Modeling a Multi-Choice Game Based on the Spirit of Equal Job Opportunities (New)

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Abstract. The H&R Shapley value defined by Hsiao and Raghavan for multi-choice cooperative game is redundant free. If the H&R Shapley value is used as the solution of a game, there won't be any objection to a player's taking redundant actions. Therefore, the spirit of the law on equal job opportunities is automatically fulfilled. Also, if the H&R Shapley value is used as the solution of a game, it makes no difference to the players whether they have the same number of options or not. Moreover, the D&P Shapley value, the P&Z Shapley value and the WAC value are linear combinations of the H&R Shapley value, hence, they have all the same dummy free properties and the independent property as does the H&R Shapley value. Finally the N&P Shapley value is not redundant free.

JEL Classification Numbers. C71, D46, D61, D72, K31.

Keywords. *Shapley value, multi-choice cooperative game, redundant free, independent of non-essential player.*

1. Introduction. In real-life, a player might work diligently or work lazily in a coalition. But, a traditional cooperative game can not reflect the above truth. In order to remedy that weak point of the traditional cooperative games, Hsiao and Raghavan(1993) extended the traditional cooperative game to a multi-choice cooperative game and extended the traditional Shapley value to a multi-choice Shapley value. In short, we call the shapley value for multi-choice cooperative games the **multi-choice Shapley value**. Some authors call the multi-choice Shapley value defined by Hsiao and Raghavan the **H&R** Shapley value.

Based on the spirit of the law on equal job opportunities, Hsiao and Raghavan(1993) allowed players to have the same number of levels of actions. Some authors slightly extended Hsiao and Raghavan(1993) to a multi-choice game where the players have different numbers of options. However, in this article, we will prove that the H&R Shapley value is **redundant free**. If the H&R Shapley value is used as the solution of a game, it makes no difference to the players whether they have the same number of options or not.

We may see, in the reference of Derk and Peters(1993), that in 1990, before Hsiao and Raghavan(1993) appeared, Raghavan presented the multi-choice game and H&R Shapley value to some other authors. They proposed some other extensions of the Shapley vale right after Raghavan's presentation. So far, there are mainly three other extensions of the Shapley value for multi-choice games proposed by Derks and Peters(1993)(the **D&P** Shapley value), Nouweland et al.(1995)(the **N&P** Shapley value) and Peters and Zank(2005)(the **P&Z** Shapley value), respectively. In the 2nd(1991) and the 4th(1993) international conference on game theory at Stony-Brook, Hsiao asked " Is there any solution of the multi-choice games, other than the H&R Shapley value, is *dummy free of actions* or *dummy free of players* ?" Please note that *dummy free of players* is a special case of *independent of non-essential players.*

When players are playing a game, first thing first, they have to decide who are allowed to play the game, what kinds of games they are playing, how many actions they are allowed to have and which solution they will adopt. If a solution which is not independent of non-essential players is adopted, then the players will have trouble in deciding who is allowed to play the game. If a non-essential player's participating in the game may make some players get more (or pay less), then the other players who may get less (or pay more) will refuse the non-essential player's participating in the game, and the law on equal job opportunities will be violated.

In his article, we will rewrite the definitions and the formula in Hsiao and Raghaven(1993) by allowing the players to have different numbers of actions, and show that the H&R Shapley value is not only *dummy free of actions* and *independent of non-essential players* but also *redundant free.* Therefore, if the H&R Shapley value is adopted as the solution of a game, the spirit of the law on equal job opportunities will be automatically fulfilled.

Also, in this article, we will prove explicitly that the (**D&P** Shapley value), the (**P&Z** Shapley value) and the **WAC value** proposed in Hwang and Liao(2008a) are linear combinations of the **H&R Shapley value** with special weight functions, therefore the values are also redundant free and independent of non-essential players as is the **H&R Shapley value**. Moreover, we will show that the **V&D** Shapley value is not *redundant free* by their own example in Nouweland et al.(1995). Since the N&P Shapley value is not redundant free, a player's redundant action will make some player get more (or play less) and make the other players get less (or pay more), if the value is adopted, the players will have trouble in deciding how many actions a player can have. Finally, we will prove explicitly that the **N&P** Shapley value is in a sense of *independent of non-essential players*.

2. Definitions and Notations.

We believe that all the readers are familiar with the traditional mathematical symbols, for example in most of mathematics text books, the bold face letter $\mathbf{x}$ denotes a vector. Therefore, from cognitive point of view, in this article, we will use the traditional mathematical symbols and notations to modify the multi-choice game in order to acquire better meta-cognition.

Let U be the universal set of players. Without loss of generality, given a finite set of n players $N \subset U$ where $N = \{1, 2, ..., n\}$, we have the following definitions and notations. Any subset $S \subset N$ is called a coalition. Other than what we did in Hsiao and Raghavan(1993), we now allow players to have different numbers of actions. We allow player j to have $(m_j + 1)$ actions, say σ_0, σ_1, σ_2, ..., σ_{m_j}, where σ_0 is the action to do nothing, while σ_k is the choice to work at level k, which has higher level than σ_{k-1}. In this article, we assume that there are finitely many players with finitely many choices.

For convenience, we will use non-negative integers to denote the players' actions. Let I_+ denote the set of all finite non-negative integers. Let $\boldsymbol{\beta}_j = \{0, 1, \ldots, m_j\}$, with $m_j > 0$, be the action space of player j. Given $\mathbf{m} = (m_1, m_2, ..., m_n) \in I_+^n$, with $m_j > 0$ for all j, the action space of N is defined by $\Gamma(\mathbf{m}) = \prod_{j \in N} \boldsymbol{\beta}_j = \{(x_1, ..., x_n) \mid x_j \leq m_j$ and $x_j \in I_+$, for all $j \in N\}$. Thus $\mathbf{x} = (x_1, ..., x_n)$ is called an action vector of N, and $x_j = k$ if and only if player j takes action σ_k. When $m_j = 0$, we won't regard j as a player.

Definition 1. A multi-choice cooperative game in characteristic function form is the pair $(\mathbf{m}, v)$ defined by, $v : \Gamma(\mathbf{m}) \to R$, such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, 0..., 0)$.

We may consider $v(\mathbf{x})$ as the payoff or the cost for the players whenever the players take action vector $\mathbf{x}$. Sometimes, we will denote $v(\mathbf{x})$ by $(\mathbf{m}, v)(\mathbf{x})$ in order to emphasis that the domain of v is $\Gamma(\mathbf{m})$.

In general a multi-choice cooperative game need not be non-decreasing. When too many players overwork there can be a total system breakdown. Now, we consider the solution of multi-choice cooperative games.

Let G be the set of all multi-choice cooperative games with finitely many players and finitely many actions. Instead of regarding the power index of a game as a vector, we regard the power index or value of a game as a matrix-type table, of course essentially a vector. Let ψ be a function defined on G such that assigns each $(\mathbf{m}, v) \in G$ a $\sum_{j=1}^{n} m_j$ dimensional matrix-type table as follows.

$$\psi(v) = \begin{pmatrix} \psi_{1,1}(v) & \psi_{1,2}(v) & \dots & \psi_{1,n}(v) \\ \psi_{2,1}(v) & \psi_{2,2}(v) & \dots & \psi_{2,n}(v) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \psi_{m_2,2}(v) & \vdots & \vdots \\ \psi_{m_1,1}(v) & & \ddots & \vdots \\ & & & \psi_{m_n,n}(v) \end{pmatrix}$$

$$= (\vec{\psi}_1(v), \dots, \vec{\psi}_n(v)) \tag{1}$$

and

$$\vec{\psi}_j(v) = \begin{pmatrix} \psi_{1,j}(v) \\ \psi_{2,j}(v) \\ \vdots \\ \psi_{m_j,j}(v) \end{pmatrix}$$

Essentially,

$$\psi(v) = (\psi_{1,1}(v), ..., \psi_{m_1,1}(v), \psi_{1,2}(v), ..., \psi_{m_2,2}(v), ..., \psi_{1,n}(v), ..., \psi_{m_n,n}(v)) \tag{2}$$

The vector (2) looks much more concise than (1). However, (1) gives us the motivation of redundant free property.

Here $\psi_{i,j}(v)$ is the power index or the value of player j when he takes action σ_i in game v.

In fact, we neglect $\psi_{0,j}(v)$ and assign $\psi_{0,j}(v) = 0$, for all $j \in N$ as does the traditional Shapley value (1953).

Remark 1. Please note that, to be consistent with the traditional notation of a matrix, throughout this article, we use the row index i (or k or s)to denote the action with level i (or k or s) and the column index j (or ℓ or t or r)to denote player j (or ℓ or t or r).

On the contrary, some authors use the column index to denote the levels of actions and use the row index to denote the players. We don't think it is a good idea to do that, because readers are familiar with traditional mathematical notation of matrix and matrix-tye table, from cognitive and metacognitive viewpoint, the existing theorems and properties of matrix theory and reliability theory are helpful in studying the multi-choice games, we had better use notations which are consistent with traditional mathematics.

Since we do not assume that the difference between σ_{k-1} and σ_k is the same as the difference between σ_k and σ_{k+1}, etc., giving weights (discrimination) to actions is necessary.

Let $w : I_+ \to R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \leq w(2) \leq \ldots$, then w is called a **weight function** and $w(i)$ is said to be the **weight** of σ_i.

Remark 2. The weight function w has different meanings in different fields. In military sciences, we may treat $w(i)$s' as parameters to modify the differences due to different levels of military actions.

Most of all, the fact which different players with the same action may have different contributions to a coalition **has been counted** in the characteristic function, therefore it is fair to give weights w on the actions rather than the players.

Rewrite Hsiao and Raghavan(1993), we can show that when w is given, there exists a unique solution ϕ^w defined on G satisfying the following four axioms.

Axiom 1. Suppose a weight function w is given. If v is of the form

$$v(\mathbf{y}) = \begin{cases} c > 0 & \text{if } \mathbf{y} \geq \mathbf{x} \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^w_{x_j,j}(v)$ is proportional to $w(x_j)$.

Axiom 1 states that for binary valued (0 or c) games that stipulate a minimal exertion from players, the reward, for players using the minimal exertion level is proportional to the weight of his minimal level action.

We denote $(\mathbf{x} \mid x_j = k)$ as an action vector with $x_j = k$. Given $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbf{m})$, we define $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$ where $x_j \vee y_i = \max\{x_j, y_j\}$ for each j. Similarly, we define $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$ where $x_i \wedge y_j = \min\{x_j, y_j\}$ for each j.

Definition 2. A vector $\mathbf{x}^* \in \Gamma(\mathbf{m})$ is called a **carrier** of v, if $v(\mathbf{x}^* \wedge \mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$. We call $\mathbf{x}^0$ a **minimal carrier** of v if $\sum x^0_j = \min\{\sum x_j \mid \mathbf{x} \text{ is a carrier of } v\}$.

Definition 3. Player j is said to be a **dummy player** if $v((\mathbf{x} \mid x_j = k)) = v((\mathbf{x} \mid x_j = 0))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and for all $k = 0, 1, 2, ..., m_j$.

The following is a version of the usual efficiency axiom that combines the carrier and the notions of dummy player.

Axiom 2. If $\mathbf{x}^*$ is a carrier of v then, for $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ we have

$$\sum_{\substack{x_j^* \neq 0 \\ x_j^* \in \mathbf{x}^*}} \phi^w_{x_j^*, j}(v) = v(\mathbf{m}).$$

By $x_j^* \in \mathbf{x}^*$ we mean x_j^* is the j-th component of $\mathbf{x}^*$.

Axiom 3. $\phi^w(v^1 + v^2) = \phi^w(v^1) + \phi^w(v^2)$, where $(v^1 + v^2)(\mathbf{x}) = v^1(\mathbf{x}) + v^2(\mathbf{x})$.

Axiom 4. Given $\mathbf{x}^0 \in \Gamma(\mathbf{m})$ if $v(\mathbf{x}) = 0$, whenever $\mathbf{x} \not\geq \mathbf{x}^0$, then for each $j \in N$ $\phi^w_{k,j}(v) = 0$, for all $k < x_j^0$.

Axiom 4 states that in games that stipulate a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded.

Definition 4. Given $\mathbf{x} \in \Gamma(\mathbf{m})$, let $S(\mathbf{x}) = \{j \mid x_j \neq 0, x_j \text{ is a component of } \mathbf{x}\}$. Given $S \subseteq N$, let $\mathbf{e}(S)$ be the binary vector with components $e_j(S)$ satisfying

$$e_j(S) = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we let the standard unit vectors $\mathbf{e}(\{j\}) = \mathbf{e}_j$, for all $j \in N$, and let $|S|$ be the number of elements of S.

Definition 5. Given $\Gamma(\mathbf{m})$ and a weight function, for any $\mathbf{x} \in \Gamma(\mathbf{m})$, we define $\|\mathbf{x}\|_w = \sum\limits_{r=1}^{n} w(x_r)$.

Definition 6. Given $\mathbf{x} \in \Gamma(\mathbf{m})$ and $j \in N = \{1, 2, \ldots, n\}$, we define $M_j(\mathbf{x}; \mathbf{m}) = \{t \mid x_t \neq m_t,\ t \neq j\}$.

Following Hsiao and Raghavan(1993), we have

$$\phi^w_{i,j}(v) = \sum_{k=1}^{i} \sum_{\substack{x_j = k \\ \mathbf{x} \neq \mathbf{0} \\ \mathbf{x} \in \Gamma(\mathbf{m})}} \left[\sum_{T \subseteq M_j(\mathbf{x};\mathbf{m})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum\limits_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \times [v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_j)]. \qquad (\star)$$

Remark 3. It is well-known that the traditional Shapley value has applications in many fields such as economics, political sciences, accounting and even military sciences. Of course, our multi-choice Shapley value also has the same applications as the traditional Shapley value does.

3. Main Results. The matrix-type table (1) of the multi-choice value and the law on equal job opportunities give us the motivation that we should avoid discrimination among the players and allow the players to try the same number of actions. We have Definitions as follow.

Definition 7. Given a game $(\mathbf{m}, v)$, the action σ_{m_t} is said to be a **redundant action** for player t if $v((\mathbf{x} \mid x_t = m_t)) = v((\mathbf{x} \mid x_t = m_t - 1))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$.

Given a solution ψ for $(\mathbf{m}, v)$, suppose we allow player t to have one more action which is redundant for player t, say σ_{m_t+1},

Let $\mathbf{m}^* = (m_1, m_2, ..., m_{t-1}, (m_t + 1), m_{t+1}, ..., m_n)$, then we have a new action vector space $\Gamma(\mathbf{m}^*) = \{(x_1, \cdots, x_t, \cdots, x_n) \mid x_j \le m_j, x_j \in I_+$ for all $j \ne t$, and $x_t = 0, 1, 2, \cdots, m_t + 1\}$. We may extend $(\mathbf{m}, v)$ to $(\mathbf{m}^*, v^R)$ such that $v^R(\mathbf{x}) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and $v^R((\mathbf{x} \mid x_t = m_t + 1)) = v((\mathbf{x} \mid x_t = m_t))$, for all for all $\mathbf{x} \in \Gamma(\mathbf{m}^*)$. The solution ψ is said to be **redundant free** if and only if $\psi_{k,\ell}(v^R) = \psi_{k,\ell}(v)$ for all $\ell \in N$, and $k = 1, 2, ..., m_\ell$, and $\psi_{(m_t+1),t}(v^R) = \psi_{m_t,t}(v)$. Otherwise, the solution is say to be dependent on redundant action.

Note 1. Since Hsiao and Raghavan(1992) assumed that players have same number of actions, please note that the definition of **redundant free** in this article is quite different from the definition of **dummy free of action** in Hsiao and Raghavan(1992).

Theorem 1. Given a weight function w, the **H& R** value ϕ^w is redundant free.

Proof. Omitted.

Remark 4. In the above proof, without loss of generality, we may assume $m_t + 1 \le m = \max\{m_\ell | \ell = 1, 2, ..., n\}$. Therefore, if the H&R Shapley value is adopted as the solutions of $(\mathbf{m}, v)$, then by a few times of non-essential extensions, we may extend $(\mathbf{m}, v)$ to a game where all the players have the same number, say m, of choices, without objections from the players. The spirit of the law on equal job opportunities is automatically fulfilled.

We now consider **dummy free of player** properties. Following Hsiao and Raghavan(1992), we have the definition as follows.

Definition 8. Given $N = \{1, 2, \ldots, n\}$, $\mathbf{m} = (m_1, \ldots, m_n)$, and a multi-choice cooperative game $(\mathbf{m}, v)$, suppose $\psi_{i,j}(v) = a_{i,j}$ for feasible $i \in \boldsymbol{\beta}_j$. Now, allow a dummy player, say $(n+1)$ with $m_{n+1} > 0$ and $\boldsymbol{\beta}_{n+1} = \{0, 1, \ldots, m_{n+1}\}$ to join the game. Let $N^D = \{1, \ldots, n, n+1\}$ and $\mathbf{m}^D = (m_1, \ldots, m_n, m_{n+1})$, then we have a new game $(\mathbf{m}^D, v^D)$ such that $v^D((\mathbf{x} \mid x_{n+1} = i)) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and all $i \in \boldsymbol{\beta}_{n+1}$. $(\mathbf{m}^D, v^D)$ is called a dummy player extension of $(\mathbf{m}, v)$.

Suppose $\psi_{i,j}(v^D) = b_{i,j}$, for feasible $i \in \boldsymbol{\beta}_j$ and $j \in N^D$, we could ask whether $a_{i,j} = b_{i,j}$ for all $i \in \boldsymbol{\beta}_j$ and all $j \in N$. A solution of a multi-choice cooperative game is said to be **dummy free of players** if (i) $b_{i,n+1} = 0$ for all $i \in \boldsymbol{\beta}_{n+1}$ and (ii) $a_{i,j} = b_{i,j}$ for all $i \in \boldsymbol{\beta}_j$ and all $j \in N$; otherwise the solution is said to be dummy dependent of players. Hsiao and Raghavan(1992) showed that the H&R Shapley value is dummy free of players.

Definition 9. A multi-choice cooperative game $(\mathbf{m}, v)$ is called a non-essential game if

$$v(\mathbf{x}) = \sum_{j=1}^{n} v((\mathbf{0} \mid x_j)),$$

where $(\mathbf{0} \mid x_j)$ is an action tuple where player j takes action σ_{x_j} and all the other players take action σ_0.

Definition 10. Player j in the game $(\mathbf{m}, v)$ is called a non-essential player if

$$v(\mathbf{x}) = v((\mathbf{x} \mid x_j = 0)) + v((\mathbf{0} \mid x_j))$$

for all $\mathbf{x} \in \Gamma(\mathbf{m})$. Here $\mathbf{x}$ and $(\mathbf{x} \mid x_j = 0)$ are action tuples such that player j takes action σ_{x_j} in $\mathbf{x}$ and takes σ_0 in $(\mathbf{x} \mid x_j = 0)$, and all the other players take the same actions in both $\mathbf{x}$ and $(\mathbf{x} \mid x_j = 0)$.

Definition 11. Given a multi-choice cooperative game $(\mathbf{m}, v)$ with $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, allow a new player, say $(n+1)$, to join the game, then we have a new set of player $N^o = N \cup \{n+1\} = \{1, 2, \ldots, n, n+1\}$ and a new action space $\Gamma(\mathbf{m}^0)$ where $\mathbf{m}^0 = (m_1, m_2, m_n, m_{n+1})$.

Let $v^0((\mathbf{x} \mid x_{n+1} = 0)) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m}^0)$ and all $(\mathbf{x} \mid x_{n+1} = 0) = (x_1 \ldots x_m, 0) \in \Gamma(\mathbf{m}^0)$. Assign each $v^0((\mathbf{0} \mid x_{n+1} = k))$, $k \neq 0$, a value not necessarily zero. Then we can define a new game $(\mathbf{m}^0, v^0)$, such that $n+1$ is a non-essential player in $(\mathbf{m}^0, v^0)$. We call $(\mathbf{m}^0, v^0)$ a non-essential extension of $(\mathbf{m}, v)$. A solution ψ of $(\mathbf{m}, v)$ is said to be independent of non-essential players $\psi_{i,j}(v) = \psi_{i,j}(v^0)$, for all $i \in \boldsymbol{\beta}_j$ and $j \in N$. Otherwise, ψ is said to be dependent of players.

Remark 5. It is obvious that dummy extension is a special case of non-essential extension and dummy free of players is a special case of independent non-essential players.

4. The D&P Shapley value, the P&Z Shapley value and the WAC Value.

In this section we will show explictly that the D&P Shapley value, the P&Z Shapley value and the WAC Value are linear combinations of the H&R Shapley value, hence they have all the same dummy free properties and independent property as does the H&R Shapley value. Following Hsiao et al.(1994), we have the definitions and notations as follows. Given $j \in N$ and $v(\mathbf{x})$, we define

$$d_j v(\mathbf{x}) = v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_j),$$

then d_j is associative, i.e., $d_j(d_\ell v(\mathbf{x})) = d_\ell(d_j v(\mathbf{x}))$. For convenience, we denote $d_j d_\ell = d_{j,\ell}$, $d_{j_1,j_2,j_3} = d_{j_1} d_{j_2} d_{j_3}$, ..., etc. We also denote $d_{j_1,j_2,\ldots,j_\ell} = d_T$ whenever $\{j_1, j_2, ..., j_\ell\} = T$. Furthermore, for brevity, we denote $d_{S(\mathbf{x})}$by $d_{\mathbf{x}}$.

Note 2. It is easy to see that for each $\mathbf{y} \in \Gamma(\mathbf{m})$, we have

$$\sum_{T \subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r \in T} \mathbf{e}_r) = d_{\mathbf{y}} v(\mathbf{y})$$

Here, we copy a reformulation of the H&R Shapley value provided in the proof of Theorem 1 in Hsiao(1995), the first formula in page 428 in Hsiao(1995), as follows.

$$\begin{aligned}
\phi^w_{i,j}(\mathbf{m}, v) &= \sum_{k=1}^{i} w(k) \cdot [\sum_{\substack{y_j = k \\ \mathbf{y} \in \Gamma(\mathbf{m})}} \frac{1}{||\mathbf{y}||_w} \sum_{T \subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r \in T} \mathbf{e}_r)] \\
&= \sum_{k=1}^{i} w(k) \cdot [\sum_{\substack{y_j = k \\ \mathbf{y} \in \Gamma(\mathbf{m})}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}} v(\mathbf{y})] \qquad (4.1)
\end{aligned}$$

Although, in this article we allow the players to have different numbers of actions and in Hsiao(1995) we let the players to have the same number of actions, by Theorem 1 in this article and by Theorem 3 in Hsiao(1995), we have the following Theorem.

Theorem A. The H&R Shapley value $(\star)$=(4.1) is independent of non-essential player.

Definition 11. (Hwang and Liao(2008a)) Taking the following part of (4.1) as the value for player j with action σ_k, we get the so called weighted associated consistent value (**WAC value**)in Hwang and Liao(2008a). We denote the value as follows.

$$\psi^{HL}_{k,j}(\mathbf{m},v) = w(k)\cdot\Bigg[\sum_{\substack{y_j=k\\ \mathbf{y}\in\Gamma(\mathbf{m})}}\frac{1}{||\mathbf{y}||_w}\sum_{T\subseteq S(\mathbf{y})}(-1)^{|T|}v(\mathbf{y}-\sum_{r\in T}\mathbf{e}_r)\Bigg]. \tag{4.2}$$

We would like to remind readers that in this article, k denote action σ_k and j denote player j.

Corollary 1. The WAC value ψ^{HL} is both redundant free and independent of non-essential players. Since dummy free of players is a special case of independent of non-essential players, WAC value is also dummy free of players.

Proof. Omitted.

Given a multi-choice game $(\mathbf{m},v)$, Hwang and Liao(2008b) gave the D&P Shapley value and the P&Z Shapley value explicit formulas as the following.

$$\psi^{DP}_{i,j}(\mathbf{m},v) = \sum_{\substack{\mathbf{y}\in\Gamma(\mathbf{m})\\ y_j\geq i}}\frac{1}{||\mathbf{y}||}d_{\mathbf{y}}v(\mathbf{y}),\ \text{where}\ \|\mathbf{y}\| = \sum_{j\in N}y_j, \tag{4.3}$$

and the P&Z Shapley value is given by

$$\psi^{PZ}_{i,j}(\mathbf{m},v) = \sum_{\substack{\mathbf{y}\in\Gamma(\mathbf{m})\\ x_j=i}}\frac{1}{|S(\mathbf{y})|}d_{\mathbf{y}}v(\mathbf{y}), \tag{4.4}$$

where $|S(\mathbf{y})|$ is the number of players in $S(\mathbf{y})$.

Corollary 2. The D&P Shapley value ψ^{DP} is both redundant free and independent of non-essential players. Since dummy free of players is a special case of independent of non-essential players, the value is also dummy free of players.

Proof. Omitted.

Corollary 3. The P&Z Shapley value ψ^{PZ} is both redundant free and independent of non-essential players. Since dummy free of players is a special case of independent of non-essential players, the value is also dummy free of players.

Proof. Omitted. As we see in the above proofs, the explicit formulas play a central role in getting the insights of the values. We now check the N&P Shapley value by an elegant explicit formula provided by Calvo and Santos(2000).

5. The N&P Shapley value is not Redundant Free. As a matter of fact, in 1990 Raghavan presented Hsiao and Raghavan(1993) to the authors in Nouweland et al.(1995), right after Raghavan's presentation, they allowed the players to have different numbers of actions and proposed the **N&P Shapley value** defined on G which associates with each $(\mathbf{m}, v) \in G$ and each player $j \in N$ a value $\psi_j^{NP}(\mathbf{m}, v)$ called the N&P Shapley value.

In the 2nd(1991) and the 4th(1993) international conference on game theory at Stony-Brook, Hsiao asked "Is the N&P Shapley value *dummy free of actions* or *dummy free of players* ? " The authors were not able to answer the question. The possible reason was that they did not have the following explicit formula given by Calvo and Santos(2000). Here, we rewrite the N&P Shapley value formulated by Calvo and Santos(2000) as follows.

$$\psi_j^{NP}(\mathbf{m}, v) = \sum_{\substack{\mathbf{x} \leq \mathbf{m} \\ x_j \neq 0}} m_j \cdot \left(\frac{(|\mathbf{x}|-1)!(|\mathbf{m}|-|\mathbf{x}|)!}{|\mathbf{m}|!}\right) \cdot \left[\prod_{\substack{r \in S(\mathbf{m}) \\ r \neq j}} \binom{m_r}{x_r}\right] \cdot \binom{m_j - 1}{x_j - 1} \cdot [v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_j)], \tag{5.1}$$

where $|\mathbf{x}| = \sum_{j \in N} x_j$, $S(\mathbf{m}) = \{j | m_j \neq 0\}$ and the traditional mathematical notation $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

Here, we show that the N&P Shapley vale is not redundant free by an example in Nouweland et al.(1995) as follows.

Example 1. Let $N = \{1, 2\}$, $\mathbf{m} = (2, 1)$ and $(\mathbf{m}, v)$ be such that $v((0, 0)) = v((1, 0)) = v((2, 0)) = v((0, 1)) = 0$, $v((1, 1)) = 2$ and $v((2, 1)) = 3$. Then by formula (5.1), the N&P Shapley value $\psi^{NP}(\mathbf{m}, v) = (\psi_1^{NP}(\mathbf{m}, v), \psi_2^{NP}(\mathbf{m}, v)) = (\frac{4}{3}, \frac{5}{3})$.

Now, suppose $N = \{1, 2\}$, $\mathbf{m}^* = (2, 2)$, and $(\mathbf{m}*, v^R)$ be such that $v^R((0, 0)) = v^R((1, 0)) = v^R((2, 0)) = v^R((0, 1)) = v^R((0, 2)) = 0$, $v^R((1, 1)) = v^R((1, 2)) = 2$ and $v^R((2, 1)) = v^R((2, 2)) = 3$. Then by formula (5.1), the N&P Shapley value $\psi^{NP}(\mathbf{m}^*, v^R)$ $= (\psi_1^{NP}(\mathbf{m}^*, v^R), \psi_2^{NP}(\mathbf{m}^*, v^R)) = (\frac{11}{6}, \frac{7}{6})$.

Hence, the N&P Shapley value is **not redundant free**. However, the N&P Shapley value is, in some sense, independent of non-essential of players as follows. For notational convenience, we adopt the following notation in reliability theory.

Theorem 2. Given $N = \{1, \ldots, n\}$, $\mathbf{m} = (m_1, \ldots, m_n)$, a game $(\mathbf{m}, v)$ and its N&P Shapley value $\psi^{NP}(\mathbf{m}, v) = (\psi_1^{NP}(\mathbf{m}, v), \ldots, \psi_n^{NP}(\mathbf{m}, v))$, let$(\mathbf{m}^0, v^0)$ be a non-essential extension of $(\mathbf{m}, v)$ with $N^0 = \{1, \ldots, n, n+1\}$, $\mathbf{m}^0 = (m_1, \ldots, m_n, m_{n+1})$, then $\psi_j^{NP}(\mathbf{m}, v) = \psi_j^{NP}(\mathbf{m}^0, v^0)$, for $j = 1, \ldots, n$ and $\psi_{n+1}^{NP}(\mathbf{m}^0, v^0) = v^0((\mathbf{0}|x_{n+1} = m_{n+1}))$.

We need the following equality to prove Theorem 2.

Lemma 1. For all non-negative integers $k, x, m \in I_+$ with $x \leq m$ the following equality holds

$$\sum_{i=0}^{k} \frac{(x+i-1)![m+k-(x+i)]!}{(m+k)!} \binom{k}{i} = \frac{(x-1)!(m-x)!}{m!} \tag{5.2}$$

Proof. Omitted.

Proof of Theorem 2. Omitted.

Since the dummy extension is a special case of non-essential extension, we have the following Corollary.

Corollary 4. Given $N = \{1, \ldots, n\}$, $\mathbf{m} = (m_1, \ldots, m_n)$, a game $(\mathbf{m}, v)$ and its N&P Shapley value $\psi^{NP}(\mathbf{m}, v) = (\psi_1^{NP}(\mathbf{m}, v), \ldots, \psi_n^{NP}(\mathbf{m}, v))$, let $(\mathbf{m}^D, v^D)$ be a dummy player extension of $(\mathbf{m}, v)$ with $N^D = \{1, \ldots, n, n+1\}$, $\mathbf{m}^D = (m_1, \ldots, m_n, m_{n+1})$, then $\psi_j^{NP}(\mathbf{m}, v) = \psi_j^{NP}(\mathbf{m}^D, v^D)$, for $j = 1, \ldots, n$ and $\psi_{n+1}^{NP}(\mathbf{m}^D, v^D) = 0$.

Conclusion and Suggestion. The real world is full of discrimination, therefore, we need the law to amend the discrimination. Based on the spirit of the law on equal job opportunities, when modeling a multi-choice game and its solution, we have to focus on dummy free properties, independence of non-essential players and redundant free property.

If a consultant in the real world proposes a solution to his/or her clients (players) and the solution is not redundant free or independent of non-essential platers, then the solution will be very controversial or even against the law on equal job opportunities.

In the field of multi-choice games, there are still much more insight to be discovered by cognitive and meta-cognitive tools from the traditional mathematics. For example, if we regard the action vector $\mathbf{x}$ as a status of an international coalition to fight against an international pandemic and regard $v(\mathbf{x})$ as an uncertain number of lives that can be saved by the action vector $\mathbf{x}$, then we are studying a dynamic process of action vector (status) formation with uncertain payoff. The traditional control theory could be helpful in studying the disease control multi-choice cooperative game.

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