}\mathbf{v}(\mathbf{I} - \mathbf{H}\widehat{\pi})^{-1}\mathbf{H} \tag{3.13}$$

Of course, the solutions for quantities are much simpler than those for prices. While quantities simply depend on final demand, each price depends on *all* other prices.

4 A non-stationary system

In what follows, we will make the assumption that population is growing at the positive rate g, while per-capita demand in each sector is growing at the sector-specific rates r_i. This means that sectoral demands are growing at the sectoral-specific rates $(g + r_i)$. We arrange this growth rates in a diagonal matrix which we call $\widehat{\mathbf{C}}$.

4.1 The quantity system

As sectoral demands are growing, the total quantities to be produced must now satisfy final demand, replace worn out productive capacity and also expand it accordingly with the expansion of demand itself.

Equations system (2.1) then becomes:

$$\mathbf{q} = \mathbf{A}\mathbf{q} + \mathbf{A}\widehat{\mathbf{C}}\mathbf{q} + \mathbf{x} = \mathbf{A}(\mathbf{I} + \widehat{\mathbf{C}})\mathbf{q} + \mathbf{x} = \mathbf{A}_c\mathbf{q} + \mathbf{x} \tag{4.1}$$

and therefore:

$$\mathbf{q} = (\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{x} \tag{4.2}$$

Accordingly, equation (2.3) in this case becomes:

$$\mathbf{s}_c = \mathbf{A}_c\mathbf{q} = \mathbf{A}_c(\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{x} = \mathbf{H}_c\mathbf{x} \tag{4.3}$$

while equation (2.4) becomes:

$$x_N = \mathbf{a^T}_c\mathbf{q} = \mathbf{a^T}_c(\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{x} = \mathbf{v^T}_c\mathbf{x} \tag{4.4}$$

where $\mathbf{H}_c = [h^c_{ij}]$ is a matrix whose m columns represent the units of vertically hyper-integrated productive capacity and $\mathbf{v^T}_c = [v^c_{Ni}]$ is the (row) vector of vertically hyper-integrated labour coefficients.

The difference with respect to section 4 is that now we have matrix $\mathbf{H}_c$ — of vertically hyper-integrated productive capacity — instead of matrix $\mathbf{H}$ — of vertically integrated productive capacity — and vector $\mathbf{v^T}_c$ — of vertically hyper-integrated labour — instead of vector $\mathbf{v^T}$ — of vertically integrated labour. Hence, system (3.2) becomes:

$$\begin{cases} x_1 = a_{1N}x_N \\ \vdots \\ x_m = a_{mN}x_N \\ \mathbf{x_{k_1}} = \mathbf{a^c_{k_1 1}}x_1 \\ \vdots \\ \mathbf{x_{k_m}} = \mathbf{a^c_{k_m m}}x_m \\ x_N = \sum_{i=1}^{m} v^c_{Ni}x_i \end{cases} \tag{4.5}$$

i.e.:

$$\begin{cases} x_1 = a^c_{1N} x_N \\ \vdots \\ x_m = a^c_{mN} x_N \\ x_{1k_1} = h^c_{11} x_1 \\ \vdots \\ x_{mk_1} = h^c_{m1} x_1 \\ x_{1k_m} = h^c_{1m} x_m \\ \vdots \\ x_{mk_m} = h^c_{mm} x_m \\ x_N = \sum_{i=1}^{m} v^c_{Ni} x_i \end{cases} \tag{4.6}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|ccc|c} -1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a^c_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a^c_{mN} \\ \hline h^c_{11} & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h^c_{m1} & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & h^c_{1m} & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & h^c_{mm} & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ \hline v^c_{N1} & \cdots & v^c_{Nm} & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \\ \hline x_{1k_1} \\ \vdots \\ x_{mk_1} \\ \hline x_{1k_m} \\ \vdots \\ x_{mk_m} \\ \hline x_N \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array}\right] \tag{4.7}$$

This is again a linear and homogeneous system, and therefore, in order for non-trivial solutions to exist, the determinant of the coefficient matrix must be equal to zero. This condition has the same form, and meaning, as equation 3.4:

$$\sum_{i=1}^{m} v^c_{Ni} a_{iN} = 1 \tag{4.8}$$

4.2 The price system

Again assuming rates of profits non-homogeneous among sectors, equations system (3.6) here becomes:

$$\mathbf{p}^{\mathbf{T}} = w\mathbf{a}^{\mathbf{T}}{}_c(\mathbf{I} - \mathbf{A}_c)^{-1} + \mathbf{p}^{\mathbf{T}}\mathbf{A}_c\widehat{\mathbf{\Pi}}(\mathbf{I} - \mathbf{A}_c)^{-1} \tag{4.9}$$

and therefore:

$$\mathbf{p^T} = w\mathbf{v^T}_c + \mathbf{p^T}\tilde{\mathbf{H}} \tag{4.10}$$

The diagonal matrix of the rates of profit of the m vertically hyper-integrated sectors sectors is given by:

$$\widehat{\boldsymbol{\pi}} = [\widehat{\pi}_i] = \left(\widehat{\mathbf{p^T H}}\right)^{-1}\left(\widehat{\mathbf{p^T}\tilde{\mathbf{H}}}\right) = (\widehat{\mathbf{p_k}})^{-1}\left(\widehat{\mathbf{p^T}\tilde{\mathbf{H}}}\right) \tag{4.11}$$

System (4.10) becomes:

$$\mathbf{p^T} = w\mathbf{v^T}_c + \mathbf{p_k^T}\widehat{\boldsymbol{\pi}} = w\mathbf{v^T}_c + \mathbf{p^T H}\widehat{\boldsymbol{\pi}} \tag{4.12}$$

and the price system is:

$$\begin{cases} p_1 = wv^c_{n1} + p_{k_1}\widehat{\pi}_1 \\ \vdots \\ p_m = wv^c_{nm} + p_{k_m}\widehat{\pi}_m \\ p_{k_1} = \sum_{i=1}^m h^c_{i1}p_i \\ \vdots \\ p_{k_m} = \sum_{i=1}^m h^c_{im}p_i \\ w = \sum_{i=1}^m a_{iN}p_i - \sum_{i=1}^m \widehat{\pi}_i a_{iN} p_{k_i} \end{cases} \tag{4.13}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|c} -1 & \cdots & 0 & \widehat{\pi}_1 & \cdots & 0 & v^c_{N1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & \widehat{\pi}_m & v^c_{Nm} \\ \hline h^c_1 1 & \cdots & h^c_{m1} & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h^c_{1m} & \cdots & h^c_{mm} & 0 & \cdots & -1 & 0 \\ \hline a_{1N} & \cdots & a_{mN} & -\hat{\pi}_1 a_{1N} & \cdots & -\widehat{\pi}_m a_{mN} & -1 \end{array}\right] \left[\begin{array}{c} p_1 \\ \vdots \\ p_m \\ \hline p_{k_1} \\ \vdots \\ p_{k_m} \\ \hline w \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array}\right] \tag{4.14}$$

Again, we are dealing with a linear non-homogeneous system, whose condition for non-trivial solutions to exist is still (4.8)

$$\sum_{i=1}^m v^c_{Ni} a_{iN} = 1$$

4.3 Solutions for physical quantities and commodity prices

Again considering total population as exogenous, and then fixing $x_N = \overline{x}_N$, the solutions for physical quantities are:

$$\begin{cases} x_1 = a_{1N}\overline{x}_N \\ \vdots \\ x_m = a_{mN}\overline{x}_N \\ \mathbf{x_{k_1}} = \mathbf{a^c_{k_1 1}} x_1 = \mathbf{a^c_{k_1 1}} a_{1N}\overline{x}_N \\ \vdots \\ \mathbf{x_{k_m}} = \mathbf{a^c_{k_m m}} x_m = \mathbf{a^c_{k_m m}} a_{mN}\overline{x}_N \end{cases} \tag{4.15}$$

To solve the price system, let us again arbitrarily fix $w = \overline{w}$; the solutions for consumption goods prices are therefore given by:

$$\mathbf{p^T} = \overline{w}\mathbf{v^T}_c \left(\mathbf{I} - \mathbf{H}_c \widehat{\boldsymbol{\pi}}\right)^{-1} \tag{4.16}$$

while the solutions for the prices of the units of vertically hyper-integrated productive capacity are:

$$\mathbf{p_k^T} = \mathbf{p^T}\mathbf{H}_c = \overline{w}\mathbf{v^T}_c \left(\mathbf{I} - \mathbf{H}_c \widehat{\boldsymbol{\pi}}\right)^{-1} \mathbf{H}_c \tag{4.17}$$

5 The natural rates of profit

Up to now, the rates of profit have been taken as exogenously given. As pointed out by Pasinetti (1981, Chapter II, section 6 and Chapter VII, section 4), looking at expressions (4.16), this gives rise to a theory of value in terms of *labour equivalents*.

However, following the same argument as in Pasinetti (1981, Chapter VII, section 3), we can easily conclude that the very theoretical framework set up hitherto already implies a whole series of *natural rate of profits*, emerging from the dynamic — i.e growing — nature of the economic system considered so far.

Actually, the rates of profit must provide those resources necessary for the investments required for the growth of the productive structure. We know that the whole growth process is driven by demand, more specifically by the sector-specific *rates of growth* of demand. This means first of all that the natural rate of profits may be uniform only in the (quite unrealistic) case in which demand varies at the same pace in all vertically integrated sectors. In all other cases, each vertically integrated sector has its own natural rate of profit, given by:

$$\widehat{\pi}_i^* = g + r_i \tag{5.1}$$

and therefore:

$$\widehat{\boldsymbol{\pi}}^* = \widehat{\mathbf{C}} \tag{5.2}$$

By substituting expression (5.2) into (4.16) we get:

$$\mathbf{p}^{\mathbf{T}} = \overline{w}\mathbf{v}^{\mathbf{T}}{}_c \left(\mathbf{I} - \mathbf{H}_c\widehat{\mathbf{C}}\right)^{-1} \tag{5.3}$$

By choosing for the wage rate the specific value $\overline{w} = 1$, as the Classics did, the previous expression reduces to:

$$\mathbf{p}^{\mathbf{T}} = \mathbf{v}^{\mathbf{T}}{}_c \left(\mathbf{I} - \mathbf{H}_c\widehat{\mathbf{C}}\right)^{-1} \tag{5.4}$$

which expresses commodity prices as the weighted average of (dated) quantities of labour: we have a *pure labour theory of value*.

6 Generalisation: not all produced goods are consumed

Let us now generalise the analytical framework presented in the above sections.

Usually, in an actual economic system, not all goods are consumed. Some of them are simply used as (circulating) capital. The immediate consequence is that, according to the definition of vertically hyper-integrated sector given above, the number of industries and that of vertically hyper-integrated sectors themselves is different.

Let us therefore suppose that the economic system is made up by m industries, and hence that m goods are produced. However, only the first h ones enter the final demand, while the remaining $m - h = s$ are only used as (circulating) capital.

It is immediately clear that while we have m industries in the economic system, we have only h vertically hyper-integrated sectors.

6.1 The quantity system

Let's first of all write down the physical quantity system:

$$\begin{cases} x_1 = a_{1N} x_N \\ \vdots \\ x_h = a_{hN} x_N \\ \mathbf{x_{k_1}} = \mathbf{a^c_{k_1 1}} x_1 \\ \vdots \\ \mathbf{x_{k_h}} = \mathbf{a^c_{k_h h}} x_h \\ x_N = \sum_{i=1}^{h} v^c_{Ni} x_i \end{cases} \tag{6.1}$$

i.e.:

$$\begin{cases} x_1 = a_{1N} x_N \\ \vdots \\ x_m = a_{mN} x_N \\ x_{1k_1} = h^c_{11} x_1 \\ \vdots \\ x_{mk_1} = h^c_{m1} x_1 \\ x_{1k_m} = h^c_{1m} x_m \\ \vdots \\ x_{mk_m} = h^c_{mm} x_m \\ x_N = \sum_{i=1}^{m} v^c_{Ni} x_i \end{cases} \tag{6.2}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|ccc|c} -1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{mN} \\ \hline h^c_{11} & \cdots & 0 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h^c_{m1} & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & h^c_{1m} & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & h^c_{mm} & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 \\ \hline v^c_{N1} & \cdots & v^c_{Nm} & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_m \\ \hline x_{1k_1} \\ \vdots \\ x_{mk_1} \\ \hline x_{1k_m} \\ \vdots \\ x_{mk_m} \\ \hline x_N \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array}\right] \tag{6.3}$$

Of course, while we have only h goods entering final demand, the units of vertically hyper-integrated productive capacity will be made up by those goods used as circulating capital as well.

Again, the condition for non-trivial solutions to this linear and homogeneous system to exist is the same as (4.8):

$$\sum_{i=1}^{h} v_{Ni}^{c} a_{iN}$$

6.2 The price system

With the price system, things become much more complicated. As we have already said, each commodity price depends on the prices of *all* other commodities, included, of course, the non-consumption goods' ones — which however enter the production costs.

Therefore, the last s prices will not enter the price system explicitly, as it contains only the prices of consumption goods, but *implicitly* through the prices of the corresponding units of vertically hyper-integrated productive capacity.

It should be clear that consumption good's prices are still the same both in industry and in sectors terms. The price system still is:

$$\mathbf{p}^{\mathrm{T}} = w\mathbf{v}^{\mathrm{T}}{}_{c} + \mathbf{p}^{\mathrm{T}}\mathbf{A}_{c}\widehat{\mathbf{\Pi}}\left(\mathbf{I} - \mathbf{A}_{c}\right)^{-1} = w\mathbf{v}^{\mathrm{T}}{}_{c} + \mathbf{p}^{\mathrm{T}}\mathbf{A}_{c}\tilde{\mathbf{\Pi}} \qquad (6.4)$$

Moreover, we can of course still find the values of units of vertically hyper-integrated productive capacity by using expression $\mathbf{p}_{\mathbf{k}}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{H}_{c}$. However, we are interested only in the first h elements of the resulting vector, as we only have h vertically hyper-integrated sectors.

But even considering only the relevant elements of vector $\mathbf{p}_{\mathbf{k}}^{\mathrm{T}}$, we have to notice that their analytical expressions still depend on *all* commodity prices:[6]

$$\mathbf{p}^{\mathrm{T}}_{\mathbf{k_h}} = \mathbf{p}^{\mathrm{T}}_{\mathbf{h}}\mathbf{H_{hh}} + \mathbf{p_s}\mathbf{H_{sh}} \tag{6.5}$$

In order to write down the price system in the same form as system (3.10) and system (4.14) we need to convert it into a homogeneous system, thus eliminating from such expressions the last s prices.

In order to do so, let us first of all re-write expression (6.4) as follows:

$$\begin{cases} \mathbf{p}^{\mathrm{T}}_{\mathbf{h}} = w\mathbf{v}^{\mathrm{T}}_{\mathbf{h}} + \mathbf{p}^{\mathrm{T}}_{\mathbf{h}}\mathbf{A}^{\mathbf{r}}_{\mathbf{h}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}} + \mathbf{p_s}\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}} \\ \mathbf{p_s} = w\mathbf{v}^{\mathrm{T}}_{\mathbf{s}} + \mathbf{p}^{\mathrm{T}}_{\mathbf{h}}\mathbf{A}^{\mathbf{r}}_{\mathbf{h}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}} + \mathbf{p_s}\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}} \end{cases} \tag{6.6}$$

where $\mathbf{A}^{\mathbf{r}}_{\mathbf{h}}$ is a rectangular, $(h \times m)$, matrix made up by the first h rows of matrix $\mathbf{A}_c$ and, symmetrically, $\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}$ is a rectangular, $(s \times m)$, matrix made up by the last s rows of the same matrix; $\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}$ is a rectangular, $(m \times h)$, matrix made up by the first h columns of matrix $\tilde{\mathbf{\Pi}}$ and $\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}}$ is a rectangular, $(m \times s)$, matrix made up by the last s columns of the same matrix.

Our aim is that of re-writing the second expression in system (6.6) only in terms of prices $\mathbf{p}^{\mathrm{T}}_{\mathbf{h}}$, thus eliminating vector $\mathbf{p_s}$.

Let us therefore solve the first expression in system (6.6) for $\mathbf{p_s}$:

$$\mathbf{p_s} = \mathbf{p}^{\mathrm{T}}_{\mathbf{h}}\left(\mathbf{I} - \mathbf{A}^{\mathbf{r}}_{\mathbf{h}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)\left(\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)^{-1} - w\mathbf{v}^{\mathrm{T}}_{\mathbf{h}}\left(\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)^{-1} \tag{6.7}$$

and substitute it into the second one, to get:

$$\begin{aligned} \mathbf{p_s} =& \mathbf{p}^{\mathrm{T}}_{\mathbf{h}}\left[\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}} + \left(\mathbf{I} - \mathbf{A}^{\mathbf{r}}_{\mathbf{h}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)\left(\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)^{-1}\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}}\right] + \\ &+ w\left[\mathbf{v}^{\mathrm{T}}_{\mathbf{s}} - \mathbf{v}^{\mathrm{T}}_{\mathbf{h}}\left(\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{h}}\right)^{-1}\mathbf{A}^{\mathbf{r}}_{\mathbf{s}}\boldsymbol{\pi}^{\boldsymbol{c}}_{\boldsymbol{s}}\right] \end{aligned} \tag{6.8}$$

[6] For analytical purposes, we have partitioned matrix $\mathbf{H}_c$ as follows:

$$\mathbf{H}_c = \left[\begin{array}{ccc|ccc} h^c_{11} & \cdots & h^c_{1h} & h^c_{1s} & \cdots & h^c_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h^c_{h1} & \cdots & h^c_{hh} & h^c_{hs} & \cdots & h^c_{hm} \\ \hline h^c_{s1} & \cdots & h^c_{sh} & h^c_{ss} & \cdots & h^c_{sm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h^c_{m1} & \cdots & h^c_{mh} & h^c_{ms} & \cdots & h^c_{mm} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H_{hh}} & \mathbf{H_{hs}} \\ \hline \mathbf{H_{sh}} & \mathbf{H_{ss}} \end{array}\right]$$

Now we can substitute system (6.8), which is an expression of prices $\mathbf{p_s}$ in terms of prices $\mathbf{p_h^T}$, into expression (6.5), to get:

$$\begin{aligned}\mathbf{p_{k_h}^T} =&\mathbf{p_h^T}\left\{\mathbf{H_{hh}}+\left[\mathbf{A_s^r}\boldsymbol{\pi_s^c}+\left(\mathbf{I}-\mathbf{A_h^r}\boldsymbol{\pi_h^c}\right)\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]\mathbf{H_{sh}}\right\}+\\ &+w\left[\mathbf{v_s^T}-\mathbf{v_h^T}\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]\mathbf{H_{sh}}\end{aligned} \tag{6.9}$$

For ease of notation, we can re-write expression (6.9) as:

$$\mathbf{p_{k_h}^T}=\mathbf{p_h^T}\mathbf{M_h}+w\mathbf{z_h} \tag{6.10}$$

where

$$\mathbf{M_h}=[m_{ij}]=\mathbf{H_{hh}}+\left[\mathbf{A_s^r}\boldsymbol{\pi_s^c}+\left(\mathbf{I}-\mathbf{A_h^r}\boldsymbol{\pi_h^c}\right)\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]\mathbf{H_{sh}}$$

and

$$\mathbf{z_h}=[z_i]=\left[\mathbf{v_s^T}-\mathbf{v_h^T}\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]\mathbf{H_{sh}}$$

while for consumption goods we have:

$$\mathbf{p_h^T}=w\mathbf{v_h^T}+\mathbf{p_{k_h}^T}\widehat{\boldsymbol{\pi}} \tag{6.11}$$

To get the expression for $\widehat{\boldsymbol{\pi}}$, i.e. for the vertical integrated rates of profit, we have first of all to substitute system (6.8) into the first system in (6.6), in order to get rid of $\mathbf{p_s}$ in the expression for $\mathbf{p_h^T}$, thus obtaining:

$$\begin{aligned}\mathbf{p_h^T} =&w\mathbf{v_h^T}+\mathbf{p_h^T}\left[\mathbf{A_h^r}\boldsymbol{\pi_h^c}+\mathbf{A_s^r}\boldsymbol{\pi_s^c}+\left(\mathbf{I}-\mathbf{A_h^r}\boldsymbol{\pi_h^c}\right)\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]+\\ &+w\left[\mathbf{v_s^T}-\mathbf{v_h^T}\left(\mathbf{A_s^r}\boldsymbol{\pi_h^c}\right)^{-1}\mathbf{A_s^r}\boldsymbol{\pi_s^c}\right]\end{aligned} \tag{6.12}$$

The sum of the second and third addendum, let us call it $\tilde{\mathbf{p}}_\mathbf{h}^\mathbf{T}\tilde{\mathbf{H}}$, gives us total profits in the h vertical integrated sectors, and therefore:

$$\widehat{\boldsymbol{\pi}}=[\widehat{\pi}_i]=\left(\widehat{\mathbf{p_{k_h}}}\right)^{-1}\left(\widehat{\tilde{\mathbf{p}}_\mathbf{h}^\mathbf{T}\tilde{\mathbf{H}}}\right) \tag{6.13}$$

Finally, we can write down the linear and homogeneous price system:

$$\begin{cases} p_1 = wv^c_{N1} + p_{k_1}\widehat{\pi}_1 \\ \vdots \\ p_h = wv^c_{Nh} + p_{k_h}\widehat{\pi}_h \\ p_{k_1} = \sum_{i=1}^{h} p_i m_{i1} + wz_1 \\ \vdots \\ p_{k_h} = \sum_{i=1}^{h} p_i m_{ih} + wz_h \\ w = \sum_{i=1}^{h} p_i a_{iN} - \sum_{i=1}^{h} p_{k_i} a_{iN}\widehat{\pi}_i \end{cases} \tag{6.14}$$

In matrix form:

$$\left[\begin{array}{ccc|ccc|c} -1 & \cdots & 0 & \widehat{\pi}_1 & \cdots & 0 & v^c_{N1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & \widehat{\pi}_h & v^c_{Nh} \\ \hline m_{11} & \cdots & m_{1h} & -1 & \cdots & 0 & z_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{h1} & \cdots & m_{hh} & 0 & \cdots & -1 & z_h \\ \hline a_{1N} & \cdots & a_{hN} & -\widehat{\pi}_1 a_{1N} & \cdots & -\widehat{\pi}_h a_{hN} & -1 \end{array}\right] \tag{6.15}$$

The condition for non-trivial solutions to system (3.10) to exist has again the same as form as (4.8):

$$\sum_{i=1}^{m} v^c_{Ni} a_{iN} = 1$$

6.3 Solutions for physical quantities and commodity prices

The solutions for the physical quantity system are:

$$\begin{cases} x_1 = a_{1N}\overline{x}_N \\ \vdots \\ x_h = a_{hN}\overline{x}_N \\ \mathbf{x_{k_1}} = \mathbf{a^c_{k_1 1}} x_1 = \mathbf{a^c_{k_1 1}} a_{1N}\overline{x}_N \\ \vdots \\ \mathbf{x_{k_h}} = \mathbf{a^c_{k_h h}} x_h = \mathbf{a^c_{k_h h}} a_{hN}\overline{x}_N \end{cases} \tag{6.16}$$

As to the price system, the solutions for consumption goods prices are simply the first h elements of vector $\mathbf{p^T} = w\mathbf{v^T}_c\left(\mathbf{I} - \mathbf{H}_c\widehat{\boldsymbol{\pi}}\right)^{-1}$.

However, in order to continue using the same notation as in the previous section, let us substitute solve system (6.10)–(6.11), thus obtaining:

$$\mathbf{p_h^T} = \overline{w}\left(\mathbf{v_h^T} + \mathbf{z_h}\widehat{\boldsymbol{\pi}}\right)\left(\mathbf{I} - \mathbf{M_h}\widehat{\boldsymbol{\pi}}\right)^{-1} \tag{6.17}$$

and

$$\mathbf{p_{k_h}^T} = \overline{w}\left[\left(\mathbf{v_h^T} + \mathbf{z_h}\widehat{\boldsymbol{\pi}}\right)\left(\mathbf{I} - \mathbf{M_h}\widehat{\boldsymbol{\pi}}\right)^{-1}\mathbf{M_h} + \mathbf{z_h}\right] \tag{6.18}$$

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